# Global existence for reaction-diffusion systems with nonlinear diffusion and control of mass 

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Received 7 January 2015; received in revised form 20 November 2015; accepted 1 March 2016
Available online 17 March 2016


#### Abstract

We prove here global existence in time of weak solutions for some reaction-diffusion systems with natural structure conditions on the nonlinear reactive terms which provide positivity of the solutions and uniform control of the total mass. The diffusion operators are nonlinear, in particular operators of the porous media type $u_{i} \mapsto-d_{i} \Delta u_{i}^{m_{i}}$. Global existence is proved under the assumption that the reactive terms are bounded in $L^{1}$. This extends previous similar results obtained in the semilinear case when the diffusion operators are linear of type $u_{i} \mapsto-d_{i} \Delta u_{i}$.


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MSC: 35K10; 35K40; 35K57
Keywords: Reaction-diffusion systems; Nonlinear diffusion; Porous media equation; Global existence

## 1. Introduction

The goal of this paper is the study of global existence in time of solutions to reaction-diffusion systems of the following type

$$
\left\{\begin{array}{lll}
\text { for all } i=1, \ldots, m, & &  \tag{1}\\
\partial_{t} u_{i}-\Delta \varphi_{i}\left(u_{i}\right) & =f_{i}\left(u_{1}, u_{2}, \cdots, u_{m}\right) & \\
\text { in }] 0,+\infty[\times \Omega \\
u_{i}(t, .) & =0, & \text { on }] 0,+\infty[\times \partial \Omega, \\
u_{i}(0, .) & =u_{i 0} \geq 0 & \text { in } \Omega .
\end{array}\right.
$$

Here $\Omega$ is a bounded open subset of $\mathbb{R}^{N}$ with a regular boundary, $\varphi_{i}, i=1, \ldots, m$ are continuous increasing functions from $[0,+\infty)$ into $[0,+\infty)$ with $\varphi_{i}(0)=0$ and the $f_{i}$ are regular functions such that the two following main properties occur:

[^0]- $(P)$ : the nonnegativity of the solutions is preserved for all time;
- $(M)$ : the total mass of the components is controlled for all time (sometimes even exactly preserved).

Properties $(P)$ and $(M)$ are natural in applications: these systems are mathematical models for evolution phenomena undergoing at the same time spatial diffusion and (bio-)chemical type of reactions. The unknown functions are generally densities, concentrations, temperature so that their nonnegativity is required. Moreover, often a control of the total mass, sometimes even preservation of the total mass, is naturally guaranteed by the model. Interest has increased recently for these models in particular for applications in biology, ecology, environment and population dynamics.

Mathematically speaking, $(P)$ is satisfied (like for systems of ordinary differential equations) if and only if $f=$ $\left(f_{i}\right)_{1 \leq i \leq m}$ is quasipositive whose meaning is recalled in (5).

Condition ( $M$ ) is satisfied if for instance

$$
\begin{equation*}
\sum_{1 \leq i \leq m} f_{i} \leq 0 \tag{2}
\end{equation*}
$$

or, more generally, if this sum is reasonably controlled (see (6) for precise assumption).
These two conditions imply $L^{1}(\Omega)$-bounds on the solutions which are uniform on each finite time interval (see Lemma 2.3):

$$
\forall i=1, \ldots, m, \forall T>0, \sup _{t \in[0, T]}\left\|u_{i}(t)\right\|_{L^{1}(\Omega)}<+\infty
$$

Unlike uniform $L^{\infty}(\Omega)$-estimates on each finite interval, such $L^{1}$-estimates are not enough to imply existence of global solution on $(0,+\infty)$. More structure is needed for global existence.

Actually, many results of global existence are known for these systems in the semilinear case when the diffusions are linear and given for instance by $\varphi_{i}\left(u_{i}\right)=d_{i} u_{i}, d_{i} \in(0,+\infty)$. Existence of regular bounded solutions on $(0,+\infty)$ may be found for example in $[25,16,27,26,19,18,36,10,9,22,39,14,17,3,4]$ and in several other articles whose references may be found in the survey [31] or in the book [37]. However, it is well-known that the solutions may blow up in $L^{\infty}(\Omega)$ in finite time as proved in $[33,34]$ where explicit finite time blow up in $L^{\infty}(\Omega)$ are given. Thus, even in the semilinear case, it is necessary to deal with weak solutions if one expects global existence.

Our main goal here is to exploit the good " $L^{1}$-framework" provided by the two conditions $(P),(M)$ and to see how the main results of global existence of weak solutions extend from the semilinear case to the case when the $\varphi_{i}$ are nonlinear, in particular of the porous media type, namely $\varphi_{i}\left(u_{i}\right)=d_{i} u_{i}^{m_{i}}, m_{i} \geq 1$. In this case, degeneracy of the diffusion occurs at the same time for small $u_{i}$ and for large $u_{i}$.

We are interested in looking for extensions to these nonlinear diffusions of the two following main results proved in the semilinear case:

- first the global existence result of weak solutions for (1) when $(P),(M)$ hold and when moreover an a priori $L^{1}$-estimate holds for the nonlinear reactive part, namely (see [29,30] and the survey [31]).

$$
\begin{equation*}
\forall i=1, \ldots, m, \forall T>0, \int_{(0, T) \times \Omega}\left|f_{i}\left(u_{1}, \ldots, u_{m}\right)\right|<+\infty \tag{3}
\end{equation*}
$$

- next the fact that global existence of weak solutions hold for quadratic nonlinearities $f_{i}$ satisfying only $(P),(M)$. This is a consequence of the latter result and of a main a priori $L^{2}((0, T) \times \Omega)$-estimate on the solutions implied by $(P)+(M)$ and which is interesting for itself. This estimate was noticed in [12,28,33,34] and then widely exploited, see for example [11,6,7,35,32,8,5,9,31].

We will first see that these two results extend to the case when the $\varphi_{i}$ are nonlinear but nondegenerate (that is when $\varphi_{i}^{\prime}$ is bounded from below and from above, see Proposition 2.4). But, the situation is more complicated and not so clear in the degenerate case $\varphi_{i}\left(u_{i}\right)=d_{i}\left(u_{i}\right)^{m_{i}}$. More precisely:

1. We are able to prove global existence of solutions under the a priori estimate (3) if $m_{i} \in\left(\frac{(N-2)^{+}}{N}, 2\right)$ for all $i$. We do not know whether the restriction $m_{i}<2$ is only technical or due to deeper phenomena. But, at least, it appears
as being necessary to extend the approach of the semilinear case as such. This is explained in more details next (see Theorem 2.6, Corollary 2.11 and their proofs).
2. On the other hand, we can prove that the a priori $L^{2}$-estimate of the semilinear case has a natural extension to the degenerate case, this for any $m_{i} \geq 1$. Indeed, under the only assumptions $(P),(M)$, the solutions $u_{i}$ are a priori bounded in $L^{m_{i}+1}((0, T) \times \Omega)$ for all $T>0$. This allows in particular to prove global existence for System (1) with quadratic reactive terms or with growth less than $m_{i}+1$ and some other classical reactive terms (see Theorem 2.7, Corollary 2.8 and Corollary 2.9).

A main reason to try to exploit the " $L^{1}$-framework" provided by $(P),(M)$ for System (1) is that, like in the semilinear case, the operator $u_{i} \rightarrow \partial_{t} u_{i}-d_{i} \Delta u_{i}^{m_{i}}$ has good $L^{1}$-compactness properties in the sense that the following mapping is compact when $m>(N-2)^{+} / N$ :

$$
\left(w_{0}, F\right) \in L^{1}(\Omega) \times L^{1}((0, T) \times \Omega) \mapsto w \in L^{1}((0, T) \times \Omega),
$$

where $w$ is the solution of

$$
\partial_{t} w-d \Delta w^{m}=F \text { in }(0, T) \times \Omega, w=0 \text { on }(0, T) \times \partial \Omega, w(0)=w_{0} .
$$

This provides compactness for the solutions of the adequate approximations of System (1). Next, the main difficultywhich is actually serious-is to show that the limit of these approximate solutions is indeed solution of the limit system.

Note that, besides the semilinear case, this kind of $L^{1}$-approach was also used with success in [20] for such systems with nonlinear diffusions of the $p$-Laplacian type $\partial_{t} u_{i}-\nabla \cdot\left(\left|\nabla u_{i}\right|^{p-2} \nabla u_{i}\right)$. Let us also mention some global existence and finite time blow up in [15,27] and [21] for $2 \times 2$ systems with nonlinear diffusion $\varphi_{i}\left(u_{i}\right)=u_{i}^{m_{i}}, i=1,2$ and with growth conditions on the reactive terms like

$$
f_{1}\left(u_{1}, u_{2}\right)=u_{1}^{\alpha}+u_{2}^{\beta}+C_{1}, f_{2}\left(u_{1}, u_{2}\right)=u_{1}^{\delta}+u_{2}^{\gamma}, 1 \leq \alpha, \delta \leq m_{1}, 1 \leq \beta, \gamma \leq m_{2} .
$$

A particular example of System (1) with

$$
m=2, \varphi_{1}\left(u_{1}\right)=u_{1}^{m_{1}}, \varphi_{2}\left(u_{2}\right)=d_{2} u_{2}, f_{1} \leq 0, f_{2}=-f_{1}
$$

was also shown in [20] to have weak solutions for $m_{1} \in[1,2)$ and initial data $\left(u_{10}, u_{20}\right) \in L^{m_{1}+1}(\Omega) \times L^{2}(\Omega)$ and as well strong bounded global solutions for bounded initial data and polynomial growth of $f_{1}$ (even for general $\varphi_{1}$ in this case). Nondegenerate nonlinear diffusions were also considered in [12] and [32] with quadratic reactive terms.

## 2. Main results

Throughout this paper, we denote $Q:=(0,+\infty) \times \Omega, Q_{T}:=(0, T) \times \Omega, \Sigma:=(0,+\infty) \times \partial \Omega, \Sigma_{T}:=(0, T) \times \partial \Omega$ and, for $p \in[1,+\infty)$

$$
\begin{aligned}
& \|u(t)\|_{L^{p}(\Omega)}=\left(\int_{\Omega}|u(t, x)|^{p} d x\right)^{1 / p}, \quad\|u\|_{L^{p}\left(Q_{T}\right)}=\left(\int_{0}^{T} \int_{\Omega}|u(t, x)|^{p} d t d x\right)^{1 / p}, \\
& \|u(t)\|_{L^{\infty}(\Omega)}=\operatorname{ess} \sup _{x \in \Omega}|u(t, x)|, \quad\|u\|_{L^{\infty}\left(Q_{T}\right)}=\operatorname{ess} \sup _{(t, x) \in Q_{T}}|u(t, x)|
\end{aligned}
$$

For $i=1, \ldots, m$, let $f_{i}: Q \times[0,+\infty)^{m} \rightarrow \mathbb{R}$ be such that

$$
\text { Regularity: }\left\{\begin{array}{l}
f_{i} \text { is measurable, }  \tag{4}\\
\forall T>0, f(., ., 0), g(., ., 0) \in L^{1}\left(Q_{T}\right), \\
\exists K:[0,+\infty) \rightarrow[0,+\infty) \text { nondecreasing such that: } \\
\left|f_{i}(t, x, r)-f_{i}(t, x, \hat{r})\right| \leq K(M)\|r-\hat{r}\|, \\
\text { for all } M>0, \text { for all } r, \hat{r} \in(0, M)^{m} \text { and a.e. }(t, x) \in Q,
\end{array}\right.
$$

where $\|r\|=\sum_{1 \leq i \leq m}\left|r_{i}\right|$ is the norm chosen in $\mathbb{R}^{m}$.

Quasipositivity: ( $P$ ) $\left\{\begin{array}{l}f_{i}\left(t, x, r_{1}, \ldots, r_{i-1}, 0, r_{i+1}, \ldots, r_{m}\right) \geq 0, \\ \text { for all } r=\left(r_{i}\right)_{1 \leq i \leq m} \geq 0, \text { a.e. }(t, x) \in Q .\end{array}\right.$
Control of mass: (M) $\left\{\begin{array}{l}\forall r \in[0,+\infty)^{m}, \text { for a.e. }(t, x), \sum_{i} f_{i}(t, x, r) \leq \sigma\|r\|+h(t, x) \\ \text { for some } \sigma \in[0,+\infty), h \in L^{1}\left(Q_{T}\right)^{+} \text {for all } T>0 .\end{array}\right.$
...These three above properties will be assumed throughout the paper...
Remark 2.1. Note that all results given in this paper immediately extend if $(M)$ is replaced by the existence of $\alpha_{i} \in(0,+\infty)$ such that

$$
\forall r \in[0,+\infty)^{m} \text {, a.e. }(t, x), \sum_{i} \alpha_{i} f_{i}(t, x, r) \leq \sigma\|r\|+h(t, x) .
$$

Indeed we may multiply each $i$-th equation by $\alpha_{i}$ and changing $u_{i}$ into $v_{i}:=\alpha_{i} u_{i}$. For simplicity, and without loss of generality, we will work here with $(M)$ as above.

For $i=1, \ldots, m$, let $\varphi_{i}:[0,+\infty) \rightarrow[0,+\infty)$ be increasing, continuously differentiable on $(0,+\infty)$ with $\varphi_{i}(0)=0$. We will mainly consider two situations:

The nondegenerate case:

$$
\begin{equation*}
\exists a_{i}, b_{i} \in(0,+\infty), \forall s \in(0,+\infty), 0<a_{i} \leq \varphi_{i}^{\prime}(s) \leq b_{i}<+\infty . \tag{7}
\end{equation*}
$$

The possibly degenerate case:

$$
\begin{equation*}
\forall s \in[0,+\infty), \varphi_{i}(s)=d_{i} s^{m_{i}}, m_{i} \in(0,+\infty), d_{i} \in(0,+\infty) \tag{8}
\end{equation*}
$$

We consider the associated System (1) where the weak solution of each equation is understood in the sense of nonlinear semigroups in $L^{1}(\Omega)$ (see [40] for various definitions of solutions). More precisely, if $\varphi$ denotes one of the $\varphi_{i}$ and if $\left(w_{0}, F\right) \in L^{\infty}(\Omega) \times L^{\infty}\left(Q_{T}\right)$, we will use, especially in the approximation processes, the following notion of bounded solutions:

$$
\left\{\begin{array}{l}
w \in \mathcal{C}\left([0, T) ; L^{1}(\Omega)\right) \cap L^{\infty}\left(Q_{T}\right), \varphi(w) \in L^{2}\left(0, T ; H_{0}^{1}(\Omega)\right),  \tag{9}\\
\partial_{t} w-\Delta \varphi(w)=F \text { in the sense of distributions in } Q_{T}, \\
w(0)=w_{0}
\end{array}\right.
$$

If $\varphi$ satisfies one of the conditions (7) or (8), then for $\left(w_{0}, F\right)$ given in $L^{\infty}(\Omega) \times L^{\infty}\left(Q_{T}\right)$, such a solution exists and is unique (see e.g. [40, Chapters 5 and 6]). Moreover, if $\hat{w}$ is the solution associated with $\left(\hat{w}_{0}, \widehat{F}\right) \in L^{\infty}(\Omega) \times L^{\infty}\left(Q_{T}\right)$, we have

$$
\begin{equation*}
\|w(t)-\hat{w}(t)\|_{L^{1}(\Omega)} \leq\left\|w_{0}-\hat{w}_{0}\right\|_{L^{1}(\Omega)}+\int_{0}^{t}\|F(s)-\widehat{F}(s)\|_{L^{1}(\Omega)} d s \tag{10}
\end{equation*}
$$

so that

$$
\left(w_{0}, F\right) \in L^{1}(\Omega) \times L^{1}\left(Q_{T}\right) \mapsto w \in \mathcal{C}\left([0, T] ; L^{1}(\Omega)\right)
$$

is a contraction. This allows to extend by density, in a unique way, the notion of solution to any $\left(F, w_{0}\right) \in L^{1}\left(Q_{T}\right) \times$ $L^{1}(\Omega)$ and we will denote it by

$$
\begin{equation*}
w:=S_{\varphi}\left(w_{0}, F\right) . \tag{11}
\end{equation*}
$$

This is the notion of solution that will mainly be used in this paper. Note that it satisfies

$$
\left\{\begin{array}{l}
w \in \mathcal{C}\left([0, T) ; L^{1}(\Omega)\right), \varphi(w) \in L^{1}\left(Q_{T}\right) \text { and } \forall \psi \in \mathcal{C}_{T},  \tag{12}\\
-\int_{\Omega} \psi(0) w_{0}-\int_{Q_{T}} \partial_{t} \psi w+\varphi(w) \Delta \psi=\int_{Q_{T}} \psi F,
\end{array}\right.
$$

where

$$
\begin{equation*}
\mathcal{C}_{T}=\left\{\psi:[0, T] \times \bar{\Omega} \rightarrow \mathbb{R} ; \psi, \partial_{t} \psi, \partial_{x_{i} x_{j}}^{2} \psi \text { are continuous, } \psi=0 \text { on } \Sigma_{T}, \psi(T)=0\right\} . \tag{13}
\end{equation*}
$$

The latter property (12) corresponds to the notion of very weak solution in Definition 6.2 of [40].

Solutions in the sense of (11) satisfy the maximum principle and order properties:

$$
\left[w_{0} \geq 0, F \geq 0\right] \Rightarrow\left[S_{\varphi}\left(w_{0}, F\right) \geq 0\right],\left[w_{1} \geq w_{2}, F_{1} \geq F_{2}\right] \Rightarrow\left[S_{\varphi}\left(w_{1}, F_{1}\right) \geq S_{\varphi}\left(w_{2}, F_{2}\right)\right]
$$

Recall also that $w:=S_{\varphi}\left(w_{0}, F\right)$ satisfies (see e.g. [40])

$$
\begin{equation*}
\forall p \in[1,+\infty], \forall t \in[0, T],\|w(t)\|_{L^{p}(\Omega)} \leq\left\|w_{0}\right\|_{L^{p}(\Omega)}+\int_{0}^{t}\|F(s)\|_{L^{p}(\Omega)} d s . \tag{14}
\end{equation*}
$$

We now define what we mean by a solution to our System (1).
Definition 2.2. Given $u_{i 0} \in L^{1}(\Omega), u_{i 0} \geq 0, i=1, \ldots, m$, by global weak solution to System (1), we mean $u=$ $\left(u_{1}, u_{2}, \ldots, u_{m}\right):(0,+\infty) \times \Omega \rightarrow[0,+\infty)^{m}$ such that, for all $i=1, \ldots, m$ and for all $T>0$

$$
\left\{\begin{array}{l}
u_{i} \in \mathcal{C}\left([0,+\infty) ; L^{1}(\Omega)\right), \varphi_{i}\left(u_{i}\right) \in L^{1}\left(0, T ; W_{0}^{1,1}(\Omega)\right)  \tag{15}\\
u_{i}=S_{\varphi_{i}}\left(u_{0 i}, f_{i}(u)\right)
\end{array}\right.
$$

Note that we only deal with nonnegative solutions.
The approximate reaction-diffusion system Next we consider the following approximation of System (1) with solution $u^{n}:=\left(u_{1}^{n}, \ldots, u_{m}^{n}\right)$ in the sense of (9) for each equation, that is

$$
\left\{\begin{array}{l}
\text { for all } i=1, \ldots, m,  \tag{16}\\
\text { for all } T>0, u_{i}^{n} \in L^{\infty}\left(Q_{T}\right) u_{i} \geq 0, \varphi_{i}\left(u_{i}^{n}\right) \in L^{2}\left(0, T ; H_{0}^{1}(\Omega)\right), \\
\partial_{t} u_{i}^{n}-\Delta \varphi_{i}\left(u_{i}^{n}\right)=f_{i}^{n}\left(u^{n}\right) \text { in } Q, \\
u_{i}^{n}(t, .)=0 \text { on } \Sigma, \\
u_{i}^{n}(0, .)=u_{i 0}^{n} \geq 0 \text { in } \Omega,
\end{array}\right.
$$

where $u_{i 0}^{n} \in L^{\infty}(\Omega)^{+}$converges to $u_{i 0}$ in $L^{1}(\Omega)$ and the approximate nonlinearities $f_{i}^{n}$ satisfy (4) with $K(\cdot)$ independent of $n$, (5), (6) with $\sigma, h$ independent of $n$ and are in $L^{\infty}\left(Q_{T} \times \mathbb{R}^{m}\right)$ for each $n$. The convergence of $f_{i}^{n}$ toward $f_{i}$ is defined as follows. Let us denote

$$
\begin{equation*}
\epsilon_{M}^{n}:=\max _{1 \leq i \leq m} \sup _{0 \leq\|r\| \leq M}\left|f_{i}^{n}(t, x, r)-f_{i}(t, x, r)\right| . \tag{17}
\end{equation*}
$$

We will assume that

$$
\begin{equation*}
\epsilon_{M}^{n} \rightarrow 0 \text { in } L^{1}\left(Q_{T}\right) \text { and a.e. as } n \rightarrow+\infty . \tag{18}
\end{equation*}
$$

As a typical example, we may choose

$$
\begin{equation*}
f_{i}^{n}:=\frac{f_{i}}{1+\frac{1}{n} \sum_{1 \leq j \leq m}\left|f_{j}\right|} \tag{19}
\end{equation*}
$$

Note that, with this choice, $\left\|f_{i}^{n}\right\|_{L^{\infty}(Q)} \leq n$ and the other properties may easily been checked (in (4), $K(M)$ has to be replaced by $(2+m) K(M))$.

Lemma 2.3. Assume that, for $1 \leq i \leq m, \varphi_{i}$ satisfies either (7) or (8) with $m_{i}>0$. Then the approximate system (16) has a (global and regular) solution $u^{n}$ and there exists $C:[0,+\infty) \rightarrow[0,+\infty)$, independent of $n$ such that

$$
\forall n, \forall T>0, \sup _{t \in[0, T]} \sum_{1 \leq i \leq m}\left\|u_{i}^{n}(t)\right\|_{L^{1}(\Omega)} \leq C(T)\left[1+\sum_{1 \leq i \leq m}\left\|u_{i 0}\right\|_{L^{1}(\Omega)}\right] .
$$

Now, let us assume, like in the semilinear case that, for whatever reason, an a priori $L^{1}$-estimate holds for the solution $u^{n}$ of the approximate System (16), namely

$$
\begin{equation*}
\forall T>0, \sup _{n} \sum_{1 \leq i \leq m}\left\|f_{i}^{n}\left(u^{n}\right)\right\|_{L^{1}\left(Q_{T}\right)}<+\infty \tag{20}
\end{equation*}
$$

Examples of such situations and applications will be given later (see also the survey [31]). The question is to decide whether, like in the semilinear case, $u^{n}$ converges to a global weak solution of (1).

A first-not surprising-result is that, when the nonlinearities $\varphi_{i}$ are nondegenerate, then this convergence property does hold. Moreover, the a priori $L^{2}$-estimate holds as well. Indeed we have the following proposition (and this is a particular case of Theorem 2.6 and Theorem 2.7 below):

Proposition 2.4. Assume all functions $\varphi_{i}$ are nondegenerate in the sense of (7). Then, up to a subsequence, $u^{n}$ converges in $\left[L^{1}\left(Q_{T}\right)\right]^{m}$ for all $T>0$ to a global weak solution of (1) in the sense of Definition 2.2. If moreover $u_{0} \in L^{2}(\Omega)^{m}$ and $h \in L_{\text {loc }}^{1}\left([0,+\infty) ; L^{2}(\Omega)\right)$ in the assumption (6), then there exists $C:[0,+\infty) \rightarrow[0,+\infty)$ such that

$$
\sup _{n} \sum_{1 \leq i \leq n}\left\|u_{i}^{n}\right\|_{L^{2}\left(Q_{T}\right)} \leq C(T)\left[1+\sum_{1 \leq i \leq m}\left\|u_{i 0}\right\|_{L^{2}(\Omega)}\right] .
$$

Remark 2.5. Versions of the above $L^{2}$-estimate may also be found in $[12,28,32]$ where they were used to prove global existence results for systems of type (1) with nondegenerate $\varphi_{i}$ and quadratic reactive terms. Global existence with general right-hand side bounded in $L^{1}$ seems however to be new.

Now the question is to decide what happens in the degenerate case. We can prove the following.
Theorem 2.6. Assume that, for $1 \leq i \leq m, \varphi_{i}$ satisfies either (7) or (8) with $m_{i} \in\left((N-2)^{+} / N, 2\right)$. Assume $L^{1}$-estimate (20) holds. Then, up to a subsequence, $u^{n}$ converges in $\left[L^{1}\left(Q_{T}\right)\right]^{m}$ for all $T>0$ to a global weak solution of (1) in the sense of Definition 2.2.

As commented in the introduction, we do not know whether the restriction $m_{i}<2$ is necessary or not. We will explain where it naturally appears in the proof and suggest some possible reasons. We will deduce a global existence result for System (1) below in Corollary 2.11.

On the other hand, it turns out that the a priori $L^{2}$-estimate does have a natural extension no matter the value of the $m_{i}$.

Theorem 2.7. Assume that, for $1 \leq i \leq m, \varphi_{i}$ satisfies either (7) or (8) with $m_{i}>0$. If moreover $h \in L_{\text {loc }}^{1}([0,+\infty)$; $\left.L^{2}(\Omega)\right)$ in the assumption (6), then there exists $C:[0,+\infty) \rightarrow[0,+\infty)$ such that

$$
\sup _{n} \sum_{1 \leq i \leq m}\left\|u_{i}^{n}\right\|_{L^{m_{i}+1}\left(Q_{T}\right)} \leq C(T)\left[1+\sum_{1 \leq i \leq m}\left\|u_{i 0}^{n}\right\|_{L^{2}(\Omega)}\right],
$$

where we set $m_{i}:=1$ in case (7).
We deduce the following global existence result.
Corollary 2.8. Assume that, for $1 \leq i \leq m, \varphi_{i}$ satisfies either (7) or (8) with $m_{i} \geq 1$. Assume there exists $\epsilon>0$ such that

$$
\begin{equation*}
\sum_{1 \leq i \leq m}\left|f_{i}(u)\right| \leq C\left[1+\sum_{1 \leq i \leq m} u_{i}^{m_{i}+1-\epsilon}\right] . \tag{21}
\end{equation*}
$$

Then, for all $u_{0} \in L^{2}(\Omega)^{m}, u_{0} \geq 0$, the system (1) has a global weak solution in the sense of Definition 2.2.
As we will see in the proof, the main point of the " $-\epsilon$ " in the above assumption is that it makes the nonlinearities $f_{i}^{n}\left(u^{n}\right)$ not only bounded in $L^{1}\left(Q_{T}\right)$, but uniformly integrable. This is the main tool to pass to the limit in the reactive terms. Actually, any other assumption guaranteeing this uniform integrability of the $f_{i}^{n}\left(u^{n}\right)$ will lead to global existence. For instance, it follows from this theorem that global existence holds for the typical system modeling the chemical reaction

$$
U_{1}+U_{3} \rightleftharpoons U_{2}+U_{4}
$$

Indeed, applying the mass action law for the reactive terms and a Darcy's law for the diffusion lead to the following $4 \times 4$ system for the concentrations $u_{i}=u_{i}(t, x)$ of the components $U_{i}, 1 \leq i \leq 4$ :

$$
\left\{\begin{array}{l}
\text { for } 1 \leq i \leq 4  \tag{22}\\
\partial_{t} u_{i}-d_{i} \Delta \varphi_{i}\left(u_{i}\right)=(-1)^{i}\left[u_{1} u_{3}-u_{2} u_{4}\right] \text { in } Q \\
u_{i}=0 \text { on } \Sigma, \quad u_{i}(0)=u_{i 0} \geq 0
\end{array}\right.
$$

The following result is a direct consequence of Corollary 2.8 when $m_{i}>1$ for all $1 \leq i \leq 4$. If some of the $m_{i}$ are equal to 1 , then an extra argument is needed to prove that the reactive terms are not only bounded in $L^{1}$, but uniformly integrable. This is coming from the entropy inequality and from an $L^{1}$-estimate that it provides on $u_{i}^{m_{i}+1}\left(\log u_{i}\right)^{2}$ (extending the $L^{2}$-techniques of [11]).

Corollary 2.9. Assume that, for $1 \leq i \leq 4$, $\varphi_{i}$ satisfies either (7) or (8) with $m_{i} \geq 1$. Then, for all $u_{0}=\left(u_{i 0}\right)_{1 \leq i \leq 4}$ with $u_{i 0} \geq 0$ and $u_{i 0} \log u_{i 0} \in L^{2}(\Omega)$ for $1 \leq i \leq 4$, System (22) has a global weak solution in the sense of Definition 2.2.

Remark 2.10. We may also consider more general reversible chemical reactions of the form

$$
p_{1} U_{1}+p_{2} U_{2}+\ldots+p_{m} U_{m} \rightleftharpoons q_{1} U_{1}+q_{2} U_{2}+\ldots+q_{m} U_{m}
$$

where $p_{i}, q_{i}$ are nonnegative integers. According to the usual mass action kinetics and with Darcy's laws for the diffusion, the evolution of the concentrations $u_{i}$ of $U_{i}$ may be modeled by the following system

$$
\begin{equation*}
\partial_{t} u_{i}-d_{i} \Delta u_{i}^{m_{i}}=\left(p_{i}-q_{i}\right)\left(k_{2} \Pi_{j=1}^{m} u_{j}^{q_{j}}-k_{1} \Pi_{j=1}^{m} u_{j}^{p_{j}}\right), i=1 \ldots m \tag{23}
\end{equation*}
$$

where $k_{1}, k_{2}$ are positive diffusion coefficients and where a stoichiometric law holds like $\sum_{i} \alpha_{i} p_{i}=\sum_{i} \alpha_{i} q_{i}$ for some $\alpha_{i} \in(0,+\infty)$. Similarly to Corollary 2.9, global existence of weak solutions may be proven when

$$
\begin{equation*}
\sum_{i} \frac{p_{i}}{m_{i}+1} \leq 1, \quad \sum_{i} \frac{q_{i}}{m_{i}+1} \leq 1 \tag{24}
\end{equation*}
$$

Indeed, together with Theorem 2.7, this guarantees that the reactive terms are bounded in $L^{1}\left(Q_{T}\right)$. The entropy inequality, which is valid for this system as well as for (22), allows to prove their uniform integrability in all cases when (24) holds. This approach is the same as for Corollary 2.9 and relies also on an $L^{1}$-estimate on $u_{i}^{m_{i}+1}\left(\log u_{i}\right)^{2}$. For completeness, we also give the main steps of the proof of this remark after the proof of Corollary 2.9.

It is known that, for instance for a $2 \times 2$ system, an a priori $L^{1}$ bound of type (20) holds as soon as two linear relations between $f_{1}, f_{2}$ hold rather than only one, like

$$
f_{1}+f_{2} \leq 0, f_{1}+\lambda f_{2} \leq 0, \lambda \in[0,1)
$$

More generally, if there are $m$ linearly independent similar inequalities in an $m \times m$ system, then estimate (20) holds. Actually, by coupling Theorem 2.6 and Theorem 2.7, we may even prove the following.

Corollary 2.11. Assume that for all $1 \leq i \leq m, \varphi_{i}$ satisfies either (7) or (8) with $m_{i} \in\left((N-2)^{+} / N, 2\right)$. Assume moreover that there exists an invertible $m \times m$ matrix $P$ with nonnegative entries and $\mathbf{b} \in \mathbb{R}^{m}$ such that

$$
\begin{equation*}
\forall r \in[0,+\infty)^{m}, P f(r) \leq \mathbf{b}\left[1+\sum_{i} r_{i}^{1+m_{i}}\right] \tag{25}
\end{equation*}
$$

where again $m_{i}:=1$ in case (7). Then, for all $u_{0} \in L^{1}(\Omega)^{m}, u_{0} \geq 0$, the System (1) has a global weak solution in the sense of Definition 2.2.

Remark 2.12. We emphasize the fact that any $L^{1}(\Omega)$-initial data is allowed in this result. As particular standard situations covered by Corollary 2.11, we have the $2 \times 2$ systems where the nonlinearities are as in the two following examples:

1) $f_{1} \geq 0, f_{2}=-f_{1}$.
2) $f_{1}\left(u_{1}, u_{2}\right)=\lambda u_{1}^{p} u_{2}^{q}-u_{1}^{\alpha} u_{2}^{\beta}, f_{2}\left(u_{1}, u_{2}\right)=-u_{1}^{p} u_{2}^{q}+u_{1}^{\alpha} u_{2}^{\beta}, \lambda \in[0,1), p, q, \alpha, \beta \geq 1$.

Indeed, (25) is satisfied with $\mathbf{b}=(0,0)$ and successively

$$
P=\left(\begin{array}{ll}
1 & 1 \\
1 & 0
\end{array}\right), P=\left(\begin{array}{ll}
1 & 1 \\
1 & \lambda
\end{array}\right) .
$$

Note that in these two examples, there is no restriction on the growth of $f_{1}, f_{2}$, but as stated in Corollary 2.11, it is required that $m_{i}<2$. On the other hand, when applying Corollary 2.8 to this system (see also Remark 2.10), we obtain global existence of weak solution, no matter the values of the $m_{1}, m_{2}$, but with the growth conditions

$$
\frac{p}{m_{1}+1}+\frac{q}{m_{2}+1}<1, \frac{\alpha}{m_{1}+1}+\frac{\beta}{m_{2}+1}<1 .
$$

The case $m_{1}=m_{2}=3, p=\beta=5, q=\alpha=2$ is for instance not covered (except may be in small space dimensions) by any of the above results although the reactive terms are a priori bounded in $L^{1}\left(Q_{T}\right)$ (see the proof of Corollary 2.11) and even if $\lambda=0$. This is an interesting open problem.

## 3. The proofs

Proof of Lemma 2.3. Since the $f_{i}^{n}$ are bounded for each $n$, existence of a (unique) bounded global solution $u^{n}$ is classical. Let us recall a procedure without too many details. Given $T \in(0,+\infty)$, we consider the set

$$
\mathcal{W}:=\left\{v \in \mathcal{C}\left([0, T] ; L^{1}(\Omega)^{m}\right) ; \forall i=1, \ldots, m, v_{i}(0)=u_{i 0}^{n},\left\|v_{i}\right\|_{L^{\infty}\left(Q_{T}\right)} \leq R\right\}
$$

where $R=\left\|u_{i 0}^{n}\right\|_{L^{\infty}(\Omega)}+n T$ (recall that $f_{i}^{n}$ is uniformly bounded by $n$ ). We equip $\mathcal{W}$ with the norm : $\|v\|:=$ $\max _{i} \sup _{t \in[0, T]}\left\|v_{i}(t)\right\|_{L^{1}(\Omega)}$. Then, we consider the mapping $\mathcal{F}$ which to $v^{n}=\left(v_{1}^{n}, \cdots, v_{m}^{n}\right) \in \mathcal{W}$ associates the solution $u^{n}=\left(u_{1}^{n}, \cdots, u_{m}^{n}\right) \in \mathcal{W}, u_{i}^{n}=S_{\varphi_{i}}\left(u_{i 0}^{n}, f_{i}^{n}\left(\pi\left(v^{n}\right)\right)\right)$ where $\pi: \mathbb{R}^{m} \rightarrow[0,+\infty)^{m}$ is the projection onto the positive cone, that is $\pi\left(r_{1}, \ldots, r_{m}\right)=\left(r_{1}^{+}, \ldots, r_{m}^{+}\right)$. Using the estimates (10)-(14), it is easy to prove that $\mathcal{F}$ send $\mathcal{W}$ into itself and that some iterate of $\mathcal{F}$ is a strict contraction. Whence the existence of a fixed point $u^{n}$. The $L^{2}\left(0, T ; H_{0}^{1}(\Omega)\right.$ )-regularity holds by construction for these bounded solutions (see [40]). Next, multiplying each equation by $\left(u_{i}^{n}\right)^{-}=-\inf \left\{u_{i}^{n}, 0\right\}$, integrating on $Q_{T}$ and summing over $i$, thanks to the quasipositivity of $f^{n}$ we deduce that $\left(u_{i}^{n}\right)^{-} \equiv 0$, whence the nonnegativity of $u^{n}$. We refer e.g. to [20] for more details.

Next, summing all the $m$ equations of (16) and integrating on $\Omega$ gives, using ( $M$ ):

$$
\partial_{t} \int_{\Omega} \sum_{1 \leq i \leq m} u_{i}^{n}(t) \leq \int_{\Omega} \sum_{1 \leq i \leq m} f_{i}^{n}\left(u^{n}\right) \leq \sigma \sum_{1 \leq i \leq m}\left\|u_{i}^{n}(t)\right\|_{L^{1}(\Omega)}+h=\sigma \int_{\Omega} \sum_{1 \leq i \leq m} u_{i}^{n}(t)+h .
$$

Integrating this Gronwall's inequality gives for all $t \in[0, T]$

$$
\sum_{1 \leq i \leq m}\left\|u_{i}^{n}(t)\right\|_{L^{1}(\Omega)} \leq e^{\sigma T}\left[\sum_{1 \leq i \leq m}\left\|u_{i 0}\right\|_{L^{1}(\Omega)}+\|h\|_{L^{1}\left(Q_{T}\right)}\right]
$$

Whence the estimate of Lemma 2.3.
Let us now recall the main compactness properties of the solutions of (11). Here $\varphi:[0,+\infty) \rightarrow[0,+\infty)$ denotes one of the functions $\varphi_{i}$.

Lemma 3.1. Assume $\varphi$ satisfies (7) or (8) with $m_{i}>\frac{(N-2)^{+}}{N}$. Then the mapping

$$
\left(w_{0}, F\right) \in L^{1}(\Omega) \times L^{1}\left(Q_{T}\right) \mapsto S_{\varphi}\left(w_{0}, F\right) \in L^{1}\left(Q_{T}\right)
$$

is compact.
Proof. For a proof, see [2].

Lemma 3.2. Let $\varphi(w)=d w^{q}, d \in(0,+\infty), q>(N-2)^{+} / N$. Then, for $\left(w_{0}, F\right) \in L^{1}(\Omega) \times L^{1}\left(Q_{T}\right), w=S_{\varphi}\left(w_{0}, F\right)$ of (11) satisfies

$$
\begin{align*}
& \int_{Q_{T}}|w|^{q \alpha} \leq C \text { for } 0<\alpha<1+\frac{2}{q N},  \tag{26}\\
& \int_{Q_{T}}\left|\nabla w^{q}\right|^{\beta} \leq C_{2} \text { for } 1 \leq \beta<1+\frac{1}{1+q N}, \tag{27}
\end{align*}
$$

where $C=C\left(T, \alpha, \beta, q,\left\|w_{0}\right\|_{L^{1}(\Omega)},\|F\|_{L^{1}\left(Q_{T}\right)}\right)$.
If $\varphi$ is nondegenerate in the sense of (7), then the estimates (26) and (27) are valid with $q=1$.
Proof. For a proof, see Lukkari [23, Lemma 4.7] for the case $q>1$ and Lukkari [24, Lemma 3.5] for the case $\frac{(N-2)^{+}}{N}<q<1$. In these two references, the proof is given with zero initial data, but with right-hand side a bounded measure. We may use the measure $\delta_{t=0} \otimes w_{0} d x$ to include the case of initial data $w_{0}$. We may also use the results in [1, Theorem 2.9]). The estimate in the nondegenerate case may be obtained in a similar way.

In several of the proofs below, we will use the famous Vitali's Lemma (see e.g. [13, theorem 2.24, page 150], [38, chapter 16]).

Lemma 3.3 (Vitali). Let $(E, \mu)$ be a measured space such that $\mu(E)<+\infty$, let $1 \leq p<+\infty$ and let $\left\{f_{n}\right\}_{n} \subset L^{p}(E)$ such that $f_{n} \rightarrow f$ a.e. If $\left\{f_{n}^{p}\right\}_{n}$ is uniformly integrable over $E$, then $f \in L^{p}(E)$ and $f_{n} \rightarrow f$ in $L^{p}(E)$.

We now deduce various compactness properties of the approximate solution $u^{n}$ of (16).
Lemma 3.4. Assume that $\varphi_{i}$ satisfy (7) or (8) with $m_{i}>(N-2)^{+} / N$ and that the $L^{1}$-estimate (20) holds for the solution $u^{n}$ of (16). Then, up to a subsequence, and for all $T>0$ and $1 \leq i \leq m$,

- $u_{i}^{n}$ converge in $L^{1}\left(Q_{T}\right)$ and a.e. to some $u_{i} \in L^{1}\left(Q_{T}\right)$,
$-\varphi_{i}\left(u_{i}^{n}\right)$ converge in $L^{\alpha}\left(Q_{T}\right)$ and a.e. to $\varphi_{i}\left(u_{i}\right)$ for all $\alpha \in\left[1,1+2 /\left(m_{i} N\right)\right)$ in case (8) and all $\alpha \in[1,1+2 / N)$ in case (7),
- $\varphi_{i}\left(u_{i}\right) \in L^{\beta}\left(0, T ; W_{0}^{1, \beta}(\Omega)\right)$ for all $\beta \in\left[1,1+1 /\left(1+m_{i} N\right)\right)$ in case (8) and for all $\beta \in[1,1+1 /(1+N))$ in case (7),
- $f_{i}^{n}\left(u^{n}\right)$ converges a.e. to $f_{i}(u) \in L^{1}\left(Q_{T}\right)$.

Proof of Lemma 3.4. By the estimate (20), $f_{i}^{n}\left(u^{n}\right)$ is bounded in $L^{1}\left(Q_{T}\right)$. According to Lemma 3.1, $u_{i}^{n}$ is relatively compact in $L^{1}\left(Q_{T}\right)$ for all $T>0$. Therefore, up to a subsequence, we may assume that $u_{i}^{n}$ converge in $L^{1}\left(Q_{T}\right)$ for all $T>0$ and a.e. in $Q$ as well to some limit $u_{i} \in L^{1}\left(Q_{T}\right)$.

Next, by Lemma 3.2, $\varphi_{i}\left(u_{i}^{n}\right)$ is bounded in $L^{\alpha}\left(Q_{T}\right)$ for $\alpha \in\left[1,1+2 /\left(m_{i} N\right)\right)$ [even for $\alpha \in[1,1+2 / N)$ in the nondegenerate case] and for all $T>0$. By arbitrarity of $\alpha$ in this interval open to the right, $\varphi_{i}\left(u_{i}^{n}\right)^{\alpha}$ is even uniformly integrable. Since it also converges a.e. to $\varphi_{i}\left(u_{i}\right)$, by the Vitali's Lemma 3.3, the convergence holds strongly in $L^{\alpha}\left(Q_{T}\right)$ to $\varphi_{i}\left(u_{i}\right)$.

Next, thanks to the estimate of the gradient in Lemma 3.2, $\varphi_{i}\left(u_{i}^{n}\right)$ stays bounded in the space $L^{\beta}\left(0, T ; W_{0}^{1, \beta}(\Omega)\right)$ for all $\beta \in\left[1,1+1 /\left(1+m_{i} N\right)\right)$ [even all $\beta \in[1,1+1 /(1+N))$ in the nondegenerate case]. These spaces being reflexive (for $\beta>1$ ), it follows that $\varphi_{i}\left(u_{i}\right)$ also belongs to these same spaces.

Finally, due to the definition of the $f_{i}^{n}$ and to the a.e. convergence of $u^{n}$ to $u=\left(u_{i}\right)_{1 \leq i \leq m}$, it is clear that $f_{i}^{n}\left(u^{n}\right)$ converges a.e. to $f_{i}(u)$. By Fatou's Lemma, $f_{i}(u) \in L^{1}\left(Q_{T}\right)$.

Remark 3.5. To prove that the limit $u$ is solution of the limit problem, we would "only need" to prove that the convergence of $f_{i}^{n}\left(u^{n}\right)$ to $f_{i}(u)$ holds in the sense of distributions and not only a.e. But this is where the main difficulty of the proof lies. Indeed, $f_{i}^{n}\left(u^{n}\right)$ is bounded in $L^{1}\left(Q_{T}\right)$. Therefore it converges in the sense of measures to $f_{i}(u)+\mu$ where $\mu$ is a bounded measure. The point is to prove that this measure is equal to zero.

An easy situation is when the $f_{i}^{n}\left(u^{n}\right)$ are uniformly integrable and not only bounded in $L^{1}\left(Q_{T}\right)$. Then, using Vitali's Lemma 3.3, we deduce that the convergence of $f_{i}^{n}\left(u^{n}\right)$ to $f_{i}(u)$ holds in $L^{1}\left(Q_{T}\right)$ and therefore in the sense of distributions. It follows that $u$ is solution of the limit problem.

Actually, our method here, similar to the one in [30], will be to first prove that $u$ is a supersolution of the limit system. This is where the main difficulty is concentrated. The main result is stated in the next proposition. It is interesting to emphasize that the conclusion of this proposition is valid without the structure property $(M)$. This property ( $M$ ) will only be used later to prove the reverse inequality.

Proposition 3.6. Under the assumptions of Lemma 3.4, the limit $u$ is a supersolution of the limit system, which means that, for all $\psi \in \mathcal{C}_{T}$ as defined in(13), $\psi \geq 0$ and for $1 \leq i \leq m$ :

$$
-\int_{\Omega} \psi(0) u_{i 0}+\int_{Q_{T}}-\partial_{t} \psi u_{i}+\nabla \psi \nabla \varphi_{i}\left(u_{i}\right) \geq \int_{Q_{T}} \psi f_{i}(u)
$$

where $u_{i} \in L^{\infty}\left((0, T) ; L^{1}(\Omega)\right), \varphi_{i}\left(u_{i}\right) \in L^{1}\left((0, T) ; W_{0}^{1,1}(\Omega)\right)$.
Preliminary remark about Proposition 3.6 and its proof. Note that the result of this Proposition is interesting for itself and, as we already remark, is valid without the structure assumption $(M)$ on the nonlinearities $f_{i}^{n}$. The ideas of the proof of Proposition 3.6 are taken from [30]. A first idea is that, if $w$ is a solution of the heat equation, then $T_{k}(w)$ is a supersolution of the heat equation where $T_{k}$ is a regular approximation of the truncation function $r \in[0,+\infty) \mapsto \inf \{r, k\}$ as defined below. Here, we first prove that $T_{k}\left(u_{i}\right)$ is indeed a supersolution for all $k$ : by letting $k$ go to $+\infty$, it will follow that $u_{i}$ itself is a supersolution, whence Proposition 3.6.

In order to obtain that $T_{k}\left(u_{i}\right)$ is a supersolution, we pass to the limit as $n \rightarrow+\infty$ in the inequation satisfied by an adequate approximation of $T_{k}\left(u_{i}^{n}\right)$. But to pass to the limit in the sense of distributions in the nonlinear reaction terms (which a priori converge only a.e.), each truncation of the $i$-th equation must also involve all the $u_{j}^{n}, j \neq i$ : more precisely, in the semilinear case, the method was to write for each $i$, the inequation satisfied by $T_{k}\left(u_{i}^{n}+\eta \sum_{j \neq i} u_{j}^{n}\right)$ with $\eta>0$, then first to let $n \rightarrow+\infty$ for $\eta, k$ fixed, and next to let $\eta \rightarrow 0$, then $k \rightarrow+\infty$ (see [30]). The main work was to justify the step $\eta \rightarrow 0$ which involves estimates on the gradient of the solutions.

Here, the ideas are the same, but we have to adapt them to nonlinear diffusions. Besides and because of the degeneracy due this nonlinearity, gradient estimates are not as good as for linear diffusions, especially near $u_{i}=0$. Moreover, the nonlinear diffusion requires more complex truncations than in the linear case. This is why we consider the truncating process (29) below.

To prepare the proof of Proposition 3.6, let us introduce the truncating functions $T_{k}:[0,+\infty) \rightarrow[0,+\infty)$ of class $\mathcal{C}^{3}$ which satisfy the following for all $k \geq 1$ :

$$
\left\{\begin{array}{l}
T_{k}(r)=r \text { if } r \in[0, k-1]  \tag{28}\\
T_{k}(r) \leq k ; \\
T_{k}^{\prime}(r)=0 \text { if } r \geq k \\
0 \leq T_{k}^{\prime}(r) \leq 1,-1 \leq T_{k}^{\prime \prime}(r) \leq 0 \text { for all } r \geq 0
\end{array}\right.
$$

Next, for all $i=1, \ldots, m$ and for $(n, \eta, k) \in \mathbb{N}^{*} \times(0,1) \times[1,+\infty[$, we introduce

$$
\begin{equation*}
A_{i, \eta, k}^{n}=\partial_{t}\left(T_{k}\left(u_{i}^{n}\right) T_{k}^{\prime}\left(\eta V_{i}^{n}\right)\right)-\nabla \cdot\left(T_{k}^{\prime}\left(u_{i}^{n}\right) T_{k}^{\prime}\left(\eta V_{i}^{n}\right) \nabla \varphi_{i}\left(u_{i}^{n}\right)\right), \quad V_{i}^{n}=\sum_{j \neq i} u_{j}^{n} . \tag{29}
\end{equation*}
$$

Remark 3.7. To give some light on the choice of the above expression, note that, when $\eta \rightarrow 0$, then $T_{k}^{\prime}\left(\eta V_{i}^{n}\right) \rightarrow 1$ and when $k \rightarrow+\infty$, then $T_{k}$ tends to the identity so that this expression approximates $\partial_{t} u_{i}^{n}-\nabla \cdot\left(\nabla \varphi_{i}\left(u_{i}^{n}\right)\right)=$ $\partial_{t} u_{i}^{n}-\Delta \varphi_{i}\left(u_{i}^{n}\right)$.

We check that

$$
\left\{\begin{array}{l}
A_{i, \eta, k}^{n}=T_{k}^{\prime}\left(u_{i}^{n}\right) T_{k}^{\prime}\left(\eta V_{i}^{n}\right) f_{i}^{n}\left(u^{n}\right)+A_{i}^{n}+B_{i}^{n} \text { where }  \tag{30}\\
A_{i}^{n}=\eta T_{k}\left(u_{i}^{n}\right) T_{k}^{\prime \prime}\left(\eta V_{i}^{n}\right)\left(V_{i}^{n}\right)_{t}=\eta T_{k}\left(u_{i}^{n}\right) T_{k}^{\prime \prime}\left(\eta V_{i}^{n}\right) \sum_{j \neq i}\left[\Delta \varphi_{j}\left(u_{j}^{n}\right)+f_{j}^{n}\left(u^{n}\right)\right], \\
\text { that we write as an obvious sum: } A_{i}^{n}=: \sum_{j \neq i} X_{j}^{n}+Y_{j}^{n}, \\
B_{i}^{n}=-\nabla \varphi_{i}\left(u_{i}^{n}\right) \nabla\left[T_{k}^{\prime}\left(u_{i}^{n}\right) T_{k}^{\prime}\left(\eta V_{i}^{n}\right)\right] .
\end{array}\right.
$$

The proof of Proposition 3.6 will mainly rely on the following estimate.
Lemma 3.8. There exist $\delta>0, C>0$ independent of $n$ and $\eta$ such that, for all $i=1, \ldots, m$ and for all $\psi \in \mathcal{C}_{T}, \psi \geq 0$ :

$$
\begin{equation*}
\int_{Q_{T}} A_{i, \eta, k}^{n} \psi \geq \int_{Q_{T}} T_{k}^{\prime}\left(u_{i}^{n}\right) T_{k}^{\prime}\left(\eta V_{i}^{n}\right) f_{i}^{n}\left(u_{n}\right) \psi-C D(\psi) \eta^{\delta} \tag{31}
\end{equation*}
$$

where $D(\psi)=\|\psi\|_{L^{\infty}\left(Q_{T}\right)}+\|\nabla \psi\|_{L^{\infty}\left(Q_{T}\right)}$.
Proof of Lemma 3.8. It is a direct consequence of formula (30) and of Lemmas 3.12 and 3.13 below.
The proof of Lemmas 3.12 and 3.13 below will require the following preliminary estimate:
Lemma 3.9. Let $F \in L^{1}\left(Q_{T}\right)^{+}$, $w_{0} \in L^{1}(\Omega)^{+}$. Then $w=S_{\varphi}\left(w_{0}, F\right)$ as defined in (11) satisfies the following: there exists $C=C\left(\int_{Q_{T}} F, \int_{\Omega} w_{0}\right)$ such that, for all nondecreasing $\theta:(0,+\infty) \rightarrow(0,+\infty)$ of class $\mathcal{C}^{1}$ and with $\theta\left(0^{+}\right)=0$

$$
\begin{equation*}
\int_{[\theta(w) \leq k]}|\nabla \theta(w)||\nabla \varphi(w)|=\int_{[\theta(w) \leq k]} \nabla \theta(w) \nabla \varphi(w) \leq C k . \tag{32}
\end{equation*}
$$

In particular,

$$
\begin{equation*}
\int_{[\varphi(w) \leq k]}|\nabla \varphi(w)|^{2} \leq C k, \int_{[w \leq k]}|\nabla w|^{2} \leq C k^{2-\bar{m}} \tag{33}
\end{equation*}
$$

with $\bar{m}=1$ in case (7) and with $\bar{m}=m_{i}$ in case (8) assuming $m_{i}<2$.
Remark 3.10. The main restriction $m_{i}<2$ discussed in the introduction appears in the above statement. The proof of Theorem 2.6 requires to control the $L^{2}$-norm of $\nabla u_{i}^{n}$ on the level sets $\left[u_{i}^{n} \leq k\right]$. This $L^{2}$-norm is not bounded if $m_{i} \geq 2$ because of the degeneracy around the points where $u_{i}^{n}=0$. It is however valid for the large values of $u_{i}^{n}$. But this does not seem to be sufficient for the proof.

Proof of Lemma 3.9. As usual, we make the computations for regular enough solutions and they are preserved by approximation for all semigroup solutions.

Multiply equation $\partial_{t} w-\Delta \varphi(w)=F$ by $T_{k+1}(\theta(w))$. We obtain

$$
\int_{\Omega} J_{k}(w)(T)+\int_{Q_{T}} T_{k+1}^{\prime}(\theta(w)) \nabla \theta(w) \nabla \varphi(w)=\int_{Q_{T}} T_{k+1}(\theta(w)) F+\int_{\Omega} J_{k}\left(w_{0}\right),
$$

where $J_{k}^{\prime}(r)=T_{k+1}(\theta(r)), J_{k}(0)=0$. Since $T_{k+1} \leq k+1$, we have $J_{k}(r) \leq(k+1) r$ so that

$$
\int_{[\theta(w) \leq k]}|\nabla \theta(w)||\nabla \varphi(w)| \leq(k+1)\left(\int_{Q_{T}} F+\int_{\Omega} w_{0}\right) \leq C k .
$$

Choosing $\theta:=\varphi$ gives the first estimate of (33). The second one is clear in the nondegenerate case (7). If $\varphi_{i}(r)=d_{i} r^{m_{i}}$ with $m_{i}<2$, we choose $\theta(r):=r^{2-m_{i}}$ to obtain

$$
d_{i}\left(2-m_{i}\right) m_{i} \int_{\left[w^{2-m_{i}} \leq k\right]}|\nabla w|^{2} \leq C k
$$

which gives the second estimate of (33) by changing $k$ into $k^{2-m_{i}}$.

Remark 3.11. The two next lemmas provide the expected estimates for $B_{i}^{n}$, then $A_{i}^{n}$. We will often use that, for some $C$ independent of $n$ and $\eta$, it follows from (32) that, for $i, j=1, \ldots, m$

$$
\begin{equation*}
\int_{\left[\eta \varphi_{j}\left(u_{j}^{n}\right) \leq k\right]}\left|\nabla \varphi_{j}\left(u_{j}^{n}\right)\right|^{2} \leq C k / \eta, \int_{\left[\eta V_{i}^{n} \leq k\right]}\left|\nabla V_{i}^{n}\right|^{2} \leq C[k / \eta]^{2-M}, \quad M=\max \left\{1, \max _{i} m_{i}\right\} \tag{34}
\end{equation*}
$$

where we used the inclusion: $\forall j \neq i,\left[\eta V_{i}^{n} \leq k\right] \subset\left[\eta u_{j}^{n} \leq k\right]$.
Lemma 3.12. There exist $C \geq 0, \delta>0$ independent of $n$ and $\eta$ such that, for all $i=1, \ldots, m$ and for all $\psi \in \mathcal{C}_{T}$, $\psi \geq 0$

$$
\begin{equation*}
\int_{Q_{T}} \psi B_{i}^{n} \geq-\eta^{\delta} C\|\psi\|_{L^{\infty}\left(Q_{T}\right)} \tag{35}
\end{equation*}
$$

Proof of Lemma 3.12. We have

$$
\begin{aligned}
\int_{Q_{T}} \psi B_{i}^{n} & =-\int_{Q_{T}} \psi \nabla \varphi_{i}\left(u_{i}^{n}\right) \nabla\left[T_{k}^{\prime}\left(u_{i}^{n}\right) T_{k}^{\prime}\left(\eta V_{i}^{n}\right)\right] \\
& =-\int_{Q_{T}} \psi \nabla \varphi_{i}\left(u_{i}^{n}\right)\left[\nabla u_{i}^{n} T_{k}^{\prime \prime}\left(u_{i}^{n}\right) T_{k}^{\prime}\left(\eta V_{i}^{n}\right)+T_{k}^{\prime}\left(u_{i}^{n}\right) T_{k}^{\prime \prime}\left(\eta V_{i}^{n}\right) \eta \nabla V_{i}^{n}\right] \\
& \geq-\eta \int_{Q_{T}} \psi T_{k}^{\prime}\left(u_{i}^{n}\right) T_{k}^{\prime \prime}\left(\eta V_{i}^{n}\right) \nabla \varphi_{i}\left(u_{i}^{n}\right) \nabla V_{i}^{n},
\end{aligned}
$$

the last inequality coming from $T_{k}^{\prime \prime} \leq 0, \varphi_{i}^{\prime} \geq 0, \psi \geq 0$. By Schwarz's inequality and for some $C=C(k)$

$$
\begin{aligned}
\int_{Q_{T}}\left|\psi T_{k}^{\prime}\left(u_{i}^{n}\right) T_{k}^{\prime \prime}\left(\eta V_{i}^{n}\right) \nabla \varphi_{i}\left(u_{i}^{n}\right) \nabla V_{i}^{n}\right| & \leq C\|\psi\|_{L^{\infty}\left(Q_{T}\right)}\left(\int_{\left[u_{i}^{n} \leq k\right]}\left|\nabla \varphi_{i}\left(u_{i}^{n}\right)\right|^{2}\right)^{1 / 2}\left(\int_{\left[\eta V_{i}^{n} \leq k\right]}\left|\nabla V_{i}^{n}\right|^{2}\right)^{1 / 2} \\
& \leq C\|\psi\|_{L^{\infty}\left(Q_{T}\right)} \sqrt{\varphi_{i}(k)}[k / \eta]^{1-M / 2}, M:=\max \left\{1, \max _{i} m_{i}\right\},
\end{aligned}
$$

where the last inequality is obtained through (32) and (34). Thus, $\int_{Q_{T}} \psi B_{i}^{n} \geq-C D(\psi) \eta^{M / 2}$ for some $C=C(k)$. Whence (35) with $\delta=M / 2$.

Lemma 3.13. There exist $\delta>0, C \geq 0$ independent of $n$ and $\eta$ such that, for all $i=1, \ldots, m$ and for all $\psi \in \mathcal{C}_{T}$, $\psi \geq 0$ :

$$
\begin{equation*}
\int_{Q_{T}} \psi A_{i}^{n} \geq-\eta^{\delta} C D(\psi) \tag{36}
\end{equation*}
$$

Proof of Lemma 3.13. We will need several steps. Recall that $A_{i}^{n}=X_{i}^{n}+Y_{i}^{n}$.

- Let us bound $\int_{Q_{T}} Y_{j}^{n} \psi$. We have

$$
\int_{Q_{T}} Y_{j}^{n} \psi=\eta \int_{Q_{T}} \psi T_{k}\left(u_{i}^{n}\right) T_{k}^{\prime \prime}\left(\eta V_{i}^{n}\right) f_{j}^{n}\left(u^{n}\right),
$$

so that, using the $L^{1}$-bound on $f_{i}^{n}$, we obtain

$$
\begin{equation*}
\int_{Q_{T}} Y_{j}^{n} \cdot \psi \geq-\eta C(k)\|\psi\|_{L^{\infty}\left(Q_{T}\right)} \tag{37}
\end{equation*}
$$

- Let us bound $\int_{Q_{T}} X_{j}^{n} \psi$. We have

$$
\int_{Q_{T}} X_{j}^{n} \psi=\int_{Q_{T}} \eta \psi T_{k}\left(u_{i}^{n}\right) T_{k}^{\prime \prime}\left(\eta V_{i}^{n}\right) \Delta \varphi_{j}\left(u_{j}^{n}\right)=-I^{n}-J^{n}
$$

where

$$
I^{n}=\eta \int_{\left[\eta V_{i}^{n} \leq k\right]} \nabla \varphi_{j}\left(u_{j}^{n}\right) \nabla \psi T_{k}\left(u_{i}^{n}\right) T_{k}^{\prime \prime}\left(\eta V_{i}^{n}\right) \quad \text { and } \quad J^{n}=K_{1, n}+K_{2, n}
$$

with

$$
\begin{aligned}
& K_{1, n}=\eta \int_{\left[\eta V_{i}^{n} \leq k\right] \cap\left[u_{i}^{n} \leq k\right]} \psi \nabla \varphi_{j}\left(u_{j}^{n}\right) T_{k}^{\prime}\left(u_{i}^{n}\right) T_{k}^{\prime \prime}\left(\eta V_{i}^{n}\right) \nabla u_{i}^{n}, \\
& K_{2, n}=\eta^{2} \int_{\left[\eta V_{i}^{n} \leq k\right]} \psi \nabla \varphi_{j}\left(u_{j}^{n}\right) T_{k}\left(u_{i}^{n}\right) T_{k}^{\prime \prime \prime}\left(\eta V_{i}^{n}\right) \nabla V_{i}^{n},
\end{aligned}
$$

- Let us bound $I^{n}$. By (27):

$$
\left|I^{n}\right| \leq C(k) D(\psi) \eta \int_{Q_{T}}\left|\nabla \varphi_{j}\left(u_{j}^{n}\right)\right| \leq C \eta .
$$

- Let us bound $K_{1, n}$. By Schwarz's inequality, (32)-(34) and $\left[\eta V_{i}^{n} \leq k\right] \subset\left[\eta u_{j}^{n} \leq k\right]$

$$
\begin{aligned}
\left|K_{1, n}\right| & \leq \eta \int_{\left[\eta u_{j}^{n} \leq k\right] \cap\left[u_{i}^{n} \leq k\right]} \psi\left|\nabla \varphi_{j}\left(u_{j}^{n}\right)\right| T_{k}^{\prime}\left(u_{i}^{n}\right)\left|T_{k}^{\prime \prime}\left(\eta V_{i}^{n}\right)\right|\left|\nabla u_{i}^{n}\right| \\
& \leq C(k) \eta\|\psi\|_{L^{\infty}\left(Q_{T}\right)}\left(\int_{\left[\eta u_{j}^{n} \leq k\right]}\left|\nabla \varphi_{j}\left(u_{j}^{n}\right)\right|^{2}\right)^{1 / 2}\left(\int_{\left[u_{i}^{n} \leq k\right]}\left|\nabla u_{i}^{n}\right|^{2}\right)^{1 / 2} \\
& \leq C D(\psi) \eta \sqrt{\varphi_{j}(k / \eta)} k^{1-m_{i} / 2} \leq C D(\psi) \eta^{1-m_{j} / 2},
\end{aligned}
$$

where we used $\varphi_{j}(r) \leq C r^{m_{j}}$ for $r \geq 1$.

- Let us bound $K_{2, n}$. Using again Schwarz's inequality, (32)-(34) and $\left[\eta V_{i}^{n} \leq k\right] \subset\left[\eta u_{j}^{n} \leq k\right]$, we obtain:

$$
\begin{aligned}
\left|K_{2, n}\right| & \leq \eta^{2} \int_{\left[\eta V_{i}^{n} \leq k\right]} \psi\left|\nabla \varphi_{j}\left(u_{j}^{n}\right)\left\|\nabla V_{i}^{n} \mid T_{k}\left(u_{i}^{n}\right)\right\| T_{k}^{\prime \prime \prime}\left(\eta V_{i}^{n}\right)\right) \mid \\
& \leq C \eta^{2}\|\psi\|_{L^{\infty}\left(Q_{T}\right)}\left[\int_{\left[\eta u_{j}^{n} \leq k\right]}\left|\nabla \varphi_{j}\left(u_{j}^{n}\right)\right|^{2}\right]^{1 / 2}\left[\left.\int_{\left[\eta V_{i}^{n} \leq k\right]}\left|\nabla V_{i}^{n}\right|^{2}\right|^{1 / 2},\right. \\
& \leq C \eta^{2} D(\psi) \sqrt{\varphi_{j}(k / \eta)}[k / \eta]^{1-M / 2} \\
& \leq C \eta^{2} D(\psi)[k / \eta]^{1-m_{j} / 2}[k / \eta]^{1-M / 2} \\
& \leq C D(\psi) \eta^{\left(m_{j}+M\right) / 2} . \quad \square
\end{aligned}
$$

Proof of Proposition 3.6. Recall that, by Lemma 3.8, we have for all $\psi \in \mathcal{C}_{T}, \psi \geq 0$

$$
\begin{equation*}
\int_{Q_{T}} A_{i, \eta, k}^{n} \psi \geq \int_{Q_{T}} T_{k}^{\prime}\left(u_{i}^{n}\right) T_{k}^{\prime}\left(\eta V_{i}^{n}\right) f_{i}^{n}\left(u_{n}\right) \psi-C D(\psi) \eta^{\delta} \tag{38}
\end{equation*}
$$

where

$$
A_{i, \eta, k}^{n}=\partial_{t}\left(T_{k}\left(u_{i}^{n}\right) T_{k}^{\prime}\left(\eta V_{i}^{n}\right)\right)-\nabla \cdot\left(T_{k}^{\prime}\left(u_{i}^{n}\right) T_{k}^{\prime}\left(\eta V_{i}^{n}\right) \nabla \varphi_{i}\left(u_{i}^{n}\right)\right), \quad V_{i}^{n}=\sum_{j \neq i} u_{j}^{n}
$$

Note also that

$$
\left\{\begin{array}{l}
\int_{Q_{T}} A_{i, n, k}^{n} \psi=-\int_{\Omega} T_{k}\left(u_{i 0}^{n}\right) T_{k}^{\prime}\left(\eta V_{i}^{n}(0)\right) \psi(0)  \tag{39}\\
+\int_{Q_{T}}-T_{k}\left(u_{i}^{n}\right) T_{k}^{\prime}\left(\eta V_{i}^{n}\right) \partial_{t} \psi+T_{k}^{\prime}\left(u_{i}^{n}\right) T_{k}^{\prime}\left(\eta V_{i}^{n}\right) \nabla \varphi_{i}\left(u_{i}^{n}\right) \nabla \psi .
\end{array}\right.
$$

The main point is to pass to the limit in (38) and (39). We do it in the following order: first $n \rightarrow+\infty$, then $\eta \rightarrow 0$, finally $k \rightarrow+\infty$.

- Let $n \rightarrow+\infty$ along the subsequence introduced in Lemma 3.4 ( $\eta$ and $k$ are fixed). Since $u_{i 0}^{n} \rightarrow u_{i 0}$ in $L^{1}(\Omega)$ and since $T_{k}, T_{k}^{\prime}$ are Lipschitz continuous

$$
\int_{\Omega} T_{k}\left(u_{i 0}^{n}\right) T_{k}^{\prime}\left(\eta V_{i}^{n}(0)\right) \psi(0) \rightarrow \int_{\Omega} T_{k}\left(u_{i 0}\right) T_{k}^{\prime}\left(\eta V_{i}(0)\right) \psi(0)
$$

For the last integral in (39), since, for all $j=1, \ldots, m, u_{j}^{n}$ converges in $L^{1}\left(Q_{T}\right)$ and a.e. to $u_{j}$, it follows that $T_{k}\left(u_{i}^{n}\right) T_{k}^{\prime}\left(\eta V_{i}^{n}\right) \rightarrow T_{k}\left(u_{i}\right) T_{k}^{\prime}\left(\eta V_{i}\right)$ in $L^{1}\left(Q_{T}\right)$ where we set $V_{i}:=\sum_{j \neq i} u_{j}$. It also follows that $T_{k}^{\prime}\left(\eta V_{i}^{n}\right)$ converges in $L^{2}\left(Q_{T}\right)$ to $T_{k}^{\prime}\left(\eta V_{i}\right)$. Next, $T_{k}^{\prime}\left(u_{i}^{n}\right) \nabla \varphi\left(u_{i}^{n}\right)$ is bounded in $L^{2}\left(Q_{T}\right)$ by (33) in Lemma 3.9. Therefore it converges weakly in $L^{2}\left(Q_{T}\right)$. Its limit is necessarily $T_{k}^{\prime}\left(u_{i}\right) \nabla \varphi\left(u_{i}\right)$. Indeed, $T_{k}^{\prime}\left(u_{i}^{n}\right) \nabla \varphi\left(u_{i}^{n}\right)=\nabla S_{k}\left(u_{i}^{n}\right)$ where we set $S_{k}(r):=\int_{0}^{r} T_{k}^{\prime}(s) \varphi_{i}^{\prime}(s) d s$. Since $S_{k}\left(u_{i}^{n}\right)$ converges a.e. to $S_{k}\left(u_{i}\right)$ and is bounded, the convergence holds in the sense of distributions. Therefore the distribution limit of $\nabla S_{k}\left(u_{i}^{n}\right)$ is $\nabla S_{k}\left(u_{i}\right)=T_{k}^{\prime}\left(u_{i}\right) \nabla \varphi_{i}\left(u_{i}\right)$. This ends the proof of the passing to the limit in (39).
Now, to pass to the limit in the right-hand side of (38), let us denote

$$
W_{n}:=T_{k}^{\prime}\left(u_{i}^{n}\right) T_{k}^{\prime}\left(\eta V_{i}^{n}\right) f_{i}^{n}\left(u^{n}\right), \quad W:=T_{k}^{\prime}\left(u_{i}\right) T_{k}^{\prime}\left(\eta V_{i}\right) f_{i}(u)
$$

and let us show that $W_{n}$ converges to $W$ in $L^{1}\left(Q_{T}\right)$. Since $W_{n}=0$ outside the set $\left[u_{i}^{n} \leq k\right] \cup\left[V_{i}^{n} \leq k / \eta\right]$, if $M:=\max \{k, k / \eta\}$, we may write (see the definition (17) and property (4) and recall that $\left|T_{k}^{\prime}\right| \leq 1$ )

$$
\left|W_{n}\right| \leq\left|f_{i}^{n}\left(t, x, u^{n}\right)\right| \leq\left|f_{i}(t, x, 0)\right|+\epsilon_{0}^{n}+K(M) \| u^{n}(t, x)| | .
$$

By assumption (see (18)), as $n \rightarrow+\infty, \epsilon_{0}^{n}$ tends to 0 in $L^{1}\left(Q_{T}\right)$. Moreover, $u^{n}$ converges in $L^{1}\left(Q_{T}\right)^{m}$ to $u$. Therefore, to prove the convergence of $W_{n}$ in $L^{1}\left(Q_{T}\right)$, it is sufficient to prove that it converges a.e. We know that, for all $j, u_{j}^{n}$ converges a.e. to $u_{j}$. Therefore, $T_{k}^{\prime}\left(u_{i}^{n}\right) T_{k}^{\prime}\left(\eta V_{i}^{n}\right)$ converges a.e. to $T_{k}^{\prime}\left(u_{i}\right) T_{k}^{\prime}\left(\eta V_{i}\right)$. It remains to check that

$$
\begin{equation*}
f_{i}^{n}\left(t, x, u^{n}(t, x)\right) \text { converges a.e. }(t, x) \text { to } f_{i}(t, x, u(t, x)) . \tag{40}
\end{equation*}
$$

Let $D$ be the subset of $(t, x) \in Q_{T}$ such that, at the same time, $u^{n}(t, x)$ converges to $u(t, x)$ with $\|u(t, x)\|<+\infty$ and $\epsilon_{p}^{n}(t, x)$ converges to 0 for all positive integer $p$ as $n \rightarrow+\infty$ along the subsequence introduced in Lemma 3.4. We know that $Q_{T} \backslash D$ is of zero Lebesgue measure. Now let $(t, x) \in D$ and let $p>\|u(t, x)\|$. For $n$ large enough, $\left\|u^{n}(t, x)\right\|<p$ and we may write for all $i=1, \ldots, m$ (using the definition (17) and property (4)):

$$
\left\{\begin{array}{l}
\left|f_{i}^{n}\left(t, x, u^{n}(t, x)\right)-f_{i}(t, x, u(t, x))\right| \leq \epsilon_{p}(t, x)+\left|f_{i}\left(t, x, u^{n}(t, x)\right)-f_{i}(t, x, u(t, x))\right| \\
\leq \epsilon_{p}(t, x)+K(p)\left\|u^{n}(t, x)-u(t, x)\right\| .
\end{array}\right.
$$

The right-hand side of this inequality tends to 0 by definition of $D$.
According to the above analysis, we can pass to the limit as $n \rightarrow+\infty$ in (38) and (39) and we obtain that

$$
\left\{\begin{array}{l}
-\int_{\Omega} T_{k}\left(u_{i}\right) T_{k}^{\prime}\left(\eta V_{i}(0)\right) \psi(0)+\int_{Q_{T}}-T_{k}\left(u_{i}\right) T_{k}^{\prime}\left(\eta V_{i}\right) \partial_{t} \psi+T_{k}^{\prime}\left(u_{i}\right) T_{k}^{\prime}\left(\eta V_{i}\right) \nabla \varphi_{i}\left(u_{i}\right) \nabla \psi  \tag{41}\\
\geq \int_{Q_{T}} T_{k}^{\prime}\left(u_{i}\right) T_{k}^{\prime}\left(\eta V_{i}\right) f_{i}(u) \psi-C D(\psi) \eta^{\delta} .
\end{array}\right.
$$

- We now let $\eta \rightarrow 0$ for fixed $k$ in (41). Since $f_{i}^{n}\left(u^{n}\right)$ converges a.e. to $f_{i}(u)$ (see (40)) and is bounded in $L^{1}\left(Q_{T}\right)$, Fatou's lemma implies that $f_{i}(u) \in L^{1}\left(Q_{T}\right)$. As $\left.\eta \rightarrow 0, T_{k}^{\prime}\left(\eta v_{i}\right)\right) \rightarrow 1$ a.e. and stays bounded by 1 , then by dominated convergence, we can replace at the limit $T_{k}^{\prime}\left(\eta V_{i}\right)$ in all integrals of (41). Thanks to $\delta>0$, we then obtain

$$
\begin{equation*}
-\int_{\Omega} T_{k}\left(u_{i 0}\right) \psi(0)+\int_{Q_{T}}-T_{k}\left(u_{i}\right) \partial_{t} \psi+T_{k}^{\prime}\left(u_{i}\right) \nabla \varphi_{i}\left(u_{i}\right) \nabla \psi \geq \int_{Q_{T}} T_{k}^{\prime}\left(u_{i}\right) f_{i}(u) \psi \tag{42}
\end{equation*}
$$

- Finally, we let $k \rightarrow+\infty$ in this inequality (42). Then $T_{k}\left(u_{i}\right)$ increases to $u_{i}$ and $T_{k}^{\prime}\left(u_{i}\right)$ increases to $1, \nabla \varphi_{i}\left(u_{i}\right)$ is at least in $L^{1}\left(Q_{T}\right)$ (see (27)) and $f_{i}(u) \in L^{1}\left(Q_{T}\right)$. Therefore, we easily pass to the limit in (42) to obtain

$$
\begin{equation*}
-\int_{\Omega} u_{i 0} \psi(0)+\int_{Q_{T}}-u_{i} \partial_{t} \psi+\nabla \varphi_{i}\left(u_{i}\right) \nabla \psi \geq \int_{Q_{T}} f_{i}(u) \psi \tag{43}
\end{equation*}
$$

And this ends the proof of Proposition 3.6.
Proof of Theorem 2.6. By Proposition 3.6, we already know that the limit $u$ is a supersolution in the sense that (43) is satisfied for all $\psi \in \mathcal{C}_{T}, \psi \geq 0$ and for all $i=1, \ldots, m$. We will show with the help of the $(M)$ structure property (6) that the inverse inequality is satisfied for the sum of these $m$ expressions, namely

$$
\begin{equation*}
-\int_{\Omega}\left[\sum_{i} u_{i 0}\right] \psi(0)+\int_{Q_{T}}-\left[\sum_{i} u_{i}\right] \partial_{t} \psi+\left[\sum_{i} \nabla \varphi_{i}\left(u_{i}\right)\right] \nabla \psi \leq \int_{Q_{T}}\left[\sum_{i} f_{i}(u)\right] \psi . \tag{44}
\end{equation*}
$$

This will imply that equality holds in each of the inequalities (43).
Going back to the approximate system (16) and adding the $m$ equations lead to the fact that, for all $\psi$ as above,

$$
-\int_{\Omega}\left[\sum_{i} u_{i 0}^{n}\right] \psi(0)+\int_{Q_{T}}-\left[\sum_{i} u_{i}^{n}\right] \partial_{t} \psi+\left[\sum_{i} \nabla \varphi_{i}\left(u_{i}^{n}\right)\right] \nabla \psi=\int_{Q_{T}}\left[\sum_{i} f_{i}^{n}\left(u^{n}\right)\right] \psi
$$

We already know that, along an adequate subsequence of $n \rightarrow+\infty, u_{i}^{n}$ converges in $L^{1}\left(Q_{T}\right)$ to $u_{i}$ and that $\nabla \varphi\left(u_{i}^{n}\right)$ converges weakly in $L^{2}\left(Q_{T}\right)$ to $\nabla \varphi_{i}\left(u_{i}\right)$ (see the proof of Proposition 3.6). Hence, the left-hand side of this equality converges to the expected limit as $n \rightarrow+\infty$.

For the right-hand side, the assumption (6) on the $f_{i}^{n}$ says that

$$
\sigma\left\|u^{n}\right\|+h-\sum_{i} f_{i}^{n}\left(u^{n}\right) \geq 0 .
$$

We know that $u^{n}$ converges in $L^{1}\left(Q_{T}\right)$ to $u$ and, according to (40), that $f_{i}^{n}\left(u^{n}\right)$ converges a.e. to $f_{i}(u)$. By Fatou's Lemma

$$
\int_{Q_{T}}\left[\sigma\|u\|+h-\sum_{i} f_{i}(u)\right] \psi \leq \int_{Q_{T}}(\sigma\|u\|+h) \psi+\liminf _{n \rightarrow+\infty} \int_{Q_{T}}-\left[\sum_{i} f_{i}^{n}\left(u^{n}\right)\right] \psi .
$$

Therefore

$$
\limsup _{n \rightarrow \infty} \int_{Q_{T}}\left[\sum_{i} f_{i}^{n}\left(u^{n}\right)\right] \psi \leq \int_{Q_{T}}\left[\sum_{i} f_{i}(u)\right] \psi,
$$

whence (44). And as explained above, this implies that equality holds in (43). We will use below the version obtained after integration by parts, namely that, for all $\psi \in \mathcal{C}_{T}$

$$
\begin{equation*}
-\int_{\Omega} u_{i 0} \psi(0)+\int_{Q_{T}}-u_{i} \partial_{t} \psi-\varphi_{i}\left(u_{i}\right) \Delta \psi=\int_{Q_{T}} f_{i}(u) \psi \tag{45}
\end{equation*}
$$

We know that, at least $\varphi_{i}\left(u_{i}\right) \in L^{1}\left(0, T ; W_{0}^{1,1}(\Omega)\right)$ by Lemma 3.4. To conclude the proof of Theorem 2.6, it remains to show that we exactly have

$$
\begin{equation*}
u_{i}=S_{\varphi_{i}}\left(u_{i 0}, f_{i}(u)\right) \tag{46}
\end{equation*}
$$

in the sense of (11).
To this end, we first go back to (42). Note that by approximation, this inequality remains valid if one replaces $T_{k}$ by the "exact" truncation function

$$
\forall r \in[0, k], \quad \Sigma_{k}(r)=r, \quad \forall r \in[k,+\infty), \quad \Sigma_{k}(r)=k
$$

and since $\Sigma_{k}^{\prime}\left(u_{i}\right) \nabla \varphi_{i}\left(u_{i}\right)=\nabla \varphi_{i}\left(\Sigma_{k}\left(u_{i}\right)\right)$ a.e., then (42) may be written

$$
\begin{equation*}
-\int_{\Omega} \Sigma_{k}\left(u_{i 0}\right) \psi(0)+\int_{Q_{T}}-\Sigma_{k}\left(u_{i}\right) \partial_{t} \psi+\nabla \varphi_{i}\left(\Sigma_{k}\left(u_{i}\right)\right) \nabla \psi \geq \int_{Q_{T}} \Sigma_{k}^{\prime}\left(u_{i}\right) f_{i}(u) \psi \tag{47}
\end{equation*}
$$

Inequality (47) says in some sense that $\Sigma_{k}\left(u_{i}\right)$ is a bounded "supersolution" of $\partial_{t} \Sigma_{k}\left(u_{i}\right)-\Delta \varphi_{i}\left(\Sigma_{k}\left(u_{i}\right)\right) \geq$ $\Sigma_{k}^{\prime}\left(u_{i}\right) f_{i}(u)$. By the comparison Theorem 6.5 in [40] (see also Proposition 6.4 in [40]), we may deduce that

$$
\Sigma_{k}\left(u_{i}\right) \geq S_{\varphi_{i}}\left(\Sigma_{k}\left(u_{i 0}\right), \Sigma_{k}^{\prime}\left(u_{i}\right) f_{i}(u)\right)
$$

Now, passing to the limit as $k \rightarrow+\infty$ and using the contraction property (10), we deduce (note that $\left(\Sigma_{k}\left(u_{i 0}\right)\right.$, $\left.\Sigma_{k}\left(u_{i}\right) f_{i}(u)\right)$ converges to $\left(u_{i 0}, f_{i}(u)\right)$ in $\left.L^{1}(\Omega) \times L^{1}\left(Q_{T}\right)\right)$ :

$$
u_{i} \geq S_{\varphi_{i}}\left(u_{i 0}, f_{i}(u)\right)=: U_{i}
$$

But, since $u_{i}$ satisfies (45) and since so does $U_{i}$ by (12), we have

$$
\int_{Q_{T}}\left(u_{i}-U_{i}\right) \partial_{t} \psi+\left[\varphi_{i}\left(u_{i}\right)-\varphi_{i}\left(U_{i}\right)\right] \Delta \psi=0
$$

Choosing $\psi(t, x)=(T-t) \zeta(x)$ where $-\Delta \zeta=1$ in $\Omega, \zeta=0$ on $\partial \Omega$, we obtain

$$
\int_{Q_{T}}\left(u_{i}-U_{i}\right) \zeta+\left[\varphi_{i}\left(u_{i}\right)-\varphi_{i}\left(U_{i}\right)\right](T-t)=0
$$

Since $\zeta>0, u_{i} \geq U_{i}, \varphi_{i}$ increasing, we deduce $u_{i} \equiv U_{i}$ whence (46).
Proof of Theorem 2.7. We add the $m$ equations of System (16) to obtain

$$
\partial_{t}\left(\sum_{i} u_{i}^{n}\right)-\Delta\left(\sum_{i} \varphi_{i}\left(u_{i}^{n}\right)\right)=\sum_{i} f_{i}^{n}\left(u^{n}\right) \leq \sigma\left\|u^{n}\right\|+h=\sigma \sum_{i} u_{i}^{n}+h
$$

We rewrite this as

$$
\partial_{t}\left(e^{-\sigma t} \sum_{i} u_{i}^{n}\right)-\Delta\left(e^{-\sigma t} \sum_{i} \varphi_{i}\left(u_{i}^{n}\right)\right) \leq e^{-\sigma t} h \leq h
$$

Let us set $W(t):=e^{-\sigma t} \sum_{i} u_{i}^{n}, Z(t):=\int_{0}^{t} e^{-\sigma s} \sum_{i} \varphi_{i}\left(u_{i}^{n}(s)\right) d s$. Integrating the last inequality in time leads to

$$
\begin{equation*}
W(t)-\Delta Z(t) \leq W(0)+\int_{0}^{t} h(s) d s \tag{48}
\end{equation*}
$$

We now multiply this inequality by $\partial_{t} Z(\geq 0)$ and we integrate over $Q_{T}$ :

$$
\int_{Q_{T}}\left(\partial_{t} Z\right) W+\int_{Q_{T}} \nabla \partial_{t} Z \cdot \nabla Z \leq \int_{Q_{T}} \partial_{t} Z\left[W(0)+\int_{0}^{t} h\right] \leq \int_{\Omega}\left[W(0)+\int_{0}^{T} h\right] Z(T) .
$$

We have $\int_{Q_{T}} \nabla \partial_{t} Z \cdot \nabla Z=\frac{1}{2} \int_{\Omega}|\nabla Z(T)|^{2} \geq 0$. Moreover, the above right hand-side is bounded for all $T>0$. To see it, we may introduce the solution of

$$
-\Delta \theta_{0}=W(0)+\int_{0}^{T} h \text { in } \Omega, \theta_{0}=0 \text { on } \partial \Omega, \theta_{0} \geq 0
$$

And we multiply the equation (48) at time $t=T$ by $\theta_{0}$ to find, after integration by parts

$$
\int_{\Omega} W(T) \theta_{0}(T)+\int_{\Omega}\left[W(0)+\int_{0}^{T} h\right] Z(T) \leq \int_{\Omega}\left|\nabla \theta_{0}\right|^{2} \leq C\left\|W(0)+\int_{0}^{T} h\right\|_{L^{2}(\Omega)} .
$$

Finally, for all $T>0$, we obtained $C(T) \in(0,+\infty)$ such that

$$
\int_{Q_{T}} e^{-2 \sigma t}\left[\sum_{i} u_{i}^{n}\right]\left[\sum_{i} \varphi_{i}\left(u_{i}^{n}\right)\right] \leq C(T) .
$$

In particular, if $\varphi_{i}\left(u_{i}\right)=d_{i} u_{i}^{m_{i}}$, we obtain

$$
d_{i} \int_{Q_{T}} u_{i}^{m_{i}+1} \leq e^{2 \sigma T} C(T)
$$

And if $\varphi_{i}$ is nondegenerate as in (7), this estimate is also valid with $m_{i}=1$.
Proof of Corollary 2.8. For all $i=1, \ldots, m$, we set

$$
u_{i 0}^{n}:=\inf \left\{u_{i 0}, n\right\}, \quad \forall r \in[0,+\infty)^{m}, \text { a.e. }(t, x) \in Q, f_{i}^{n}(t, x, r)=\frac{f_{i}(t, x, r)}{1+\frac{1}{n} \sum_{j}\left|f_{j}(t, x, r)\right|}
$$

As already stated (see the comments following (19)), these approximations $f_{i}^{n}$ satisfy (4), (5), (6) with values independent of $n$. Thus, we may consider the solutions of the approximate system (16) and apply Theorem 2.7 which implies that, for all $i=1, \ldots, m, u_{i}^{n}$ is bounded in $L^{m_{i}+1}\left(Q_{T}\right)$. Together with the assumption (21), it follows that $f_{i}^{n}\left(u^{n}\right)$ is uniformly integrable on $Q_{T}$. Indeed, for all measurable set $K \subset Q_{T}$ with Lebesgue measure denoted by $|K|$, we have (recall that $\left|f_{i}^{n}\right| \leq\left|f_{i}\right|$ )

$$
\int_{K} \sum_{i}\left|f_{i}^{n}\left(u^{n}\right)\right| \leq C\left[|K|+\sum_{i} \int_{K}\left(u_{i}^{n}\right)^{m_{i}+1-\epsilon}\right] \leq C\left[|K|+\sum_{i}\left(\int_{Q_{T}}\left(u_{i}^{n}\right)^{m_{i}+1}\right)^{\frac{m_{i}+1-\epsilon}{m_{i}+1}}|K|^{\frac{\epsilon}{m_{i}+1}}\right] .
$$

Since $\sup _{n} \int_{Q_{T}}\left(u_{i}^{n}\right)^{m_{i}+1}<+\infty$, this implies that $\int_{K} \sum_{i}\left|f_{i}^{n}\left(u^{n}\right)\right|$ may be made uniformly small by taking $|K|$ small enough. This is exactly the uniform integrability of the $f_{i}^{n}\left(u^{n}\right)$.

Moreover, $f_{i}^{n}\left(u^{n}\right)$ converges a.e. to $f_{i}(u)$. Therefore, at least up to a subsequence, by Vitali's Lemma 3.3, we may deduce that $f_{i}^{n}\left(u^{n}\right)$ converges in $L^{1}\left(Q_{T}\right)$ for all $T<+\infty$ to $f_{i}\left(u_{i}\right)$. This implies that $u_{i}^{n}=S_{\varphi_{i}}\left(u_{i 0}^{n}, f_{i}^{n}\left(u^{n}\right)\right)$ converges to $u_{i}=S_{\varphi_{i}}\left(u_{i 0}, f_{i}(u)\right)$.

Finally, by the estimate (27) in Lemma 3.2 of $\nabla \varphi_{i}\left(u_{i}^{n}\right)$ in $L^{\beta}\left(Q_{T}\right)$ with $\beta>1$, it follows that $\varphi_{i}\left(u_{i}\right)$ is (at least) in $L^{1}\left(0, T ; W_{0}^{1,1}(\Omega)\right)$. This ends the proof of Corollary 2.8.

Proof of Corollary 2.9. Note first that the reactive terms in System (22) satisfy the three assumptions (4), (5) and (6) with $\sigma=0, h=0$. If $\varphi_{i}\left(u_{i}\right)=d_{i} u_{i}^{m_{i}}$ for at least one odd and one even value of $i \in\{1, \ldots, 4\}$, then the assumptions of Corollary 2.8 are satisfied: indeed, if for instance $m_{1}>1$, we may write Young's inequality

$$
u_{1} u_{3} \leq \frac{1}{p} u_{1}^{p}+\frac{1}{q} u_{2}^{q}, p=\left(m_{1}+3\right) / 2<m_{1}+1, q=\left(m_{1}+3\right) /\left(m_{1}+1\right)<2,
$$

and similarly for $u_{2} u_{4}$. Whence global existence of weak solutions. With $m_{i} \geq 1, i=1, \ldots, 4$, the strict condition (21) is not necessarily satisfied. We need an extra argument to obtain strong compactness in $L^{1}\left(Q_{T}\right)$ of the reactive terms. We could use the $L^{2}$-compactness approach used in [31] and [8, Lemma 5].

Here, as in [11], we can more easily use the entropy structure of the system which would apply as well to general reversible reactions (see Remark 2.10). This will provide uniform integrability of the approximate reactive terms and, together with the a.e. convergence and Vitali's Lemma 3.3, strong $L^{1}\left(Q_{T}\right)$ compactness as well, whence the result of Corollary 2.9.

We use the same approximation as in the proof of Corollary 2.8. For $i=1, \ldots, 4$, let us set

$$
\begin{equation*}
w_{i}^{n}:=u_{i}^{n} \log u_{i}^{n}-u_{i}^{n}+1(\geq 0), \quad z_{i}^{n}=\int_{1}^{u_{i}^{n}} \log r \varphi_{i}^{\prime}(r) d r \geq 0 . \tag{49}
\end{equation*}
$$

We have

$$
\partial_{t} w_{i}^{n}-\Delta z_{i}^{n}=\log u_{i}^{n} f_{i}^{n}\left(u^{n}\right)-\frac{\varphi_{i}^{\prime}\left(u_{i}^{n}\right)}{u_{i}^{n}}\left|\nabla u_{i}^{n}\right|^{2} .
$$

The main point is that

$$
\sum_{1 \leq i \leq 4} \log u_{i}^{n} f_{i}^{n}\left(u^{n}\right)=-\left(u_{1}^{n} u_{3}^{n}-u_{2}^{n} u_{4}^{n}\right)\left(\log \left(u_{1}^{n} u_{3}^{n}\right)-\log \left(u_{2}^{n} u_{4}^{n}\right)\right) /\left[\left.1+\frac{1}{n} \sum_{1 \leq i \leq 4} \right\rvert\, f_{i}\left(u^{n}\right)\right] \leq 0 .
$$

We deduce that

$$
\begin{equation*}
\partial_{t}\left(\sum_{1 \leq i \leq 4} w_{i}^{n}\right)-\Delta \sum_{1 \leq i \leq 4} z_{i}^{n} \leq 0 . \tag{50}
\end{equation*}
$$

We now make the same computation as in the proof of Theorem 2.7. We integrate this inequality in time, we multiply by $\sum_{i} z_{i}^{n}(\geq 0)$ and we integrate over $Q_{T}$. We obtain

$$
\begin{equation*}
\int_{Q_{T}}\left(\sum_{1 \leq i \leq 4} w_{i}^{n}\right)\left(\sum_{1 \leq i \leq 4} z_{i}^{n}\right)+\frac{1}{2} \int_{\Omega}\left|\nabla \int_{0}^{T} \sum_{1 \leq i \leq 4} z_{i}^{n}\right|^{2} \leq \int_{\Omega} \sum_{1 \leq i \leq 4} w_{i}^{n}(0) \int_{0}^{T} \sum_{1 \leq i \leq 4} z_{i}^{n} . \tag{51}
\end{equation*}
$$

From

$$
-\Delta \int_{0}^{T} \sum_{1 \leq i \leq 4} z_{i}^{n} \leq \sum_{1 \leq i \leq 4} w_{i}^{n}(0) \leq \sum_{1 \leq i \leq 4} u_{i 0}\left|\log u_{i 0}\right|+4 \in L^{2}(\Omega), \sum_{1 \leq i \leq 4} z_{i}^{n}=0 \text { on } \partial \Omega,
$$

we deduce that $\int_{0}^{T} \sum_{i} z_{i}^{n}$ is bounded in $L^{2}(\Omega)$ independently of $n$. Thus, it follows from (51) that for some $C(T) \in$ ( $0,+\infty$ )

$$
\int_{Q_{T}}\left(\sum_{1 \leq i \leq 4} w_{i}^{n}\right)\left(\sum_{1 \leq i \leq 4} z_{i}^{n}\right) \leq C(T) .
$$

Now, in the nondegenerate case, $\varphi_{i}^{\prime}\left(u_{i}^{n}\right) \geq a_{i}$ for some $a_{i}>0$ so that $z_{i}^{n} \geq a_{i} w_{i}^{n}$ and the above last estimate implies $\int_{Q_{T}} a_{i}\left(\log u_{i}^{n}\right)^{2}\left(u_{i}^{n}\right)^{2} \leq C(T)$. If $\varphi_{i}\left(u_{i}^{n}\right)=d_{i}\left(u_{i}^{n}\right)^{m_{i}}$, we have $z_{i}^{n}=d_{i} \log u_{i}^{n}\left(u_{i}^{n}\right)^{m_{i}}-\left(m_{i}\right)^{-1}\left[\left(u_{i}^{n}\right)^{m_{i}}-1\right]$. From the same estimate above, we deduce

$$
\int_{Q_{T}}\left(\log u_{i}^{n}\right)^{2}\left(u_{i}^{n}\right)^{m_{i}+1} \leq C(T) .
$$

In all cases, we obtain that $\left(u_{i}^{n}\right)^{2}$ are uniformly integrable on $Q_{T}$. Thus we can pass to the limit in $L^{1}\left(Q_{T}\right)$ in the quadratic terms $f_{i}^{n}\left(u^{n}\right)$.

Proof of Remark 2.10. Let us first assume $k_{1}=k_{2}=: k$ in (23). With the same notation as in the just above proof of Corollary 2.9 , we have

$$
\begin{aligned}
& \sum_{i=1}^{m} \log u_{i}^{n} f_{i}^{n}\left(u^{n}\right)=-k\left[\prod_{i} u_{i}^{q_{i}}-\prod_{i} u_{i}^{p_{i}}\right] \log \frac{\prod_{i} u_{i}^{q_{i}}}{\prod_{i} u_{i}^{p_{i}}} \leq 0 \\
& \partial_{t}\left(\sum_{i} w_{i}^{n}\right)-\Delta \sum_{i} z_{i}^{n} \leq 0
\end{aligned}
$$

We now multiply this last inequality by $\sum_{i} z_{i}^{n}$ and, by the same computation as in (51) and in the lines which follow (51), we deduce as well that $\int_{Q_{T}}\left(\log u_{i}^{n}\right)^{2}\left(u_{i}^{n}\right)^{m_{i}+1} \leq C(T)$ for all $i=1, \ldots, m$. Therefore $\left(u_{i}^{n}\right)^{m_{i}+1}$ is uniformly integrable.

Now let $r_{i}:=\left(m_{i}+1\right) / q_{i}$ for $i=1, \ldots, m$ and $s:=1-\sum_{i=1}^{m} q_{i} /\left(m_{i}+1\right)$, this last number being nonnegative by assumption (24). Then, using $\sum_{i}\left(r_{i}\right)^{-1}+s=1$, by Young's inequality we have

$$
\prod_{i=1}^{m}\left(u_{i}^{n}\right)^{q_{i}} \leq \sum_{i=1}^{m} r_{i}^{-1}\left(u_{i}^{n}\right)^{m_{i}+1}+s
$$

This implies that the product $\prod_{i}\left(u_{i}^{n}\right)^{q_{i}}$ is itself uniformly integrable and similarly for $\prod_{i}\left(u_{i}^{n}\right)^{p_{i}}$. Therefore, as in Corollary 2.9, we can pass to the limit in $L^{1}\left(Q_{T}\right)$ for the nonlinear reaction terms of the approximate problem to System (23).

Finally, to treat the case $k_{1} \neq k_{2}$, if for instance $p_{1}-q_{1} \neq 0$, we may just change the definition of the functions $w_{1}^{n}, z_{1}^{n}$ as

$$
w_{1}^{n}:=u_{1}^{n} \log \left(\lambda u_{1}^{n}\right)-u_{1}^{n}+1 / \lambda, z_{1}^{n}=\int_{1}^{u_{1}^{n}} \log (\lambda r) \varphi_{1}^{\prime}(r) d r,
$$

with $\lambda^{p_{1}-q_{1}}:=k_{1} / k_{2}$. The rest is unchanged.
Proof of Corollary 2.11. Again, we consider the same approximation as in the proof of Corollary 2.8. By Theorem 2.6, it is sufficient to prove that the $L^{1}\left(Q_{T}\right)$-estimate (20) holds. Let us denote $P=\left(p_{i j}\right)_{1 \leq i, j \leq m}$. By Assumption (25), and using (19), we have

$$
\forall i=1, \ldots, m, \sum_{j} p_{i j} f_{j}^{n}\left(u^{n}\right)=\frac{\sum_{j} p_{i j} f_{j}\left(u^{n}\right)}{1+\frac{1}{n} \sum_{p}\left|f_{p}\left(u^{n}\right)\right|} \leq b_{i}\left[1+\sum_{j}\left(u_{j}^{n}\right)^{m_{j}+1}\right] .
$$

Since the right-hand side is nonnegative, we can even write

$$
\forall i=1, \ldots, m,\left[\sum_{j} p_{i j} f_{j}^{n}\left(u^{n}\right)\right]^{+} \leq b_{i}\left[1+\sum_{j}\left(u_{j}^{n}\right)^{m_{j}+1}\right] .
$$

But, by Theorem 2.7, $u_{j}^{n}$ is bounded in $L^{m_{j}+1}\left(Q_{T}\right)$ independently of $n$. Therefore, for some $C(T) \in(0,+\infty)$

$$
\int_{Q_{T}}\left[\sum_{j} p_{i j} f_{j}^{n}\left(u^{n}\right)\right]^{+} \leq C(T)
$$

Now multiplying each $j$-th equation of the approximate System (16) by $p_{i j}$ and summing over $j$ leads, for all $i=$ $1, \ldots, m$, to

$$
\sum_{j} p_{i j}\left[\partial_{t} u_{j}^{n}-\Delta \varphi_{j}\left(u_{j}^{n}\right)\right]+\left[\sum_{j} p_{i j} f_{j}^{n}\left(u^{n}\right)\right]^{-}=\left[\sum_{j} p_{i j} f_{j}^{n}\left(u^{n}\right)\right]^{+}
$$

Integrating over $Q_{T}$ and using positivity of the various terms gives

$$
\int_{Q_{T}}\left[\sum_{j} p_{i j} f_{j}^{n}\left(u^{n}\right)\right]^{-} \leq \int_{Q_{T}}\left[\sum_{j} p_{i j} f_{j}^{n}\left(u^{n}\right)\right]^{+}+\sum_{j} p_{i j} u_{j 0} .
$$

We deduce that for some $C(T) \in(0,+\infty)$

$$
\sum_{i} \int_{Q_{T}}\left|\sum_{j} p_{i j} f_{j}^{n}\left(u^{n}\right)\right| \leq C(T) \text { or } \int_{Q_{T}}\left\|P f^{n}\left(u^{n}\right)\right\| \leq C(T),
$$

where $\forall r \in \mathbb{R}^{m},\|r\|=\sum_{i}\left|r_{i}\right|$. If we denote also by $\|\cdot\|$ the induced norm on $m \times m$ matrices, then we have

$$
\int_{Q_{T}}\left\|f^{n}\left(u^{n}\right)\right\|=\int_{Q_{T}}\left\|P^{-1} P f^{n}\left(u^{n}\right)\right\| \leq\left\|P^{-1}\right\| \int_{Q_{T}}\left\|P f^{n}\left(u^{n}\right)\right\| \leq C(T) .
$$

## Conflict of interest statement

## No conflict of interest.

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