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Global weak solutions in a three-dimensional chemotaxis–Navier–Stokes system

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Abstract

The chemotaxis–Navier–Stokes system

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \nabla\cdot(n\chi(c)\nabla c), \\ c_t + u\cdot\nabla c = \Delta c - nf(c), \\ u_t + (u\cdot\nabla)u = \Delta u + \nabla P + n\nabla\Phi, \\ \nabla\cdot u = 0, \end{cases} \qquad (\star) \qquad (0.1)$$

is considered under homogeneous boundary conditions of Neumann type for n and c, and of Dirichlet type for u, in a bounded convex domain $\Omega\subset\mathbb{R}^3$ with smooth boundary, where $\Phi\in W^{2,\infty}(\Omega)$, and where $f\in C^1([0,\infty))$ and $\chi\in C^2([0,\infty))$ are nonnegative with $f(0)=0$. Problems of this type have been used to describe the mutual interaction of populations of swimming aerobic bacteria with the surrounding fluid. Up to now, however, global existence results seem to be available only for certain simplified variants such as e.g. the two-dimensional analogue of $(\star)$, or the associated chemotaxis–Stokes system obtained on neglecting the nonlinear convective term in the fluid equation.

The present work gives an affirmative answer to the question of global solvability for $(\star)$ in the following sense: Under mild assumptions on the initial data, and under modest structural assumptions on f and χ, inter alia allowing for the prototypical case when

$$f(s)=s \quad \text{for all } s\geq 0 \qquad \text{and} \qquad \chi\equiv const.,$$

the corresponding initial–boundary value problem is shown to possess a globally defined weak solution.

This solution is obtained as the limit of smooth solutions to suitably regularized problems, where appropriate compactness properties are derived on the basis of a priori estimates gained from an energy-type inequality for $(\star)$ which in an apparently novel manner combines the standard L^2 dissipation property of the fluid evolution with a quasi-dissipative structure associated with the chemotaxis subsystem in $(\star)$.

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1. Introduction

We consider the chemotaxis–Navier–Stokes system

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \nabla\cdot(n\chi(c)\nabla c), & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c = \Delta c - nf(c), & x\in\Omega,\ t>0,\\ u_t + (u\cdot\nabla)u = \Delta u + \nabla P + n\nabla\Phi, & x\in\Omega,\ t>0,\\ \nabla\cdot u = 0, & x\in\Omega,\ t>0, \end{cases} \tag{1.2}$$

in a domain $\Omega\subset\mathbb{R}^N$, where the main focus of this work will be on the case when $N=3$ and Ω is bounded and convex with smooth boundary.

As described in [8], problems of this type arise in the modeling of populations of swimming aerobic bacteria in situations when besides their chemotactically biased movement toward oxygen as their nutrient, a buoyancy-driven effect of bacterial mass on the fluid motion is not negligible. Indeed, striking experimental findings indicate that such a mutual chemotaxis–fluid interaction may lead to quite complex types of collective behavior, even in markedly simple settings such as present when populations of *Bacillus subtilis* are suspended in sessile drops of water [8,35,21].

In particular, in (1.2) it is assumed that the presence of bacteria, with density denoted by $n=n(x,t)$, affects the fluid motion, as represented by its velocity field $u=u(x,t)$ and the associated pressure $P=P(x,t)$, through buoyant forces. Moreover, it is assumed that both cells and oxygen, the latter with concentration $c=c(x,t)$, are transported by the fluid and diffuse randomly, that the cells partially direct their movement toward increasing concentrations of oxygen, and that the latter is consumed by the cells.

The regularity problem in the Navier–Stokes and chemotaxis subsystems. The mathematical understanding of such types of interplay is yet quite rudimentary only, which may be viewed as reflecting the circumstance that (1.2) joins two delicate subsystems which themselves are far from understood even when decoupled from each other: Indeed, as is well-known, the three-dimensional Navier–Stokes system is still lacking a satisfactory existence theory even in absence of external forcing terms [37]: Global weak solutions for initial data in $L^2(\Omega)$ have been known to exist since Leray's celebrated pioneering work [18,28], but despite intense research over the past decades it cannot be decided up to now whether the nonlinear convective term may enforce the spontaneous emergence of singularities in the sense of blow-up with respect to e.g. the norm in $L^\infty(\Omega)$, or whether such phenomena are entirely ruled out by diffusion; in contrast to this, the latter is known to be the case in the two-dimensional analogue in which unique global smooth solutions exist for all reasonably regular initial data to the corresponding Dirichlet problem in bounded domains, for instance [28].

A similar criticality of the spatially three-dimensional setting with respect to rigorous analytical evidence can be observed for the chemotaxis subsystem obtained upon neglecting the fluid interaction in (1.2). In fact, e.g. for the prototypical system

$$\begin{cases} n_t = \Delta n - \nabla\cdot(n\nabla c), & x\in\Omega,\ t>0,\\ c_t = \Delta c - nc, & x\in\Omega,\ t>0, \end{cases} \tag{1.3}$$

it is known that the Neumann initial–boundary value problem in planar bounded convex domains is uniquely globally solvable for all suitably smooth nonnegative initial data, whereas in the three-dimensional counterpart only certain weak solutions are known to exist globally, with the question whether or not blow-up may occur being undecided yet [31]. Anyhow, a highly destabilizing potential of cross-diffusive terms of the type in (1.3), at relative strength increasing with the spatial dimension, is indicated by known results on the related classical Keller–Segel system of chemotaxis, as obtained by replacing the second equation in (1.3) with $c_t=\Delta c - c + n$: While all classical solutions to the corresponding initial–boundary value problem remain bounded when either $N=1$, or $N=2$ and the total mass $\int_\Omega n_0$ of cells is small [26,25], it is known that finite-time blow-up does occur for large classes of radially symmetric initial data when either $N=2$ and $\int_\Omega n_0$ is large, or $N\ge 3$ and $\int_\Omega n_0$ is an arbitrarily small prescribed number [24,40].

Existence results for chemotaxis–fluid systems. Accordingly, the literature on coupled chemotaxis–fluid systems is yet quite fragmentary, and beyond very interesting numerical findings [5,22], most rigorous analytical results available so far concentrate either on special cases involving somewhat restrictive assumptions, or on variants of (1.2) which contain additional regularizing effects. For instance, a considerable simplification consists in removing the convective term $(u \cdot \nabla)u$ from the third equation in (1.2), thus assuming the fluid motion to be governed by the linear Stokes equations. The correspondingly modified system is indeed known to possess global solutions at least in a certain weak sense under suitable initial and boundary conditions in smoothly bounded three-dimensional convex domains, provided that the coefficient functions in (1.2) are adequately smooth, and that χ and f satisfy some mild structural conditions (cf. (1.9) below) generalizing the prototypical choices

$$f(s) = s \quad \text{for all } s \geq 0 \qquad \text{and} \qquad \chi \equiv const. \tag{1.4}$$

(see [39]); it is not known, however, whether these solutions are sufficiently regular so as to avoid phenomena of unboundedness, e.g. with respect to the norm of n in $L^\infty(\Omega)$, in either finite or infinite time (cf. also [4] for some refined extensibility criteria for local-in-time smooth solutions).

A further regularization can be achieved by assuming the diffusion of cells to be nonlinearly enhanced at large densities. Indeed, if in the first equation the term Δn is replaced by Δn^m for $m > 1$, then, firstly, for any such choice of m it is again possible to construct global weak solutions under appropriate assumptions on χ and f [10], but beyond this one can secondly prove local-in-time and even global boundedness of these solutions in the cases $m > \frac{8}{7}$ and $m > \frac{7}{6}$, respectively, and thereby rule out the occurrence of blow-up in finite time, and also in infinite time (cf. [33] and [42]). If the range of m is further restricted by assuming $m > \frac{4}{3}$, then global existence of, possibly unbounded, solutions can be derived even in presence of the full nonlinear Navier–Stokes equations [36].

As alternative blow-up preventing mechanisms, the authors in [2] and [36] identify certain saturation effects at large cell densities in the cross-diffusive term, as well as the inclusion of logistic-type cell kinetics with quadratic death terms in (1.2), in both cases leading to corresponding results on global existence of weak solutions.

In the spatially two-dimensional case, the knowledge on systems of type (1.2) is expectedly much further developed. Even in the original chemotaxis–Navier Stokes system (1.2) containing nonlinear convection in the fluid evolution, the regularizing effect of the diffusive mechanisms turns out to be strong enough so as to allow for the construction of unique global bounded classical solutions under the mild assumptions (1.8) and (1.9) on χ, f and Φ [39]; (see also [20]), and to furthermore enforce stabilization of these solutions toward spatially homogeneous equilibria in the large time limit [41]. Corresponding results on global existence in presence of porous medium type cell diffusion, or of additional logistic terms, can be found in [6,32,36,1] and [15], for instance, and recently statements on global existence and boundedness have been derived in [7] and [30] for a two-dimensional chemotaxis–Stokes variant of (1.2) involving signal production by cells and a quadratic death term in the cell evolution, as proposed in a different modeling context in [16].

Main results. For the full three-dimensional chemotaxis–Navier–Stokes system (1.2), even at the very basic level of global existence in generalized solution frameworks, a satisfactory solution theory is entirely lacking. The only global existence results we are aware of concentrate on the construction of solutions near constant steady states [9], or on the particular case when χ precisely coincides with a multiple of f [3], where the latter not only excludes the situation determined by (1.4), but under the natural assumption that $f(0) = 0$ apparently also rules out any choice of χ which is consistent with standard approaches in the modeling of chemotaxis phenomena [14].

It is the purpose of the present work to undertake a first step toward a comprehensive existence theory for (1.2) under mild assumptions of the coefficient functions therein, and for widely general initial data. In order to formulate our main results in this direction, let us specify the precise evolution problem addressed in the sequel by considering (1.2) along with the initial conditions

$$n(x,0) = n_0(x), \quad c(x,0) = c_0(x) \quad \text{and} \quad u(x,0) = u_0(x), \qquad x \in \Omega, \tag{1.5}$$

and under the boundary conditions

$$\frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = 0 \quad \text{and} \quad u = 0 \qquad \text{on } \partial\Omega, \tag{1.6}$$

in a bounded convex domain $\Omega \subset \mathbb{R}^3$ with smooth boundary, where we assume that

$$\begin{cases} n_0 \in L\log L(\Omega) \quad \text{is nonnegative with } n_0 \not\equiv 0, \quad \text{that} \\ c_0 \in L^\infty(\Omega) \quad \text{is nonnegative and such that } \sqrt{c_0} \in W^{1,2}(\Omega), \quad \text{and that} \\ u_0 \in L^2_\sigma(\Omega), \end{cases} \tag{1.7}$$

with $L^2_\sigma(\Omega) := \{\varphi \in L^2(\Omega) \mid \nabla\cdot\varphi = 0\}$ denoting the Hilbert space of all solenoidal vector fields in $L^2(\Omega)$.

With regard to the chemotactic sensitivity χ, the signal consumption rate f and the potential Φ in (1.2), throughout this paper we shall require that

$$\begin{cases} \chi \in C^2([0,\infty)) \quad \text{is positive on } [0,\infty), \\ f \in C^1([0,\infty)) \quad \text{is nonnegative on } [0,\infty) \text{ with } f(0)=0, \text{ and that} \\ \Phi \in W^{2,\infty}(\Omega), \end{cases} \tag{1.8}$$

and moreover we will need the structural hypotheses

$$\Big(\frac{f}{\chi}\Big)' > 0, \quad \Big(\frac{f}{\chi}\Big)'' \le 0 \quad \text{and} \quad (\chi\cdot f)' \ge 0 \qquad \text{on } [0,\infty). \tag{1.9}$$

We shall see that within this framework, there exists at least one globally defined triple (n,c,u) of functions solving (1.2) in a natural generalized sense specified in Definition 2.1 below. Apart from satisfying the respective weak formulations associated with the PDEs in (1.2), this solution will enjoy further properties in that it fulfills two energy-type inequalities. The first of these will be the standard estimate (1.13) reflecting energy dissipation in the Navier–Stokes system, as satisfied by any so-called turbulent solution thereof [37,28], while the second will refer to the functional $\mathcal{F}_\kappa$ with appropriate $\kappa > 0$, where we have set

$$\mathcal{F}_\kappa[n,c,u] := \int_\Omega n\ln n + \frac{1}{2}\int_\Omega \frac{\chi(c)}{f(c)}|\nabla c|^2 + \kappa\int_\Omega |u|^2 \tag{1.10}$$

for $\kappa > 0$ whenever $n \in L\log L(\Omega)$ and $c \in W^{1,2}(\Omega)$ are nonnegative and such that $\frac{\chi(c)}{f(c)}|\nabla c|^2 \in L^1(\Omega)$, and $u \in L^2(\Omega;\mathbb{R}^3)$.

Now our main result reads as follows.

Theorem 1.1. *Let (1.8) and (1.9) hold. Then for all n_0, c_0 and u_0 fulfilling (1.7), there exist*

$$\begin{aligned} & n \in L^\infty((0,\infty); L^1(\Omega)) \cap L^{\frac{5}{4}}_{loc}([0,\infty); W^{1,\frac{5}{4}}(\Omega)), \\ & c \in L^\infty(\Omega\times(0,\infty)) \cap L^4_{loc}([0,\infty); W^{1,4}(\Omega)), \\ & u \in L^\infty_{loc}([0,\infty); L^2_\sigma(\Omega)) \cap L^2_{loc}([0,\infty); W^{1,2}_0(\Omega)), \end{aligned} \tag{1.11}$$

such that (n,c,u) is a global weak solution of the problem (1.2), (1.5), (1.6) in the sense of Definition 2.1. This solution can be obtained as the pointwise limit a.e. in $\Omega\times(0,\infty)$ of a suitable sequence of classical solutions to the regularized problems (2.9) below. Moreover, (n,c,u) has the additional properties that

$$\begin{aligned} & n^{\frac{1}{2}} \in L^2_{loc}([0,\infty); W^{1,2}(\Omega)) \quad \text{and} \\ & c^{\frac{1}{4}} \in L^4_{loc}([0,\infty); W^{1,4}(\Omega)), \end{aligned} \tag{1.12}$$

and there exist $\kappa > 0$, $K > 0$ and a null set $N \subset (0,\infty)$ such that

$$\frac{1}{2}\int_\Omega |u(\cdot,t)|^2 + \int_{t_0}^t\int_\Omega |\nabla u|^2 \le \frac{1}{2}\int_\Omega |u(\cdot,t_0)|^2 + \int_{t_0}^t\int_\Omega nu\cdot\nabla\Phi \qquad \text{for all } t_0 \in [0,\infty)\setminus N \text{ and all } t > t_0, \tag{1.13}$$

as well as

$$\frac{d}{dt}\mathcal{F}_\kappa[n,c,u] + \frac{1}{K}\int_\Omega \Big\{ \frac{|\nabla n|^2}{n} + \frac{|\nabla c|^4}{c^3} + |\nabla u|^2 \Big\} \le K \qquad \text{in } \mathcal{D}'([0,\infty)). \tag{1.14}$$

Remark. The property $c^{\frac{1}{4}} \in L^4_{loc}([0,\infty); W^{1,4}(\Omega))$ along with the boundedness of c and the fact that $\frac{f}{\chi}$ is nonnegative and belongs to $C^1([0,\infty))$, with nonvanishing derivative at zero, ensures that $\frac{\chi(c)}{f(c)}|\nabla c|^2 \in L^1(\Omega)$ for a.e. $t > 0$ by the Cauchy–Schwarz inequality. Along with the regularity features of n and u in (1.11) this implies that $\mathcal{F}_\kappa[n,c,u](t)$ and $\int_\Omega |u(\cdot,t)|^2$ are well-defined for a.e. $t > 0$, whence the statements (1.14) and (1.13) are indeed meaningful.

The plan of this paper is as follows. In Section 2.2 we shall specify the generalized solution concept considered thereafter, and introduce a family of regularized problems each of which allows for smooth solutions at least locally in time. Section 3.1 will be devoted to an analysis of the functional obtained on letting $\kappa = 0$ in (1.10), evaluated at these approximate solutions. As known from previous studies, the assumptions in (1.9) ensure that the time evolution of this two-component functional involves, besides certain dissipated quantities, expressions containing the fluid velocity. An apparently novel way to treat the latter by making appropriate use of the standard energy dissipation in the Navier–Stokes equations will allow for absorbing these suitably in Section 3.2. This will entail a series of a priori estimates which will firstly be used in Section 3.3 to make sure that all the approximate solutions are actually global in time, and which secondly enable us to derive further ε-independent bounds in Section 3.4. On the basis of the compactness properties thereby implied, in Section 4 we shall finally pass to the limit along an adequate sequence of numbers $\varepsilon = \varepsilon_j \searrow 0$ and thereby verify Theorem 1.1.

2. Preliminaries

2.1. A weak solution concept

We first specify the notion of weak solution to which we will refer in the sequel. Here for candidates of solutions we require the apparently weakest possible regularity properties which ensure that all expressions in the weak identities (2.3), (2.4) and (2.5) are meaningful. As already announced in the formulation of Theorem 1.1, the solution we shall construct below will actually be significantly more regular.

Throughout the sequel, for vectors $v \in \mathbb{R}^3$ and $w \in \mathbb{R}^3$ we let $v \otimes w$ denote the matrix $(a_{ij})_{i,j\in\{1,2,3\}} \in \mathbb{R}^{3\times 3}$ defined on setting $a_{ij} := v_i w_j$ for $i, j \in \{1,2,3\}$.

Definition 2.1. By a *global weak solution* of (1.2), (1.5), (1.6) we mean a triple (n,c,u) of functions

$$n \in L^1_{loc}([0,\infty); W^{1,1}(\Omega)), \quad c \in L^1_{loc}([0,\infty); W^{1,1}(\Omega)), \quad u \in L^1_{loc}([0,\infty); W^{1,1}_0(\Omega;\mathbb{R}^3)), \tag{2.1}$$

such that $n \geq 0$ and $c \geq 0$ a.e. in $\Omega \times (0,\infty)$,

$$\begin{aligned} &nf(c) \in L^1_{loc}(\bar\Omega \times [0,\infty)), \qquad u \otimes u \in L^1_{loc}(\bar\Omega \times [0,\infty);\mathbb{R}^{3\times 3}), \qquad \text{and} \\ &n\chi(c)\nabla c, nu \text{ and } cu \text{ belong to } L^1_{loc}(\bar\Omega \times [0,\infty);\mathbb{R}^3), \end{aligned} \tag{2.2}$$

that $\nabla \cdot u = 0$ a.e. in $\Omega \times (0,\infty)$, and that

$$-\int_0^\infty\int_\Omega n\phi_t - \int_\Omega n_0\phi(\cdot,0) = -\int_0^\infty\int_\Omega \nabla n \cdot \nabla\phi + \int_0^\infty\int_\Omega n\chi(c)\nabla c \cdot \nabla\phi + \int_0^\infty\int_\Omega nu \cdot \nabla\phi \tag{2.3}$$

for all $\phi \in C_0^\infty(\bar\Omega \times [0,\infty))$,

$$-\int_0^\infty\int_\Omega c\phi_t - \int_\Omega c_0\phi(\cdot,0) = -\int_0^\infty\int_\Omega \nabla c \cdot \nabla\phi - \int_0^\infty\int_\Omega nf(c)\phi + \int_0^\infty\int_\Omega cu \cdot \nabla\phi \tag{2.4}$$

for all $\phi \in C_0^\infty(\bar\Omega \times [0,\infty))$ as well as

$$-\int_0^\infty\int_\Omega u \cdot \phi_t - \int_\Omega u_0 \cdot \phi(\cdot,0) = -\int_0^\infty\int_\Omega \nabla u \cdot \nabla\phi + \int_0^\infty\int_\Omega u \otimes u \cdot \nabla\phi + \int_0^\infty\int_\Omega n\nabla\Phi \cdot \phi \tag{2.5}$$

for all $\phi \in C_0^\infty(\Omega \times [0,\infty);\mathbb{R}^3)$ satisfying $\nabla \cdot \phi \equiv 0$.

2.2. A family of regularized problems

In order to suitably regularize the original problem (1.2), (1.5), (1.6), let us consider families of approximate initial data $n_{0\varepsilon}, c_{0\varepsilon}$ and $u_{0\varepsilon}$, $\varepsilon \in (0,1)$, with the properties that

$$\begin{cases} n_{0\varepsilon} \in C_0^\infty(\Omega), \quad n_{0\varepsilon} \ge 0 \text{ in } \Omega, \quad \int_\Omega n_{0\varepsilon} = \int_\Omega n_0 \quad \text{for all } \varepsilon \in (0,1) \qquad \text{and} \\ n_{0\varepsilon} \to n_0 \quad \text{in } L\log L(\Omega) \qquad \text{as } \varepsilon \searrow 0, \end{cases} \tag{2.6}$$

that

$$\begin{cases} c_{0\varepsilon} \ge 0 \text{ in } \Omega \text{ is such that } \sqrt{c_{0\varepsilon}} \in C_0^\infty(\Omega) \quad \text{and} \quad \|c_{0\varepsilon}\|_{L^\infty(\Omega)} \le \|c_0\|_{L^\infty(\Omega)} \quad \text{for all } \varepsilon \in (0,1) \qquad \text{and} \\ \sqrt{c_{0\varepsilon}} \to \sqrt{c_0} \quad \text{a.e. in } \Omega \text{ and in } W^{1,2}(\Omega) \quad \text{as } \varepsilon \searrow 0, \end{cases} \tag{2.7}$$

and that

$$\begin{cases} u_{0\varepsilon} \in C_{0,\sigma}^\infty(\Omega) \quad \text{with} \quad \|u_{0\varepsilon}\|_{L^2(\Omega)} = \|u_0\|_{L^2(\Omega)} \quad \text{for all } \varepsilon \in (0,1) \qquad \text{and} \\ u_{0\varepsilon} \to u_0 \quad \text{in } L^2(\Omega) \quad \text{as } \varepsilon \searrow 0, \end{cases} \tag{2.8}$$

where as usual $L\log L(\Omega)$ denotes the standard Orlicz space associated with the Young function $(0,\infty) \ni z \mapsto z\ln(1+z)$.

For $\varepsilon \in (0,1)$, we thereupon consider

$$\begin{cases} \partial_t n_\varepsilon + u_\varepsilon \cdot \nabla n_\varepsilon = \Delta n_\varepsilon - \nabla \cdot (n_\varepsilon F_\varepsilon'(n_\varepsilon)\chi(c_\varepsilon)\nabla c_\varepsilon), & x \in \Omega,\ t > 0, \\ \partial_t c_\varepsilon + u_\varepsilon \cdot \nabla c_\varepsilon = \Delta c_\varepsilon - F_\varepsilon(n_\varepsilon) f(c_\varepsilon), & x \in \Omega,\ t > 0, \\ \partial_t u_\varepsilon + (Y_\varepsilon u_\varepsilon \cdot \nabla) u_\varepsilon = \Delta u_\varepsilon + \nabla P_\varepsilon + n_\varepsilon \nabla\phi, & x \in \Omega,\ t > 0, \\ \nabla \cdot u_\varepsilon = 0, & x \in \Omega,\ t > 0, \\ \frac{\partial n_\varepsilon}{\partial\nu} = \frac{\partial c_\varepsilon}{\partial\nu} = 0, \quad u_\varepsilon = 0, & x \in \partial\Omega,\ t > 0, \\ n_\varepsilon(x,0) = n_{0\varepsilon}(x), \quad c_\varepsilon(x,0) = c_{0\varepsilon}(x), \quad u_\varepsilon(x,0) = u_{0\varepsilon}(x), & x \in \Omega, \end{cases} \tag{2.9}$$

where we adopt from [39] the weakly increasing approximation F_ε of $[0,\infty) \ni s \mapsto s$ determined by

$$F_\varepsilon(s) := \frac{1}{\varepsilon}\ln(1+\varepsilon s) \qquad \text{for } s \ge 0, \tag{2.10}$$

and where we utilize the standard Yosida approximation Y_ε [28,23] defined by

$$Y_\varepsilon v := (1+\varepsilon A)^{-1} v \qquad \text{for } v \in L^2_\sigma(\Omega). \tag{2.11}$$

Here and throughout the sequel, by A we mean the realization of the Stokes operator $-\mathcal{P}\Delta$ in $L^2_\sigma(\Omega)$, with domain $D(A) = W^{2,2}(\Omega) \cap W^{1,2}_{0,\sigma}(\Omega)$, where $W^{1,2}_{0,\sigma}(\Omega) := W^{1,2}_0(\Omega) \cap L^2_\sigma(\Omega) \equiv \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{W^{1,2}(\Omega)}}$ with $C^\infty_{0,\sigma}(\Omega) := C_0^\infty(\Omega) \cap L^2_\sigma(\Omega)$, and where $\mathcal{P}$ denotes the Helmholtz projection in $L^2(\Omega)$. It is well-known that A is self-adjoint and positive due to the fact that Ω is bounded, and hence in particular possesses fractional powers A^α for arbitrary $\alpha \in \mathbb{R}$ [28, Ch. III.2].

We remark that in contrast to the case of the pure Navier–Stokes equations without chemotactic coupling, where global existence of weak solutions can be proved employing the less regularizing operators $(1+\varepsilon A^{\frac{1}{2}})^{-1}$ instead of Y_ε [28, Ch. V.2], the use of our stronger regularization in (2.11) will turn out to be more convenient in the present context, because in conjunction with the properties of F_ε it will allow for a comparatively simple proof of global solvability in (2.9) due to the fact that Y_ε acts as a bounded operator from $L^2(\Omega)$ into $L^\infty(\Omega)$ (cf. Lemma 3.9).

Let us furthermore note that our choice of F_ε ensures that

$$0 \le F_\varepsilon'(s) = \frac{1}{1+\varepsilon s} \le 1 \quad \text{and} \quad 0 \le F_\varepsilon(s) \le s \qquad \text{for all } s \ge 0 \text{ and } \varepsilon \in (0,1), \tag{2.12}$$

and that

$$F_\varepsilon'(s) \nearrow 1 \quad \text{and} \quad F_\varepsilon(s) \nearrow s \quad \text{as } \varepsilon \searrow 0 \qquad \text{for all } s \ge 0. \tag{2.13}$$

All the above approximate problems admit for local-in-time smooth solutions:

Lemma 2.2. *For each $\varepsilon \in (0,1)$, there exist $T_{max,\varepsilon} \in (0,\infty]$ and uniquely determined functions*

$$n_\varepsilon \in C^{2,1}(\bar\Omega \times [0,T_{max,\varepsilon})), \quad c_\varepsilon \in C^{2,1}(\bar\Omega \times [0,T_{max,\varepsilon})) \quad and \quad u_\varepsilon \in C^{2,1}(\bar\Omega \times [0,T_{max,\varepsilon}); \mathbb{R}^3) \tag{2.14}$$

which are such that $n_\varepsilon > 0$ and $c_\varepsilon > 0$ in $\bar\Omega \times (0,T_{max,\varepsilon})$, and such that with some $P_\varepsilon \in C^{1,0}(\Omega \times (0,T_{max,\varepsilon}))$, the quadruple $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ solves (2.9) classically in $\Omega \times (0,T_{max,\varepsilon})$. Moreover,

$$\text{if } T_{max,\varepsilon} < \infty, \text{ then } \|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|c_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)} + \|A^\alpha u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \to \infty \quad \text{as } t \nearrow T_{max,\varepsilon}$$
$$\text{for all } q > 3 \text{ and } \alpha > \frac{3}{4}. \tag{2.15}$$

Proof. A proof for this, based e.g. on the contraction mapping principle and standard regularity theories for the heat equation and the Stokes system [17,27,29,28], can be copied almost word by word from [39, Lemma 2.1], where minor modifications, mainly due to the presence of the Yosida approximation operator Y_ε, may be left to the reader. □

We shall later see in Lemma 3.9 that each of these solutions is in fact global in time. This will be a particular consequence of a series of a priori estimates for (2.9), as the first two of which one may view the following two basic properties.

Lemma 2.3. *For any $\varepsilon \in (0,1)$, we have*

$$\int_\Omega n_\varepsilon(\cdot,t) = \int_\Omega n_0 \qquad \text{for all } t \in (0,T_{max,\varepsilon}) \tag{2.16}$$

and

$$\|c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le s_0 := \|c_0\|_{L^\infty(\Omega)} \qquad \text{for all } t \in (0,T_{max,\varepsilon}). \tag{2.17}$$

Proof. In (2.9), we only need to integrate the first equation over Ω and apply the maximum principle to the second equation. □

3. A priori estimates

3.1. An energy functional for the chemotaxis subsystem

A key role in the derivation of further estimates will be played by the following identity which was stated in [39] for the case when Y_ε in (2.9) is replaced by the identity operator, but which readily extends to the present situation, because it actually only relies on the first two equations in (2.9) and the fact that the fluid component in the transport terms therein is solenoidal. The novelty of the present reasoning, as compared to previous approaches based on this or similar identities (cf. also [9]), appears to consist in the particular manner in which (3.1) will subsequently be related to the natural dissipative properties of the Navier–Stokes subsystem in (2.9).

Lemma 3.1. *Given any $\varepsilon \in (0,1)$, the solution of (2.9) satisfies*

$$\begin{aligned}
&\frac{d}{dt}\left\{\int_\Omega n_\varepsilon \ln n_\varepsilon + \frac12 \int_\Omega |\nabla\Psi(c_\varepsilon)|^2\right\} + \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + \int_\Omega g(c_\varepsilon)|D^2\rho(c_\varepsilon)|^2 \\
&= -\frac12 \int_\Omega \frac{g'(c_\varepsilon)}{g^2(c_\varepsilon)}|\nabla c_\varepsilon|^2 (u_\varepsilon\cdot\nabla c_\varepsilon) + \int_\Omega \frac{1}{g(c_\varepsilon)}\Delta c_\varepsilon (u_\varepsilon \cdot \nabla c_\varepsilon) + \int_\Omega F_\varepsilon(n_\varepsilon)\left\{\frac{f(c_\varepsilon)g'(c_\varepsilon)}{2g^2(c_\varepsilon)} - \frac{f'(c_\varepsilon)}{g(c_\varepsilon)}\right\}\cdot|\nabla c_\varepsilon|^2 \\
&\quad + \frac12 \int_\Omega \frac{g''(c_\varepsilon)}{g^2(c_\varepsilon)}|\nabla c_\varepsilon|^4 + \frac12 \int_{\partial\Omega} \frac{1}{g(c_\varepsilon)}\cdot\frac{\partial|\nabla c_\varepsilon|^2}{\partial\nu} \qquad \text{for all } t \in (0,T_{max,\varepsilon}),
\end{aligned} \tag{3.1}$$

where we have set

$$g(s) := \frac{f(s)}{\chi(s)}, \quad \Psi(s) := \int_1^s \frac{d\sigma}{\sqrt{g(\sigma)}} \quad \text{and} \quad \rho(s) := \int_1^s \frac{d\sigma}{g(\sigma)} \qquad \text{for } s > 0. \tag{3.2}$$

Proof. This can be obtained by straightforward computation on the basis of the first two equations in (2.9) and the fact that $\nabla \cdot u_\varepsilon \equiv 0$ (see [39, Lemma 3.2] for details). □

In order to take full advantage of the dissipated quantities on the left of (3.1), we recall the following functional inequality from [39, Lemma 3.3].

Lemma 3.2. *Suppose that $h \in C^1((0,\infty))$ is positive, and let $\Theta(s) := \int_1^s \frac{d\sigma}{h(\sigma)}$ for $s > 0$. Then*

$$\int_\Omega \frac{h'(\varphi)}{h^3(\varphi)} |\nabla\varphi|^4 \le (2+\sqrt{3})^2 \int_\Omega \frac{h(\varphi)}{h'(\varphi)} |D^2\Theta(\varphi)|^2 \tag{3.3}$$

holds for all $\varphi \in C^2(\bar\Omega)$ satisfying $\varphi > 0$ in $\bar\Omega$ and $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega$.

For our application of (3.3) to (3.1), let us state a consequence of our hypotheses (1.8) and (1.9) on the function g from (3.2) which appears as a weight function in several expressions in (3.1).

Lemma 3.3. *Let s_0 be as in (2.17). Then there exist $C_g^+ > 0$ and $C_g^- > 0$ such that the function $g = \frac{f}{\chi}$ in (3.2) satisfies*

$$C_g^- \cdot s \le g(s) \le C_g^+ \cdot s \qquad \text{for all } s \in [0, s_0]. \tag{3.4}$$

Proof. This is an immediate consequence of the first assumption in (1.9), which together with (1.8) entails that g belongs to $C^1([0,s_0])$ with $g(0) = 0$, $g'(0) > 0$ and $g > 0$ on $(0, s_0]$. □

We can thereby turn (3.1) into an inequality only involving the Dirichlet integral of the fluid velocity on its right-hand side.

Lemma 3.4. *There exists $K_0 \ge 1$ such that for all $\varepsilon \in (0,1)$ we have*

$$\frac{d}{dt}\left\{ \int_\Omega n_\varepsilon \ln n_\varepsilon + \frac{1}{2}\int_\Omega |\nabla\Psi(c_\varepsilon)|^2 \right\} + \frac{1}{K_0} \cdot \left\{ \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + \int_\Omega \frac{|D^2 c_\varepsilon|^2}{c_\varepsilon} + \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} \right\}$$
$$\le K_0 \int_\Omega |\nabla u_\varepsilon|^2 \tag{3.5}$$

for all $t \in (0, T_{max,\varepsilon})$, where Ψ is as in (3.2).

Proof. We first follow an argument presented in [39, Lemma 3.4] to infer from the third and the second inequality in (1.9) that with $g = \frac{f}{\chi}$ we have

$$\frac{fg'}{2g^2} - \frac{f'}{g} = -\frac{(\chi\cdot f)'}{2f} \le 0 \quad \text{and} \quad g'' \le 0 \qquad \text{on } [0,\infty). \tag{3.6}$$

Apart from that, we note that with s_0 as in (2.17), the first two assumptions in (1.9) along with (2.17) imply that $g'(c_\varepsilon) \ge g'(s_0) > 0$ in $\Omega \times (0, T_{max,\varepsilon})$, whence Lemma 3.2 combined with (3.4) shows that if we take ρ from (3.2), then

$$\frac{g'(s_0)}{(C_g^+)^3} \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} \le \int_\Omega \frac{g'(c_\varepsilon)}{g^3(c_\varepsilon)} |\nabla c_\varepsilon|^4$$

$$\le (2+\sqrt{3})^2 \int_\Omega \frac{g(c_\varepsilon)}{g'(c_\varepsilon)} |D^2\rho(c_\varepsilon)|^2$$
$$\le \frac{(2+\sqrt{3})^2}{g'(s_0)} \int_\Omega g(c_\varepsilon)|D^2\rho(c_\varepsilon)|^2 \qquad \text{for all } t\in(0,T_{max,\varepsilon}). \tag{3.7}$$

Next, for $i,j\in\{1,2,3\}$ we may again rely on Lemma 3.3 and the concavity and positivity of g and use that $(a+b)^2 \ge \frac{1}{2}a^2 - b^2$ for $a,b\in\mathbb{R}$ to obtain the pointwise inequality

$$\begin{aligned} g(c_\varepsilon)|\partial_{ij}\rho(c_\varepsilon)|^2 &= g(c_\varepsilon)\Big|\rho'(c_\varepsilon)\partial_{ij}c_\varepsilon + \rho''(c_\varepsilon)\partial_i c_\varepsilon \partial_j c_\varepsilon\Big|^2 \\ &\ge \frac{1}{2} g(c_\varepsilon)\rho'^2(c_\varepsilon)|\partial_{ij}c_\varepsilon|^2 - g(c_\varepsilon)\rho''^2(c_\varepsilon)|\partial_i c_\varepsilon\partial_j c_\varepsilon|^2 \\ &= \frac{1}{2g(c_\varepsilon)}|\partial_{ij}c_\varepsilon|^2 - \frac{g'^2(c_\varepsilon)}{g^3(c_\varepsilon)}|\partial_i c_\varepsilon \partial_j c_\varepsilon|^2 \\ &\ge \frac{1}{2C_g^+}\cdot\frac{|\partial_{ij}c_\varepsilon|^2}{c_\varepsilon} - \frac{g'^2(0)}{(C_g^-)^3}\cdot\frac{|\partial_i c_\varepsilon\partial_j c_\varepsilon|^2}{c_\varepsilon^3} \qquad \text{in } \Omega\times(0,T_{max,\varepsilon}). \end{aligned} \tag{3.8}$$

Summarizing, from (3.7) and (3.8) we thus infer the existence of positive constants C_1, C_2 and C_3 such that

$$\int_\Omega g(c_\varepsilon)|D^2\rho(c_\varepsilon)|^2 \ge C_1\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} \quad \text{and} \quad \int_\Omega g(c_\varepsilon)|D^2\rho(c_\varepsilon)|^2 \ge C_2\int_\Omega\frac{|D^2 c_\varepsilon|^2}{c_\varepsilon} - C_3\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3}$$

for all $t\in(0,T_{max,\varepsilon})$, which in combination can easily be seen to imply that

$$\int_\Omega g(c_\varepsilon)|D^2\rho(c_\varepsilon)|^2 \ge C_4\int_\Omega \frac{|D^2c_\varepsilon|^2}{c_\varepsilon} + C_4\int_\Omega\frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} \tag{3.9}$$

holds for all $t\in(0,T_{max,\varepsilon})$ if we let $C_4 := \min\{\frac{C_1}{4}, \frac{C_2}{2}, \frac{C_1}{8C_3}\}$.

Since finally $\frac{\partial|\nabla c_\varepsilon|^2}{\partial\nu} \le 0$ on $\partial\Omega$ according to the convexity of Ω [19], (3.1), (3.6) and (3.9) imply that

$$\begin{aligned} &\frac{d}{dt}\Big\{\int_\Omega n_\varepsilon \ln n_\varepsilon + \frac{1}{2}\int_\Omega |\nabla\Psi(c_\varepsilon)|^2\Big\} + \int_\Omega\frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + C_4\int_\Omega\frac{|D^2c_\varepsilon|^2}{c_\varepsilon} + C_4\int_\Omega\frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} \\ &\quad\le -\frac{1}{2}\int_\Omega \frac{g'(c_\varepsilon)}{g^2(c_\varepsilon)}|\nabla c_\varepsilon|^2(u_\varepsilon\cdot\nabla c_\varepsilon) + \int_\Omega\frac{1}{g(c_\varepsilon)}\Delta c_\varepsilon(u_\varepsilon\cdot\nabla c_\varepsilon) \qquad \text{for all } t\in(0,T_{max,\varepsilon}). \end{aligned} \tag{3.10}$$

We now adopt an idea from [3] and integrate by parts in the rightmost integral herein to see that

$$\begin{aligned} \int_\Omega \frac{1}{g(c_\varepsilon)}\Delta c_\varepsilon(u_\varepsilon\cdot\nabla c_\varepsilon) &= \int_\Omega \frac{g'(c_\varepsilon)}{g^2(c_\varepsilon)}|\nabla c_\varepsilon|^2(u_\varepsilon\cdot\nabla c_\varepsilon) - \int_\Omega\frac{1}{g(c_\varepsilon)}\nabla c_\varepsilon\cdot(\nabla u_\varepsilon\cdot\nabla c_\varepsilon) \\ &\quad - \int_\Omega \frac{1}{g(c_\varepsilon)} u_\varepsilon\cdot(D^2c_\varepsilon\cdot\nabla c_\varepsilon) \end{aligned}$$

for all $t\in(0,T_{max,\varepsilon})$, where another integration by parts yields

$$\begin{aligned} -\int_\Omega\frac{1}{g(c_\varepsilon)}u_\varepsilon\cdot(D^2c_\varepsilon\cdot\nabla c_\varepsilon) &= -\frac{1}{2}\int_\Omega\frac{1}{g(c_\varepsilon)}u_\varepsilon\cdot\nabla|\nabla c_\varepsilon|^2 \\ &= -\frac{1}{2}\int_\Omega\frac{g'(c_\varepsilon)}{g^2(c_\varepsilon)}|\nabla c_\varepsilon|^2(u_\varepsilon\cdot\nabla c_\varepsilon) \qquad \text{for all } t\in(0,T_{max,\varepsilon}), \end{aligned}$$

because $\nabla \cdot u_\varepsilon \equiv 0$. Thereby the first integral on the right-hand side of (3.10) can be canceled, so that altogether we obtain

$$\frac{d}{dt}\left\{\int_\Omega n_\varepsilon \ln n_\varepsilon + \frac{1}{2}\int_\Omega |\nabla\Psi(c_\varepsilon)|^2\right\} + \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + C_4\int_\Omega \frac{|D^2 c_\varepsilon|^2}{c_\varepsilon} + C_4\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3}$$
$$\le -\int_\Omega \frac{1}{g(c_\varepsilon)}\nabla c_\varepsilon \cdot (\nabla u_\varepsilon \cdot \nabla c_\varepsilon) \tag{3.11}$$

for all $t \in (0, T_{max,\varepsilon})$, where by Young's inequality, (3.4) and (2.17) we see that with some $C_5 > 0$ we have

$$-\int_\Omega \frac{1}{g(c_\varepsilon)}\nabla c_\varepsilon \cdot (\nabla u_\varepsilon \cdot \nabla c_\varepsilon) \le \frac{C_4}{2}\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + C_5\int_\Omega \frac{c_\varepsilon^3}{g^2(c_\varepsilon)}|\nabla u_\varepsilon|^2$$
$$\le \frac{C_4}{2}\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac{C_5}{(C_g^-)^2}\int_\Omega c_\varepsilon|\nabla u_\varepsilon|^2$$
$$\le \frac{C_4}{2}\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac{C_5 s_0}{(C_g^-)^2}\int_\Omega |\nabla u_\varepsilon|^2 \qquad \text{for all } t \in (0, T_{max,\varepsilon}).$$

The claimed inequality (3.5) thus results from (3.11) if we let $K_0 := \max\left\{1\,,\,\frac{2}{C_4}\,,\,\frac{C_5 s_0}{(C_g^-)^2}\right\}$. □

3.2. *Involving fluid dissipation*

In order to absorb the source on the right of (3.5) appropriately, we recall the following standard energy inequality for the fluid component (cf. also [28]).

Lemma 3.5. *For each $\varepsilon \in (0,1)$, the solution of (2.9) satisfies*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 = \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla\Phi \qquad \textit{for all } t \in (0, T_{max,\varepsilon}). \tag{3.12}$$

Proof. Testing the third equation in (2.9) by u_ε and writing $v_\varepsilon := Y_\varepsilon u_\varepsilon$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 = -\int_\Omega (v_\varepsilon \cdot \nabla)u_\varepsilon \cdot u_\varepsilon + \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla\Phi \qquad \text{for all } t \in (0, T_{max,\varepsilon}). \tag{3.13}$$

Here since $\nabla \cdot u_\varepsilon \equiv 0$ and also $\nabla \cdot (I + \varepsilon A)^{-1} u_\varepsilon \equiv 0$, twice integrating by parts shows that

$$\int_\Omega (v_\varepsilon \cdot \nabla)u_\varepsilon \cdot u_\varepsilon = -\int_\Omega (\nabla \cdot v_\varepsilon)|u_\varepsilon|^2 - \frac{1}{2}\int_\Omega v_\varepsilon \cdot \nabla|u_\varepsilon|^2 = -\frac{1}{2}\int_\Omega (\nabla \cdot v_\varepsilon)|u_\varepsilon|^2 = 0$$

for all $t \in (0, T_{max,\varepsilon})$, whence (3.13) implies (3.12). □

Now a suitable combination of Lemma 3.4 with Lemma 3.5 yields the following energy-type inequality which simultaneously involves all the components n_ε, c_ε and u_ε.

Lemma 3.6. *Let Ψ be as given by (3.2). Then there exist $\kappa > 0$ and $K > 0$ such that for all $\varepsilon \in (0,1)$, the solution of (2.9) satisfies*

$$\frac{d}{dt}\left\{\int_\Omega n_\varepsilon \ln n_\varepsilon + \frac{1}{2}\int_\Omega |\nabla\Psi(c_\varepsilon)|^2 + \kappa\int_\Omega |u_\varepsilon|^2\right\} + \frac{1}{K}\left\{\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + \int_\Omega \frac{|D^2 c_\varepsilon|^2}{c_\varepsilon} + \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \int_\Omega |\nabla u_\varepsilon|^2\right\}$$
$$\le K \qquad \textit{for all } t \in (0, T_{max,\varepsilon}). \tag{3.14}$$

In particular, with $\mathcal{F}_\kappa$ as defined in (1.10) we have

$$-\int_0^\infty \mathcal{F}_\kappa[n_\varepsilon, c_\varepsilon, u_\varepsilon](t)\cdot\phi'(t)dt + \frac{1}{K}\int_0^\infty\int_\Omega \left\{\frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + |\nabla u_\varepsilon|^2\right\}(x,t)\cdot\phi(t)dxdt$$
$$\le \mathcal{F}_\kappa[n_{0\varepsilon}, c_{0\varepsilon}, u_{0\varepsilon}]\cdot\phi(0) + K\int_0^\infty \phi(t)dt \tag{3.15}$$

for each nonnegative $\phi \in C_0^\infty([0,\infty))$ and all $\varepsilon \in (0,1)$.

Proof. We first combine (3.5) with (3.12) to see that with K_0 as introduced in Lemma 3.4,

$$\frac{d}{dt}\left\{\int_\Omega n_\varepsilon \ln n_\varepsilon + \frac{1}{2}\int_\Omega |\nabla\Psi(c_\varepsilon)|^2 + K_0\int_\Omega |u_\varepsilon|^2\right\}$$
$$+\frac{1}{K_0}\cdot\left\{\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + \int_\Omega \frac{|D^2 c_\varepsilon|^2}{c_\varepsilon} + \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3}\right\} + K_0\int_\Omega |\nabla u_\varepsilon|^2$$
$$\le K_0\int_\Omega n_\varepsilon u_\varepsilon\cdot\nabla\Phi \quad \text{for all } t\in(0,T_{max,\varepsilon}). \tag{3.16}$$

Here we use the Hölder inequality, (1.8) and the continuity of the embedding $W^{1,2}(\Omega)\hookrightarrow L^6(\Omega)$ to find $C_1>0$ such that

$$K_0\int_\Omega n_\varepsilon u_\varepsilon\cdot\nabla\Phi \le K_0\|\nabla\Phi\|_{L^\infty(\Omega)}\|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)}\|u_\varepsilon\|_{L^6(\Omega)}$$
$$\le C_1\|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)}\|\nabla u_\varepsilon\|_{L^2(\Omega)} \qquad \text{for all } t\in(0,T_{max,\varepsilon}),$$

where the Gagliardo–Nirenberg inequality provides $C_2>0$ and $C_3>0$ such that

$$\|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)} = \|n_\varepsilon^{\frac{1}{2}}\|^2_{L^{\frac{12}{5}}(\Omega)}$$
$$\le C_2\|\nabla n_\varepsilon^{\frac{1}{2}}\|^{\frac{1}{2}}_{L^2(\Omega)}\|n_\varepsilon^{\frac{1}{2}}\|^{\frac{3}{2}}_{L^2(\Omega)} + C_2\|n_\varepsilon^{\frac{1}{2}}\|^2_{L^2(\Omega)}$$
$$\le C_3\cdot\left\{\|\nabla n_\varepsilon^{\frac{1}{2}}\|_{L^2(\Omega)}+1\right\}^{\frac{1}{2}} \qquad \text{for all } t\in(0,T_{max,\varepsilon}),$$

because $\|n_\varepsilon^{\frac{1}{2}}(\cdot,t)\|^2_{L^2(\Omega)} = \int_\Omega n_\varepsilon(\cdot,t) = \int_\Omega n_0$ for all $t\in(0,T_{max,\varepsilon})$ by (2.16). Twice applying Young's inequality, we hence infer that with some $C_4>0$ and $C_5>0$ we have

$$K_0\int_\Omega n_\varepsilon u_\varepsilon\cdot\nabla\Phi \le C_1C_3\cdot\left\{\|\nabla n_\varepsilon^{\frac{1}{2}}\|_{L^2(\Omega)}+1\right\}^{\frac{1}{2}}\cdot\|\nabla u_\varepsilon\|_{L^2(\Omega)}$$
$$\le \frac{K_0}{2}\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + C_4\cdot\left\{\|\nabla n_\varepsilon^{\frac{1}{2}}\|_{L^2(\Omega)}+1\right\}$$
$$\le \frac{K_0}{2}\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{2K_0}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + C_5 \qquad \text{for all } t\in(0,T_{max,\varepsilon}).$$

Since $K_0\ge 1$ and hence $\frac{K_0}{2}\ge\frac{1}{2K_0}$, inserted into (3.16) this readily yields (3.14) if we let $\kappa := K_0$ and $K := \max\{2K_0, C_5\}$. Finally, (3.15) can be obtained in a straightforward manner on multiplying (3.14) by ϕ and integrating the resulting inequality over $(0,\infty)$, dropping a nonnegative term on its left-hand side. □

In order to derive suitable estimates from this, let us make sure that the energy functional $\mathcal{F}_\kappa$ therein, when evaluated at the initial time, approaches its expected limit as $\varepsilon \searrow 0$. Since in this respect the integrals involving $n_{0\varepsilon}$ and $u_{0\varepsilon}$ clearly have the desired behavior due to (2.6) and (2.8), this actually reduces to proving the following lemma, which for later purpose asserts a slightly more general statement.

Lemma 3.7. *Let Ψ be as in (3.2), and suppose that $(\varphi_j)_{j\in\mathbb{N}} \subset C^2(\bar\Omega;(0,\infty))$ and $\varphi:\Omega\to[0,\infty)$ are such that $\sqrt{\varphi}\in W^{1,2}(\Omega)$ and*

$$\sqrt{\varphi_j} \to \sqrt{\varphi} \quad \text{in } W^{1,2}(\Omega) \text{ and a.e. in } \Omega \quad \text{as } j\to\infty \tag{3.17}$$

as well as $\|\varphi_j\|_{L^\infty(\Omega)} \le s_0$ with s_0 given by (2.17). Then $\Psi(\varphi)\in W^{1,2}(\Omega)$ and

$$\Psi(\varphi_j) \to \Psi(\varphi) \quad \text{in } W^{1,2}(\Omega) \qquad \text{as } j\to\infty. \tag{3.18}$$

Proof. Since (3.4) and the inequality $\varphi_j \le s_0$ ensure that

$$\varphi_j \Psi'^2(\varphi_j) = \frac{\varphi_j}{g(\varphi_j)} \le \frac{1}{C_g^-} \quad \text{in } \Omega \qquad \text{for all } \varepsilon\in(0,1),$$

our assumption that $\varphi_j\to\varphi$ a.e. in Ω as $j\to\infty$ entails that $\varphi_j\Psi'^2(\varphi_j) \overset{\star}{\rightharpoonup} \varphi\Psi'^2(\varphi)$ in $L^\infty(\Omega)$ as $j\to\infty$. Combined with the fact that by (3.17) we have $|\nabla\sqrt{\varphi_j}|^2 \to |\nabla\sqrt{\varphi}|^2$ in $L^1(\Omega)$ as $j\to\infty$, this shows that

$$\int_\Omega |\nabla\Psi(\varphi_j)|^2 = \int_\Omega \varphi_j\Psi'^2(\varphi_j)|\nabla\sqrt{\varphi_j}|^2 \to \int_\Omega \varphi\Psi'^2(\varphi)|\nabla\sqrt{\varphi}|^2 = \int_\Omega |\nabla\Psi(\varphi)|^2$$

as $j\to\infty$. Since the estimates in Lemma 3.7 along with (3.17) and the dominated convergence theorem readily imply that $\Psi(\varphi_j)\to\Psi(\varphi)$ in $L^1(\Omega)$ as $j\to\infty$, this proves (3.18). □

We can thereby draw the following consequence of Lemma 3.6.

Lemma 3.8. *There exists $C>0$ such that with Ψ as in (3.2) we have*

$$\int_\Omega n_\varepsilon(\cdot,t)\ln n_\varepsilon(\cdot,t) + \int_\Omega |\nabla\Psi(c_\varepsilon(\cdot,t))|^2 + \int_\Omega |u_\varepsilon(\cdot,t)|^2 \le C \qquad \text{for all } t\in(0,T_{max,\varepsilon}) \tag{3.19}$$

and

$$\int_0^T\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + \int_0^T\int_\Omega \frac{|D^2c_\varepsilon|^2}{c_\varepsilon} + \int_0^T\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \int_0^T\int_\Omega |\nabla u_\varepsilon|^2 \le C\cdot(T+1) \qquad \text{for all } T\in(0,T_{max,\varepsilon}) \tag{3.20}$$

whenever $\varepsilon\in(0,1)$.

Proof. With $\kappa>0$ and $K>0$ as provided by Lemma 3.6, an application of the latter shows that if we take Ψ from (3.2), then for each $\varepsilon\in(0,1)$,

$$y_\varepsilon(t) := \mathcal{F}_\kappa[n_\varepsilon,c_\varepsilon,u_\varepsilon](t) \equiv \int_\Omega \left\{ n_\varepsilon\ln n_\varepsilon + \frac{1}{2}|\nabla\Psi(c_\varepsilon)|^2 + \kappa|u_\varepsilon|^2 \right\}(\cdot,t), \qquad t\in[0,T_{max,\varepsilon}),$$

satisfies

$$y_\varepsilon'(t) + \frac{1}{K}h_\varepsilon(t) \le K \qquad \text{for all } t\in(0,T_{max,\varepsilon}), \tag{3.21}$$

where

$$h_\varepsilon(t) := \int_\Omega \left\{ \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + \frac{|D^2c_\varepsilon|^2}{c_\varepsilon} + \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + |\nabla u_\varepsilon|^2 \right\}(\cdot,t) \qquad \text{for } t\in(0,T_{max,\varepsilon}).$$

In order to estimate $y_\varepsilon(t)$ in terms of $h_\varepsilon(t)$, we first use the Poincaré inequality to find $C_1 > 0$ such that

$$\int_\Omega |u_\varepsilon|^2 \le C_1 \int_\Omega |\nabla u_\varepsilon|^2 \qquad \text{for all } t \in (0, T_{max,\varepsilon}), \tag{3.22}$$

and next recall the definitions of Ψ and g in (3.2) as well as (3.4) and (2.17) to see employing Young's inequality that

$$\begin{aligned}
\frac{1}{2}\int_\Omega |\nabla \Psi(c_\varepsilon)|^2 &= \frac{1}{2}\int_\Omega \frac{|\nabla c_\varepsilon|^2}{g(c_\varepsilon)} \\
&\le \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac{1}{16}\int_\Omega \frac{c_\varepsilon^3}{g^2(c_\varepsilon)} \\
&\le \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac{1}{16(C_g^-)^2}\int_\Omega c_\varepsilon \\
&\le \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac{s_0|\Omega|}{16(C_g^-)^2} \qquad \text{for all } t \in (0, T_{max,\varepsilon}).
\end{aligned} \tag{3.23}$$

We finally make use of the elementary inequality $z \ln z \le \frac{3}{2} z^{\frac{5}{3}}$, valid for all $z \ge 0$, and invoke the Gagliardo–Nirenberg inequality together with (2.16) to infer that with some $C_2 > 0$ and $C_3 > 0$ we have

$$\begin{aligned}
\int_\Omega n_\varepsilon \ln n_\varepsilon &\le \frac{3}{2}\int_\Omega n_\varepsilon^{\frac{5}{3}} \\
&= \frac{3}{2}\|n_\varepsilon^{\frac{1}{2}}\|_{L^{\frac{10}{3}}(\Omega)}^{\frac{10}{3}} \\
&\le C_2\|\nabla n_\varepsilon^{\frac{1}{2}}\|_{L^2(\Omega)}^2 \|n_\varepsilon^{\frac{1}{2}}\|_{L^2(\Omega)}^{\frac{4}{3}} + C_2\|n_\varepsilon^{\frac{1}{2}}\|_{L^2(\Omega)}^{\frac{10}{3}} \\
&\le C_3\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + C_3
\end{aligned}$$

for all $t \in (0, T_{max,\varepsilon})$. In conjunction with (3.22) and (3.23), this provides $C_4 > 0$ such that

$$y_\varepsilon(t) \le C_4 h_\varepsilon(t) + C_4 \qquad \text{for all } t \in (0, T_{max,\varepsilon}),$$

so that (3.21) implies that y_ε satisfies the ODI

$$y_\varepsilon'(t) + \frac{1}{2K}h_\varepsilon(t) + \frac{1}{2KC_4}y_\varepsilon(t) \le C_5 := K + \frac{1}{2K} \qquad \text{for all } t \in (0, T_{max,\varepsilon}). \tag{3.24}$$

This firstly warrants that

$$y_\varepsilon(t) \le C_6 := \max\Big\{\sup_{\varepsilon\in(0,1)} y_\varepsilon(0)\,,\, 2KC_4C_5\Big\} \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \tag{3.25}$$

and thus proves (3.19), because $\sup_{\varepsilon\in(0,1)} y_\varepsilon(0)$ is finite thanks to (2.6), (2.7), (2.8) and Lemma 3.7. Secondly, another integration of (3.24) thereupon shows that

$$\frac{1}{2K}\int_0^T h_\varepsilon(t)dt \le y_\varepsilon(0) + C_5 T \le C_6 + C_5 T \qquad \text{for all } T \in (0, T_{max,\varepsilon}),$$

which in view of the definition of h_ε establishes (3.20). □

3.3. Global existence in the regularized problems

In light of the fact that our specific choice (2.10) of F_ε warrants that $[0,\infty)\ni s\mapsto sF'_\varepsilon(s)$ is bounded for any fixed $\varepsilon\in(0,1)$, the bound for ∇c_ε in $L^4_{loc}(\bar\Omega\times[0,T_{max,\varepsilon}))$ implied by Lemma 3.8 and (2.17) is sufficient to guarantee that each of our approximate solutions is indeed global in time:

Lemma 3.9. *For all $\varepsilon\in(0,1)$, the solution of (2.9) is global in time; that is, we have $T_{max,\varepsilon}=\infty$.*

Proof. Assuming that $T_{max,\varepsilon}$ be finite for some $\varepsilon\in(0,1)$, we first note that as a particular consequence of Lemma 3.8 and (2.17) we can then find $C_1>0$ and $C_2>0$ such that

$$\int_0^{T_{max,\varepsilon}}\int_\Omega|\nabla c_\varepsilon|^4\le C_1\qquad\text{and}\qquad\int_\Omega|u_\varepsilon(\cdot,t)|^2\le C_2\quad\text{for all } t\in(0,T_{max,\varepsilon}).\tag{3.26}$$

To deduce a contradiction from this, we firstly multiply the equation for n_ε in (2.9) by n_ε^3, integrate by parts and use Young's inequality together with the fact that $n_\varepsilon F'_\varepsilon(n_\varepsilon)\le\frac{1}{\varepsilon}$ by (2.12) to obtain $C_3>0$, as all constants below possibly depending on ε, such that

$$\frac14\frac{d}{dt}\int_\Omega n_\varepsilon^4+3\int_\Omega n_\varepsilon^2|\nabla n_\varepsilon|^2\le\int_\Omega n_\varepsilon^2|\nabla n_\varepsilon|^2+\int_\Omega|\nabla c_\varepsilon|^4+C_3\int_\Omega n_\varepsilon^4\qquad\text{for all } t\in(0,T_{max,\varepsilon}),$$

so that thanks to the first inequality in (3.26) we see that

$$\int_\Omega n_\varepsilon^4(\cdot,t)\le C_4\qquad\text{for all } t\in(0,T_{max,\varepsilon})\tag{3.27}$$

with some $C_4>0$.

We next observe that $D(1+\varepsilon A)=W^{2,2}(\Omega)\cap W^{1,2}_{0,\sigma}(\Omega)\hookrightarrow L^\infty(\Omega)$, according to the second estimate in (3.26) there exist $C_5>0$ and $C_6>0$ such that $v_\varepsilon:=Y_\varepsilon u_\varepsilon$ satisfies

$$\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}=\left\|(1+\varepsilon A)^{-1}u_\varepsilon(\cdot,t)\right\|_{L^2(\Omega)}\le C_4\|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}\le C_6\qquad\text{for all } t\in(0,T_{max,\varepsilon}).\tag{3.28}$$

Therefore, testing the projected Stokes equation $u_{\varepsilon t}+Au_\varepsilon=h_\varepsilon(x,t):=\mathcal{P}[-(v_\varepsilon\cdot\nabla)u_\varepsilon+n_\varepsilon\nabla\Phi]$ by Au_ε shows that

$$\begin{aligned}\frac12\frac{d}{dt}\int_\Omega|A^{\frac12}u_\varepsilon|^2+\int_\Omega|Au_\varepsilon|^2&=\int_\Omega Au_\varepsilon\cdot h_\varepsilon\\&\le\int_\Omega|Au_\varepsilon|^2+\frac14\int_\Omega|h_\varepsilon|^2\\&\le\int_\Omega|Au_\varepsilon|^2+\frac12\int_\Omega|(v_\varepsilon\cdot\nabla)u_\varepsilon|^2+\frac12\int_\Omega|n_\varepsilon\nabla\Phi|^2\\&\le\int_\Omega|Au_\varepsilon|^2+C_7\cdot\left\{\int_\Omega|\nabla u_\varepsilon|^2+\int_\Omega n_\varepsilon^2\right\}\qquad\text{for all } t\in(0,T_{max,\varepsilon})\end{aligned}$$

with some $C_7>0$, because $\|\mathcal{P}\varphi\|_{L^2(\Omega)}\le\|\varphi\|_{L^2(\Omega)}$ for all $\varphi\in L^2(\Omega)$. As $\int_\Omega|A^{\frac12}\varphi|^2=\int_\Omega|\nabla\varphi|^2$ for all $\varphi\in D(A)$, in view of (3.27) this implies the existence of $C_8>0$ fulfilling

$$\int_\Omega|\nabla u_\varepsilon(\cdot,t)|^2\le C_8\qquad\text{for all } t\in(0,T_{max,\varepsilon}).\tag{3.29}$$

Along with (3.28) and again (3.27), this in turn provides $C_9>0$ such that $\|h_\varepsilon(\cdot,t)\|_{L^2(\Omega)}\le C_9$ for all $t\in(0,T_{max,\varepsilon})$. Thus if we pick an arbitrary $\alpha\in(\frac34,1)$, then known smoothing properties of the Stokes semigroup [12, p. 201] entail that for some $C_{10}>0$ we have

$$\|A^\alpha u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} = \left\| A^\alpha e^{-tA} u_{0\varepsilon} + \int_0^t A^\alpha e^{-(t-s)A} h_\varepsilon(\cdot,s)ds \right\|_{L^2(\Omega)}$$
$$\le C_{10} t^{-\alpha} \|u_{0\varepsilon}\|_{L^2(\Omega)} + C_{10} \int_0^t (t-s)^{-\alpha} \|h_\varepsilon(\cdot,s)\|_{L^2(\Omega)} ds$$
$$\le C_{10} t^{-\alpha} \|u_{0\varepsilon}\|_{L^2(\Omega)} + \frac{C_9 C_{10} T_{max,\varepsilon}^{1-\alpha}}{1-\alpha} \qquad \text{for all } t \in (0, T_{max,\varepsilon}).$$

Since also $D(A^\alpha)$ is continuously embedded into $L^\infty(\Omega)$ due to our choice of α [13, Thm. 1.6.1], [11], we thereby obtain $C_{11} > 0$ and $C_{12} > 0$ such that with $\tau := \frac{1}{2} T_{max,\varepsilon}$,

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C_{11} \|A^\alpha u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C_{12} \qquad \text{for all } t \in (\tau, T_{max,\varepsilon}). \tag{3.30}$$

Thereafter, standard smoothing estimates for the Neumann heat semigroup [27,38], an application of the Hölder inequality and (2.17), (2.12), (3.27), (3.30) and (3.26) yield positive constants C_{13}, C_{14} and C_{15} satisfying

$$\|\nabla c_\varepsilon(\cdot,t)\|_{L^4(\Omega)} = \left\| \nabla e^{(t-\tau)\Delta} c_\varepsilon(\cdot,\tau) - \int_\tau^t \nabla e^{(t-s)\Delta} \Big\{ F_\varepsilon(n_\varepsilon) f(c_\varepsilon) + u_\varepsilon \cdot \nabla c_\varepsilon \Big\}(\cdot,s) ds \right\|_{L^4(\Omega)}$$
$$\le C_{13} (t-\tau)^{-\frac{1}{2}} \|c_\varepsilon(\cdot,\tau)\|_{L^4(\Omega)}$$
$$+ C_{13} \int_\tau^t (t-s)^{-\frac{1}{2}} \Big\{ \Big\| F_\varepsilon(n_\varepsilon(\cdot,s)) f(c_\varepsilon(\cdot,s)) \Big\|_{L^4(\Omega)} + \Big\| u_\varepsilon(\cdot,s) \cdot \nabla c_\varepsilon(\cdot,s) \Big\|_{L^4(\Omega)} \Big\} ds$$
$$\le C_{13} (t-\tau)^{-\frac{1}{2}} \|c_\varepsilon(\cdot,\tau)\|_{L^4(\Omega)}$$
$$+ C_{14} \int_\tau^t (t-s)^{-\frac{1}{2}} ds + C_{14} \int_\tau^t (t-s)^{-\frac{1}{2}} \|\nabla c_\varepsilon(\cdot,s)\|_{L^4(\Omega)} ds$$
$$\le C_{13} (t-\tau)^{-\frac{1}{2}} \|c_\varepsilon(\cdot,\tau)\|_{L^4(\Omega)} + 2 C_{14} T_{max,\varepsilon}^{\frac{1}{2}}$$
$$+ C_{14} \left\{ \int_0^{T_{max,\varepsilon}} \sigma^{-\frac{2}{3}} d\sigma \right\}^{\frac{3}{4}} \cdot \left\{ \int_0^{T_{max,\varepsilon}} \|\nabla c_\varepsilon(\cdot,s)\|_{L^4(\Omega)}^4 ds \right\}^{\frac{1}{4}}$$
$$\le C_{15} \qquad \text{for all } t \in (2\tau, T_{max,\varepsilon}). \tag{3.31}$$

Similarly, combining (2.17), (2.12), (3.27) and (3.30) shows that there exist $C_{16} > 0$, $C_{17} > 0$ and $C_{18} > 0$ such that

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}$$
$$= \left\| e^{(t-\tau)\Delta} n_\varepsilon(\cdot,\tau) - \int_\tau^t e^{(t-s)\Delta} \nabla \cdot \Big\{ n_\varepsilon(\cdot,s) F'_\varepsilon(n_\varepsilon(\cdot,s)) \chi(c_\varepsilon(\cdot,s)) \nabla c_\varepsilon(\cdot,s) + n_\varepsilon(\cdot,s) u_\varepsilon(\cdot,s) \Big\} ds \right\|_{L^\infty(\Omega)}$$
$$\le C_{16} (t-\tau)^{-\frac{3}{2}} \|n_\varepsilon(\cdot,\tau)\|_{L^1(\Omega)} + C_{16} \int_\tau^t (t-s)^{-\frac{1}{2} - \frac{3}{2} \cdot \frac{1}{4}} \|n_\varepsilon(\cdot,s)\|_{L^4(\Omega)} \Big\{ 1 + \|u_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)} \Big\} ds$$
$$\le C_{17} \qquad \text{for all } t \in (2\tau, T_{max,\varepsilon}).$$

Together with (3.31) and (3.30), this contradicts the extensibility criterion (2.15) in Lemma 2.2 and thereby entails that actually $T_{max,\varepsilon} = \infty$, as claimed. □

3.4. Further a priori estimates. Time regularity

By interpolation, the estimates from Lemma 3.8 imply bounds for further spatio-temporal integrals.

Lemma 3.10. *There exists $C > 0$ such that for each $\varepsilon \in (0,1)$ we have*

$$\int_0^T\int_\Omega n_\varepsilon^{\frac{5}{3}} \le C\cdot(T+1) \qquad \text{for all } T>0 \tag{3.32}$$

and

$$\int_0^T\int_\Omega |\nabla n_\varepsilon|^{\frac{5}{4}} \le C\cdot(T+1) \qquad \text{for all } T>0 \tag{3.33}$$

as well as

$$\int_0^T\int_\Omega |u_\varepsilon|^{\frac{10}{3}} \le C\cdot(T+1) \qquad \text{for all } T>0. \tag{3.34}$$

Proof. According to Lemma 3.8, there exists $C_1 > 0$ such that

$$\int_0^T\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} \le C_1\cdot(T+1) \qquad \text{for all } T>0. \tag{3.35}$$

Thus, invoking the Gagliardo–Nirenberg inequality along with (2.16) we obtain $C_2 > 0$ and $C_3 > 0$ such that

$$\begin{aligned}
\int_0^T\int_\Omega n_\varepsilon^{\frac{5}{3}} &= \int_0^T \|n_\varepsilon^{\frac{1}{2}}(\cdot,t)\|_{L^{\frac{10}{3}}(\Omega)}^{\frac{10}{3}}\,dt \\
&\le C_2\int_0^T \Big\{\|\nabla n_\varepsilon^{\frac{1}{2}}(\cdot,t)\|_{L^2(\Omega)}^2\cdot\|n_\varepsilon^{\frac{1}{2}}(\cdot,t)\|_{L^2(\Omega)}^{\frac{4}{3}} + \|n_\varepsilon^{\frac{1}{2}}(\cdot,t)\|_{L^2(\Omega)}^{\frac{10}{3}}\Big\}dt \\
&\le C_2\cdot\Big\{\frac{C_1\cdot(T+1)}{4}\cdot\|n_0\|_{L^1(\Omega)}^{\frac{2}{3}} + \|n_0\|_{L^1(\Omega)}^{\frac{5}{3}}T\Big\} \\
&\le C_3\cdot(T+1) \qquad \text{for all } T>0.
\end{aligned} \tag{3.36}$$

Employing the Hölder inequality, again by (2.16) we furthermore conclude from (3.35) together with (3.36) that

$$\begin{aligned}
\int_0^T\int_\Omega |\nabla n_\varepsilon|^{\frac{5}{4}} &= \int_0^T\int_\Omega \frac{|\nabla n_\varepsilon|^{\frac{5}{4}}}{n_\varepsilon^{\frac{5}{8}}}\cdot n_\varepsilon^{\frac{5}{8}} \\
&\le \Big(\int_0^T\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon}\Big)^{\frac{5}{8}}\cdot\Big(\int_0^T\int_\Omega n_\varepsilon^{\frac{5}{3}}\Big)^{\frac{3}{8}} \\
&\le C_1^{\frac{5}{8}}C_3^{\frac{3}{8}}\cdot(T+1) \qquad \text{for all } T>0.
\end{aligned}$$

Since once more relying on Lemma 3.8 we can find $C_4 > 0$ and $C_5 > 0$ such that

$$\int_\Omega |u_\varepsilon(\cdot,t)|^2 \le C_4 \quad \text{for all } t>0 \qquad \text{and} \qquad \int_0^T\int_\Omega |\nabla u_\varepsilon|^2 \le C_5\cdot(T+1) \qquad \text{for all } T>0,$$

upon another application of the Gagliardo–Nirenberg inequality in precisely the same way as in (3.36) we see that with some $C_6 > 0$ we have

$$\int_0^T \|u_\varepsilon(\cdot,t)\|_{L^{\frac{10}{3}}(\Omega)}^{\frac{10}{3}} dt \le C_6 \int_0^T \Big\{ \|\nabla u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}^2 \cdot \|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}^{\frac{4}{3}} + \|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}^{\frac{10}{3}} \Big\} dt$$
$$\le C_4^{\frac{2}{3}} C_5 C_6 \cdot (T+1) + C_4^{\frac{5}{3}} C_6 T \qquad \text{for all } T > 0,$$

whereby the proof is completed. □

In a straightforward manner, from Lemma 3.8 and Lemma 3.10 we can moreover deduce certain regularity features of the time derivatives in (2.9). Since in Lemma 4.1 these will mainly be used to warrant pointwise convergence we refrain from pursuing here the question which are the smallest spaces within which such derivative bounds can be obtained.

Lemma 3.11. *There exists $C > 0$ such that*

$$\int_0^T \|\partial_t n_\varepsilon(\cdot,t)\|_{(W^{1,10}(\Omega))^\star}^{\frac{10}{9}} dt \le C \cdot (T+1) \qquad \text{for all } T > 0 \tag{3.37}$$

and

$$\int_0^T \|\partial_t \sqrt{c_\varepsilon}(\cdot,t)\|_{(W^{1,\frac{5}{2}}(\Omega))^\star}^{\frac{5}{3}} dt \le C \cdot (T+1) \qquad \text{for all } T > 0 \tag{3.38}$$

as well as

$$\int_0^T \|\partial_t u_\varepsilon(\cdot,t)\|_{(W_{0,\sigma}^{1,5}(\Omega))^\star}^{\frac{5}{4}} dt \le C \cdot (T+1) \qquad \text{for all } T > 0. \tag{3.39}$$

Proof. For arbitrary $t > 0$ and $\varphi \in C^\infty(\bar\Omega)$, multiplying the first equation in (2.9) by φ, integrating by parts and using the Hölder inequality we obtain

$$\Big| \int_\Omega \partial_t n_\varepsilon(\cdot,t)\varphi \Big| = \Big| -\int_\Omega \nabla n_\varepsilon \cdot \nabla\varphi + \int_\Omega n_\varepsilon F_\varepsilon'(n_\varepsilon)\chi(c_\varepsilon)\nabla c_\varepsilon \cdot \nabla\varphi + \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla\varphi \Big|$$
$$\le \Big\{ \|\nabla n_\varepsilon\|_{L^{\frac{10}{9}}(\Omega)} + \|n_\varepsilon F_\varepsilon'(n_\varepsilon)\chi(c_\varepsilon)\nabla c_\varepsilon\|_{L^{\frac{10}{9}}(\Omega)} + \|n_\varepsilon u_\varepsilon\|_{L^{\frac{10}{9}}(\Omega)} \Big\} \cdot \|\varphi\|_{W^{1,10}(\Omega)},$$

so that with some $C_1 > 0$ we have

$$\int_0^T \|\partial_t n_\varepsilon(\cdot,t)\|_{(W^{1,10}(\Omega))^\star}^{\frac{10}{9}} dt \le \int_0^T \Big\{ \|\nabla n_\varepsilon\|_{L^{\frac{10}{9}}(\Omega)} + \|n_\varepsilon F_\varepsilon'(n_\varepsilon)\chi(c_\varepsilon)\nabla c_\varepsilon\|_{L^{\frac{10}{9}}(\Omega)} + \|n_\varepsilon u_\varepsilon\|_{L^{\frac{10}{9}}(\Omega)} \Big\}^{\frac{10}{9}} dt$$
$$\le C_1 \int_0^T\int_\Omega |\nabla n_\varepsilon|^{\frac{10}{9}} + C_1 \int_0^T\int_\Omega |n_\varepsilon \nabla c_\varepsilon|^{\frac{10}{9}} + C_1 \int_0^T\int_\Omega |n_\varepsilon u_\varepsilon|^{\frac{10}{9}} \tag{3.40}$$

for all $T > 0$, because $F_\varepsilon'(n_\varepsilon) \le 1$ by (2.12) and $\chi(c_\varepsilon) \le \|\chi\|_{L^\infty((0,s_0))}$ with $s_0 = \|c_0\|_{L^\infty(\Omega)}$ according to (2.17). Here several applications of Young's inequality show that

$$\int_0^T\int_\Omega |\nabla n_\varepsilon|^{\frac{10}{9}} \le \int_0^T\int_\Omega |\nabla n_\varepsilon|^{\frac{5}{4}} + |\Omega| T \qquad \text{for all } T > 0$$

and

$$\int_0^T\int_\Omega |n_\varepsilon \nabla c_\varepsilon|^{\frac{10}{9}} \le \int_0^T\int_\Omega n_\varepsilon^{\frac{5}{3}} + \int_0^T\int_\Omega |\nabla c_\varepsilon|^{\frac{10}{3}} \le \int_0^T\int_\Omega n_\varepsilon^{\frac{5}{3}} + \int_0^T\int_\Omega |\nabla c_\varepsilon|^4 + |\Omega| T \qquad \text{for all } T>0$$

as well as

$$\int_0^T\int_\Omega |n_\varepsilon u_\varepsilon|^{\frac{10}{9}} \le \int_0^T\int_\Omega n_\varepsilon^{\frac{5}{3}} + \int_0^T\int_\Omega |u_\varepsilon|^{\frac{10}{3}} \qquad \text{for all } T>0,$$

whence in light of Lemma 3.10, Lemma 3.8 and (2.17), (3.37) results from (3.40).

Likewise, given any $\varphi \in C^\infty(\bar\Omega)$ we may test the second equation in (2.9) against $\frac{\varphi}{\sqrt{c_\varepsilon}}$ to see that

$$\begin{aligned}
&\left| \int_\Omega \partial_t \sqrt{c_\varepsilon}(\cdot,t)\varphi \right| \\
&= \left| -\frac{1}{2}\int_\Omega \frac{\nabla c_\varepsilon}{\sqrt{c_\varepsilon}}\cdot\nabla\varphi + \frac{1}{4}\int_\Omega \frac{|\nabla c_\varepsilon|^2}{\sqrt{c_\varepsilon}^3}\varphi - \frac{1}{2}\int_\Omega F_\varepsilon(n_\varepsilon)\frac{f(c_\varepsilon)}{\sqrt{c_\varepsilon}}\varphi + \int_\Omega \sqrt{c_\varepsilon} u_\varepsilon\cdot\nabla\varphi \right| \\
&\le \left\{ \frac{1}{2}\left\| \frac{\nabla c_\varepsilon}{\sqrt{c_\varepsilon}} \right\|_{L^{\frac{5}{3}}(\Omega)} + \frac{1}{4}\left\| \frac{|\nabla c_\varepsilon|^2}{\sqrt{c_\varepsilon}^3} \right\|_{L^{\frac{5}{3}}(\Omega)} + \left\| F_\varepsilon(n_\varepsilon)\frac{f(c_\varepsilon)}{\sqrt{c_\varepsilon}} \right\|_{L^{\frac{5}{3}}(\Omega)} + \|\sqrt{c_\varepsilon}u_\varepsilon\|_{L^{\frac{5}{3}}(\Omega)} \right\} \cdot \|\varphi\|_{W^{1,\frac{5}{2}}(\Omega)}
\end{aligned}$$

for all $t>0$, and that hence by Young's inequality we can find $C_2>0$ such that

$$\begin{aligned}
\int_0^T \|\partial_t\sqrt{c_\varepsilon}(\cdot,t)\|_{(W^{1,\frac{5}{2}}(\Omega))^\star}^{\frac{5}{3}} dt &\le C_2\int_0^T\int_\Omega c_\varepsilon^{-\frac{5}{6}}|\nabla c_\varepsilon|^{\frac{5}{3}} + C_2\int_0^T\int_\Omega c_\varepsilon^{-\frac{5}{2}}|\nabla c_\varepsilon|^{\frac{10}{3}} + C_2\int_0^T\int_\Omega n_\varepsilon^{\frac{5}{3}} + C_2\int_0^t\int_\Omega |u_\varepsilon|^{\frac{5}{3}} \\
&\le C_2\int_0^T\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + C_2 M^{\frac{5}{7}}|\Omega| T + C_2\int_0^T\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + C_2|\Omega| T \\
&\quad + C_2\int_0^T\int_\Omega n_\varepsilon^{\frac{5}{3}} + C_2\int_0^T\int_\Omega |u_\varepsilon|^{\frac{10}{3}} + C_2|\Omega| T
\end{aligned}$$

for all $T>0$, since $F_\varepsilon(n_\varepsilon)\le n_\varepsilon$ due to (2.12), and since once more by (2.17) we have $c_\varepsilon \le s_0$ and $\frac{f(c_\varepsilon)}{\sqrt{c_\varepsilon}} \le \|f\|_{C^{\frac{1}{2}}([0,s_0])}$ in $\Omega\times(0,\infty)$. Again by Lemma 3.10 and Lemma 3.8, this implies (3.38).

Finally, given $\varphi\in C_{0,\sigma}^\infty(\Omega;\mathbb{R}^3)$ we infer from the third equation in (2.9) that

$$\begin{aligned}
\left| \int_\Omega \partial_t u_\varepsilon(\cdot,t)\cdot\varphi \right| &= \left| -\int_\Omega \nabla u_\varepsilon\cdot\nabla\varphi - \int_\Omega (Y_\varepsilon u_\varepsilon\otimes u_\varepsilon)\cdot\nabla\varphi + \int_\Omega n_\varepsilon\nabla\Phi\cdot\varphi \right| \\
&\le \left\{ \|\nabla u_\varepsilon\|_{L^{\frac{5}{4}}(\Omega)} + \|Y_\varepsilon u_\varepsilon\otimes u_\varepsilon\|_{L^{\frac{5}{4}}(\Omega)} + \|n_\varepsilon\nabla\Phi\|_{L^{\frac{5}{4}}(\Omega)} \right\}\cdot\|\varphi\|_{W^{1,5}(\Omega)}
\end{aligned} \tag{3.41}$$

for all $t>0$. In view of Young's inequality, (3.41) implies that there exists $C_3>0$ fulfilling

$$\begin{aligned}
\int_0^T \|\partial_t u_\varepsilon(\cdot,t)\|_{(W_{0,\sigma}^{1,5}(\Omega))^\star}^{\frac{5}{4}} dt &\le C_3\int_0^T\int_\Omega |\nabla u_\varepsilon|^{\frac{5}{4}} + C_3\int_0^T\int_\Omega |Y_\varepsilon u_\varepsilon\otimes u_\varepsilon|^{\frac{5}{4}} + C_3\int_0^T\int_\Omega n_\varepsilon^{\frac{5}{4}} \\
&\le C_3\int_0^T\int_\Omega |\nabla u_\varepsilon|^2 + C_3\int_0^T\int_\Omega |Y_\varepsilon u_\varepsilon|^2 + C_3\int_0^T\int_\Omega |u_\varepsilon|^{\frac{10}{3}}
\end{aligned}$$

$$+ C_3 \int_0^T \int_\Omega n_\varepsilon^{\frac{5}{3}} + 2C_3|\Omega|T \qquad \text{for all } T > 0,$$

because $\nabla\Phi \in L^\infty(\Omega)$. Since $\|Y_\varepsilon v\|_{L^2(\Omega)} \le \|v\|_{L^2(\Omega)}$ for all $v \in L^2_\sigma(\Omega)$ and hence $\int_0^T \int_\Omega |Y_\varepsilon u_\varepsilon|^2 \le \int_0^T \int_\Omega |u_\varepsilon|^{\frac{10}{3}} + |\Omega|T$ for all $T > 0$, (3.39) results from this upon another application of Lemma 3.10 and Lemma 3.8. □

4. Passing to the limit. Proof of Theorem 1.1

With the above compactness properties at hand, by means of a standard extraction procedure we can now derive the following lemma which actually contains our main existence result already.

Lemma 4.1. *There exists $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$, and such that as $\varepsilon = \varepsilon_j \searrow 0$ we have*

$$n_\varepsilon \to n \qquad \text{in } L^{\frac{5}{3}}_{loc}(\bar\Omega \times [0,\infty)) \text{ and a.e. in } \Omega \times (0,\infty), \tag{4.1}$$

$$\nabla n_\varepsilon \rightharpoonup \nabla n \qquad \text{in } L^{\frac{5}{4}}_{loc}(\bar\Omega \times [0,\infty)), \tag{4.2}$$

$$c_\varepsilon \to c \qquad \text{a.e. in } \Omega \times (0,\infty), \tag{4.3}$$

$$c_\varepsilon \overset{\star}{\rightharpoonup} c \qquad \text{in } L^\infty(\Omega \times (0,\infty)), \tag{4.4}$$

$$\nabla c_\varepsilon^{\frac{1}{4}} \rightharpoonup \nabla c^{\frac{1}{4}} \qquad \text{in } L^4_{loc}(\bar\Omega \times [0,\infty)), \tag{4.5}$$

$$u_\varepsilon \to u \qquad \text{in } L^2_{loc}(\bar\Omega \times [0,\infty)) \text{ and a.e. in } \Omega \times (0,\infty), \tag{4.6}$$

$$u_\varepsilon \overset{\star}{\rightharpoonup} u \qquad \text{in } L^\infty([0,\infty); L^2_\sigma(\Omega)), \tag{4.7}$$

$$u_\varepsilon \rightharpoonup u \qquad \text{in } L^{\frac{10}{3}}_{loc}(\bar\Omega \times [0,\infty)) \qquad \text{and} \tag{4.8}$$

$$\nabla u_\varepsilon \rightharpoonup \nabla u \qquad \text{in } L^2_{loc}(\bar\Omega \times [0,\infty)) \tag{4.9}$$

with some limit functions n, c and u such that (n, c, u) is a global weak solution of (1.2), (1.5), (1.6) in the sense of Definition 2.1.

Proof. Using the pointwise identity

$$\partial_{ij}\sqrt{c_\varepsilon} = \frac{1}{2\sqrt{c_\varepsilon}}\partial_{ij}c_\varepsilon - \frac{1}{4\sqrt{c_\varepsilon}^3}\partial_i c_\varepsilon \partial_j c_\varepsilon,$$

valid in $\Omega \times (0, T_{max,\varepsilon})$ for all $i, j \in \{1,2,3\}$, we see that according to Lemma 3.8, Lemma 3.10 and Lemma 3.11, an application of the Aubin–Lions lemma [34, Ch. III.2.2] provides a sequence $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ and limit functions n, c and u such that $\varepsilon_j \searrow 0$ as $j \to \infty$ and such that (4.2)–(4.9) hold as well as

$$n_\varepsilon \rightharpoonup n \qquad \text{in } L^{\frac{5}{3}}_{loc}(\bar\Omega \times [0,\infty)), \tag{4.10}$$

$$n_\varepsilon \to n \qquad \text{in } L^{\frac{5}{4}}_{loc}(\bar\Omega \times [0,\infty)) \text{ and a.e. in } \Omega \times (0,\infty), \tag{4.11}$$

$$n_\varepsilon(\cdot,t) \to n(\cdot,t) \qquad \text{in } L^{\frac{5}{4}}(\Omega) \text{ for all } t \in (0,\infty) \setminus N, \tag{4.12}$$

$$\sqrt{n_\varepsilon} \rightharpoonup \sqrt{n} \qquad \text{in } L^2_{loc}([0,\infty); W^{1,2}(\Omega)), \tag{4.13}$$

$$\sqrt{c_\varepsilon} \to \sqrt{c} \qquad \text{in } L^2_{loc}([0,\infty); W^{1,2}(\Omega)), \tag{4.14}$$

$$\sqrt{c_\varepsilon(\cdot,t)} \to \sqrt{c(\cdot,t)} \qquad \text{in } W^{1,2}(\Omega) \text{ for all } t \in (0,\infty) \setminus N, \qquad \text{and} \tag{4.15}$$

$$u_\varepsilon(\cdot,t) \to u(\cdot,t) \qquad \text{in } L^2(\Omega) \text{ for all } t \in (0,\infty) \setminus N, \tag{4.16}$$

as $\varepsilon = \varepsilon_j \searrow 0$ with some null set $N \subset (0,\infty)$. Since (4.10) entails that for each $T > 0$ we have

$$\int_0^T\int_\Omega n_\varepsilon^{\frac{5}{3}} = \int_0^\infty\int_\Omega \mathbf{1}_{\Omega\times(0,T)} n_\varepsilon^{\frac{5}{3}} \to \int_0^\infty\int_\Omega \mathbf{1}_{\Omega\times(0,T)} n^{\frac{5}{3}} = \int_0^T\int_\Omega n^{\frac{5}{3}}$$

as $\varepsilon = \varepsilon_j \searrow 0$, it follows that also (4.1) holds.

Now in order to verify (1.14), given a nonnegative $\phi \in C_0^\infty([0,\infty))$ we recall (3.15) to obtain

$$-\int_0^\infty \mathcal{F}_\kappa[n_\varepsilon, c_\varepsilon, u_\varepsilon](t)\cdot\phi'(t)dt + \frac{1}{K}\int_0^\infty\int_\Omega \left\{\frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + |\nabla u_\varepsilon|^2\right\}(x,t)\cdot\phi(t)dxdt$$
$$\le \mathcal{F}_\kappa[n_{0\varepsilon}, c_{0\varepsilon}, u_{0\varepsilon}]\cdot\phi(0) + K\int_0^\infty \phi(t)dt \tag{4.17}$$

for all $\varepsilon \in (0,1)$. Here combining (4.12), (4.15) and (4.16) with Lemma 3.7 shows that as $\varepsilon = \varepsilon_j \searrow 0$ we have

$$\int_\Omega n_\varepsilon(\cdot,t)\ln n_\varepsilon(\cdot,) \to \int_\Omega n(\cdot,t)\ln n(\cdot,t), \qquad \int_\Omega |\nabla\Psi(c_\varepsilon(\cdot,t))|^2 \to \int_\Omega |\nabla\Psi(c(\cdot,t))|^2 \quad \text{and}$$
$$u_\varepsilon(\cdot,t) \to u(\cdot,t) \quad \text{in } L^2(\Omega) \qquad \text{for all } t\in(0,\infty)\setminus N \tag{4.18}$$

and hence

$$\mathcal{F}_\kappa[n_\varepsilon, c_\varepsilon, u_\varepsilon](t) \to \mathcal{F}_\kappa[n,c,u](t) \qquad \text{for all } t\in(0,\infty)\setminus N,$$

so that since clearly $\mathcal{F}_\kappa[n_\varepsilon, c_\varepsilon, u_\varepsilon](t) \ge -\frac{|\Omega|}{e}$ for all $\varepsilon\in(0,1)$ and $t>0$ and

$$\sup_{\varepsilon\in(0,1)}\sup_{t>0} \mathcal{F}_\kappa[n_\varepsilon, c_\varepsilon, u_\varepsilon](t) < \infty$$

according to Lemma 3.8, we may invoke the dominated convergence theorem to infer that

$$\int_0^\infty \mathcal{F}_\kappa[n_\varepsilon, c_\varepsilon, u_\varepsilon](t)\cdot\phi'(t)dt \to \int_0^\infty \mathcal{F}_\kappa[n,c,u](t)\cdot\phi'(t)dt \qquad \text{as } \varepsilon=\varepsilon_j\searrow 0.$$

Since (2.6), (2.7), Lemma 3.7 and (2.8) warrant that similarly

$$\mathcal{F}_\kappa[n_{0\varepsilon}, c_{0\varepsilon}, u_{0\varepsilon}] \to \mathcal{F}_\kappa[n_0, c_0, u_0] \qquad \text{as } \varepsilon\searrow 0,$$

by nonnegativity of ϕ we conclude from (4.17) and the weak convergence properties in (4.13), (4.5) and (4.9) that

$$-\int_0^\infty \mathcal{F}_\kappa[n,c,u](t)\cdot\phi'(s)dt + \frac{1}{K}\int_0^\infty\int_\Omega \left\{\frac{|\nabla n|^2}{n} + \frac{|\nabla c|^4}{c^3} + |\nabla u|^2\right\}(x,t)\cdot\phi(t)dxdt$$
$$\le \mathcal{F}_\kappa[n_0, c_0, u_0]\cdot\phi(0) + K\int_0^\infty \phi(t)dt$$

for any such ϕ, and that hence (1.14) holds.

Next, to deduce (1.13) we integrate (3.12) in time to see that

$$\frac{1}{2}\int_\Omega |u_\varepsilon(\cdot,t)|^2 + \int_{t_0}^t\int_\Omega |\nabla u_\varepsilon|^2 = \frac{1}{2}\int_\Omega |u_\varepsilon(\cdot,t_0)|^2 + \int_{t_0}^t\int_\Omega n_\varepsilon u_\varepsilon\cdot\nabla\Phi \qquad \text{for all } t_0\ge 0 \text{ and } t>t_0, \tag{4.19}$$

where for any such t_0 and t we have

$$\int_{t_0}^t\int_\Omega n_\varepsilon u_\varepsilon\cdot\nabla\Phi \to \int_{t_0}^t\int_\Omega nu\cdot\nabla\Phi \qquad \text{as } \varepsilon=\varepsilon_j\searrow 0,$$

because

$$n_\varepsilon u_\varepsilon \to nu \qquad \text{in } L^1_{loc}(\bar\Omega \times [0,\infty)) \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0 \tag{4.20}$$

due to (4.1), (4.8) and the fact that $\frac{3}{5} + \frac{3}{10} = \frac{9}{10} < 1$. Therefore, (4.9) together with the last property in (4.18) ensures that if $t_0 \in [0,\infty) \setminus N$ and $t > t_0$ then indeed

$$\begin{aligned}
\frac{1}{2}\int_\Omega |u(\cdot,t)|^2 + \int_{t_0}^t\int_\Omega |\nabla u|^2 &\le \lim_{\varepsilon=\varepsilon_j\searrow 0} \left\{ \frac{1}{2}\int_\Omega |u_\varepsilon(\cdot,t_0)|^2 + \int_{t_0}^t\int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla\Phi \right\} \\
&= \frac{1}{2}\int_\Omega |u(\cdot,t_0)|^2 + \int_{t_0}^t\int_\Omega nu \cdot \nabla\Phi,
\end{aligned}$$

as desired.

Now in order to verify that (n, c, u) is a global weak solution of (1.2), (1.5), (1.6) in the sense of Definition 2.1, we first note that the regularity properties (2.1) therein are obvious from (4.1), (4.2), (4.4), (4.5), (4.6) and (4.9), that clearly n and c inherit nonnegativity from n_ε and c_ε, and that $\nabla \cdot u = 0$ a.e. in $\Omega \times (0,\infty)$ according to (2.9) and (4.9). To prepare a derivation of (2.2) and (2.3)–(2.5), we observe that in view of the dominated convergence theorem,

$$F'_\varepsilon(n_\varepsilon)\chi(c_\varepsilon)c_\varepsilon^{\frac{3}{4}} \to \chi(c)c^{\frac{3}{4}} \qquad \text{in } L^{\frac{20}{3}}_{loc}(\bar\Omega \times [0,\infty)) \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0$$

as a consequence of (4.1) and (4.3), the boundedness of $(c_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^\infty(\Omega \times (0,\infty))$ and the fact that $F'_\varepsilon \nearrow 1$ on $(0,\infty)$ as $\varepsilon \searrow 0$ by (2.13). Combining this with (4.1) and (4.5) shows that

$$n_\varepsilon F'_\varepsilon(n_\varepsilon)\chi(c_\varepsilon)\nabla c_\varepsilon = 4n_\varepsilon \cdot F'_\varepsilon(n_\varepsilon)\chi(c_\varepsilon)c_\varepsilon^{\frac{3}{4}} \cdot \nabla c_\varepsilon^{\frac{1}{4}} \rightharpoonup 4n \cdot \chi(c)c^{\frac{3}{4}} \cdot \nabla c^{\frac{1}{4}} = n\chi(c)\nabla c \qquad \text{in } L^1_{loc}(\bar\Omega \times [0,\infty)) \tag{4.21}$$

as $\varepsilon = \varepsilon_j \searrow 0$. We proceed to make sure that

$$F_\varepsilon(n_\varepsilon) \to n \qquad \text{in } L^{\frac{5}{3}}_{loc}(\bar\Omega \times [0,\infty)) \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0. \tag{4.22}$$

Indeed, for each fixed $T > 0$ we have

$$\begin{aligned}
\|F_\varepsilon(n_\varepsilon) - n\|_{L^{\frac{5}{3}}(\Omega\times(0,T))} &\le \|F_\varepsilon(n_\varepsilon) - F_\varepsilon(n)\|_{L^{\frac{5}{3}}(\Omega\times(0,T))} + \|F_\varepsilon(n) - n\|_{L^{\frac{5}{3}}(\Omega\times(0,T))} \\
&\le \|F'_\varepsilon\|_{L^\infty((0,\infty))}\|n_\varepsilon - n\|_{L^{\frac{5}{3}}(\Omega\times(0,T))} + \|F_\varepsilon(n) - n\|_{L^{\frac{5}{3}}(\Omega\times(0,T))},
\end{aligned} \tag{4.23}$$

where $\|n_\varepsilon - n\|_{L^{\frac{5}{3}}(\Omega\times(0,T))} \to 0$ as $\varepsilon = \varepsilon_j \searrow 0$ by (4.1), and where since

$$\left\| F_\varepsilon(n(\cdot,t)) - n(\cdot,t) \right\|^{\frac{5}{3}}_{L^{\frac{5}{3}}(\Omega)} \le 2^{\frac{5}{3}}\|n(\cdot,t)\|^{\frac{5}{3}}_{L^{\frac{5}{3}}(\Omega)} \qquad \text{for a.e. } t > 0$$

due to (2.12), (4.1) guarantees that also

$$\int_0^T \left\| F_\varepsilon(n(\cdot,t)) - n(\cdot,t) \right\|^{\frac{5}{3}}_{L^{\frac{5}{3}}(\Omega)} dt \to 0 \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0$$

by the dominated convergence theorem. As $0 \le F'_\varepsilon \le 1$ by (2.12), (4.23) and (4.1) prove (4.22), from which we particularly obtain that

$$F_\varepsilon(n_\varepsilon)f(c_\varepsilon) \to nf(c) \qquad \text{in } L^1_{loc}(\bar\Omega \times [0,\infty)) \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0, \tag{4.24}$$

because again Lebesgue's theorem along with (4.3) and (2.17) ensures that $f(c_\varepsilon) \to f(c)$ in $L^{\frac{5}{2}}_{loc}(\bar\Omega \times [0,\infty))$ as $\varepsilon = \varepsilon_j \searrow 0$.

Next, since (4.3) and (2.17) furthermore imply that $c_\varepsilon \to c$ in $L^{\frac{10}{7}}_{loc}(\bar\Omega \times [0,\infty))$ as $\varepsilon = \varepsilon_j \searrow 0$, from (4.8) we infer that

$$c_\varepsilon u_\varepsilon \to cu \qquad \text{in } L^1_{loc}(\bar\Omega \times [0,\infty)) \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0. \tag{4.25}$$

Finally, following an argument from [28, Theorem V.3.1.1] we use that for each $\varphi \in L^2_\sigma(\Omega)$ we have $\|Y_\varepsilon\varphi\|_{L^2(\Omega)} \le \|\varphi\|_{L^2(\Omega)}$ and $Y_\varepsilon\varphi \to \varphi$ in $L^2(\Omega)$ as $\varepsilon \searrow 0$ to infer from (4.18) that for each $t \in (0,\infty) \setminus N$ we have

$$\begin{aligned}
\Big\|Y_\varepsilon u_\varepsilon(\cdot,t) - u(\cdot,t)\Big\|_{L^2(\Omega)} &\le \Big\|Y_\varepsilon\Big(u_\varepsilon(\cdot,t) - u(\cdot,t)\Big)\Big\|_{L^2(\Omega)} + \Big\|Y_\varepsilon u(\cdot,t) - u(\cdot,t)\Big\|_{L^2(\Omega)} \\
&\le \Big\|u_\varepsilon(\cdot,t) - u(\cdot,t)\Big\|_{L^2(\Omega)} + \Big\|Y_\varepsilon u(\cdot,t) - u(\cdot,t)\Big\|_{L^2(\Omega)} \\
&\to 0 \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0,
\end{aligned}$$

and that moreover

$$\begin{aligned}
\Big\|Y_\varepsilon u_\varepsilon(\cdot,t) - u(\cdot,t)\Big\|^2_{L^2(\Omega)} &\le \Big(\|Y_\varepsilon u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} + \|u(\cdot,t)\|_{L^2(\Omega)}\Big)^2 \\
&\le \Big(\|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} + \|u(\cdot,t)\|_{L^2(\Omega)}\Big)^2 \\
&\le 4 \sup_{\varepsilon' \in (0,1)} \|u_{\varepsilon'}\|^2_{L^2(\Omega\times(0,\infty))} \qquad \text{for all } t \in (0,\infty) \setminus N \text{ and } \varepsilon \in (0,1).
\end{aligned}$$

In view of (3.19), once more thanks to the dominated convergence theorem this entails that for all $T > 0$ we obtain

$$\int_0^T \Big\|Y_\varepsilon u_\varepsilon(\cdot,t) - u(\cdot,t)\Big\|^2_{L^2(\Omega)} dt \to 0 \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0.$$

Thus,

$$Y_\varepsilon u_\varepsilon \to u \quad \text{in } L^2_{loc}(\bar\Omega \times [0,\infty)) \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0,$$

which in conjunction with (4.6) entails that

$$Y_\varepsilon u_\varepsilon \otimes u_\varepsilon \to u \otimes u \quad \text{in } L^1_{loc}(\bar\Omega \times [0,\infty)) \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0. \tag{4.26}$$

Now (4.20), (4.21), (4.24), (4.25) and (4.26) firstly warrant that the integrability requirements in (2.2) are satisfied, and secondly, together with (4.1)–(4.9) and (2.6)–(2.8), allow for passing to the limit in the respective weak formulations associated with the equations in (2.9). In fact, if for $\varepsilon \in (0,1)$ we multiply the first equation in (2.9) by an arbitrary $\phi \in C_0^\infty(\bar\Omega \times [0,\infty))$ and integrate by parts, then in the resulting identity

$$-\int_0^\infty\int_\Omega n_\varepsilon\phi_t - \int_\Omega n_{0\varepsilon}\phi(\cdot,0) = -\int_0^\infty\int_\Omega \nabla n_\varepsilon\cdot\nabla\phi + \int_0^\infty\int_\Omega n_\varepsilon F'_\varepsilon(n_\varepsilon)\chi(c_\varepsilon)\nabla c_\varepsilon\cdot\nabla\phi + \int_0^\infty\int_\Omega n_\varepsilon u_\varepsilon\cdot\nabla\phi$$

we may apply (4.1), (2.6), (4.2), (4.21) and (4.20) to take $\varepsilon = \varepsilon_j \searrow 0$ in the first, second, third, fourth and fifth integral, respectively, to conclude that (2.3) holds. Likewise, since for all $\phi \in C_0^\infty(\bar\Omega \times [0,\infty))$ and $\varepsilon \in (0,1)$ we have

$$-\int_0^\infty\int_\Omega c_\varepsilon\phi_t - \int_\Omega c_{0\varepsilon}\phi(\cdot,0) = -\int_0^\infty\int_\Omega \nabla c_\varepsilon\cdot\nabla\phi - \int_0^\infty\int_\Omega F_\varepsilon(n_\varepsilon)f(c_\varepsilon)\phi + \int_0^\infty\int_\Omega c_\varepsilon u_\varepsilon\cdot\nabla\phi,$$

invoking (4.3), (2.7), (4.5), (4.24) and (4.25) and again applying the dominated convergence theorem along with (2.17) establishes (2.4). Finally, given any $\phi \in C_0^\infty(\Omega \times [0,\infty); \mathbb{R}^3)$ satisfying $\nabla\cdot\phi \equiv 0$, from (2.9) we obtain

$$-\int_0^\infty\int_\Omega u_\varepsilon\cdot\phi_t - \int_\Omega u_{0\varepsilon}\cdot\phi(\cdot,0) = -\int_0^\infty\int_\Omega \nabla u_\varepsilon\cdot\nabla\phi + \int_0^\infty\int_\Omega Y_\varepsilon u_\varepsilon\otimes u_\varepsilon\cdot\nabla\phi + \int_0^\infty\int_\Omega n_\varepsilon\nabla\Phi\cdot\phi$$

for all $\varepsilon \in (0, 1)$, so that taking $\varepsilon = \varepsilon_j \searrow 0$ and using (4.6), (2.8), (4.9), (4.26) and (4.1) yields (2.5) and thereby completes the proof. □

Proof of Theorem 1.1. The statement is evidently implied by Lemma 4.1. □

Conflict of interest statement

We confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

References

[1] X. Cao, S. Ishida, Global-in-time bounded weak solutions to a degenerate quasilinear Keller–Segel system with rotation, preprint.

[2] X. Cao, Y. Wang, Global classical solutions of a 3D chemotaxis–Stokes system with rotation, preprint.

[3] M. Chae, K. Kang, J. Lee, Existence of smooth solutions to coupled chemotaxis–fluid equations, Discrete Contin. Dyn. Syst., Ser. A 33 (6) (2013) 2271–2297.

[4] M. Chae, K. Kang, J. Lee, Global existence and temporal decay in Keller–Segel models coupled to fluid equations, Commun. Partial Differ. Equ. 39 (2014) 1205–1235.

[5] A. Chertock, K. Fellner, A. Kurganov, A. Lorz, P.A. Markowich, Sinking, merging and stationary plumes in a coupled chemotaxis–fluid model: a high-resolution numerical approach, J. Fluid Mech. 694 (2012) 155–190.

[6] M. DiFrancesco, A. Lorz, P.A. Markowich, Chemotaxis–fluid coupled model for swimming bacteria with nonlinear diffusion: global existence and asymptotic behavior, Discrete Contin. Dyn. Syst., Ser. A 28 (2010) 1437–1453.

[7] E.E. Espejo, T. Suzuki, Reaction terms avoiding aggregation in slow fluids, Nonlinear Anal., Real World Appl. 21 (2015) 110–126.

[8] C. Dombrowski C, L. Cisneros, S. Chatkaew, R.E. Goldstein, J.O. Kessler, Self-concentration and large-scale coherence in bacterial dynamics, Phys. Rev. Lett. 93 (2004), 098103-1-4.

[9] R.J. Duan, A. Lorz, P.A. Markowich, Global solutions to the coupled chemotaxis–fluid equations, Commun. Partial Differ. Equ. 35 (2010) 1635–1673.

[10] R. Duan, Z. Xiang, A note on global existence for the chemotaxis–Stokes model with nonlinear diffusion, Int. Math. Res. Not. 2012 (2012), http://dx.doi.org/10.1093/imrn/rns270, rns270, 20 pages.

[11] Y. Giga, The Stokes operator in L_r spaces, Proc. Jpn. Acad. 2 (1981) 85–89.

[12] Y. Giga, Solutions for semilinear parabolic equations in L_p and regularity of weak solutions of the Navier–Stokes system, J. Differ. Equ. 61 (1986) 186–212.

[13] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Mathematics, vol. 840, Springer, Berlin, Heidelberg, New York, 1981.

[14] T. Hillen, K.J. Painter, A user's guide to PDE models for chemotaxis, J. Math. Biol. 58 (2009) 183–217.

[15] S. Ishida, Global existence for chemotaxis–Navier–Stokes systems with rotation in 2D bounded domains, preprint.

[16] A. Kiselev, L. Ryzhik, Biomixing by chemotaxis and enhancement of biological reactions, Commun. Partial Differ. Equ. 37 (1–3) (2012) 298–318.

[17] O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural'ceva, Linear and Quasi-Linear Equations of Parabolic Type, Amer. Math. Soc. Transl., vol. 23, Amer. Math. Soc., Providence, RI, 1968.

[18] J. Leray, Sur le mouvement d'un liquide visqueus amplissant l'espace, Acta Math. 63 (1934) 193–248.

[19] P.L. Lions, Résolution de problèmes elliptiques quasilinéaires, Arch. Ration. Mech. Anal. 74 (1980) 335–353.

[20] J.-G. Liu, A. Lorz, A coupled chemotaxis–fluid model: global existence, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 28 (5) (2011) 643–652.

[21] A. Lorz, Coupled chemotaxis fluid model, Math. Models Methods Appl. Sci. 20 (2010) 987–1004.

[22] A. Lorz, A coupled Keller–Segel–Stokes model: global existence for small initial data and blow-up delay, Commun. Math. Sci. 10 (2012) 555–574.

[23] T. Miyakawa, H. Sohr, On energy inequality, smoothness and large time behaviour in L^2 for weak solutions of the Navier–Stokes equations, Math. Z. 199 (1988) 465–478.

[24] N. Mizoguchi, M. Winkler, Blow-up in the two-dimensional parabolic Keller–Segel system, preprint.

[25] T. Nagai, T. Senba, K. Yoshida, Application of the Trudinger–Moser inequality to a parabolic system of chemotaxis, Funkc. Ekvacioj, Ser. Int. 40 (1997) 411–433.

[26] K. Osaki, A. Yagi, Finite dimensional attractor for one-dimensional Keller–Segel equations, Funkc. Ekvacioj 44 (2001) 441–469.

[27] P. Quittner, Ph. Souplet, Superlinear Parabolic Problems. Blow-up, Global Existence and Steady States, Birkhäuser Advanced Texts, Birkhäuser, Basel/Boston/Berlin, 2007.

[28] H. Sohr, The Navier–Stokes Equations. An Elementary Functional Analytic Approach, Birkhäuser, Basel, 2001.

[29] V.A. Solonnikov, Schauder estimates for the evolutionary generalized Stokes problem, in: Nonlinear Equations and Spectral Theory, in: Amer. Math. Soc. Transl., Series 2, vol. 220, Amer. Math. Soc., Providence, RI, 2007, pp. 165–200.

[30] Y. Tao, Boundedness in a Keller–Segel–Stokes system modeling the process of coral fertilization, preprint.

[31] Y. Tao, M. Winkler, Eventual smoothness and stabilization of large-data solutions in a three-dimensional chemotaxis system with consumption of chemoattractant, J. Differ. Equ. 252 (2012) 2520–2543.

[32] Y. Tao, M. Winkler, Global existence and boundedness in a Keller–Segel–Stokes model with arbitrary porous medium diffusion, Discrete Contin. Dyn. Syst., Ser. A 32 (5) (2012) 1901–1914.
[33] Y. Tao, M. Winkler, Locally bounded global solutions in a three-dimensional chemotaxis–Stokes system with nonlinear diffusion, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 30 (1) (2013) 157–178.
[34] R. Temam, Navier–Stokes Equations. Theory and Numerical Analysis, Studies in Mathematics and its Applications, vol. 2, North-Holland, Amsterdam, 1977.
[35] I. Tuval, L. Cisneros, C. Dombrowski, C.W. Wolgemuth, J.O. Kessler, R.E. Goldstein, Bacterial swimming and oxygen transport near contact lines, Proc. Natl. Acad. Sci. USA 102 (2005) 2277–2282.
[36] D. Vorotnikov, Weak solutions for a bioconvection model related to Bacillus subtilis, Commun. Math. Sci. 12 (2014) 545–563.
[37] M. Wiegner, The Navier–Stokes equations – a neverending challenge?, Jahresber. Dtsch. Math.-Ver. 101 (1999) 1–25.
[38] M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller–Segel model, J. Differ. Equ. 248 (2010) 2889–2905.
[39] M. Winkler, Global large-data solutions in a chemotaxis–(Navier–)Stokes system modeling cellular swimming in fluid drops, Commun. Partial Differ. Equ. 37 (2012) 319–351.
[40] M. Winkler, Finite-time blow-up in the higher-dimensional parabolic–parabolic Keller–Segel system, J. Math. Pures Appl. 100 (2013) 748–767, arXiv:1112.4156v1.
[41] M. Winkler, Stabilization in a two-dimensional chemotaxis–Navier–Stokes system, Arch. Ration. Mech. Anal. 211 (2) (2014) 455–487.
[42] M. Winkler, Boundedness and large time behavior in a three-dimensional chemotaxis–Stokes system with nonlinear diffusion and general sensitivity, preprint.