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Finite time blowup for the parabolic–parabolic Keller–Segel system with critical diffusion

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Abstract

The present paper is concerned with the parabolic–parabolic Keller–Segel system

$$\partial_t u = \mathrm{div}\left(\nabla u^{q+1} - u\nabla v\right), \quad t > 0, \ x \in \Omega,$$
$$\partial_t v = \Delta v - \alpha v + u, \quad t > 0, \ x \in \Omega,$$
$$(u, v)(0) = (u_0, v_0) \geq 0, \quad x \in \Omega,$$

with degenerate critical diffusion $q = q_\star := (N-2)/N$ in space dimension $N \geq 3$, the underlying domain Ω being either $\Omega = \mathbb{R}^N$ or the open ball $\Omega = B_R(0)$ of $\mathbb{R}^N$ with suitable boundary conditions. It has remained open whether there exist solutions blowing up in finite time, the existence of such solutions being known for the parabolic–elliptic reduction with the second equation replaced by $0 = \Delta v - \alpha v + u$. Assuming that $N = 3, 4$ and $\alpha > 0$, we prove that radially symmetric solutions with negative initial energy blow up in finite time in $\Omega = \mathbb{R}^N$ and in $\Omega = B_R(0)$ under mixed Neumann–Dirichlet boundary conditions. Moreover, if $\Omega = B_R(0)$ and Neumann boundary conditions are imposed on both u and v, we show the existence of a positive constant C depending only on N, Ω, and the mass of u_0 such that radially symmetric solutions blow up in finite time if the initial energy does not exceed $-C$. The criterion for finite time blowup is satisfied by a large class of initial data.

MSC: 35B44; 35B33; 35K15; 35K65; 35Q92

Keywords: Blowup; Radial symmetry; Nonlinear diffusion; Parabolic–parabolic Keller–Segel system

1. Introduction

This paper is concerned with the generalized parabolic–parabolic Keller–Segel system

$$\partial_t u = \mathrm{div}\left(\nabla u^{q+1} - u\nabla v\right), \quad t > 0, \ x \in \Omega, \tag{1.1}$$

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$$\tau \partial_t v = \Delta v - \alpha v + u \,, \quad t > 0 \,,\ x \in \Omega \,, \tag{1.2}$$

$$(u, v)(0) = (u_0, v_0) \geq 0 \,, \quad x \in \Omega \,, \tag{1.3}$$

where τ is a positive constant, and q and α are non-negative parameters. When Ω is a bounded domain of $\mathbb{R}^N$, the system (1.1)–(1.3) is supplemented with either Neumann boundary conditions

$$\partial_\nu u^{q+1} = \partial_\nu v = 0 \,, \quad t > 0 \,,\ x \in \partial\Omega \,, \tag{1.4}$$

or mixed Neumann–Dirichlet boundary conditions

$$\partial_\nu u^{q+1} - u\partial_\nu v = v = 0 \,, \quad t > 0 \,,\ x \in \partial\Omega \,. \tag{1.5}$$

A salient feature of non-negative solutions to (1.1)–(1.3) satisfying the boundary conditions (1.4) or (1.5) or a suitable decay condition at spatial infinity is the conservation of mass of u throughout time evolution, that is,

$$\|u(t)\|_1 = \|u_0\|_1 \qquad \text{for } t \in (0, T_{\max}) \,, \tag{1.6}$$

where $T_{\max}$ is the maximal existence time of the solution and $\|\cdot\|_p$ denotes the L^p-norm for $p \in [1,\infty]$. It is simply obtained by integrating (1.1) over the domain and using Green's formula, the boundary terms vanishing as a consequence of the boundary behavior. Another noteworthy property of (1.1)–(1.3) is that the energy $F[u,v]$ given by

$$F[u,v] := \int_\Omega \left(\frac{u^{q+1}}{q} + \frac{|\nabla v|^2}{2} + \frac{\alpha}{2} v^2 - uv \right) dx \,, \tag{1.7}$$

is a Liapunov functional, the term u^{q+1}/q being replaced by $u \ln u$ when $q = 0$.

The system (1.1)–(1.3) with $q = 0$ and $N = 2$ supplemented with Neumann boundary conditions (1.4) in a bounded domain was originally derived by Keller and Segel [17] as a model of aggregation of cells moving towards higher concentration gradients of a chemical substance generated by the cells. From a mathematical viewpoint, the aggregation of cells is defined as the blowup of $\|u(t)\|_\infty$ in finite time, that is,

$$\limsup_{t \to T_{\max}} \|u(t)\|_\infty = \infty \quad \text{for some finite} \quad T_{\max} \in (0,\infty) \,.$$

Since it was too difficult to treat the blowup issue in the parabolic–parabolic system, a simplified version with $\tau = 0$ was introduced. Its generalized form reads

$$\partial_t u = \mathrm{div}\left(\nabla u^{q+1} - u \nabla v \right) \,, \quad t > 0 \,,\ x \in \Omega \,, \tag{1.8}$$

$$0 = \Delta v - \alpha v + u \,, \quad t > 0 \,,\ x \in \Omega \,, \tag{1.9}$$

$$u(0) = u_0 \geq 0 \,, \quad x \in \Omega \,, \tag{1.10}$$

supplemented with the boundary conditions (1.4) or (1.5) when Ω is a bounded domain of $\mathbb{R}^N$. The system (1.8)–(1.10) is now a parabolic–elliptic system and can actually be reduced to a single nonlocal parabolic equation by expressing v in terms of u with the help of the Green function associated to the Laplace operator. This particular feature marks a serious difference between the "parabolic–parabolic" and "parabolic–elliptic" versions of the Keller–Segel system from a mathematical point of view.

These two systems have attracted considerable interest since they not only reproduce qualitatively some observed biological phenomena such as the aggregation of cells (also referred to as chemotactic collapse in the literature) but also display a wide variety of dynamical behaviors. Indeed, it is by now well-known that, when $N \geq 2$, there is a critical value $q_\star := (N-2)/N$ of the parameter q which separates two different behaviors: when $q > q_\star$, the diffusion term $\mathrm{div}\left(\nabla u^{q+1}\right)$ dominates the attractive drift term $-\mathrm{div}\,(u\nabla v)$ and the initial value problems (1.1)–(1.3) and (1.8)–(1.10) are globally well-posed for a large class of integrable and non-negative initial data. When $q < q_\star$, the dynamics is rather governed by the attractive drift term leading to unbounded solutions, the diffusion term still allowing for the existence of global solutions for sufficiently small initial data. The critical case $q = q_\star$ offers an interesting novelty as a new parameter, the mass $\|u_0\|_1$ of the initial condition, comes into play: there is a threshold value $M_c(N)$ of this parameter (with $M_c(2) = 8\pi$) below which solutions exist globally and above which finite time blowup is expected to occur, at least for some initial conditions, a proof being only available for the parabolic–elliptic reduction (1.8)–(1.10) until recently. We refer to [2,13,25,24] and the references therein for a more complete and accurate description of the available results.

For the parabolic–parabolic system (1.1)–(1.3), the global existence issue can be handled in more or less the same way as (1.8)–(1.10), see [3,5,9,14,16,19,22,26] and the references therein, though some peculiar phenomenon might occur when $q = 0$ and $N = 2$ [1]. The occurrence of finite time blowup turns out to be much more difficult, which contrasts markedly with the parabolic–elliptic system (1.8)–(1.10). In particular, no valuable information seem to be provided directly by the time evolution of the second moment $\int_\Omega |x - x_0|^2 u(t, x)dx$ around some point $x_0 \in \overline{\Omega}$ of u for the parabolic–parabolic system. It also does not seem to be possible to reduce (1.1)–(1.3) to a single equation even in the radially symmetric case. These two approaches being at the heart of the blowup results for the parabolic–elliptic system (1.8)–(1.10), only a few blowup results are therefore available for the parabolic–parabolic system (1.1)–(1.3). Solutions blowing up in finite time have been constructed in the non-critical case $q < q_\star = (N - 2)/N$ for $N \geq 2$ [7,8,27], see also [4,6] for $N = 1$ with different diffusion coefficients.

In the present paper, we focus on the parabolic–parabolic system (1.1)–(1.3) in the critical case $q = q_\star = (N - 2)/N$ and first recall that it gives rise to a more complex dynamics, even for the simplified parabolic–elliptic system (1.8)–(1.10). More generally, peculiar phenomena generated by the critical relation between the equation and the dimension appear in a variety of partial differential equations, and it is shared the common belief that criticality brings various difficulties to mathematical treatment.

For many years, the only known solution to (1.1)–(1.3) blowing up in finite time in the critical case $q = q_\star$ was a particular radially symmetric solution with $\|u_0\|_1 > 8\pi$ constructed in [12] for $q = 0$ and $N = 2$, which was based on their previous result for the corresponding parabolic–elliptic system [11]. Though of great interest, this result is not fully satisfactory. According to the biological experiments that motivated the modeling by Keller and Segel [17] and the mathematical results already known for the simplified system (1.8)–(1.10), finite time blowup is expected to occur for a large class of initial data for the full system (1.1)–(1.3) with $q = 0$ and $N = 2$. A positive answer to this issue has been recently provided in [20,21] where it is shown that there is a large class of radially symmetric initial data such that the corresponding solutions to (1.1)–(1.3) blow up in finite time. Roughly speaking, the proof relies on a detailed study of the time evolution of the energy $F[u, v]$ defined in (1.7). The main difference between the non-critical case $q < q_\star$ handled in [7,8,27] and the critical case $q = q_\star$ considered in [20,21] is that it suffices to estimate the negative term of the energy in the former while the interplay between the positive and the negative contributions in the energy have to be taken into account in the latter. Let us finally mention that a completely different approach is used in [23] to identify the blowup profile of radially symmetric blowing up solutions to (1.1)–(1.3) in $\mathbb{R}^2$ with $q = 0$ and $N = 2$ but the result is restricted to initial data having a mass slightly above the critical mass $M_c(2) = 2\pi$.

The purpose of this paper is to study the blowup issue for the generalized parabolic–parabolic Keller–Segel system (1.1)–(1.3) in the critical case $q = q_\star = (N - 2)/N$ in higher space dimensions $N \geq 3$ which has not been considered yet as far as we know. Then $q = q_\star > 0$ and the degeneracy of the diffusion term seems to prevent the use of the approach developed in [11,12] to construct a solution blowing up in finite time since it relies on the linearization of the system. In the present paper, we instead use the energy $F[u, v]$ defined by (1.7) as done in [20,21]. Since the energy has a different form for $q = q_\star$ and $N \geq 3$ and the way to evaluate it in [20,21] was based on properties peculiar to $q = 0$ and $N = 2$ a different approach is required in our situation. In fact, when $N = 2$, the regularity of the second component v of a solution (u, v) to (1.1)–(1.3) which is derived from (1.2) and the boundedness of $\|u\|_1$ is much better than that in $N \geq 3$. Indeed it follows from standard parabolic regularity theory that for any $1 < p < 2$ and $s > 0$ there exist constants $C_1 = C_1(p, \|u_0\|_1, \|v_0\|_1) > 0$, $C_2 = C_2(s, \|u_0\|_1, \|v_0\|_1) > 0$ such that

$$\|v(t)\|_{W^{1,p}} \leq C_1 \quad \text{and} \quad \|v(t)\|_s \leq C_2 \quad \text{for } 0 < t < T_{\max}.$$

Moreover in the radial case, for each $\kappa > 0$ there exists $C_3 = C_3(\kappa, \|u_0\|_1, \|v_0\|_1) > 0$ such that

$$v(t, r) \leq C_3 r^{-\kappa} \quad \text{for } r = |x| > 0 \text{ and } 0 < t < T_{\max}.$$

This higher regularity is of great advantage in the process of evaluating the energy $F[u, v]$ as it allows one to easily dominate terms including v except $\|\nabla v\|_2$. When $N \geq 3$, lack of adequate regularity of v prevents us from bounding terms related to v as seen in our proof in the subsequent sections. We have not completely cleared the difficulty because we assume $N = 3, 4$. One would need further devices to overcome the restriction of spatial dimension. Differently from the case of non-critical degeneracy of diffusion [8], the effect of less regularity is serious in the delicate estimates in the critical case that we treat in this paper. Furthermore the following identity was essentially used to estimate the energy in [20,21]:

$$r^2u(t,r) = 2U(t,r) - \frac{1}{2}U^2(t,r) + \int_0^r \rho u(t,\rho)V(t,\rho)d\rho - \int_0^r \rho u(t,\rho)F(t,\rho)d\rho$$
$$+ \int_0^r \rho^2\sqrt{u(t,\rho)}\left(\frac{\partial_r u(t,\rho)}{\sqrt{u(t,\rho)}} - \sqrt{u(t,\rho)}\partial_r v(t,\rho)\right)d\rho$$

for $r > 0$ and $0 < t < T_{\max}$, where

$$U(t,r) := \int_0^r \rho u(t,\rho)d\rho\ ,\ V(t,r) := \int_0^r \rho v(t,\rho)d\rho\ ,\ F(t,r) := -\int_0^r \rho\partial_t v(t,\rho)d\rho$$

for $r > 0$ and $0 < t < T_{\max}$. The important value $r_0 = r_0(t)$ of r is such that $U(t,r_0) = 4$, which means that the mass of u in the ball $B_{r_0}(0)$ equals the threshold value 8π between finite time blowup and global existence of solutions. As is easily seen the first two terms in the right-hand side vanish at $r = r_0$ and turn out to be negative for $r > r_0$. This fact was very useful not only to evaluate the energy but also to show that, if the energy is initially negative, then the solution blows up in finite time. However, it seems difficult to derive a similar identity in the case of degenerate diffusion. Therefore we must take a different way from the method in [20,21].

More specifically, we assume throughout this paper that

$$N \geq 3\ , \quad q = q_\star = \frac{N-2}{N} \in (0,1)\ , \quad \tau = 1\ , \quad \alpha > 0\ , \tag{1.11}$$

and, in order to handle simultaneously the case of the whole space $\mathbb{R}^N$ and the case of the ball $B_R(0)$, we set

$$\Omega_R := B_R(0) \quad \text{for} \quad R \in (0,\infty) \quad \text{and} \quad \Omega_\infty := \mathbb{R}^N\ .$$

Let $R \in (0,\infty]$ and (u_0, v_0) be non-negative and radially symmetric functions satisfying

$$u_0 \in L^1(\Omega_R; (1+|x|^2)dx) \cap L^{q+1}(\Omega_R)\ , \quad v_0 \in W^{1,1}(\Omega_R) \cap H^1(\Omega_R)\ , \tag{1.12}$$

as well as

$$u_0 \in L^\infty(\Omega_R)\ , \quad \nabla u_0^{q+1} \in L^2(\Omega_R)\ , \quad v_0 \in W^{1,\infty}(\Omega_R)\ . \tag{1.13}$$

The conditions (1.12) and (1.13), that are imposed in [14–16], are likely to be too strong for the local existence of a weak solution. However we assume them here for simplicity since our main interest is the finite time blowup and not finding optimal conditions on the initial data guaranteeing the local existence of weak solutions. Let us emphasize here that the proof of the finite time blowup given below only requires the regularity (1.12) on (u_0, v_0). Consider a radially symmetric weak solution (u, v) to (1.1)–(1.3) supplemented with the boundary conditions (1.4) or (1.5) if $R < \infty$ and let $T_{\max}$ be its maximal existence time, see Definition 2.1. Assume further that (u, v) satisfies the energy inequality

$$\frac{d}{dt}\mathcal{F}[u(t), v(t)] + \mathcal{D}[u(t), v(t)] \leq 0\ , \quad t \in [0, T_{\max})\ , \tag{1.14}$$

where $r := |x|$,

$$\mathcal{F}[u,v] := \int_0^R \left(\frac{u^{q+1}}{q} + \frac{|\partial_r v|^2}{2} + \alpha\frac{v^2}{2} - uv\right) r^{N-1}dr \tag{1.15}$$

and

$$\mathcal{D}[u,v] := \int_0^R \left(u \left| \frac{q+1}{q} \partial_r u^q - \partial_r v \right|^2 + |\partial_t v|^2 \right) r^{N-1} dr \geq 0 . \tag{1.16}$$

The main result of this paper then reads:

Theorem 1.1. *Let $N = 3, 4$, $q = 1 - 2/N$, and $R \in (0, \infty]$. Let $\mathcal{I}_R$ be the class of pairs (u_0, v_0) of non-negative and radially symmetric functions satisfying (1.12) and (1.13). There is a non-negative constant C_0 depending only on N, α, and R such that, if $(u_0, v_0) \in \mathcal{I}_R$ satisfies*

$$\mathcal{F}[u_0, v_0] < -C_0 \|u_0\|_1^2 , \tag{1.17}$$

then a corresponding radially symmetric weak solution (u, v) to the parabolic–parabolic system (1.1)–(1.3) supplemented with (1.4) or (1.5) if $R < \infty$ blows up in finite time. Furthermore, if $R = \infty$ or $R < \infty$ and the boundary conditions are the mixed Neumann–Dirichlet ones (1.5), then one can take $C_0 = 0$ in (1.17).

When $R = \infty$, the outcome of Theorem 1.1 combined with the analysis performed in [3] confirms the threshold phenomenon already alluded to previously. Indeed, according to [2, Proposition 3.4], there is a critical mass $M_c = M_c(N) > 0$ depending only on N such that $\mathcal{F}[u_0, v_0] > 0$ for all non-negative and radially symmetric initial data (u_0, v_0) satisfying (1.12) as well as $\|u_0\|_1 < M_c$ and there is a global weak solution to (1.1)–(1.3) emanating from (u_0, v_0) by [3, Theorem 1]. On the opposite, there are non-negative and radially symmetric initial data (u_0, v_0) satisfying (1.12) as well as $\|u_0\|_1 > M_c$ and $\mathcal{F}[u_0, v_0] < 0$ and Theorem 1.1 guarantees that a weak solution to (1.1)–(1.3) emanating from (u_0, v_0) blows up in finite time.

That there are indeed initial data to which Theorem 1.1 applies is guaranteed by the next result.

Theorem 1.2. *Let $N = 3, 4$, $q = 1 - 2/N$, $R \in (0, \infty]$, and $M > M_c(N)$. Then there exist $(u_0^*, v_0^*) \in \mathcal{I}_R$ with $\|u_0^*\|_1 = M$ and $\varepsilon > 0$ such that if $(u_0, v_0) \in \mathcal{I}_R$ satisfies $\|(u_0, v_0) - (u_0^*, v_0^*)\|_{L^{q+1}(\Omega_R) \times W^{1,2}(\Omega_R)} < \varepsilon$ then a corresponding radially symmetric weak solution to (1.1)–(1.3) supplemented with (1.4) or (1.5) if $R < \infty$ blows up in finite time. Furthermore the class of initial data for which a corresponding radially symmetric weak solution to (1.1)–(1.3) supplemented with (1.4) or (1.5) if $R < \infty$ blows up in finite time is dense in $\mathcal{I}_R$ with respect to the weaker topology of $L^p(\Omega_R) \times W^{1,\sigma}(\Omega_R)$ with $0 < p < 1$ and $1 < \sigma < N/(N-1)$.*

Remark 1.3. Theorems 1.1 and 1.2 are likely to be valid also for $\alpha = 0$. It is actually rather clear that the proof given below readily extends to $\alpha = 0$ when $R < \infty$ and (1.1)–(1.3) is supplemented with the Neumann–Dirichlet boundary conditions (1.5). In the other cases, handling the case $\alpha = 0$ seems to require some technical adaptations to remedy the lack of coercivity of the Laplace operator.

The main ingredient in the proof of Theorem 1.1 is to prove that, if a solution (u, v) exists globally in time, then $t \mapsto \mathcal{F}[u(t), v(t)]$ is bounded below in $(0, \infty)$. Once the boundedness of the energy is shown, we combine it with the evolution of the second moment when $R = \infty$ to complete the proof of Theorem 1.1 while Lemma 2.5 plays a crucial role through a combination with the boundedness of the energy when $R < \infty$. In the next section, we first state the existence of a weak solution to (1.1)–(1.3) along with its properties, see Section 2.1. We next derive in Section 2.2 a differential inequality for $-\mathcal{F}[u, v]$ which reads

$$\frac{d}{dt} (-\mathcal{F}[u, v]) \geq C \left(-\mathcal{F}[u, v]\right)^{(2N-2)/(2N-3)} - 1 ,$$

see Proposition 2.3, and eventually leads to the boundedness of the energy for global solutions since $2N - 2 > 2N - 3 > 0$. When $R = \infty$, the analysis of the evolution of the second moment is performed in Section 3.1 where we also prove Theorem 1.1. We also give a different approach without using the second moment. The proof of Theorem 1.1 for $R < \infty$ is next given in Section 3.3. The last section is devoted to the proof of Theorem 1.2 and we collect some useful properties of the Bessel potentials in the appendix.

2. Free energy and its dissipation

2.1. Weak solutions

We first recall the definition of a weak solution.

Definition 2.1. Let $R \in (0, \infty]$ and (u_0, v_0) be non-negative initial conditions satisfying (1.12) and (1.13) and let $T > 0$. A weak solution to (1.1)–(1.3) (supplemented with the boundary conditions (1.4) or (1.5) when $R < \infty$) is a couple (u, v) of non-negative functions such that

$$(u, v) \in C([0, T]; L^{q+1}(\Omega_R) \times H^1(\Omega_R)) , \quad u^{q+1} \in L^2(0, T; H^1(\Omega_R)) ,$$

and $(u, v)(0) = (u_0, v_0)$, which satisfies (1.1)–(1.2) (as well as either (1.4) or (1.5) when $R < \infty$) in a weak sense and the energy inequality (1.14).

The existence of a local weak solution to (1.1)–(1.3) with homogeneous Neumann boundary conditions (1.4) in a ball is shown in [16, Theorem 1.1]. A similar result does not seem to be available for (1.1)–(1.3) in $\mathbb{R}^N$ or in the ball with the mixed Neumann–Dirichlet boundary conditions but can nevertheless be proved by adapting arguments from [2,14,16,26]. Summarizing, one has the following existence result:

Proposition 2.2. *Let $R \in (0, \infty]$ and (u_0, v_0) be non-negative initial conditions satisfying (1.12) and (1.13). There are $T_{\max} \in (0, \infty]$ and a couple (u, v) of non-negative functions defined on $[0, T_{\max}) \times \Omega_R$ such that (u, v) is a weak solution to (1.1)–(1.3) (supplemented with the boundary conditions (1.4) or (1.5) when $R < \infty$) on $[0, T]$ in the sense of Definition 2.1 for all $T < T_{\max}$. Moreover, it satisfies the alternative:*

$$\textit{either} \quad T_{\max} = \infty \quad \textit{or} \quad T_{\max} < \infty \ \ \textit{with} \ \ \lim_{t \to T_{\max}} \|u(t)\|_\infty = \infty .$$

In addition, $(u(t), v(t))$ are radially symmetric for all $t \in (0, T_{\max})$ if (u_0, v_0) are radially symmetric.

From now on, we fix $R \in (0, \infty]$, a pair of non-negative radially symmetric initial conditions (u_0, v_0) satisfying (1.12) and (1.13), and a non-negative radially symmetric weak solution (u, v) to (1.1)–(1.3) (supplemented with the boundary conditions (1.4) or (1.5) when $R < \infty$) given by Proposition 2.2. We assume in addition that

$$\mathcal{F}[u_0, v_0] < 0 \tag{2.1}$$

and recall that

$$\|u(t)\|_1 = M := \|u_0\|_1 , \quad t \in [0, T_{\max}) , \tag{2.2}$$

while (1.14), (2.1), and the non-negativity of $\mathcal{D}$ guarantee that

$$\mathcal{F}[u(t), v(t)] \le \mathcal{F}[u_0, v_0] < 0 , \quad t \in [0, T_{\max}) . \tag{2.3}$$

2.2. A differential inequality for the free energy

As announced in the introduction, the cornerstone of the proof of Theorem 1.1 is the following differential inequality for the energy:

Proposition 2.3. *Assume that $N \in \{3, 4\}$. There is $C_1 > 0$ depending only on N, M, α, R, and $\|\nabla v_0\|_1$ such that*

$$\frac{d}{dt}(-\mathcal{F}[u, v]) \ge C_1 (-\mathcal{F}[u, v])^{(2N-2)/(2N-3)} - 1 , \quad t \in [0, T_{\max}) . \tag{2.4}$$

The proof of Proposition 2.3 requires several steps. We begin with a weighted L^∞-estimate for v as in [27, Lemma 3.2].

Lemma 2.4. *Given $p \in [1, N/(N-1))$, there is $C_2(p) > 0$ depending only on p, N, α, R, $\|u_0\|_1$, $\|v_0\|_1$, and $\|\nabla v_0\|_p$ such that*

$$0 \le v(t,r) \le C_2(p) r^{(p-N)/p} \ , \quad r > 0 \ , \ t \in [0, T_{\max}) \ . \tag{2.5}$$

Furthermore,

$$\|v(t)\|_1 \le \|v_0\|_1 + \frac{M}{\alpha} \ , \ t \in [0, T_{\max}) \ . \tag{2.6}$$

Proof. Let $(G(t))_{t\ge 0}$ be the semigroup associated to the operator $-\Delta$ in either Ω_∞ or in Ω_R, $R < \infty$, supplemented with either homogeneous Neumann or Dirichlet boundary conditions. Then v is given by the variation-of-constants formula:

$$v(t) = G(t) v_0 e^{-\alpha t} + \int_0^t G(t-s) u(s) e^{\alpha(s-t)}\, ds \ , \quad t \in [0, T_{\max}) \ .$$

On the one hand, since $(G(t))_{t\ge 0}$ is a semigroup of contractions in $L^1(\Omega_R)$, the estimate (2.6) follows from (2.2) as

$$\|v(t)\|_1 \le \|v_0\|_1 e^{-\alpha t} + M \frac{1-e^{-\alpha t}}{\alpha} \le \|v_0\|_1 + \frac{M}{\alpha} \ .$$

On the other hand, if $p \in [1, N/(N-1))$, we infer from the regularizing properties of the heat semigroup and (2.2) that

$$\begin{aligned}
\|\nabla v(t)\|_p &\le e^{-\alpha t} \|\nabla G(t) v_0\|_p + \int_0^t \|\nabla G(t-s) u(s)\|_p e^{\alpha(s-t)}\, ds \\
&\le e^{-\alpha t} \|\nabla v_0\|_p + C(p) \int_0^t (t-s)^{-(1/2)-N(p-1)/(2p)} \|u(s)\|_1 e^{\alpha(s-t)}\, ds \\
&\le \|\nabla v_0\|_p + C(p) \|u_0\|_1 \int_0^t s^{-(1/2)-N(p-1)/(2p)} e^{-\alpha s}\, ds \\
&\le \|\nabla v_0\|_p + C(p) \|u_0\|_1 \ .
\end{aligned} \tag{2.7}$$

Now, let $r \in (0, R)$ and $r_0 \in (0, R)$. Arguing as in [10, Lemma 2.5], it follows from (2.6) that there is $r_1(t) \in (r_0/2, r_0)$ such that

$$v(t, r_1(t)) = \frac{2}{r_0} \int_{r_0/2}^{r_0} v(t,\rho)\, d\rho \le \left(\frac{2}{r_0}\right)^N \int_{r_0/2}^{r_0} v(t,\rho) \rho^{N-1}\, d\rho \le C\, r_0^{-N} \ .$$

Consequently,

$$v(t,r) = v(t, r_1(t)) + \int_{r_1(t)}^r \partial_r v(t,\rho)\, d\rho \le C\, r_0^{-N} + \int_{\min\{r, r_0/2\}}^R |\partial_r v(t,\rho)|\, d\rho \ . \tag{2.8}$$

For $p = 1$ we infer from (2.7) and (2.8) that

$$\begin{aligned}
v(t,r) &\le C\, r_0^{-N} + \min\{r, r_0/2\}^{1-N} \int_{\min\{r, r_0/2\}}^R |\partial_r v(t,\rho)| \rho^{N-1}\, d\rho \\
&\le C \left(r_0^{-N} + \min\{r, r_0/2\}^{1-N} \right) ,
\end{aligned}$$

while, for $p \in (1, N/(N-1))$, we deduce from (2.7), (2.8), and Hölder's inequality that

$$\begin{aligned} v(t,r) &\le C\, r_0^{-N} + \left(\int_{\min\{r,r_0/2\}}^{R} |\partial_r v(t,\rho)|^p \rho^{N-1}\, d\rho \right)^{1/p} \left(\int_{\min\{r,r_0/2\}}^{R} \rho^{-(N-1)/(p-1)}\, d\rho \right)^{(p-1)/p} \\ &\le C \left(r_0^{-N} + \min\{r, r_0/2\}^{(p-N)/p} \right) . \end{aligned}$$

We have thus shown that, for $p \in [1, N/(N-1))$, $r \in (0,R)$, and $r_0 \in (0,R)$, there holds

$$v(t,r) \le C \left(r_0^{-N} + \min\{r, r_0/2\}^{(p-N)/p} \right) . \tag{2.9}$$

Now set $R_0 := \min\{R/2, 1\}$ and consider $r \in (0,R)$. Either $r \in (0, R_0/2)$ and we infer from (2.9) with $r_0 = R_0$ that

$$v(t,r) \le C \left(\frac{2}{R_0} \right)^{(N-p)/p} R_0^{((N+1)p-N)/p} + C r^{(p-N)/p} \le C r^{(p-N)/p} .$$

Or $r \ge R_0/2$ and it follows from (2.9) with $r_0 = r$ that

$$v(t,r) \le C \left(r^{-((N+1)p-N)/p} + 1 \right) r^{(p-N)/p} \le C \left(R_0^{-((N+1)p-N)/p} + 1 \right) r^{(p-N)/p} \le C\, r^{(p-N)/p} ,$$

which completes the proof. □

We next set

$$f := -\partial_t v = -\Delta v + \alpha v - u , \tag{2.10}$$

$$g := \frac{2q+2}{2q+1} \nabla u^{(2q+1)/2} - \sqrt{u} \nabla v , \tag{2.11}$$

so that the dissipation $\mathcal{D}[u,v]$ defined in (1.16) reads

$$\mathcal{D}[u,v] = \frac{1}{\sigma_N} \left(\|f\|_2^2 + \|g\|_2^2 \right) , \tag{2.12}$$

with $\sigma_N := N|B_1(0)|$. Moreover, for any $r \in (0,R] \cap (0,\infty)$, we define $\mathcal{F}_r[u,v]$ as the contribution of the ball $B_r(0)$ to the energy, namely,

$$\mathcal{F}_r[u,v] := \int_0^r \left(\frac{u^{q+1}}{q} + \frac{|\partial_r v|^2}{2} + \alpha \frac{v^2}{2} - uv \right) \rho^{N-1} d\rho . \tag{2.13}$$

Lemma 2.5. *For all $r_1 \in (0,R] \cap (0,\infty)$,*

$$\begin{aligned} -\mathcal{F}_{r_1}[u,v] = {} & \frac{1}{N-2} \int_0^{r_1} g \sqrt{u} r^N\, dr - \frac{\alpha}{N-2} \int_0^{r_1} v^2 r^{N-1}\, dr - \frac{1}{N-2} \int_0^{r_1} f\, \partial_r v r^N\, dr \\ & - \int_0^{r_1} f v r^{N-1}\, dr - \frac{r_1^N}{2(N-2)} (\partial_r v(r_1))^2 - \frac{r_1^N}{N-2} (u(r_1))^{q+1} \\ & + \frac{\alpha r_1^N}{2(N-2)} (v(r_1))^2 - r_1^{N-1} v(r_1) \partial_r v(r_1) . \end{aligned} \tag{2.14}$$

Proof. Owing to the definition (2.10) of f,

$$\begin{aligned}\int_0^{r_1} uvr^{N-1}\,dr &= \int_0^{r_1}\left[-\frac{1}{r^{N-1}}\partial_r\left(r^{N-1}\partial_r v\right)+\alpha v - f\right] vr^{N-1}\,dr\\ &= -\int_0^{r_1} v\partial_r\left(r^{N-1}\partial_r v\right)\,dr + \int_0^{r_1}\left(\alpha v^2 - fv\right)r^{N-1}\,dr\\ &= -r_1^{N-1}v(r_1)\partial_r v(r_1) + \int_0^{r_1}\left[(\partial_r v)^2+\alpha v^2 - fv\right]r^{N-1}\,dr\ .\end{aligned}$$

Using the previous identity, $-\mathcal{F}_{r_1}[u,v]$ reads

$$\begin{aligned}-\mathcal{F}_{r_1}[u,v] = \frac{1}{2}\int_0^{r_1}\left[(\partial_r v)^2+\alpha v^2\right]r^{N-1}\,dr - \frac{1}{q}\int_0^{r_1} u^{q+1}r^{N-1}\,dr\\ -\int_0^{r_1} fvr^{N-1}\,dr - r_1^{N-1}v(r_1)\partial_r v(r_1)\ .\end{aligned} \tag{2.15}$$

Using once more (2.10) we realize that

$$\frac{1}{2}\partial_r\left[\left(r^{N-1}\partial_r v\right)^2\right] = r^{N-1}\partial_r v\ \partial_r\left(r^{N-1}\partial_r v\right) = r^{2N-2}\partial_r v\,(\alpha v - u - f)\ ,$$

so that

$$\frac{1}{2}\int_0^{r_1}\partial_r\left[\left(r^{N-1}\partial_r v\right)^2\right]r^{2-N}\,dr = \int_0^{r_1} r^N\partial_r v\,(\alpha v - u - f)\,dr\ ,$$

$$\begin{aligned}\frac{r_1^N}{2}\left(\partial_r v(r_1)\right)^2 + \frac{N-2}{2}\int_0^{r_1}(\partial_r v)^2 r^{N-1}\,dr = \frac{\alpha r_1^N}{2}v(r_1)^2 - \frac{\alpha N}{2}\int_0^{r_1} v^2 r^{N-1}\,dr\\ -\int_0^{r_1} u\partial_r v r^N\,dr - \int_0^{r_1} f\partial_r v r^N\,dr\ .\end{aligned}$$

Replacing $\sqrt{u}\partial_r v$ with (2.11) we obtain

$$\begin{aligned}\frac{1}{2}\int_0^{r_1}(\partial_r v)^2 r^{N-1}\,dr &= -\frac{r_1^N}{2(N-2)}\left(\partial_r v(r_1)\right)^2 + \frac{\alpha r_1^N}{2(N-2)}v(r_1)^2\\ &\quad - \frac{\alpha N}{2(N-2)}\int_0^{r_1} v^2 r^{N-1}\,dr + \frac{1}{N-2}\int_0^{r_1} g\sqrt{u}r^N\,dr\\ &\quad - \frac{1}{N-2}\int_0^{r_1}\partial_r u^{q+1}r^N\,dr - \frac{1}{N-2}\int_0^{r_1} f\partial_r v r^N\,dr\\ &= -\frac{r_1^N}{2(N-2)}\left(\partial_r v(r_1)\right)^2 + \frac{\alpha r_1^N}{2(N-2)}v(r_1)^2\end{aligned}$$

$$
- \frac{\alpha}{2q} \int_0^{r_1} v^2 r^{N-1}\, dr + \frac{1}{N-2} \int_0^{r_1} g\sqrt{u} r^N\, dr
$$
$$
- \frac{r_1^N}{N-2} u(r_1)^{q+1} + \frac{1}{q} \int_0^{r_1} u^{q+1} r^{N-1}\, dr - \frac{1}{N-2} \int_0^{r_1} f \partial_r v r^N\, dr .
$$

Combining (2.15) and the previous identity gives (2.14). □

In the next step, we estimate the terms involving f in (2.14). To this end, we decompose v as

$$
v = \tilde{v} + \hat{v} , \tag{2.16}
$$

where, if $R = \infty$,

$$
\tilde{v} := \mathcal{B}_\alpha \star u \quad \text{and} \quad \hat{v} := \mathcal{B}_\alpha \star f , \tag{2.17}
$$

the Bessel kernel $\mathcal{B}_\alpha$ being defined by

$$
\mathcal{B}_\alpha(x) := \int_0^\infty \exp\left\{ -\frac{|x|^2}{4s} - \alpha s \right\} \frac{ds}{(4\pi s)^{N/2}} , \quad x \in \mathbb{R}^N , \tag{2.18}
$$

and, if $R < \infty$, $(\tilde{v}, \hat{v})$ are the unique solutions to

$$
-\Delta \tilde{v} + \alpha \tilde{v} = u \quad \text{in} \quad \Omega_R , \tag{2.19}
$$
$$
-\Delta \hat{v} + \alpha \hat{v} = f \quad \text{in} \quad \Omega_R , \tag{2.20}
$$

supplemented with homogeneous Neumann or Dirichlet boundary conditions according to whether v satisfies (1.4) or (1.5).

We now handle separately the contributions from $\tilde{v}$ and $\hat{v}$.

Lemma 2.6. *For $r_1 \in (0, R] \cap (0, \infty)$ there is $C_3 > 0$ depending only on N, M, α, and R such that*

$$
\left| \int_0^{r_1} f \left(\frac{r}{N-2} \partial_r \tilde{v} + \tilde{v} \right) r^{N-1}\, dr \right| \le C_3 (1 + r_1) \left(\int_0^{r_1} f^2 r^3\, dr \right)^{1/2} . \tag{2.21}
$$

Proof. Introducing $w(r) := r^{N-2} \tilde{v}(r)$ for $r > 0$, we find that

$$
\int_0^{r_1} f \left(\frac{r}{N-2} \partial_r \tilde{v} + \tilde{v} \right) r^{N-1}\, dr = \frac{1}{N-2} \int_0^{r_1} f \partial_r w r^2\, dr . \tag{2.22}
$$

It follows from the definition (2.16) of $\tilde{v}$ that w solves

$$
r^{N-1} u(r) = -\partial_r \left(r \partial_r w(r) \right) + (N-2) \partial_r w(r) + \alpha r w(r) , \quad r > 0 .
$$

We multiply the above identity by $r \partial_r w(r)$ and integrate with respect to r over $(0, r_1)$ to obtain

$$
\int_0^{r_1} u \partial_r w r^N\, dr = -\frac{r_1^2}{2} \left(\partial_r w(r_1) \right)^2 + (N-2) \int_0^{r_1} (\partial_r w)^2 r\, dr + \alpha \int_0^{r_1} w \partial_r w r^2\, dr ,
$$

and thus

$$
(N-2) \int_0^{r_1} (\partial_r w)^2 r\, dr = \int_0^{r_1} u \partial_r w r^N\, dr + \frac{r_1^2}{2} \left(\partial_r w(r_1) \right)^2 - \alpha \int_0^{r_1} w \partial_r w r^2\, dr . \tag{2.23}
$$

We now infer from Lemma A.1 (i) and (2.2) that

$$0 \le w(r) \le C_{10}\|u\|_1 = MC_{10} \,, \tag{2.24}$$

$$|r\partial_r w(r)| \le r^{N-1}\,|\partial_r \tilde{v}(r)| + (N-2)r^{N-2}\tilde{v}(r) \le (N-1)C_{10}\|u\|_1 = (N-1)MC_{10} \,. \tag{2.25}$$

Estimating the right-hand side of (2.23) with the help of (2.2), (2.24), and (2.25) we end up with

$$\begin{aligned}(N-2)\int_0^{r_1} (\partial_r w)^2 r\, dr &\le (N-1)MC_{10}\frac{\|u\|_1}{\sigma_N} + (N-1)^2M^2C_{10}^2 + \alpha(N-1)M^2C_{10}^2r_1^2 \\ &\le C(1+r_1^2) \,. \end{aligned} \tag{2.26}$$

We finally infer from (2.22), (2.26), and Hölder's inequality that

$$\begin{aligned}\left|\int_0^{r_1} f\left(\frac{r}{N-2}\partial_r\tilde{v} + \tilde{v}\right) r^{N-1}\, dr\right| &\le \frac{1}{N-2}\left(\int_0^{r_1} (\partial_r w)^2 r\, dr\right)^{1/2}\left(\int_0^{r_1} f^2 r^3\, dr\right)^{1/2} \\ &\le C(1+r_1)\left(\int_0^{r_1} f^2 r^3\, dr\right)^{1/2} \,, \end{aligned}$$

as claimed. □

Remark 2.7. Observe that (2.26) implies that $r \mapsto (\partial_r w(r))^2 r$ belongs to $L^1(0, r_1)$, a property which cannot be deduced from (2.25). Indeed, the latter only gives $r\,(\partial_r w(r))^2 \le C/r$ which is not integrable near zero.

We now turn to the contribution of $\hat{v}$ in the same term.

Lemma 2.8. *There is $C_4 > 0$ depending only on N, M, α, and R such that*

$$\left|\int_0^{r_1} f\left(\frac{r}{N-2}\partial_r\hat{v} + \hat{v}\right) r^{N-1}\, dr\right| \le C_4\, r_1^{3/2}\left(1 + \sqrt{r_1|\ln r_1|}\right)\|f\|_2^2 \tag{2.27}$$

for $r_1 \in (0, R]$ in a ball and $r_1 \in (0, \infty)$ in the whole space.

Proof. On the one hand, it follows from (A.7) and the Cauchy–Schwarz inequality that

$$\begin{aligned}\frac{1}{N-2}\left|\int_0^{r_1} f\,\partial_r\hat{v} r^N\, dr\right| &\le \frac{\|f\|_2}{\sqrt{\sigma_N}}\left(\int_0^{r_1} (\partial_r\hat{v})^2 r^{N+1}\, dr\right)^{1/2} \\ &\le C_{11}\frac{\|f\|_2^2}{\sqrt{\sigma_N}}\left(\int_0^{r_1} r^3\, dr\right)^{1/2} \le Cr_1^2\|f\|_2^2 \,. \end{aligned}$$

On the other hand, by (A.6) and the Cauchy–Schwarz inequality,

$$\left|\int_0^{r_1} f\hat{v}r^{N-1}\, dr\right| \le \frac{\|f\|_2}{\sqrt{\sigma_N}}\left(\int_0^{r_1} (\hat{v})^2 r^{N-1}\, dr\right)^{1/2} \le Cr_1^{3/2}\left(1+\sqrt{r_1|\ln r_1|}\right)\|f\|_2^2 \,.$$

Combining the previous two estimates completes the proof of (2.27). □

Gathering the outcomes of Lemma 2.5, Lemma 2.6, and Lemma 2.8 provides the following estimate on the contribution $\mathcal{F}_{r_1}[u, v]$ of the ball $B_{r_1}(0)$, $r_1 \in (0, 1)$, to the energy.

Proposition 2.9. *Assume that $N \in \{3,4\}$. There is $C_5 > 0$ depending only on N, M, α, and R such that, for $r_1 \in (0, \min\{1, R\})$ and $t \in [0, T_{\max})$,*

$$-\mathcal{F}_{r_1}[u(t), v(t)] \le C_5 \left[r_1 \|g(t)\|_2 + r_1 \|f(t)\|_2^2 + \|f(t)\|_2 + r_1^{1-N} \right] .$$

Proof. Since $r_1 \in (0,1)$, we infer from (2.2), (2.14), (2.21), (2.27), and the Cauchy–Schwarz inequality that

$$\begin{aligned}
-\mathcal{F}_{r_1}[u, v] \le{}& \frac{r_1}{N-2} \left(\int_0^{r_1} g^2 r^{N-1}\, dr \right)^{1/2} \left(\int_0^{r_1} u r^{N-1}\, dr \right)^{1/2} \\
&+ C_3(1+r_1) \left(\int_0^{r_1} f^2 r^3\, dr \right)^{1/2} + C_4 r_1^{3/2} \left(1 + \sqrt{r_1 |\ln r_1|}\right) \|f\|_2^2 \\
&- \frac{1}{2(N-2)} \left[r_1^{N/2} \partial_r v(r_1) + (N-2) r_1^{(N-2)/2} v(r_1) \right]^2 \\
&+ \frac{N-2}{2} r_1^{N-2} v(r_1)^2 + \frac{\alpha}{2(N-2)} r_1^N v(r_1)^2 \\
\le{}& C \left[r_1 \|g\|_2 + r_1 \|f\|_2^2 + \left(\int_0^{r_1} f^2 r^3\, dr \right)^{1/2} + r_1^{N-2} v(r_1)^2 \right] .
\end{aligned}$$

Since $2N/(2N-1) < N/(N-1)$ we infer from Lemma 2.4 that

$$r_1^{N-2} v(r_1)^2 \le C_2(2N/(2N-1))\, r_1^{3-2N} r_1^{N-2} \le C\, r_1^{1-N} .$$

Consequently,

$$-\mathcal{F}_{r_1}[u, v] \le C \left[r_1 \|g\|_2 + r_1 \|f\|_2^2 + \left(\int_0^{r_1} f^2 r^3\, dr \right)^{1/2} + r_1^{1-N} \right] ,$$

and we complete the proof of Proposition 2.9 after noticing that $r^3 = r^{N-1} r^{4-N} \le r^{N-1}$ for $r \in (0,1)$ and $N \in \{3,4\}$. □

We supplement Proposition 2.9 with an estimate on the contribution of the complement of $B_{r_1}(0)$, $r_1 \in (0, \min\{1, R\})$, to the energy.

Lemma 2.10. *There is $C_6 > 0$ depending only on N, M, α, R, and $\|\nabla v_0\|_1$ such that*

$$-\mathcal{F}[u(t), v(t)] + \mathcal{F}_{r_1}[u(t), v(t)] \le C_6 r_1^{1-N} \tag{2.28}$$

for $r_1 \in (0, R]$ in a ball and $r_1 \in (0, \infty)$ in the whole space and $t \in [0, T_{\max})$.

Proof. Owing to (2.2) and (2.5) (with $p = 1$), we find

$$\begin{aligned}
\frac{1}{\sigma_N} \int_{\Omega_R \setminus B_{r_1}(0)} \left(uv - \frac{u^{q+1}}{q} - \frac{|\nabla v|^2}{2} - \alpha \frac{v^2}{2} \right) dx &\le \int_{r_1}^R u v r^{N-1} dr \le C_2(1) \int_{r_1}^R u\, dr \\
&\le C r_1^{1-N} \int_{r_1}^R u r^{N-1}\, dr \le C M r_1^{1-N} ,
\end{aligned}$$

as claimed. □

Remark 2.11. In contrast to the outcome of Proposition 2.9 note that the right-hand side of (2.28) involves neither f nor g and is thus not controlled by the dynamics of (1.1)–(1.3). It however vanishes as $r_1 \to \infty$ and will actually only be used when $R = \infty$.

We are now in a position to prove the main result of this section.

Proof of Proposition 2.3. According to (2.3), Proposition 2.9, and Lemma 2.10, there holds

$$\begin{aligned} 0 \le -\mathcal{F}[u,v] &= -\mathcal{F}_{r_1}[u,v] - \mathcal{F}[u,v] + \mathcal{F}_{r_1}[u,v] \\ &\le C_5 \left[r_1 \|g\|_2 + r_1 \|f\|_2^2 + \|f\|_2 + r_1^{2-N} \right] + C_6 r_1^{1-N} \\ &\le C \left[r_1 \left(\|g\|_2 + \|f\|_2^2 \right) + \|f\|_2 + r_1^{1-N} \right] \end{aligned}$$

for all $r_1 \in (0, \min\{1, R\})$. By Young's inequality we further obtain

$$0 \le -\mathcal{F}[u,v] \le C \left[r_1 \left(1 + \|g\|_2^2 + \|f\|_2^2 \right) + \|f\|_2 + r_1^{1-N} \right] , \quad r_1 \in (0,1) .$$

Recalling that $\mathcal{D}[u,v] = \left(\|g\|_2^2 + \|f\|_2^2 \right) / \sigma_N$ by (2.12) and choosing

$$r_1 = \left(1 + R^{-2(N-1)} + \|g\|_2^2 + \|f\|_2^2 \right)^{-1/(2N-2)} \in (0, \min\{1, R\})$$

in the previous inequality give

$$\begin{aligned} 0 \le -\mathcal{F}[u,v] &\le C \left[\left(1 + R^{-2(N-1)} + \|g\|_2^2 + \|f\|_2^2 \right)^{(2N-3)/(2N-2)} + \|f\|_2 \right. \\ &\qquad \left. + \left(1 + R^{-2(N-1)} + \|g\|_2^2 + \|f\|_2^2 \right)^{1/2} \right] \\ &\le C \left[1 + (1 + \mathcal{D}[u,v])^{(2-N)/(2N-2)} \right] (1 + \mathcal{D}[u,v])^{(2N-3)/(2N-2)} \\ &\le C \, (1 + \mathcal{D}[u,v])^{(2N-3)/(2N-2)} . \end{aligned}$$

We finally combine the above inequality with (1.14) to obtain

$$C \, (-\mathcal{F}[u,v])^{(2N-2)/(2N-3)} \le 1 + \mathcal{D}[u,v] \le 1 + \frac{d}{dt} (-\mathcal{F}[u,v]) ,$$

and thereby complete the proof of (2.4). □

For further use we report the following consequence of Proposition 2.3. Recall that $\mathcal{F}[u_0, v_0] < 0$ by (2.1).

Theorem 2.12. *Assume that $N \in \{3,4\}$. There is $C_7 > 0$ depending only on N, M, α, R, and $\|\nabla v_0\|_1$ such that, if $T_{\max} = \infty$,*

$$-\mathcal{F}[u_0, v_0] \le -\mathcal{F}[u(t), v(t)] \le C_7 , \quad t > 0 , \quad \textit{and} \quad \int_0^\infty \mathcal{D}[u(s), v(s)] \, ds \le C_7 . \tag{2.29}$$

Proof. Setting $C_7 := C_1^{-(2N-3)/(2N-2)}$, we assume for contradiction that there is $t_0 \ge 0$ such that $-\mathcal{F}[u(t_0), v(t_0)] > C_7$. Thanks to the differential inequality (2.4), a classical argument entails that $t \mapsto -\mathcal{F}[u(t), v(t)]$ is increasing on $[t_0, \infty)$ and satisfies $-\mathcal{F}[u(t), v(t)] > C_7$ for all $t \ge t_0$. Using again (2.4), we realize that, for $t \ge t_0$,

$$\begin{aligned} \frac{d}{dt} (-\mathcal{F}[u(t), v(t)]) &\ge \left(C_1 - \frac{1}{(-\mathcal{F}[u(t_0), v(t_0)])^{(2N-2)/(2N-3)}} \right) (-\mathcal{F}[u(t), v(t)])^{(2N-2)/(2N-3)} \\ &\qquad + \frac{(-\mathcal{F}[u(t), v(t)])^{(2N-2)/(2N-3)}}{(-\mathcal{F}[u(t_0), v(t_0)])^{(2N-2)/(2N-3)}} - 1 \\ &\ge \delta_0 \, (-\mathcal{F}[u(t), v(t)])^{(2N-2)/(2N-3)} \end{aligned}$$

for some $\delta_0 > 0$. Since $(2N-2)/(2N-3) > 1$, this implies that $-\mathcal{F}[u,v]$ blows up in finite time and contradicts the assumption $T_{\max} = \infty$. Recalling (2.3) we have thus shown the first statement in (2.29). Combining it with (1.14) completes the proof. □

3. Finite time blowup

3.1. Proof of Theorem 1.1: $R = \infty$

The first step is to compute the evolution of the second moment of u given by

$$M_2(t) := \frac{1}{\sigma_N} \int_{\mathbb{R}^N} u(t,x)|x|^2\, dx = \int_0^\infty u(t,r) r^{N+1}\, dr\ , \qquad t \in [0, T_{\max})\ . \tag{3.1}$$

Lemma 3.1. *There is $C_8 > 0$ depending only on N and α such that, for $t \in [0, T_{\max})$,*

$$\frac{dM_2}{dt}(t) \le 2(N-2)\mathcal{F}[u_0, v_0] + C_8 \|\partial_t v(t)\|_2 \int_0^\infty u(t,r) r^{(N+2)/2}\, dr\ . \tag{3.2}$$

Proof. As in Section 2 we decompose v as

$$v = \tilde{v} + \hat{v}$$

with $(\tilde{v}, \hat{v})$ given by (2.17) since $R = \infty$. We infer from (1.1) and Lemma A.2 that

$$\begin{aligned}
\sigma_N \frac{dM_2}{dt}(t) &= -2 \int_{\mathbb{R}^N} x \cdot \left(\nabla u^{q+1}(t,x) - u(t,x)\nabla\tilde{v}(t,x) - u(t,x)\nabla\hat{v}(t,x) \right) dx \\
&= 2N \int_{\mathbb{R}^N} u(t,x)^{q+1}\, dx + 2 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} u(t,x) x \cdot \nabla\mathcal{B}_\alpha(x-y) u(t,y)\, dydx \\
&\quad + 2 \int_{\mathbb{R}^N} u(t,x) x \cdot \nabla\hat{v}(t,x)\, dx \\
&= 2(N-2) \int_{\mathbb{R}^N} \frac{u(t,x)^{q+1}}{q}\, dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (x-y)\cdot\nabla\mathcal{B}_\alpha(x-y) u(t,x) u(t,y)\, dydx \\
&\quad + 2\sigma_N \int_0^\infty u(t,r)\partial_r\hat{v}(t,r) r^N\, dr \\
&\le 2(N-2) \int_{\mathbb{R}^N} \frac{u(t,x)^{q+1}}{q}\, dx - (N-2) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathcal{B}_\alpha(x-y) u(t,x) u(t,y)\, dydx \\
&\quad + 2\sigma_N \int_0^\infty u(t,r)\partial_r\hat{v}(t,r) r^N\, dr\ .
\end{aligned}$$

Now, the definitions of $\tilde{v}$ and of the Bessel kernel guarantee that

$$-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathcal{B}_\alpha(x-y)u(t,x)u(t,y)\,dydx = -\int_{\mathbb{R}^N} u(t,x)\tilde{v}(t,x)\,dx$$
$$= \int_{\mathbb{R}^N} \left(|\nabla\tilde{v}(t,x)|^2 + \alpha|\tilde{v}(t,x)|^2 - 2u(t,x)\tilde{v}(t,x)\right)\,dx$$
$$= 2\sigma_N \mathcal{F}[u(t),\tilde{v}(t)] - \frac{2}{q}\int_{\mathbb{R}^N} u(t,x)^{q+1}\,dx\ .$$

Therefore,

$$\frac{dM_2}{dt}(t) \le 2(N-2)\mathcal{F}[u(t),\tilde{v}(t)] + 2\int_0^\infty u(t,r)\partial_r\hat{v}(t,r)r^N\,dr\ .$$

We next infer from (A.7) that

$$\frac{dM_2}{dt}(t) \le 2(N-2)\mathcal{F}[u(t),\tilde{v}(t)] + 2C_{11}\|\partial_t v\|_2 \int_0^\infty u(t,r)r^{(N+2)/2}\,dr\ .$$

We complete the proof with the help of (1.14) and [3, Lemma 4] which guarantee that $\mathcal{F}[u(t),\tilde{v}(t)] \le \mathcal{F}[u(t),v(t)] \le \mathcal{F}[u_0,v_0]$ for $t\in[0,T_{\max})$. □

We next state a simple consequence of (2.2) and Lemma 3.1 for $N\in\{3,4\}$.

Corollary 3.2. *There is $C_9>0$ depending only on N, M, and α such that, for $t\in[0,T_{\max})$,*

$$\frac{dM_2}{dt}(t) \le 2\mathcal{F}[u_0,v_0] + C_9\|\partial_t v(t)\|_2 M_2(t)^{1/4} \quad \text{for} \quad N=3\ , \tag{3.3}$$
$$\frac{dM_2}{dt}(t) \le 4\mathcal{F}[u_0,v_0] + C_9\|\partial_t v(t)\|_2 \quad \text{for} \quad N=4\ . \tag{3.4}$$

Proof. For $N=4$, $(N+2)/2=3=N-1$ so that

$$\int_0^\infty u(t,r)r^{(N+2)/2}\,dr = \int_0^\infty u(t,r)r^{N-1}\,dr = \frac{M}{\sigma_N}$$

by (2.2) and the differential inequality (3.4) readily follows from (3.2) with $C_9 := MC_8/\sigma_N$.

For $N=3$, $(N+2)/2=5/2$ and we infer from (2.2) and Hölder's inequality that

$$\int_0^\infty u(t,r)r^{(N+2)/2}\,dr \le \left(\int_0^\infty u(t,r)r^2\,dr\right)^{3/4}\left(\int_0^\infty u(t,r)r^4\,dr\right)^{1/4}$$
$$\le \left(\frac{M}{\sigma_N}\right)^{3/4} M_2(t)^{1/4}\ ,$$

which gives the differential inequality (3.3) once combined with (3.2). □

Proof of Theorem 1.1: $R=\infty$. Assume for contradiction that $T_{\max}=\infty$. We infer from (1.14), (1.16), (1.17), and (2.29) that

$$V(t) := \int_0^t \|\partial_t v(s)\|_2^2\,ds \le \sigma_N \int_0^t \mathcal{D}[u(s),v(s)]\,ds \le C_7\ , \qquad t\ge 0\ . \tag{3.5}$$

Consequently,

$$t \mapsto \|\partial_t v(t)\|_2^2 \in L^1(0,\infty) \quad \text{with} \quad \int_0^\infty \|\partial_t v(s)\|_2^2\, ds \le C_7 \,. \tag{3.6}$$

We now handle the cases $N=3$ and $N=4$ separately.

***N* = 3**: We infer from (3.3) and Young's inequality that

$$\frac{dM_2}{dt} \le 2\mathcal{F}[u_0,v_0] + \|\partial_t v\|_2^2 M_2 + C\|\partial_t v\|_2^{2/3} \,,$$

hence, after integration with respect to time,

$$M_2(t) \le M_2(0)e^{V(t)} + \int_0^t \left[2\mathcal{F}[u_0,v_0] + C\|\partial_t v(s)\|_2^{2/3}\right] e^{V(t)-V(s)}\, ds \,, \quad t \ge 0 \,.$$

Thanks to (3.5), (3.6), and Hölder's inequality, we further obtain

$$\begin{aligned} M_2(t) &\le M_2(0)e^{C_7} + 2\mathcal{F}[u_0,v_0]t + Ce^{C_7}V(t)^{1/3}t^{2/3} \\ &\le C\left[M_2(0) + t^{2/3}\right] + 2\mathcal{F}[u_0,v_0]t \,. \end{aligned}$$

Since $\mathcal{F}[u_0,v_0] < -C_0$ by (1.17) with $C_0 = 0$, it readily follows from the above inequality that $M_2(t)$ becomes negative for sufficiently large t, contradicting the non-negativity of u. We have thus established that $T_{\max} < \infty$ in that case.

***N* = 4**: Let $t \ge 0$. A straightforward consequence of (3.4), (3.5), and the Cauchy–Schwarz inequality is

$$\begin{aligned} M_2(t) &\le M_2(0) + 4\mathcal{F}[u_0,v_0]t + C_9 V(t)^{1/2}t^{1/2} \\ &\le M_2(0) + \sqrt{C_7}C_9 t^{1/2} + 4\mathcal{F}[u_0,v_0]t \,. \end{aligned}$$

Owing to (1.17) with $C_0 = 0$, the above inequality implies that M_2 becomes negative in finite time, contradicting the non-negativity of u. Thus $T_{\max} < \infty$ and the proof of Theorem 1.1 is complete. □

3.2. Alternative proof of Theorem 1.1: $R = \infty$

We next give an alternative proof of Theorem 1.1 in the case $R = \infty$ which does not involve the second moment.

Proof of Theorem 1.1: $R = \infty$. Assume for contradiction that $T_{\max} = \infty$ and consider $t \ge 0$ and $r_1 \in (0,\infty)$. We aim at showing that, owing to the time integrability (2.29) of $\|f\|_2^2$ and $\|g\|_2^2$ (recall that $\mathcal{D}[u,v] = \left(\|f\|_2^2 + \|g\|_2^2\right)/\sigma_N$), the energy $-\mathcal{F}_{r_1}[u(t),v(t)]$ located in $B_{r_1}(0)$ behaves as a negative power of r_1 as $t \to \infty$. Since this is also true uniformly with respect to time for the remaining part of the energy according to Lemma 2.10 we will conclude that the limit of $-\mathcal{F}[u(t),v(t)]$ is bounded by a negative power of r_1 as $t \to \infty$, thereby contradicting the negativity (2.29) of $\mathcal{F}[u_0,v_0]$ since $r_1 > 0$ was arbitrarily taken. More specifically, by (2.29), Lemma 2.5, and Lemma 2.10,

$$\begin{aligned} -\mathcal{F}[u_0,v_0] &\le -\mathcal{F}[u(t),v(t)] \\ &\le -\mathcal{F}[u(t),v(t)] + \mathcal{F}_{r_1}[u(t),v(t)] - \mathcal{F}_{r_1}[u(t),v(t)] \\ &\le C_6 r_1^{1-N} + \frac{1}{N-2}\int_0^{r_1} g\sqrt{u}\, r^N\, dr - \int_0^{r_1} f\left(\frac{r}{N-2}\partial_r v + v\right) r^{N-1}\, dr \\ &\quad - \frac{1}{2(N-2)}\left[r_1^{N/2}\partial_r v(r_1) + (N-2)r_1^{(N-2)/2}v(r_1)\right]^2 \\ &\quad + \frac{N-2}{2} r_1^{N-2} v(r_1)^2 + \frac{\alpha}{2(N-2)} r_1^N v(r_1)^2 \end{aligned}$$

$$\leq C_6 r_1^{1-N} + C\|g\|_2 \left(\int_0^{r_1} u r^{N+1}\, dr \right)^{1/2} + C\left(r_1^{N-2} + r_1^N\right) v(r_1)^2$$
$$+ \left| \int_0^{r_1} f\left(\frac{r}{N-2}\partial_r \hat{v} + \hat{v}\right) r^{N-1}\, dr \right| + \left| \int_0^{r_1} f\left(\frac{r}{N-2}\partial_r \tilde{v} + \tilde{v}\right) r^{N-1}\, dr \right| .$$

Using Lemma 2.6, Lemma 2.8, (2.2), and (2.5) (with $p=1$) and recalling that $N \in \{3,4\}$, we further obtain

$$-\mathcal{F}[u_0, v_0] \leq C_6 r_1^{1-N} + C r_1 \|g(t)\|_2 + C\left(r_1^{-N} + r_1^{2-N}\right)$$
$$+ C_3(1+r_1)\left(\int_0^{r_1} f(t,r)^2 r^3\, dr\right)^{1/2} + C_4 r_1^{3/2}\left(1+\sqrt{r_1|\ln r_1|}\right)\|f(t)\|_2^2$$
$$\leq C\left(r_1^{-N} + r_1^{1-N} + r_1^{2-N}\right) + C r_1\|g(t)\|_2 + C(1+r_1) r_1^{(4-N)/2}\|f(t)\|_2$$
$$+ C r_1^{3/2}\left(1+\sqrt{r_1|\ln r_1|}\right)\|f(t)\|_2^2 .$$

Integrating the above inequality with respect to time over $(T, T+1)$ for some arbitrary $T>0$, we deduce from (2.12) and the Cauchy–Schwarz inequality that

$$-\mathcal{F}[u_0, v_0] \leq C\left(r_1^{-N} + r_1^{1-N} + r_1^{2-N}\right) + C r_1 \int_T^{T+1} \|g(t)\|_2\, dt + C(1+r_1) r_1^{(4-N)/2} \int_T^{T+1} \|f(t)\|_2\, dt$$
$$+ C r_1^{3/2}\left(1+\sqrt{r_1|\ln r_1|}\right) \int_T^{T+1} \|f(t)\|_2^2\, dt$$
$$\leq C\left(r_1^{-N} + r_1^{1-N} + r_1^{2-N}\right) + C\left(r_1 + (1+r_1) r_1^{(4-N)/2}\right)\left(\int_T^{T+1} \mathcal{D}[u(t),v(t)]\, dt\right)^{1/2}$$
$$+ C r_1^{3/2}\left(1+\sqrt{r_1|\ln r_1|}\right) \int_T^{T+1} \mathcal{D}[u(t),v(t)]\, dt .$$

Owing to (2.29), we may pass to the limit as $T\to\infty$ in the previous inequality and obtain

$$-\mathcal{F}[u_0, v_0] \leq C\left(r_1^{-N} + r_1^{1-N} + r_1^{2-N}\right) .$$

Since r_1 is arbitrary in the above inequality, we may let $r_1 \to \infty$ to conclude that $-\mathcal{F}[u_0, v_0] \leq 0$, which contradicts (1.17) with $C_0 = 0$. □

3.3. *Proof of Theorem 1.1: $R<\infty$*

Proof of Theorem 1.1: $R<\infty$ with the mixed boundary conditions (1.5). Assume for contradiction that $T_{\max} = \infty$. Choosing $r_1 = R$ in Lemma 2.5 and applying Lemma 2.6 and Lemma 2.8 (with $r_1 = R$) yield

$$-\mathcal{F}[u(t), v(t)] \leq \frac{1}{N-2}\int_0^R g(t)\sqrt{u(t)}\, r^N\, dr + C_3(1+R)\left(\int_0^R f(t)^2 r^3\, dr\right)^{1/2}$$
$$+ C_4 R^{3/2}\left(1+\sqrt{R|\ln R|}\right)\|f(t)\|_2^2$$

for $t > 0$, after noticing that the last two terms in (2.14) vanish due to (1.5). Since $-\mathcal{F}[u_0, v_0] \le -\mathcal{F}[u(t), v(t)]$ for $t > 0$ according to (2.3), we infer from (2.12), Hölder's inequality and the above estimate that

$$-\mathcal{F}[u_0, v_0] \le C\left(\sqrt{\mathcal{D}[u(s), v(s)]} + \mathcal{D}[u(s), v(s)]\right) \qquad \text{for } s > 0\,.$$

Integrating the above inequality with respect to s over $(t, t+1)$ and using Hölder's inequality give

$$-\mathcal{F}[u_0, v_0] \le C\left[\left(\int_t^{t+1} \mathcal{D}[u(s), v(s)]\, ds\right)^{1/2} + \int_t^{t+1} \mathcal{D}[u(s), v(s)]\, ds\right] \qquad \text{for } t > 0\,,$$

and it readily follows from Theorem 2.12 that the right hand side of the above inequality converges to zero as $t \to \infty$. Consequently, $-\mathcal{F}[u_0, v_0] \le 0$ which contradicts (1.17) with $C_0 = 0$ and thus implies $T_{\max} < \infty$. $\square$

Proof of Theorem 1.1: $R < \infty$ with the Neumann boundary conditions (1.4). Assume for contradiction that $T_{\max} = \infty$. Setting $r_1 = R$ in Lemma 2.5 and applying Lemma 2.6 and Lemma 2.8 (with $r_1 = R$), we find

$$\begin{aligned} -\mathcal{F}[u(t), v(t)] \le{}& \frac{1}{N-2}\int_0^R g(t)\sqrt{u(t)}r^N\, dr + C_3(1+R)\left(\int_0^{r_1} f(t)^2 r^3\, dr\right)^{1/2} \\ &+ C_4 R^{3/2}\left(1 + \sqrt{R|\ln R|}\right)\|f(t)\|_2^2 + \frac{\alpha R^N}{2(N-2)}\{\tilde{v}(t, R) + \hat{v}(t, R)\}^2 \end{aligned}$$

for $t > 0$, after noticing that the last term in (2.14) vanishes due to (1.4). Recall that $\tilde{v}$ and $\hat{v}$ are the solutions to (2.19)–(2.20) supplemented with homogeneous Neumann boundary conditions. From Lemma A.1 (for $N \in \{3, 4\}$) and (2.2),

$$R^N \tilde{v}(t, R)^2 \le C_{10}^2 M^2 R^{4-N} \quad \text{and} \quad R^N \hat{v}(t, R)^2 \le C_{11}^2 R^N \left(1 + \sqrt{|\ln R|}\right)^2 \|f(t)\|_2^2 \quad \text{for } t > 0.$$

Owing to (2.2), (2.3), the definition (1.16) of $\mathcal{D}[u, v]$, and Hölder's and Young's inequalities, we are led to

$$-\mathcal{F}[u_0, v_0] \le C\left(\sqrt{\mathcal{D}[u(s), v(s)]} + \mathcal{D}[u(s), v(s)]\right) + \frac{\alpha C_{10}^2 M^2 R^{4-N}}{N-2} \qquad \text{for } s > 0\,.$$

Arguing as in the previous case with the help of (2.29), we end up with

$$-\mathcal{F}[u_0, v_0] < \alpha C_{10}^2 M^2 R^{4-N}/(N-2)$$

which contradicts (1.17) with $C_0 := \alpha C_{10}^2 R^{4-N}/(N-2)$ and again implies $T_{\max} < \infty$. $\square$

4. Existence of blowing-up solutions

In this section, we prove Theorem 1.2 showing the existence of initial data satisfying the criterion given in Theorem 1.1. The first step towards the proof of Theorem 1.2 is the existence of a couple of functions with negative energy which requires its first component to have a sufficiently high mass. Recall that, for $N \ge 3$, $M_c(N)$ is the critical mass associated to (1.8)–(1.10) which guarantees global existence if $\|u_0\|_1 < M_c(N)$ and possible blowup if $\|u_0\|_1 > M_c(N)$ [2,3].

Lemma 4.1. *For each $M > M_c(N)$ there is $(U_M, V_M) \in \mathcal{I}_\infty$ such that $\|U_M\|_1 = M$, $\mathcal{F}^0[U_M, V_M] < 0$, and both U_M and V_M have compact support in Ω_∞, where*

$$\mathcal{F}^0[u, v] := \int_0^\infty \left(\frac{u^{q+1}}{q} + \frac{1}{2}(\partial_r v)^2 - uv\right) r^{N-1} dr\,.$$

Proof. According to [2, Proposition 3.5] there is a compactly supported and radially symmetric non-negative function $U \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ satisfying

$$U^q \in H^1(\mathbb{R}^N) , \quad \|U\|_1 = M_c(N) , \quad \mathcal{F}^0[U, \mathcal{B}_0 \star U] = 0 ,$$

and

$$\mathrm{div}\left(\nabla U^{q+1} - U\nabla(\mathcal{B}_0 \star U)\right) = 0 \quad \text{in } \mathbb{R}^N .$$

Introducing $V := \mathcal{B}_0 \star U$, we observe that

$$0 = \mathcal{F}^0[U, V] = \int_0^\infty \left(\frac{1}{q}U^{q+1} - \frac{1}{2}(\partial_r V)^2\right) r^{N-1} dr ,$$

while it follows from the regularity properties of U, the compactness of its support, and Newton's formula [18, Section 9.7] that

$$V \in L^p(\mathbb{R}^N) , \quad p \in \left(\frac{N}{N-2}, \infty\right] , \quad \text{and} \quad \nabla V \in L^2(\mathbb{R}^N) ,$$

but $V \notin L^2(\mathbb{R}^N)$ for $N = 3, 4$. Thus (U, V) does not belong to $\mathcal{I}_\infty$ and we need to truncate V. To this end, let $\vartheta \in C_0^\infty(\mathbb{R}^N)$ be a radially symmetric cut-off function satisfying $0 \le \vartheta \le 1$, $\vartheta \equiv 1$ in $B_1(0)$, and $\vartheta \equiv 0$ in $\mathbb{R}^N \setminus B_2(0)$, and define $V_k(x) := V(x)\vartheta(x/k)$ for $x \in \mathbb{R}^N$ and $k \ge 1$. We then fix $\mu > 1$ such that $\mu := M/M_c(N)$ and observe that, for $k \ge 1$, $(\mu U, \mu V_k)$ belongs to $\mathcal{I}_\infty$ with $\|\mu U\|_1 = M$, both functions have compact supports, and

$$\begin{aligned}
\mathcal{F}^0[\mu U, \mu V_k] &= \mu^{q+1}\mathcal{F}^0[U, V] + \left(\mu^2 - \mu^{q+1}\right) \int_0^\infty \left[\frac{1}{2}(\partial_r V)^2 - UV\right] r^{N-1}\, dr \\
&\quad + \mu^2 \int_0^\infty \left[\frac{1}{2}\partial_r(V + V_k)\partial_r(V - V_k) - U(V - V_k)\right] r^{N-1}\, dr \\
&= -\frac{\mu^{q+1}}{q}\left(\mu^{1-q} - 1\right) \int_0^\infty U^{q+1} r^{N-1}\, dr \\
&\quad + \mu^2 \int_0^\infty \left[\frac{1}{2}\partial_r(V + V_k)\partial_r(V - V_k) - U(V - V_k)\right] r^{N-1}\, dr .
\end{aligned}$$

Owing to the regularity of U and V, the last term in the right-hand side of the above identity converges to zero as $k \to \infty$, so that there is k_μ large enough for which

$$\mathcal{F}^0[\mu U, \mu V_{k_\mu}] \le -\frac{\mu^{q+1}}{2q}\left(\mu^{1-q} - 1\right) \int_0^\infty U^{q+1} r^{N-1}\, dr < 0 .$$

Setting $(U_M, V_M) = (\mu U, \mu V_{k_\mu})$ completes the proof. □

Proof of Theorem 1.2. Fix $M > M_c(N)$. To simplify notation, we set $\mathcal{U} = U_M$ and $\mathcal{V} = V_M$ and define

$$\mathcal{U}_\lambda(r) = \lambda^N \mathcal{U}(\lambda r) \quad \text{and} \quad \mathcal{V}_\lambda(r) = \lambda^{N-2}\mathcal{V}(\lambda r) \quad \text{for } r \in [0, R) \quad \text{and } \lambda > 0 .$$

Taking λ large enough such that the supports of $\mathcal{U}$ and $\mathcal{V}$ are included in $B_{\lambda R}(0)$ it is immediate that

$$\int_0^R \mathcal{U}_\lambda(r) r^{N-1}\, dr = \int_0^{\lambda R} \mathcal{U}(r) r^{N-1}\, dr = \int_0^\infty \mathcal{U}(r) r^{N-1}\, dr = \frac{M}{\sigma_N}$$

and

$$\mathcal{F}[\mathcal{U}_\lambda, \mathcal{V}_\lambda] = \lambda^{N-2} \int_0^{\lambda R} \left(\frac{1}{q}\mathcal{U}^{q+1} + \frac{1}{2}(\partial_r \mathcal{V})^2 - \mathcal{U}\mathcal{V} \right) r^{N-1}\, dr + \frac{\alpha}{2}\lambda^{N-4} \int_0^{\lambda R} \mathcal{V}^2 r^{N-1}\, dr$$
$$= \lambda^{N-2} \left[\mathcal{F}^0[\mathcal{U}, \mathcal{V}] + \frac{\alpha}{2\lambda^2} \int_0^{\lambda R} \mathcal{V}^2 r^{N-1}\, dr \right] . \tag{4.1}$$

Recalling that $\mathcal{F}^0[\mathcal{U}, \mathcal{V}] < 0$ by Lemma 4.1 we now choose λ_* large enough such that

$$\frac{\alpha}{2\lambda_*^2} \int_0^{\lambda R} \mathcal{V}^2 r^{N-1}\, dr < -\frac{1}{2}\mathcal{F}^0[\mathcal{U}, \mathcal{V}] ,$$

and

$$\frac{\lambda_*^{N-2}}{2}\mathcal{F}^0[\mathcal{U}, \mathcal{V}] < -C_0\, M^2 = -C_0\|\mathcal{U}_{\lambda_*}\|_1^2 < 0 ,$$

and conclude that $\mathcal{F}[\mathcal{U}_{\lambda_*}, \mathcal{V}_{\lambda_*}] < -C_0\|\mathcal{U}_{\lambda_*}\|_1^2$. Taking $(u_0^*, v_0^*) = (\mathcal{U}_{\lambda_*}, \mathcal{V}_{\lambda_*})$, the continuity of $\mathcal{F}$ in $L^{q+1}(\Omega_R) \times H^1(\Omega_R)$ and Theorem 1.1 give the first assertion of Theorem 1.2.

Next, when $0 < p < 1$ and $1 < \sigma < N/(N-1)$, we have

$$\int_0^R \mathcal{U}_\lambda(r)^p r^{N-1} dr = \lambda^{N(p-1)} \int_0^{\lambda R} \mathcal{U}(r)^p r^{N-1}\, dr \longrightarrow 0 \quad \text{as } \lambda \to \infty \tag{4.2}$$

and

$$\int_0^R \{(\partial_r \mathcal{V}_\lambda(r))^\sigma + \mathcal{V}_\lambda(r)^\sigma\} r^{N-1}\, dr$$
$$= \lambda^{\sigma(N-1)-N} \int_0^{\lambda R} \{(\partial_r \mathcal{V}(r))^\sigma + \lambda^{-\sigma}\mathcal{V}(r)^\sigma\} r^{N-1}\, dr \longrightarrow 0 \quad \text{as } \lambda \to \infty . \tag{4.3}$$

Given $(u_0, v_0) \in \mathcal{I}_R$ we put $w_{0,\lambda} = u_0 + \mathcal{U}_\lambda$ and $z_{0,\lambda} = v_0 + \mathcal{V}_\lambda$ and easily deduce from (4.2) and (4.3) that $(w_{0,\lambda}, z_{0,\lambda})$ converges towards (u_0, v_0) as $\lambda \to \infty$ in the sense stated in Theorem 1.2. In addition, owing to the non-negativity of u_0, v_0, $\mathcal{U}$, and $\mathcal{V}$, we infer from (4.1) that, for $\lambda > 1$,

$$\mathcal{F}[w_{0,\lambda}, z_{0,\lambda}] \le \mathcal{F}[\mathcal{U}_\lambda, \mathcal{V}_\lambda] + \frac{q+1}{q} \int_0^R u_0 \left(u_0^q + \mathcal{U}_\lambda^q\right) r^{N-1}\, dr$$
$$+ \int_0^R \left(\partial_r \mathcal{V}_\lambda \partial_r v_0 + \frac{1}{2}(\partial_r v_0)^2 \right) r^{N-1}\, dr$$
$$+ \alpha \int_0^R \left(\mathcal{V}_\lambda v_0 + \frac{1}{2}v_0^2 \right) r^{N-1}\, dr$$
$$\le \mathcal{F}[\mathcal{U}_\lambda, \mathcal{V}_\lambda] + C(u_0, v_0) + C\lambda^{N(q-1)}\|u_0\|_\infty \|\mathcal{U}\|_1^q |\mathrm{supp}(\mathcal{U})|^{1-q}$$
$$+ \frac{C}{\lambda}\|\partial_r v_0\|_\infty \|\partial_r \mathcal{V}\|_1 + \frac{C}{\lambda^2}\|v_0\|_\infty \|\mathcal{V}\|_1$$
$$\le \lambda^{N-2} \left[\mathcal{F}^0[\mathcal{U}, \mathcal{V}] + \frac{\alpha}{2\lambda^2} \int_0^{\lambda R} \mathcal{V}^2 r^{N-1}\, dr + \frac{C(u_0, v_0)}{\lambda^{N-2}} \right] .$$

Since $\mathcal{F}^0[\mathcal{U},\mathcal{V}] < 0$ by Lemma 4.1 and the L^1-norm of $w_{0,\lambda}$ is bounded uniformly with respect to $\lambda > 1$, we readily check that $(w_{0,\lambda}, z_{0,\lambda})$ satisfies (1.17) for suitably large $\lambda > 0$ and thereby complete the proof of Theorem 1.2. □

Conflict of interest statement

There is no conflict of interest.

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Appendix A. Bessel potentials

In this last section we collect the results on the Bessel kernels and potentials we have used in the preceding sections.

Lemma A.1. *Let $N \geq 3$, $R \in (0,\infty]$, and consider a radially symmetric function h which belongs to $L^p(\Omega_R)$ for some $p \in \{1,2\}$. Let H be the unique radially symmetric solution to*

$$-\Delta H + \alpha H = h \quad in \quad \Omega_R\,, \tag{A.1}$$

supplemented with either homogeneous Neumann boundary conditions

$$\partial_\nu H = 0 \quad on \quad \partial\Omega_R\,, \tag{A.2a}$$

or homogeneous Dirichlet boundary conditions

$$H = 0 \quad on \quad \partial\Omega_R\,, \tag{A.2b}$$

if $R < \infty$, or given by

$$H = \mathcal{B}_\alpha \star h \tag{A.3}$$

if $R = \infty$.

(i) *If $h \in L^1(\Omega_R)$, $h \geq 0$ a.e. in Ω_R, then there is $C_{10} > 0$ depending only on N, R, and α such that, for $r > 0$,*

$$0 \leq H(r) \leq C_{10}\|h\|_1 r^{2-N}\,, \tag{A.4}$$

$$|\partial_r H(r)| \leq C_{10}\,\|h\|_1 r^{1-N}\,. \tag{A.5}$$

(ii) *If $h \in L^2(\Omega_R)$ then $H \in H^2(\Omega_R)$ and there is $C_{11} > 0$ depending only on N, R, and α such that, for $r > 0$,*

$$|H(r)| \leq \begin{cases} C_{11}\|h\|_2 & for \quad N = 3\,, \\ C_{11}\|h\|_2\left(1+\sqrt{|\ln r|}\right) & for \quad N = 4\,, \\ C_{11}\|h\|_2 r^{(4-N)/2} & for \quad N \geq 5\,, \end{cases} \tag{A.6}$$

and

$$|\partial_r H(r)| \leq C_{11}\|h\|_2 r^{(2-N)/2}\,. \tag{A.7}$$

Proof. **(i)**: We first consider the case $R = \infty$ where H is given by (A.3). Since $0 \leq \mathcal{B}_\alpha \leq \mathcal{B}_0$ and h is non-negative and radially symmetric, it follows from Newton's formula [18, Section 9.7] that

$$0 \leq (\mathcal{B}_\alpha \star h)(r) \leq (\mathcal{B}_0 \star h)(r) = \frac{1}{(N-2)r^{N-2}}\int_0^r h(\rho)\rho^{N-1}\,d\rho + \frac{1}{N-2}\int_r^\infty h(\rho)\rho\,d\rho$$

$$\leq \frac{1}{r^{N-2}} \int_0^r h(\rho)\rho^{N-1}\, d\rho + \frac{1}{r^{N-2}} \int_r^\infty h(\rho)\rho^{N-1}\, d\rho$$
$$\leq \frac{\|h\|_1}{\sigma_N} r^{2-N} \,.$$

We have thus proved (A.4). Next, since h is a non-negative and integrable function, it follows from [18, Theorem 6.23] that $H = \mathcal{B}_\alpha \star h$ is also a non-negative and integrable function with $\|H\|_1 = \|h\|_1/\alpha$ and solves

$$-\partial_r \left(r^{N-1} \partial_r H(r) \right) = r^{N-1} (h - \alpha H)(r) \,.$$

Integrating over $(0, r)$ gives

$$\partial_r H(r) = r^{1-N} \int_0^r (\alpha H - h)(\rho)\rho^{N-1}\, d\rho \,, \quad r > 0 \,,$$

whence

$$|\partial_r H(r)| \leq r^{1-N} \int_0^r (\alpha H + h)(\rho)\rho^{N-1}\, d\rho \leq r^{1-N} \left(\alpha \|H\|_1 + \|h\|_1\right) \leq 2\|h\|_1 r^{1-N} \,.$$

We next turn to the case $R < \infty$. Let $r \in (0, R)$. Integrating (A.1) over $(0, r)$ gives

$$-r^{N-1} \partial_r H(r) = -\alpha \int_0^r H(\rho)\rho^{N-1}\, d\rho + \int_0^r h(\rho)\rho^{N-1}\, d\rho \,. \tag{A.8}$$

Owing to the comparison principle, the non-negativity of h implies that of H in Ω_R and we realize that $\partial_r H(R) \leq 0$ for both boundary conditions (A.2a) and (A.2b). It then follows from (A.8) with $r = R$ that

$$0 \leq \alpha \int_0^R H(\rho)\rho^{N-1}\, d\rho \leq \int_0^R h(\rho)\rho^{N-1}\, d\rho \,. \tag{A.9}$$

Combining (A.8) and (A.9) then gives

$$r^{N-1} |\partial_r H(r)| \leq \int_0^R h(\rho)\rho^{N-1}\, d\rho = \frac{\|h\|_1}{\sigma_N} \,,$$

which proves (A.5). The estimate (A.4) readily follows from (A.5) when H solves (A.1)–(A.2b) as

$$H(r) = -\int_r^R \partial_r H(\rho)\, d\rho \,.$$

When H solves (A.1)–(A.2a), the estimate (A.4) is also deduced from (A.5), arguing as in [10, Lemma 2.5], see also the end of the proof of Lemma 2.4.

(ii): Since $h \in L^2(\Omega_R)$ and $\alpha > 0$, classical elliptic regularity ensures that H belongs to $H^2(\Omega_R)$ and there is a constant $C > 0$ depending only on N, R, and α such that $\|H\|_{H^2} \leq C\|h\|_2$. On the one hand, $H^2(\Omega_R)$ is continuously embedded in $L^\infty(\Omega_R)$ for $N = 3$ which gives (A.6) for $N = 3$ while (A.6) for $N \geq 4$ follows from [10, Theorem 1.1] (after noticing that the proof extends to the case $R = \infty$). On the other hand, ∇H belongs to $H^1(\Omega_R)$ and we deduce (A.7) from [10, Theorem 1.1]. □

We next turn to some properties of the gradient of $\mathcal{B}_\alpha$ which plays an important role in the computation of the evolution of the second moment of u in Section 3.1. A similar result can be found in [25, Lemma 2.1].

Lemma A.2. *For $x \in \mathbb{R}^N$ and $\alpha > 0$, there holds $\nabla \mathcal{B}_\alpha(-x) = -\nabla \mathcal{B}_\alpha(x)$ and*

$$x \cdot \nabla \mathcal{B}_\alpha(x) \le -(N-2)\mathcal{B}_\alpha(x) . \tag{A.10}$$

Proof. Differentiating (2.18) gives

$$\nabla \mathcal{B}_\alpha(x) = -\int_0^\infty \exp\left\{-\frac{|x|^2}{4s} - \alpha s\right\} \frac{ds}{2s(4\pi s)^{N/2}}\, x , \quad x \in \mathbb{R}^N ,$$

so that

$$\begin{aligned} x \cdot \nabla \mathcal{B}_\alpha(x) &= -2\int_0^\infty \frac{|x|^2}{4s^2} \exp\left\{-\frac{|x|^2}{4s}\right\} e^{-\alpha s} s^{(2-N)/2} \frac{ds}{(4\pi)^{N/2}} \\ &= -\int_0^\infty \exp\left\{-\frac{|x|^2}{4s} - \alpha s\right\} (2\alpha s + N - 2) \frac{ds}{(4\pi s)^{N/2}} \\ &\le -(N-2)\mathcal{B}_\alpha(x) , \end{aligned}$$

as claimed. □

References

[1] P. Biler, L. Corrias, J. Dolbeault, Large mass self-similar solutions of the parabolic–parabolic Keller–Segel model of chemotaxis, J. Math. Biol. 63 (2011) 1–32.

[2] A. Blanchet, J.A. Carrillo, Ph. Laurençot, Critical mass for a Patlak–Keller–Segel model with degenerate diffusion in higher dimensions, Calc. Var. Partial Differ. Equ. 35 (2009) 133–168.

[3] A. Blanchet, Ph. Laurençot, The parabolic–parabolic Keller–Segel system with critical diffusion as a gradient flow in $\mathbb{R}^d$, $d \ge 3$, Commun. Partial Differ. Equ. 38 (2013) 658–686.

[4] J. Burczak, T. Cieślak, C. Morales-Rodrigo, Global existence vs. blowup in a fully parabolic quasilinear 1D Keller–Segel system, Nonlinear Anal. 75 (2012) 5215–5228.

[5] V. Calvez, L. Corrias, The parabolic–parabolic Keller–Segel model in $\mathbb{R}^2$, Commun. Math. Sci. 6 (2008) 417–447.

[6] T. Cieślak, Ph. Laurençot, Finite time blow-up for a one-dimensional quasilinear parabolic–parabolic chemotaxis system, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 27 (2010) 437–446.

[7] T. Cieślak, C. Stinner, Finite-time blowup and global-in-time unbounded solutions to a parabolic–parabolic quasilinear Keller–Segel system in higher dimensions, J. Differ. Equ. 252 (2012) 5832–5851.

[8] T. Cieślak, C. Stinner, Finite-time blowup in a supercritical quasilinear parabolic–parabolic Keller–Segel system in dimension 2, Acta Appl. Math. 129 (2014) 135–146.

[9] H. Gajewski, K. Zacharias, Global behavior of a reaction–diffusion system modelling chemotaxis, Math. Nachr. 195 (1998) 77–114.

[10] D. Guedes de Figueiredo, E.M. dos Santos, O.H. Miyagaki, Sobolev spaces of symmetric functions and applications, J. Funct. Anal. 261 (2011) 3735–3770.

[11] M.A. Herrero, J.J.L. Velázquez, Singularity patterns in a chemotaxis model, Math. Ann. 306 (1996) 583–623.

[12] M.A. Herrero, J.J.L. Velázquez, A blow-up mechanism for a chemotaxis model, Ann. Sc. Norm. Super. Pisa, Cl. Sci. 24 (1997) 633–683.

[13] D. Horstmann, From 1970 until present: the Keller–Segel model in chemotaxis and its consequences, I, Jahresber. Dtsch. Math.-Ver. 105 (2003) 103–165.

[14] S. Ishida, T. Yokota, Global existence of weak solutions to quasilinear degenerate Keller–Segel systems of parabolic–parabolic type, J. Differ. Equ. 252 (2012) 1421–1440.

[15] S. Ishida, T. Yokota, Global existence of weak solutions to quasilinear degenerate Keller–Segel systems of parabolic–parabolic type with small data, J. Differ. Equ. 252 (2012) 2469–2491.

[16] S. Ishida, T. Yokota, Blow-up in finite or infinite time for quasilinear degenerate Keller–Segel systems of parabolic–parabolic type, Discrete Contin. Dyn. Syst., Ser. B 18 (2013) 2569–2596.

[17] E.F. Keller, L.A. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theor. Biol. 26 (1970) 399–415.

[18] E.H. Lieb, M. Loss, Analysis, second ed., Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 2001.

[19] N. Mizoguchi, Global existence for the Cauchy problem of the parabolic–parabolic Keller–Segel system on the plane, Calc. Var. Partial Differ. Equ. 48 (2013) 491–505.

[20] N. Mizoguchi, M. Winkler, Blow-up in the two-dimensional parabolic Keller–Segel system, preprint.

[21] N. Mizoguchi, M. Winkler, Boundedness of global solutions in the two-dimensional parabolic Keller–Segel system, preprint.

[22] T. Nagai, T. Senba, K. Yoshida, Application of the Trudinger–Moser inequality to a parabolic system of chemotaxis, Funkc. Ekvacioj 40 (1997) 411–433.

[23] R. Schweyer, Stable blow-up dynamic for the parabolic–parabolic Patlak–Keller–Segel model, preprint, arXiv:1403.4975.
[24] T. Senba, T. Suzuki, Applied Analysis: Mathematical Methods in Natural Science, second edition, Imperial College Press, London, 2011.
[25] Y. Sugiyama, Blow-up criterion via scaling invariant quantities with effect on coefficient growth in Keller–Segel system, Differ. Integral Equ. 23 (2010) 619–634.
[26] Y. Sugiyama, H. Kunii, Global existence and decay properties for a degenerate Keller–Segel model with a power factor in drift term, J. Differ. Equ. 227 (2006) 333–364.
[27] M. Winkler, Finite-time blow-up in the higher-dimensional parabolic–parabolic Keller–Segel system, J. Math. Pures Appl. 100 (9) (2013) 748–767.