

# A non-dynamically coherent example on $\mathbb{T}^3$

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## Abstract

In this paper we give the first example of a non-dynamically coherent partially hyperbolic diffeomorphism with one-dimensional center bundle. The existence of such an example had been an open question since 1975 [2].

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## 1. Introduction

A diffeomorphism  $f$  of a closed manifold  $M$  is *partially hyperbolic* if the tangent bundle  $TM$  of  $M$  splits into three invariant sub-bundles:  $TM = E^s \oplus E^c \oplus E^u$  such that all unit vectors  $v^\sigma \in E_x^\sigma$  ( $\sigma = s, c, u$ ) with  $x \in M$  satisfy:

$$\|T_x f v^s\| < \|T_x f v^c\| < \|T_x f v^u\| \quad (1.1)$$

for some suitable Riemannian metric. The *stable bundle*  $E^s$  must also satisfy  $\|Tf|_{E^s}\| < 1$  and the *unstable bundle*,  $\|Tf^{-1}|_{E^u}\| < 1$ . The bundle  $E^c$  is called *center bundle*.

It is a well-known fact that the *strong bundles*,  $E^s$  and  $E^u$ , are uniquely integrable [2,5]. That is, there are invariant foliations  $\mathcal{W}^s$  and  $\mathcal{W}^u$  tangent, respectively to the invariant bundles  $E^s$  and  $E^u$ .

However, the integrability of  $E^c$  is a more delicate matter. There are examples of partially hyperbolic diffeomorphisms with non-integrable center bundle. A. Wilkinson observed in [12] that there is an Anosov diffeomorphism in a six-nilmanifold, which, when seen as a partially hyperbolic one, has a non-integrable center bundle. One can choose  $E^c$  to consist of the weakest part of the Anosov-stable bundle, and the weakest part of the Anosov unstable bundle. These bundles are non-jointly integrable, hence  $E^c$  is not integrable. This example can be found in S. Smale's survey [11], and it is attributed to A. Borel. Further discussion on these examples can be found, for instance, in [3].

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The diffeomorphism  $f$  is *cs-dynamically coherent* if there exists an  $f$ -invariant foliation tangent to  $E^s \oplus E^c$ . The *cu-dynamical coherence* is defined analogously. The diffeomorphism  $f$  is *dynamically coherent* if it is both *cs-* and *cu-dynamically coherent*.

As seen above, one obstruction to dynamical coherence is a Frobenius bracket type of obstruction, that is that the sub-bundles of  $E^c$  be the non-jointly integrable. Is this the only obstruction to dynamical coherence? What happens, for instance, if the center bundle dimension is one? Would dynamical coherence be then automatic? This question has been open since the 70's. Here we show the first example of a non-dynamically coherent partially hyperbolic diffeomorphism with a one-dimensional center bundle.

**Theorem 1.1.** *There exists a non-void  $C^1$ -open set of partially hyperbolic diffeomorphisms  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  such that*

- (1)  $f$  is non-dynamically coherent;
- (2)  $f$  admits an invariant 2-torus tangent to  $E^c \oplus E^u$ .

The existence of such examples strongly contrasts with the following result obtained by M. Brin, D. Burago and S. Ivanov:

**Theorem 1.2** (Brin, Burago, Ivanov). (See [11].) *All absolutely partially hyperbolic diffeomorphisms on  $\mathbb{T}^3$  are dynamically coherent.*

Absolute partial hyperbolicity is a more restrictive notion of partial hyperbolicity, the bounds in its definition are global, contrasting with the usually used pointwise bounds. Namely,  $f$  is *absolutely partially hyperbolic* if  $TM$  admits an invariant splitting into three sub-bundles  $TM = E^s \oplus E^c \oplus E^u$  and there are constants  $\lambda < 1 < \mu$  such that all unit vectors  $v^\sigma \in E_x^\sigma$ , with  $\sigma = s, c, u$  and  $x$  in  $M$  satisfy:

$$\|T_x f v^s\| < \lambda < \|T_x f v^c\| < \mu < \|T_x f v^u\| \quad (1.2)$$

Another important issue is the unique integrability. The following question is still open, both for absolutely and pointwise partially hyperbolic diffeomorphisms:

**Question 1.3.** *Assuming  $f$  is dynamically coherent, is there a unique invariant foliation tangent to  $E^c$ ?*

This question is open for all manifolds, even for  $\mathbb{T}^3$ . However, we are able to provide an answer at the local level. If  $E$  is a distribution,  $\mathcal{W}$  is a foliation tangent to  $E$  and  $W(x)$  is a leaf of  $\mathcal{W}$  through the point  $x$ ,  $E$  is *locally uniquely integrable* at  $x$  if any embedded arc through  $x$  and tangent to  $E$  is contained in  $W(x)$ . Local unique integrability implies unique integrability.

We prove the following:

**Theorem 1.4.** *There exists a non-void  $C^1$  open set of (pointwise) partially hyperbolic diffeomorphisms  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  satisfying that  $E^c$  is non-locally uniquely integrable.*

### 1.1. Idea of the construction

The idea of the example in [Theorem 1.1](#) occurred to us while proving that invariant foliations tangent to  $E^c \oplus E^u$  do not have compact leaves [9]. We wanted to prove in fact that there were no compact leaves tangent to  $E^c \oplus E^u$  at all. However, we found that, by perturbing an Anosov map times a Morse–Smale diffeomorphism on the circle, we could obtain a partially hyperbolic diffeomorphism in  $\mathbb{T}^3$  with a center-unstable torus  $T^{cu}$ .

It is easy to see that  $T^{cu}$  is an attractor. The example was built so that  $E^c$ , and hence  $E^c \oplus E^u$  were uniquely integrable in  $\mathbb{T}^3 \setminus T^{cu}$ . This implies that any invariant foliation tangent to  $E^c \oplus E^u$ , should contain a compact leaf, which is precluded by [9].

As a matter of fact, we claim that all non-dynamically coherent examples on 3-manifolds have this pattern. That is, they have (at least) an attracting or repelling periodic torus, and trivial dynamics on the rest of the manifold.

In fact, we claim that in “most” 3-manifolds, all partially hyperbolic diffeomorphisms are indeed dynamically coherent. More precisely,

**Conjecture 1.5** (Hertz–Hertz–Ures, 2009). *If  $f : M^3 \rightarrow M^3$  is a non-dynamically coherent partially hyperbolic diffeomorphism then it admits a periodic torus tangent to either  $E^c \oplus E^u$  or  $E^s \oplus E^c$ .*

Remarkably, a 2-torus like in Conjecture 1.5 can occur only in very few 3-manifolds. Indeed, we have the following result:

**Theorem 1.6.** (See [7].) *A partially hyperbolic diffeomorphism on a 3-manifold, admitting a 2-torus tangent to either  $E^s \oplus E^u$ ,  $E^s \oplus E^c$  or  $E^c \oplus E^u$  can only occur on the following 3-manifold:*

- the 3-torus  $\mathbb{T}^3$ ;
- the mapping torus of  $-id : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ ;
- the mapping tori of hyperbolic automorphisms on  $\mathbb{T}^2$ .

With respect to Conjecture 1.5, notice that the existence of a torus tangent to  $E^{cs}$  implies the existence of a periodic torus tangent to  $E^{cs}$ , which must be repelling, since they are transverse to the unstable foliation (see [9] for a proof of this fact). An analogous statement holds for tori tangent to  $E^{cu}$ . Therefore, if Conjecture 1.5 was true, any partially hyperbolic diffeomorphisms for which  $\Omega(f) = M$  would be dynamically coherent.

Conjecture 1.5 has been proven true in the case of the 3-torus by R. Potrie [6], and in the case of 3-solvmanifolds by A. Hammerlindl and Potrie [4]. But the general case of Conjecture 1.5 remains open. What remains to be proven, whether this conjecture true, is that all partially hyperbolic diffeomorphisms in 3-manifolds that are not covered by manifolds with solvable fundamental group, are dynamically coherent.

It is interesting to note that there is a “twin conjecture” to Conjecture 1.5, which refers to ergodicity for volume preserving partially hyperbolic diffeomorphisms, namely

**Conjecture 1.7** (Hertz–Hertz–Ures, 2009). *If a volume preserving partially hyperbolic diffeomorphism of a 3-manifold is not ergodic, then there is a 2-torus tangent to  $E^s \oplus E^u$ .*

This conjecture has been proven true only in 3-nilmanifolds other than the 3-torus [8]. It is open for all the other 3-manifolds.

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## 2. Two examples

The non-dynamically coherent example is a perturbation of a product of a linear Anosov map on  $\mathbb{T}^2$  and a Morse–Smale map on the circle. The unperturbed map is Axiom A, but not partially hyperbolic. We will perturb it to obtain partial hyperbolicity, the final diffeomorphism will also be Axiom A. We will only perturb in the stable direction of the linear map. One can make more complicated examples by allowing perturbations on the  $u$ -direction also. Still, our examples leave a  $cu$ -invariant torus. As long as we make an isotopy beginning in this example and remaining in the partially hyperbolic world, it will have an invariant  $cu$ -torus and hence an attractor (see Section 3). Hence the Shub type of construction [10] of robustly transitive systems does not apply for these examples. As we stated in the introduction we believe that in dimension 3 there are no transitive examples of non-dynamically coherent diffeomorphisms.

We start by considering the unperturbed diffeomorphism  $f_0$ , which is the product of an Anosov automorphism of the 2-torus by a Morse–Smale map on the circle, that is,  $f_0 : \mathbb{T}^2 \times S^1 \rightarrow \mathbb{T}^2 \times S^1$  is such that

$$f_0(x, \theta) = (Ax, \psi(\theta))$$

Here  $A$  is a hyperbolic matrix in  $SL(2, \mathbb{Z})$  with eigenvalues  $0 < \lambda < 1$  and  $1/\lambda$ . And  $\psi : S^1 \rightarrow S^1$  is a North Pole–South Pole map such that

$$\psi(0) = 0, \psi\left(\frac{1}{2}\right) = \frac{1}{2}, \tag{2.1}$$

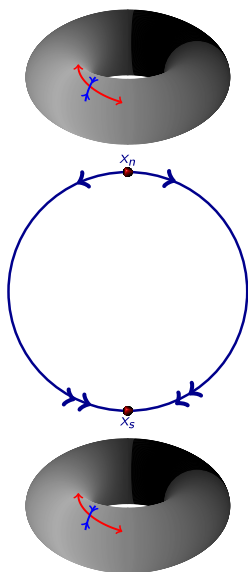
$$1 < \psi'(0) = \mu, \quad \psi'\left(\frac{1}{2}\right) = \sigma < 1, \tag{2.2}$$

$$\sigma < \lambda < 1 < \mu < \frac{1}{\lambda}. \tag{2.3}$$

That is, if  $\theta = 0$  is the North Pole and  $\theta = \frac{1}{2}$  is the South-Pole, then  $\psi$  is chosen so that  $\psi$  contracts more than the hyperbolic toral automorphism  $A$  in the South-Pole, and  $\psi$  expands less than  $A$  in the North-Pole (see the figure below).

Now let  $E_A^s$  be the contracting eigenspace of  $A$ , and consider a unit vector  $e_s$  in  $E_A^s$ . We shall consider a perturbation  $f : \mathbb{T}^2 \times S^1 \rightarrow \mathbb{T}^2 \times S^1$  of  $f_0$  of the form

$$f(x, \theta) = (Ax + v(\theta)e_s, \psi(\theta))$$



We will carefully chose the map  $v : S^1 \rightarrow \mathbb{R}$ , so that the perturbation  $f$  be partially hyperbolic and non-dynamically coherent. Our strategy will be to obtain  $f$  in the isotopy class of  $f_0$ . Such an  $f$  will be semi-conjugate to  $A : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ , via a semi-conjugacy  $h : \mathbb{T}^3 \rightarrow \mathbb{T}^2$ . The center-stable foliation will be preserved. Since we perturb only in the  $e_s$  direction, we can always assume  $h$  has the form:

$$h(x, \theta) = x - u(\theta)e_s$$

Then,  $h \circ f = A \circ h$  yields

$$\begin{aligned} h(Ax + v(\theta)e_s, \psi(\theta)) &= Ah(x, \theta) \\ Ax + v(\theta)e_s - u(\psi(\theta))e_s &= Ax - \lambda u(\theta)e_s \\ v(\theta) - u(\psi(\theta)) &= -\lambda u(\theta) \end{aligned}$$

This yields the *twisted cohomological equation*:

$$u(\psi(\theta)) - \lambda u(\theta) = v(\theta) \tag{2.4}$$

We will consider  $v(0) = v\left(\frac{1}{2}\right) = 0$ . A standard procedure gives that

$$\gamma(\theta) = \frac{1}{\lambda} \sum_{k=1}^{\infty} \lambda^k v(\psi^{-k}(\theta)) \tag{2.5}$$

is a solution to (2.4). Since  $0 < \lambda < 1$ ,  $\gamma$  is well defined and continuous, in fact, it can be seen (see Lemma 2.1) that  $\gamma$  is  $C^1$  for  $\theta \neq 1/2$ . We also have that

$$\beta(\theta) = -\frac{1}{\lambda} \sum_{k=0}^{\infty} \lambda^{-k} v(\psi^k(\theta)) \tag{2.6}$$

is a solution to (2.4). The map  $\beta$  is well defined, continuous and  $C^1$  for  $\theta \neq 0$  (see Lemma 2.2).

If we consider

$$\alpha(\theta) = \gamma(\theta) - \beta(\theta) = \frac{1}{\lambda} \sum_{k \in \mathbb{Z}} \lambda^k v(\psi^{-k}(\theta))$$

then  $\alpha$  is a solution to the equation

$$u(\psi(\theta)) - \lambda u(\theta) = 0 \tag{2.7}$$

that is  $C^1$  for  $\theta \neq 0, 1/2$ . Observe that if  $\alpha \neq 0$  then it cannot be bounded.

**Lemma 2.1.** *The map  $\gamma : S^1 \rightarrow \mathbb{R}$  is well defined and continuous. Also  $\gamma$  is  $C^1$  for  $\theta \neq 1/2$ .*

**Lemma 2.2.** *The map  $\beta : S^1 \setminus \{0\} \rightarrow \mathbb{R}$  is well defined and continuous. Also  $\beta$  is  $C^1$  for  $\theta \neq 0$ .*

Lemmas 2.1 and 2.2 are standard exercises and we leave their proofs to the reader.

### 2.1. The partially hyperbolic splitting

Let us define the partially hyperbolic splitting: First of all, choose  $E^u = E_A^u$ . Since the center-stable space  $E_A^s \times TS^1$  is preserved by  $f$ , we shall search for  $E_f^s$  and  $E_f^c$  inside  $E_A^s \times TS^1$ .

Define for  $\theta \neq 1/2$ ,

$$E_f^c(x, \theta) = [(\gamma'(\theta)e^s, 1)] \tag{2.8}$$

where  $[v]$  is the vector space spanned by  $v$ . For  $\theta = \frac{1}{2}$ , define  $E_f^c(x, 1/2) = E_A^s \times \{0\}$ .

**Remark 2.3.** Observe that, even though  $\gamma'$  is not continuous a priori in  $\frac{1}{2}$ ,  $\psi$  and  $v$  can be chosen so that  $E_f^c(x, \theta)$  is indeed continuous on all points of  $\mathbb{T}^3$ , see Lemma 2.5.

Analogously, define

$$E_f^s(x, \theta) = [(\beta'(\theta)e_s, 1)] \tag{2.9}$$

if  $\theta \neq 0$  and  $E^s(x, 0) = E_A^s \times \{0\}$ .

**Remark 2.4.** The same remark holds for  $E_f^s$ , even though  $\beta'$  is not continuous a priori in  $\theta = 0$ ,  $\psi$  and  $v$  can be chosen such that the bundle  $E_f^s(x, \theta)$  is indeed continuous on all points of  $\mathbb{T}^3$ , see Lemma 2.5.

The derivative of  $f$  is:

$$D_{(x,\theta)} f(v, t) = (Av + tv'(\theta)e_s, t\psi'(\theta)).$$

The bundles  $E_f^s$  and  $E_f^c$  only depend on the variable  $\theta$ . Hence, once we have an integral curve of  $E_f^c$  then we get the other integral curves by translating it. In a similar way we obtain the stable leaves  $W^s$ , once we get one of them.

By differentiating Eqs. (2.5) and (2.6), it is not hard to see that the bundles  $E_f^c$  and  $E_f^s$  are invariant. Moreover, using Lemmas 2.1 and 2.2 we can see that the bundles are continuous for  $\theta \neq 1/2, 0$ .

Let us prove that  $E_f^s$  and  $E_f^c$  define a splitting with bounded angle. Observe that a measure of the angle between  $E_f^c$  and  $E_f^s$  is given by  $\alpha' = \gamma' - \beta'$ .

To show that the angle between  $E_f^s$  and  $E_f^c$  is bounded, let us analyze the behavior of  $\alpha'$ :  $S^1 \setminus \{0, 1/2\}$  consists of two invariant intervals which we identify with  $(0, 1/2)$  and  $(1/2, 1)$ . On each interval one has a fundamental domain, which is also an interval. The dynamics is trivial on  $(0, \frac{1}{2})$  and on  $(\frac{1}{2}, 1)$ .

Since  $\alpha$  is a solution to (2.7), by differentiating  $\alpha$  we obtain that  $\alpha' \circ \psi \cdot \psi' = \lambda \alpha'$ . Therefore, if  $\alpha'$  is nonzero in a fundamental domain of, say,  $(0, \frac{1}{2})$  then it is nonzero everywhere on  $(0, \frac{1}{2})$ . Analogously for  $(\frac{1}{2}, 1)$ . Now, on  $\theta = 0$  and  $\theta = 1/2$  it is trivial to check that the angle between  $E_f^s$  and  $E_f^c$  is nonzero. By continuity (see Remarks 2.3 and 2.4), in a neighborhood of  $\theta = 1/2$  (and of  $\theta = 0$ ) there is a complete fundamental domain where this angle is non-zero, and then,  $\alpha'$  does not vanish. By the previous argument  $\alpha'$  does not vanish anywhere. As a consequence we have the desired splitting. Finally, the splitting is partially hyperbolic due to the fact that it is partially hyperbolic on  $\theta = 0$  and  $\theta = 1/2$ .

By choosing different functions  $v$  and  $\psi$ , there will be two cases: in one,  $\alpha'$  has opposite signs on  $(0, \frac{1}{2})$  and  $(\frac{1}{2}, 1)$ ; this gives the non-dynamically coherent example. The other case, where  $\alpha'$  has the same sign on  $(0, \frac{1}{2})$  and  $(\frac{1}{2}, 1)$  gives a dynamically coherent case, but there  $E_f^c$  is not locally uniquely integrable. This last case is treated in Section 2.3.

Let us find suitable  $v$  and  $\psi$ :

**Lemma 2.5.** *There are maps  $v$  and  $\psi$ , such that, if  $f(x, \theta) = (Ax + v(\theta)e_s, \psi(\theta))$ , then  $E_f^c$  and  $E_f^s$ , as defined in Eqs. (2.8) and (2.9) are continuous.*

*Namely,  $\lim_{\theta \rightarrow 1/2} |\gamma'(\theta)| = \infty$  and  $\lim_{\theta \rightarrow 0} |\beta'(\theta)| = \infty$ . Also,  $E^u \oplus E^c \oplus E^s$  is a partially hyperbolic splitting. We can further choose  $v$  and  $\psi$  so that either*

- (1)  $\alpha'$  has different signs on  $(0, \frac{1}{2})$  and  $(\frac{1}{2}, 1)$  (which gives a non-dynamically coherent example),
- (2) or  $\alpha'$  has the same sign on these intervals (which gives a dynamically coherent example that is locally non-uniquely integrable).

Fix some  $\epsilon_0 > 0$  and  $0 < c_0 < 1$  such that for  $\theta$  with  $|\theta - \frac{1}{2}| \leq \epsilon_0$  or  $|\theta| \leq \epsilon_0$  and for  $k \geq 0$  respectively,

$$c_0 \leq \frac{(\psi^k)'(\theta)}{\sigma^k} \leq 1/c_0 \quad \text{and} \quad c_0 \leq \frac{(\psi^{-k})'(\theta)}{\mu^{-k}} \leq 1/c_0 \tag{2.10}$$

$$c_0 \leq \frac{d(\psi^k(\theta), \frac{1}{2})}{\sigma^k d(\theta, \frac{1}{2})} \leq 1/c_0 \quad \text{and} \quad c_0 \leq \frac{d(\psi^{-k}(\theta), 0)}{\mu^{-k} d(\theta, 0)} \leq 1/c_0 \tag{2.11}$$

Observe that we can take  $\epsilon_0$  as close to 0 as we want by choosing  $c_0$  appropriately. These inequalities follow for any  $\psi$  satisfying Eqs. (2.2) and (2.3).

We treat the non-dynamically coherent case and the locally non-uniquely integrable case separately.

### 2.2. The non-dynamically coherent example

In this subsection we prove case (1) of Lemma 2.5:

**Proof of Lemma 2.5(1).** The non-dynamically coherent case is simpler, just take  $v$  such that

$$v'(\theta) < 0 \text{ on } \left(0, \frac{1}{2}\right) \quad \text{and} \quad v'(\theta) > 0 \text{ on } \left(\frac{1}{2}, 1\right); \tag{2.12}$$

$$v''(0) \neq 0 \quad \text{and} \quad v''\left(\frac{1}{2}\right) \neq 0. \tag{2.13}$$

For example  $v(\theta) = 1 + \cos 2\pi\theta$ . And take any Morse–Smale map  $\psi$  as already chosen.

This example readily satisfies the desired properties: We only need to show that  $E_f^c$  and  $E_f^s$  are continuous, that is  $\lim_{\theta \rightarrow 1/2} |\gamma'(\theta)| = \infty$  and  $\lim_{\theta \rightarrow 0} |\beta'(\theta)| = \infty$ .

In our case, i.e.  $v'(\theta) < 0$  for  $0 < \theta < \frac{1}{2}$  and  $v'(\theta) > 0$  for  $\frac{1}{2} < \theta < 1$  with  $v''(0) \neq 0$  and  $v''(\frac{1}{2}) \neq 0$ , we have that  $v'(0) = 0$  and  $v'(\frac{1}{2}) = 0$ . So for  $\theta$  with  $d(\theta, 0) \leq \epsilon_0$  or  $d(\theta, 1/2) \leq \epsilon_0$  we have respectively,

$$c_0 \leq \frac{|v'(\theta)|}{d(\theta, 0)} < 1/c_0 \quad \text{and} \quad c_0 \leq \frac{|v'(\theta)|}{d(\theta, \frac{1}{2})} < 1/c_0. \tag{2.14}$$

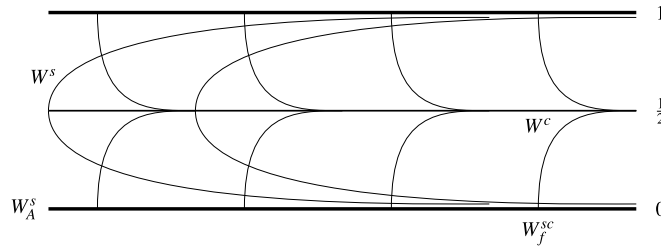


Fig. 1. A non-dynamically coherent example.

Let us see that  $\gamma'(\theta) \rightarrow \infty$  as  $\theta \rightarrow \frac{1}{2}$ . Given  $\theta$  with  $d(\theta, \frac{1}{2}) < \epsilon_0$ , let  $N = N(\theta)$  be the largest positive integer such that  $d(f^{-N}(\theta), \frac{1}{2}) \leq \epsilon_0$ . We also assume that  $c_0$  is taken so that

$$\begin{aligned} d(f^{-N}(\theta), \frac{1}{2}) &\geq c_0 d(f^{-1}(f^{-N}(\theta)), f^{-1}(\frac{1}{2})) \\ &= c_0 d(f^{-(N+1)}(\theta), \frac{1}{2}) > c_0 \epsilon_0. \end{aligned}$$

If  $d(\theta, \frac{1}{2}) \leq \epsilon_0$  and  $v'(\theta) > 0$ , then

$$\begin{aligned} \gamma'(\theta) &\geq \lambda^{(N-1)} v'(f^{-N}(\theta)) (f^{-N})'(\theta) = \lambda^{(N-1)} \frac{v'(f^{-N}(\theta))}{(f^N)'(f^{-N}(\theta))} \\ &\geq \lambda^{(N-1)} c_0 d(f^{-N}(\theta), 1/2) c_0 \sigma^{-N} \geq \lambda^{(N-1)} c_0 c_0 \epsilon_0 c_0 \sigma^{-N} = \frac{c_0^3 \epsilon_0}{\lambda} \left(\frac{\lambda}{\sigma}\right)^N. \end{aligned}$$

Finally, if  $\theta$  approaches  $\frac{1}{2}$  then  $N(\theta)$  tends to  $+\infty$  and hence  $\gamma'(\theta) \rightarrow +\infty$ . Indeed  $N(\theta)$  grows essentially as

$$\frac{\log d(\theta, \frac{1}{2})}{\log \sigma}.$$

In case  $v'(\theta) < 0$ , the computation is the same and we get that  $\gamma'(\theta) \rightarrow -\infty$  as  $\theta$  approaches  $\frac{1}{2}$ . Therefore, for some constants  $C > 0$  and  $\rho = 1 - \frac{\log \lambda}{\log \sigma} > 0$ ,  $\rho < 1$ , we have

$$|\gamma'(\theta)| \geq C d(\theta, \frac{1}{2})^{-\rho}.$$

The same computation gives that  $\lim_{\theta \rightarrow 0} |\beta'(\theta)| = \infty$ . In this case, we can see that for  $a = 1 - \frac{\log \lambda}{\log \mu} > 2$

$$|\beta'(\theta)| \geq C d(\theta, 0)^{-a}.$$

To see that  $E^c$  has no foliation tangent to it, let us explicitly compute the semiconjugacy. Remember that the semiconjugacy is  $h : \mathbb{T}^2 \times S^1 \rightarrow \mathbb{T}^2$ ,  $h(x, \theta) = x - \gamma(\theta)e^s$ .  $h$  satisfies  $h \circ f = A \circ h$  since  $\gamma$  satisfies Eq. (2.4).

Now,

$$h^{-1}(h(x_0, \theta_0)) = \{(x_0 + \gamma(\theta)e^s - \gamma(\theta_0)e^s, \theta)\}$$

so that

$$h^{-1}(h(x_0, \theta_0)) = (h(x_0, \theta_0), 0) + \{(\gamma(\theta)e^s, \theta)\}.$$

Recalling that  $E_f^c(x, \theta) = [(\gamma'(\theta)e^s, 1)]$  (Eq. (2.8)), it follows that  $h^{-1}(h(x_0, \theta_0))$  are the integral curves of  $E^c$ . From the fact that  $E^c$  is  $C^1$  for  $\theta \neq \frac{1}{2}$  we get that  $E^c$  is uniquely integrable on this domain. On the other hand, on  $\theta = \frac{1}{2}$   $E^c = E_A^s$ , hence the lines which are parallel to  $E_A^s$  inside the torus  $\theta = \frac{1}{2}$  are integral curves of  $E^c$ . See Fig. 1. Finally, close to  $\theta = \frac{1}{2}$  we have two situations: either  $\theta \in (0, \frac{1}{2})$ , and  $\gamma'$  is negative, or else  $\theta \in (\frac{1}{2}, 1)$  and  $\gamma'$  is positive. This implies the curves are like in Fig. 1, what precludes the integrability of  $E^c$ .  $\square$

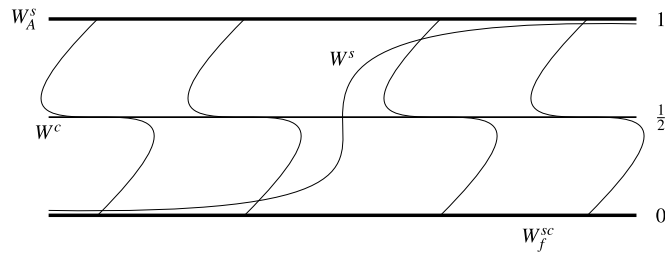


Fig. 2. A locally non-uniquely integrable example

2.3. A locally non-uniquely integrable example

In this subsection we prove case (2) of Lemma 2.5:

**Proof of Lemma 2.5(2).** The construction of a dynamically coherent example which is locally non-uniquely integrable is subtler than the previous case. In our construction, we need to choose  $v$  and  $\psi$  more carefully.

Firstly, we shall take  $\mu$ , appearing in Eq. (2.2), very close to 1; and  $\sigma$ , in the same equation, in such a way that  $\frac{\sigma}{\lambda}$  is very close to 1 too. Secondly, in order to simplify the calculations, we choose  $\psi$  to be “symmetric” with respect to  $\frac{1}{2}$ . We also choose  $\psi$  in such a way that it is affine with slope  $\sigma$  in a neighborhood of  $\frac{1}{2}$  and it is affine with slope  $\mu$  in a neighborhood of 1. Moreover, we can suppose that  $\psi$  is affine outside a fundamental domain  $D = (\theta_0, v(\theta_0))$  that depends only on  $\mu$  (recall that  $\mu$  was already chosen and  $\sigma < \lambda$ ).

Now we have to define  $v$ . Let us choose  $v$  with odd symmetry with respect to  $\frac{1}{2}$  (that is, such that  $v(\frac{1}{2} - \theta) = -v(\frac{1}{2} + \theta)$ ) and in such way  $v'(\theta) > 0$  for  $\theta \in [\frac{1}{2}, \theta^*)$  and  $v'(\theta) < 0$  for  $\theta \in (\theta^*, 1]$  where  $\theta^*$  is a point belonging to  $D$ .

Now let us show that the bundles  $E^c$  and  $E^s$  as defined above are continuous. As it was shown in the first part of the section, the fact that the angle between  $E^s$  and  $E^c$  is everywhere non-zero will follow from invariance and continuity.

The steps to show that  $\lim_{\theta \rightarrow 1/2} |\gamma'(\theta)| = \infty$  and  $\lim_{\theta \rightarrow 0} |\beta'(\theta)| = \infty$ , are very similar to those in the preceding subsection, let us make the computations for  $\gamma'$ .

Suppose that  $\theta$  is very close to  $\frac{1}{2}$  and let  $N = N(\theta)$  be such that  $f'(f^{-i}(\theta)) = \sigma$  for  $i = 0, \dots, N$ . Then

$$\begin{aligned} \gamma'(\theta) &= \frac{1}{\lambda} \sum_{k=0}^N \lambda^k v'(\psi^{-k}(\theta)) (\psi^{-k})'(\theta) + \frac{1}{\lambda} \sum_{k=N+1}^{\infty} \lambda^k v'(\psi^{-k}(\theta)) (\psi^{-k})'(\theta) \\ &\geq C_1 \left(\frac{\lambda}{\sigma}\right)^N \sum_{n=0}^N \left(\frac{\sigma}{\lambda}\right)^{N-n} - C_2 C_3 \left(\frac{\lambda}{\sigma}\right)^N \sum_{n=N+1}^{\infty} \left(\frac{\lambda}{\mu}\right)^{n-N}, \end{aligned}$$

where  $C_1$  is a lower bound for  $v'$  in its positive region, that is,  $v'(\theta) > C_1$  for  $\theta \in [\frac{1}{2}, \theta_0]$ ,  $C_2$  is an upper bound of  $|v'|$  in  $[\theta_0, 1]$  and  $C_3$  is an upper bound of  $f'$  in  $D$ . Observe that the constants  $C_i, i = 1, 2, 3$ , can be taken independent of  $\sigma$  (and of  $f$ ) if  $\sigma$  is close enough to  $\lambda$ .

Now, choosing  $\sigma$  close enough to  $\lambda$  and  $N$  sufficiently large (equivalently  $\theta$  near enough  $\frac{1}{2}$ ) we obtain

$$\begin{aligned} \gamma'(\theta) &\geq \frac{1}{\lambda} \left(\frac{\lambda}{\sigma}\right)^N \left( C_1 \sum_{n=0}^N \left(\frac{\sigma}{\lambda}\right)^{N-n} - C_2 C_3 \sum_{n=N+1}^{\infty} \left(\frac{\lambda}{\mu}\right)^{n-N} \right) \\ &\geq \frac{1}{\lambda} \left(\frac{\lambda}{\sigma}\right)^N \left( C_1 \frac{1 - (\frac{\sigma}{\lambda})^{N+1}}{1 - (\frac{\sigma}{\lambda})} - C_2 C_3 \frac{1}{1 - \lambda} \right) \end{aligned}$$

We can choose  $\sigma$  such that for a large enough  $N$ ,

$$C_1 \frac{1 - (\frac{\sigma}{\lambda})^{N+1}}{1 - (\frac{\sigma}{\lambda})} - C_2 C_3 \frac{1}{1 - \lambda} > 0.$$



This implies that  $\gamma'$  is positive for  $\theta$  close to  $\frac{1}{2}$ . The symmetries of  $v$  and  $\psi$  imply that the same is true for  $\theta$  smaller than  $\frac{1}{2}$ . The multiplying factor  $\frac{1}{\lambda}(\frac{\lambda}{\sigma})^N$  clearly forces  $\gamma'(\theta) \rightarrow +\infty$  as  $\theta \rightarrow \frac{1}{2}$ .

Finally, as in the preceding subsection, we have a semiconjugacy  $h$  and

$$h^{-1}(h(x_0, \theta_0)) = (h(x_0, \theta_0), 0) + \{(\gamma(\theta)e^s, \theta)\}.$$

Since the sign of  $\gamma'$  is the same on both sides of  $\theta = \frac{1}{2}$  this partition is a foliation.

We leave to the reader the proof that  $\lim_{\theta \rightarrow 0} |\beta'(\theta)| = \infty$  because it is very similar to the proof for  $\gamma'$ .  $\square$

### 3. Robustness of the examples and some conclusions

#### 3.1. Robustness

In the first part of this section we shall show that the non-dynamical coherence of our examples is a robust property. This is essentially a consequence of the presence of a normally hyperbolic torus tangent to the bundle  $E^{cu}$ .

**Theorem 3.1.** *There exists an open set  $\mathcal{V} \subset \text{Diff}^1(\mathbb{T}^3)$  such that,  $\forall f \in \mathcal{V}$ ,  $f$  is a non-dynamically coherent partially hyperbolic diffeomorphism. Moreover,  $f$  presents Reeb-like strips of the strong stable foliation inside the center stable leaves.*

We also have an analogous result for the non-locally uniquely integrable dynamically coherent example.

**Theorem 3.2.** *There is an open set  $\mathcal{W} \subset \text{Diff}^1(\mathbb{T}^3)$  such that,  $\forall f \in \mathcal{W}$ ,  $f$  is a dynamically coherent partially hyperbolic diffeomorphism but its center bundle is not locally uniquely integrable.*

Let us first show [Theorem 3.2](#), since it is easier.

**Proof of Theorem 3.2.** Let  $f$  be one of the examples constructed in [Subsection 2.3](#). The torus corresponding to  $\theta = \frac{1}{2}$  is a hyperbolic attractor, it is normally hyperbolic and it is tangent to the bundle  $E^{cu}$ . Moreover, the center foliation  $\mathcal{W}^c$  is also normally hyperbolic with compact leaves and  $f$  induces an expansive homeomorphisms in the space of center leaves (conjugated to  $A$ ). In particular,  $\mathcal{W}^c$  is plaque expansive. Then, if  $g$  is close enough to  $f$ ,  $g$  is partially hyperbolic, dynamically coherent (see [\[5\]](#)) and has a transitive hyperbolic attractor  $T$  that is (diffeomorphic to) a torus and tangent to  $E_g^{cu}$ . Clearly, no center leaf of  $g$  can be contained in  $T$  while for each point of  $T$  there is a complete immersed line tangent to  $E^c$  and contained in  $T$ . This implies that  $E^c$  is not locally uniquely integrable and proves the theorem.  $\square$

Now, we shall give the proof of [Theorem 3.1](#). It is a bit more involved than the proof of [Theorem 3.2](#) although the main idea is again that the presence of a center-unstable torus precludes dynamical coherence.

**Proof of Theorem 3.1.** Let  $f$  be one of the examples constructed in [Subsection 2.2](#). Observe that  $f$  satisfies Axiom A and the strong transversality condition and its nonwandering set consists of the tori corresponding to  $\theta = 0$  and  $\theta = \frac{1}{2}$ . Let  $T_1$  be the hyperbolic attractor corresponding to  $\theta = \frac{1}{2}$ .  $T_1$  is a normally hyperbolic torus tangent to  $E^{cu}$ . Let  $T_0$  be the hyperbolic repeller corresponding to  $\theta = 0 (= 1)$ . Observe that even though  $T_0$  is not normally hyperbolic, due to its hyperbolicity, it persists under perturbations. This means that a diffeomorphism  $g$  close enough to  $f$  has a hyperbolic repeller  $T_0^g$  homeomorphic to  $T_0$ . Moreover, since  $T_0^g$  is a hyperbolic repeller, it is foliated by its stable manifolds that coincide with the strong stable manifolds of its points. Of course,  $g$  also has a hyperbolic attractor  $T_0^g$  that is normally hyperbolic and tangent to the bundle  $E_g^{cu}$ .

It is not difficult to see that outside the nonwandering set (the two tori) the center leaves coincide with the intersection of stable manifolds of the attractor and unstable manifolds of the repeller. The continuous variation of these foliations implies that, for a small enough perturbation, the center leaves of  $g$  are  $C^1$ -close to the center leaves of  $f$  in the complement of a (small) neighborhood of the two tori. Iterating these curves for the future and the past we obtain the center foliation in the complement of the tori. The length of the center curves obtained in this way is bounded because they are exponentially contracted for the future and the past. The domination of the partially hyperbolic splitting

implies that this curves are tangent to  $T_1$  (if we add the limit point in this attracting torus) with the same orientation than the center leaves of  $f$ . This shows that  $g$  is nondynamically coherent.

Consider also the center-stable foliation  $\mathcal{W}^{cs}$  of  $f$ . Although  $f$  is not,  $\mathcal{W}^{cs}$  is dynamically coherent. Moreover,  $\mathcal{W}^{cs}$  is a normally hyperbolic foliation by cylinders and it is plaque expansive. Plaque expansiveness is a consequence of the fact that  $\mathcal{W}^{cs}$  projects onto the stable foliation of a hyperbolic homeomorphism under the quotient induced by the partition formed by the center circles (other way to obtain plaque expansiveness is to observe that  $\mathcal{W}^{cs}$  is  $C^1$ , even  $C^\omega$ , see [5]). Then, thanks to [5] again, we obtain that  $g$  has an invariant center-stable foliation  $\mathcal{W}_g^{cs}$  whose leaves are cylinders. Any leaf  $W^{cs}$  of  $\mathcal{W}_g^{cs}$  has a stable line  $W_0^s = W^{cs} \cap T_0$ . Cutting  $W^{cs}$  along  $W_0^s$  we obtain a strip where the strong stable foliation is Reeb-like. The reason of this is that the strong stable foliation of  $g$  is very close to the strong stable foliation of  $f$  in compact parts. Then, the strong stable foliation of  $g$  has the same shape as the strong foliation of  $f$  except in a very small neighborhood of  $W_0^s$  (the size of the neighborhood depends on closeness of  $g$  to  $f$ ). See Fig. 2 on p. 1030. Since all strong stable manifolds that are not in  $T_0$  intersect  $T_1$ , and  $T_0$  is a repeller we have that these strong manifolds are asymptotic to  $W_0^s$ . This implies that the restriction of the strong stable foliation to each center stable is Reeb-like and finishes the proof of the theorem.  $\square$

### 3.2. Connected components of partially hyperbolic diffeomorphisms

In this subsection we prove that the set of partially hyperbolic diffeomorphisms homotopic to  $A \times id$  in  $\mathbb{T}^2 \times \mathbb{S}^1 = \mathbb{T}^3$  is not connected. To be more precise, suppose that  $A$  is hyperbolic automorphisms of  $\mathbb{T}^2$  and call  $\mathcal{PH}_A \subset \text{Diff}^1(\mathbb{T}^3)$  the set of partially hyperbolic diffeomorphisms isotopic to  $A \times id$  where  $id$  is the identity map of  $\mathbb{S}^1$ .

**Theorem 3.3.**  $\mathcal{PH}_A$  is disconnected.

**Proof.** As we have already shown there are diffeomorphisms in  $\mathcal{PH}_A$  having an invariant torus tangent to the center-unstable bundle. The set  $\mathcal{PH}_A^T$  of such diffeomorphisms is obviously open. Suppose that  $f \in \mathcal{PH}_A \cap \text{closure}(\mathcal{PH}_A^T)$  and take  $f_n \in \mathcal{PH}_A^T$  converging to  $f$ . The partial hyperbolicity of  $f$  implies that the center unstable  $f_n$ -invariant tori  $T_n$  have basins of attraction of uniform size. Then, the tori  $T_n$  converge to an  $f$ -invariant tori  $T$  that is tangent to  $E^{cu}(f)$  showing that  $f \in \mathcal{PH}_A^T$ . This implies that every diffeomorphism in the connected component of one having a center-unstable torus has such a torus too. In particular, the examples of Section 2 are not in the component of  $A \times id$ .  $\square$

The same ideas can be used to prove that  $\mathcal{PH}_A$  has infinitely many components. This can be achieved by taking diffeomorphisms with more invariant center-unstable torus. It seems an interesting problem to determine the connected components in function the quantity and the rotation number of the center unstable (or stable) tori and coherence or incoherence along these tori.

### Conflict of interest statement

The authors do not have any conflict of interest.

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