Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN 31 (2014) 1175–1230

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Error bounds for the (KdV)/(KP-I) and (gKdV)/(gKP-I) asymptotic regime for nonlinear Schrödinger type equations

D. Chiron

Laboratoire J.A. Dieudonné, Université de Nice-Sophia Antipolis, Parc Valrose, 06108 Nice Cedex 02, France

Received 6 March 2013; accepted 23 August 2013

Available online 25 September 2013

Abstract

We consider the (KdV)/(KP-I) asymptotic regime for the nonlinear Schrödinger equation with a general nonlinearity. In a previous work, we have proved the convergence to the Korteweg–de Vries equation (in dimension 1) and to the Kadomtsev–Petviashvili equation (in higher dimensions) by a compactness argument. We propose a weakly transverse Boussinesq type system formally equivalent to the (KdV)/(KP-I) equation in the spirit of the work of Lannes and Saut, and then prove a comparison result with quantitative error estimates. For either suitable nonlinearities for (NLS) either a Landau–Lifshitz type equation, we derive a (mKdV)/(mKP-I) equation involving cubic nonlinearity. We then give a partial result justifying this asymptotic limit.

MSC: 35Q55; 35Q53

Keywords: Nonlinear Schrödinger equation; Gross–Pitaevskii equation; Landau–Lifshitz equation; (Generalized) Korteweg–de Vries equation; (Generalized) Kadomtsev–Petviashvili equation; Weakly transverse Boussinesq system

1. Introduction

In this paper, we consider the Nonlinear Schrödinger equation in $\mathbb{R}^d$

$$i\frac{\partial \Psi}{\partial t} + \Delta\Psi = \Psi f\big(|\Psi|^2\big), \tag{NLS}$$

with the condition at infinity

$$\big|\Psi(t,x)\big| \to r_0, \quad \text{where } r_0 > 0 \quad \text{and} \quad f\big(r_0^2\big) = 0.$$

This model appears in Nonlinear Optics (*cf.* [34]) and in Bose–Einstein condensation or superfluidity (*cf.* [46,1]). A standard well-known case is the Gross–Pitaevskii equation (GP) for which $f(\varrho) = \varrho - 1$. However, for Bose condensates, other models may be used (see [36]), such as the quintic (NLS) ($f(\varrho) = \varrho^2$ or $f(\varrho) = \varrho^2 - r_0^4$) in one space dimension and $f(\varrho) = \frac{d}{d\varrho}(\varrho^2/\ln(a\varrho))$ in two space dimension. The so-called cubic–quintic (NLS) is another relevant model (*cf.* [5]), for which

E-mail address: chiron@unice.fr.

0294-1449/$ – see front matter
http://dx.doi.org/10.1016/j.anihpc.2013.08.007

$$f(\varrho) = \alpha_1 - \alpha_3\varrho + \alpha_5\varrho^2,$$

where α_1, α_3 and α_5 are positive constants such that f has two positive roots. In Nonlinear Optics, several nonlinearities can be found in [34]:

$$f(\varrho) = \mu + \alpha\varrho^\nu + \beta\varrho^{2\nu}, \qquad f(\varrho) = \alpha\varrho\left(1 + \gamma \tanh\left(\frac{\varrho^2 - \varrho_0^2}{\sigma^2}\right)\right)$$

or (see [2]),

$$f(\varrho) = \alpha \ln(\varrho), \qquad f(\varrho) = \mu + \alpha\varrho + \beta\varrho^2 + \gamma\varrho^3,$$

and when we take into account saturation effects, one may encounter (see [34,33]):

$$f(\varrho) = \alpha\left(\frac{1}{(1+\frac{1}{\varrho_0})^\nu} - \frac{1}{(1+\frac{\varrho}{\varrho_0})^\nu}\right), \qquad f(\varrho) = 1 - \exp\left(\frac{1-\varrho}{\varrho_0}\right)$$

for some parameters $\nu > 0$, $\varrho_0 > 0$. In the study of the motion of nearly parallel vortex filaments, the (NLS) equation appears as a simplified model with $f(\varrho) = (\varrho - 1)/\varrho$ (see [4] and the references cited therein). Therefore, we shall assume f quite general and, without loss of generality, we normalize r_0 to 1. The energy associated with (NLS) is given by

$$E(\Psi) \equiv \int_{\mathbb{R}^d} |\nabla\Psi|^2 + F\big(|\Psi|^2\big)\,dx, \quad \text{where } F(\varrho) \equiv \int_\varrho^1 f.$$

If Ψ is a solution of (NLS) which does not vanish, we may use the Madelung transform

$$\Psi = A\exp(i\phi)$$

and rewrite (NLS) as a hydrodynamical system with an additional quantum pressure

$$\begin{cases} \partial_t A + 2\nabla\phi\cdot\nabla A + A\Delta\phi = 0 \\ \partial_t\phi + |\nabla\phi|^2 + f\big(A^2\big) - \dfrac{\Delta A}{A} = 0 \end{cases} \quad \text{or} \quad \begin{cases} \partial_t\rho + 2\nabla\cdot(\rho U) = 0 \\ \partial_t U + 2U\cdot\nabla U + \nabla\big(f(\rho)\big) - \nabla\left(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) = 0 \end{cases} \tag{1}$$

with $(\rho, U) \equiv (A^2, \nabla\phi)$. When neglecting the quantum pressure and linearizing this Euler type system around the particular trivial solution $\Psi = 1$ (or $(A, U) = (1, 0)$), we obtain the free wave equation

$$\begin{cases} \partial_t\bar{A} + \nabla\cdot\bar{U} = 0 \\ \partial_t\bar{U} + 2f'(1)\nabla\bar{A} = 0 \end{cases}$$

with associated speed of sound

$$\mathfrak{c}_s \equiv \sqrt{2f'(1)} > 0$$

provided $f'(1) > 0$, that is the Euler system is hyperbolic in the region $\rho \simeq 1$, which we will assume throughout the paper. For the rigorous justification of the free wave regime when (NLS) is the Gross–Pitaevskii equation, that is $f(\varrho) = \varrho - 1$, see [23] for weak convergences and more recently [10] for strong convergences. In the sequel, we shall always assume f as smooth as necessary near $\varrho = 1$.

1.1. The (KdV)/(KP-I) asymptotic regime for (NLS)

The (KdV)/(KP-I) asymptotic regime for (NLS) gives a description, as for the water waves system, of a wave of small amplitude which propagates at the speed of sound in the x_1 direction and (if $d \geqslant 2$) with a slow modulation in the transverse variables $x_\perp = (x_2, \ldots, x_d)$. More precisely, we insert the ansatz

$$\Psi(t,x) = \big(1 + \varepsilon^2 A_\varepsilon(\tau,z)\big)\exp\big(i\phi_\varepsilon(\tau,z)\big) \quad \tau = \varepsilon^3 t,\ z_1 \equiv \varepsilon(x_1 - \mathfrak{c}_s t),\ z_\perp \equiv \varepsilon^2 x_\perp \tag{2}$$

in (NLS), cancel the phase factor and separate real and imaginary parts to obtain the long wave rescaling of system (1)

$$\begin{cases} \partial_\tau A_\varepsilon - \frac{\mathfrak{c}_s}{\varepsilon^2}\partial_{z_1} A_\varepsilon + 2\partial_{z_1}\phi_\varepsilon \partial_{z_1} A_\varepsilon + 2\varepsilon^2 \nabla_{z_\perp}\phi_\varepsilon \cdot \nabla_{z_\perp} A_\varepsilon + \frac{1}{\varepsilon^2}\big(1+\varepsilon^2 A_\varepsilon\big)\big(\partial_{z_1}^2\phi_\varepsilon + \varepsilon^2 \Delta_{z_\perp}\phi_\varepsilon\big) = 0 \\ \partial_\tau \phi_\varepsilon - \frac{\mathfrak{c}_s}{\varepsilon^2}\partial_{z_1}\phi_\varepsilon + (\partial_{z_1}\phi_\varepsilon)^2 + \varepsilon^2 |\nabla_{z_\perp}\phi_\varepsilon|^2 + \frac{1}{\varepsilon^4} f\big((1+\varepsilon^2 A_\varepsilon)^2\big) - \frac{\partial_{z_1}^2 A_\varepsilon + \varepsilon^2 \Delta_{z_\perp} A_\varepsilon}{1+\varepsilon^2 A_\varepsilon} = 0. \end{cases} \tag{3}$$

In this section, we assume that f is of class $\mathcal{C}^3$ near $\varrho = 1$. On the formal level, if A_ε and ϕ_ε are indeed of order one and converge to A and ϕ, we must have, due to the singular terms in (3), for the first equation,

$$-\mathfrak{c}_s \partial_{z_1} A + \partial_{z_1}^2 \phi = 0,$$

and for the second one, using the Taylor expansion

$$f\big((1+\alpha)^2\big) = \mathfrak{c}_s^2 \alpha + \left(\frac{\mathfrak{c}_s^2}{2} + 2 f''(1)\right)\alpha^2 + f_3(\alpha), \tag{4}$$

with $f_3(\alpha) = \mathcal{O}(\alpha^3)$ as $\alpha \to 0$, we obtain

$$-\mathfrak{c}_s \partial_{z_1}\phi + \mathfrak{c}_s^2 A = 0.$$

These two constraints are actually a single one:

$$\mathfrak{c}_s A = \partial_{z_1}\phi, \tag{5}$$

and this comes from the fact that we are focusing on the wave propagating to the right. In order to cancel out the singular terms, we add $\mathfrak{c}_s^{-1}$ times the first equation of (3) to $\mathfrak{c}_s^{-2}$ times the z_1-derivative of the second one:

$$\begin{aligned} &\frac{1}{\mathfrak{c}_s}\partial_\tau\left(A_\varepsilon + \frac{\partial_{z_1}\phi_\varepsilon}{\mathfrak{c}_s}\right) + 2\frac{\partial_{z_1}\phi_\varepsilon}{\mathfrak{c}_s}\partial_{z_1} A_\varepsilon + \big(1+\varepsilon^2 A_\varepsilon\big)\Delta_{z_\perp}\left(\frac{\phi_\varepsilon}{\mathfrak{c}_s}\right) + A_\varepsilon \partial_{z_1}\left(\frac{\partial_{z_1}\phi_\varepsilon}{\mathfrak{c}_s}\right) \\ &\quad + 2\frac{\partial_{z_1}\phi_\varepsilon}{\mathfrak{c}_s}\partial_{z_1}\left(\frac{\partial_{z_1}\phi_\varepsilon}{\mathfrak{c}_s}\right) + \left(1 + \frac{4f''(1)}{\mathfrak{c}_s^2}\right) A_\varepsilon \partial_{z_1} A_\varepsilon - \frac{1}{\mathfrak{c}_s^2}\partial_{z_1}\left(\frac{\partial_{z_1}^2 A_\varepsilon + \varepsilon^2 \Delta_{z_\perp} A_\varepsilon}{1+\varepsilon^2 A_\varepsilon}\right) \\ &= -\frac{2\varepsilon^2}{\mathfrak{c}_s}\nabla_{z_\perp}\phi_\varepsilon \cdot \nabla_{z_\perp} A_\varepsilon - \frac{\varepsilon^2}{\mathfrak{c}_s^2}\partial_{z_1}\big(|\nabla_{z_\perp}\phi_\varepsilon|^2\big) - \frac{1}{\varepsilon^4 \mathfrak{c}_s^2}\partial_{z_1}\big[f_3\big(\varepsilon^2 A_\varepsilon\big)\big]. \end{aligned} \tag{6}$$

Passing to the limit $\varepsilon \to 0$ formally in (6) and using the constraint (5) (so that $\phi/\mathfrak{c}_s = \partial_{z_1}^{-1} A$), we obtain the Korteweg–de Vries equation (KdV) in dimension $d = 1$, and the Kadomtsev–Petviashvili equation (KP-I) when $d \geqslant 2$

$$\frac{2}{\mathfrak{c}_s}\partial_\tau A + \Gamma A \partial_{z_1} A - \frac{1}{\mathfrak{c}_s^2}\partial_{z_1}^3 A + \Delta_{z_\perp}\partial_{z_1}^{-1} A = 0, \tag{KdV)/(KP-I}$$

where the coefficient Γ is related to the nonlinearity f by the formula:

$$\Gamma \equiv 6 + \frac{4}{\mathfrak{c}_s^2} f''(1).$$

The (KdV)/(KP-I) flow (formally) preserves the momentum

$$\mathscr{M}(A) \equiv \int_{\mathbb{R}^d} A^2 \, dz$$

and the energy

$$\mathscr{E}(A) \equiv \int_{\mathbb{R}^d} \frac{1}{\mathfrak{c}_s^2}(\partial_{z_1} A)^2 + \big|\nabla_{z_\perp}\partial_{z_1}^{-1} A\big|^2 + \frac{\Gamma}{3} A^3 \, dz.$$

In dimension $d = 1$, the formal derivation of the (KdV) equation from the (NLS) equation in this asymptotic regime is well known in the physics literature (see, for example, [55] and [33]), and is useful in the stability analysis of dark solitons or travelling waves of small energy. We refer to [34,35] for the occurrence of the two dimensional (KP-I) in Nonlinear Optics. In [9], this (KP-I) asymptotic regime for (NLS) is formally derived for the Gross–Pitaevskii equation (*i.e.* (NLS) with $f(\varrho) = \varrho - 1$) in dimension $d = 3$, and is used to investigate the linear instability of the solitary waves of speed $\simeq \mathfrak{c}_s$.

Before turning to the mathematical justifications of this regime for (NLS), we would like to point out that the (KdV)/(KP) equation has also been rigorously derived for hyperbolic systems by W. Ben Youssef and T. Colin [6] for (KdV) and W. Ben Youssef and D. Lannes [7] for (KP). The first rigorous justifications of this long wave asymptotic regime for (NLS) are given in the papers [13] and [14], which work on the Gross–Pitaevskii equation in dimension $d = 1$. The point is that this equation is integrable, and these results rely on the higher order conservation laws of (GP). For (GP) in dimension $d = 1$, the Cauchy problem is known (see [13]) to be globally well-posed (see also [56,27,28]) in the Zhidkov space $\mathcal{Z}^\sigma(\mathbb{R}) \equiv \{v \in L^\infty(\mathbb{R}),\ \partial_x v \in H^{\sigma-1}(\mathbb{R})\}$, where σ is a positive integer. We recall the main results of [13] and [14]. In Theorems 1 and 2 below, the initial datum for (GP)

$$i\partial_t \Psi_\varepsilon + \partial_x^2 \Psi_\varepsilon = \Psi_\varepsilon\big(|\Psi_\varepsilon|^2 - 1\big)$$

is

$$\mathcal{Z}^3(\mathbb{R}) \ni \Psi_\varepsilon^{\text{in}}(x) = \big(1 + \varepsilon^2 A_\varepsilon^{\text{in}}(z)\big)\mathrm{e}^{i\varepsilon\phi_\varepsilon^{\text{in}}(z)}, \quad z = \varepsilon x,$$

and we denote $\Psi_\varepsilon \in \mathcal{C}(\mathbb{R}_+, \mathcal{Z}^3(\mathbb{R}))$ the associated solution. For the Gross–Pitaevskii nonlinearity, we have $\mathfrak{c}_s = \sqrt{2}$ and $\Gamma = 6$.

Theorem 1. *(See [13].) We assume $d = 1$, $f(\varrho) = \varrho - 1$ and that the functions $A_\varepsilon^{\text{in}}$ and $\phi_\varepsilon^{\text{in}}$ verify*

$$\big\|A_\varepsilon^{\text{in}}\big\|_{H^3(\mathbb{R})} + \big\|\partial_z\phi_\varepsilon^{\text{in}}/\sqrt{2}\big\|_{H^3(\mathbb{R})} \leqslant M.$$

Then, there exists $\varepsilon_0(M) > 0$ such that, if $0 < \varepsilon < \varepsilon_0(M)$, then Ψ_ε can be written

$$\Psi_\varepsilon(t,x) = \big(1 + \varepsilon^2 A_\varepsilon(\tau,z)\big)\mathrm{e}^{i\varepsilon\phi_\varepsilon(\tau,z)}, \quad \tau = \varepsilon^3 t,\ z = \varepsilon(x - \sqrt{2}t),$$

with $A_\varepsilon, \phi_\varepsilon : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ continuous, $(A_\varepsilon, \phi_\varepsilon)_{|\tau=0} = (A_\varepsilon^{\text{in}}, \phi_\varepsilon^{\text{in}})$, and we have, for any $\tau \geqslant 0$,

$$\big\|A_\varepsilon(\tau) - \zeta_\varepsilon(\tau)\big\|_{L^2(\mathbb{R})} \leqslant C_M\left(\left\|A_\varepsilon^{\text{in}} - \frac{\partial_z\phi_\varepsilon^{\text{in}}}{\sqrt{2}}\right\|_{H^3(\mathbb{R})} + \varepsilon\right)\mathrm{e}^{C_M\tau},$$

where ζ_ε stands for the solution of the (KdV) *equation*

$$2\sqrt{2}\partial_\tau\zeta + 12\zeta\partial_z\zeta - \partial_z^3\zeta = 0$$

with initial datum

$$(\zeta_\varepsilon)_{|\tau=0} = A_\varepsilon^{\text{in}}.$$

The error in ε is not natural, since only ε^2 appears in (3) (for $d = 1$). This is in particular due to the fact that the use, in [13], of the first three pairs of nontrivial conservation laws for (GP) yields

$$\|A_\varepsilon - \partial_z\phi_\varepsilon/\sqrt{2}\|_{H^3(\mathbb{R})} \leqslant C\big(\big\|A_\varepsilon^{\text{in}} - \partial_z\phi_\varepsilon^{\text{in}}/\sqrt{2}\big\|_{H^3(\mathbb{R})} + \varepsilon\big). \tag{7}$$

In [14] the authors improve this first result by replacing the ε above by ε^2. The price to pay is the loss of more derivatives.

Theorem 2. *(See [14].) Let $s \in \mathbb{N} \cup \{0\}$, $K_0 > 0$ and $0 < \varepsilon < 1$ be given and assume that*

$$\big\|A_\varepsilon^{\text{in}}\big\|_{H^{s+5}(\mathbb{R})} + \varepsilon\big\|\partial_z^{s+6}A_\varepsilon^{\text{in}}\big\|_{L^2(\mathbb{R})} + \big\|\partial_z\phi_\varepsilon^{\text{in}}\big\|_{H^{s+5}(\mathbb{R})} \leqslant K_0.$$

Let $\mathcal{A}_\varepsilon$ and $\mathcal{U}_\varepsilon$ denote the solutions to the (KdV) *equations*

$$2\sqrt{2}\partial_\tau\zeta + 12\zeta\partial_z\zeta - \partial_z^3\zeta = 0$$

with initial data $A_\varepsilon^{\rm in}$ *and* $\partial_z\phi_\varepsilon^{\rm in}/\sqrt{2}$ *respectively. Then, there exist* $\varepsilon_0 = \varepsilon_0(K_0, s) \in (0,1)$ *and* $K = K(K_0, s) > 0$ *such that if* $\varepsilon \leqslant \varepsilon_0(K_0, s)$, Ψ_ε *never vanishes and thus can be written*

$$\Psi_\varepsilon(t,x) = \big(1 + \varepsilon^2 A_\varepsilon(\tau, z)\big)\mathrm{e}^{i\varepsilon\phi_\varepsilon(\tau,z)}, \quad \tau = \varepsilon^3 t,\ z = \varepsilon(x - \sqrt{2}t).$$

Furthermore, for any $\tau \geqslant 0$,

$$\|A_\varepsilon - \mathcal{A}_\varepsilon\|_{H^s(\mathbb{R})} + \left\| \frac{\partial_z\phi_\varepsilon}{\sqrt{2}} - \mathcal{U}_\varepsilon \right\|_{H^s(\mathbb{R})} \leqslant K\left(\varepsilon^2 + \left\| A_\varepsilon^{\rm in} - \frac{\partial_z\phi_\varepsilon^{\rm in}}{\sqrt{2}} \right\|_{H^s(\mathbb{R})}\right)\mathrm{e}^{K\tau}.$$

The improvement in their proof is due to the fact that they take into account waves going to the right and to the left: see Theorem 1 in [14] for a precise statement. In this paper, we shall treat only the right-going wave.

In [20], we have investigated the (KdV)/(KP-I) limit in arbitrary dimension d and for a general nonlinearity f satisfying $f'(1) > 0$. Here is one of our results (see also in [20] a result in the energy space when $d = 1$, and a result for non-well-prepared data). Here, for $A_\varepsilon^{\rm in}$, $\phi_\varepsilon^{\rm in}$, we consider the initial datum for (NLS)

$$\Psi_\varepsilon^{\rm in}(x) \equiv \big(1 + \varepsilon^2 A_\varepsilon^{\rm in}(z)\big)\exp\big(i\varepsilon\phi_\varepsilon^{\rm in}(z)\big),\ z_1 = \varepsilon x_1,\ z_\perp = \varepsilon^2 x_\perp,$$

and denote $\Psi_\varepsilon \in \Psi_\varepsilon^{\rm in} + \mathcal{C}([0,T_\varepsilon), H^{s+1}(\mathbb{R}^d))$ the corresponding H^{s+1} maximal solution. Let us recall that for initial data $A^{\rm in}$ in H^s with $s > 1 + d/2$, the (KdV)/(KP-I) equation has a unique local in time weak solution (in the distributional sense) $A \in L^\infty([0,\tau_0], H^s(\mathbb{R}^d))$, as can be easily seen (for $d \geqslant 2$) by cutting off low frequencies and passing to the limit. If moreover the antiderivative $\partial_{z_1}^{-1}A^{\rm in}$ exists in the sense that $(1+|\xi|)^s\xi_1^{-1}\mathscr{F}(A^{\rm in}) \in L^2(\mathbb{R}^d)$ (where $\mathscr{F}$ is the Fourier transform), then, from the result of [32], we know that A actually belongs to $\mathcal{C}([0,\tau_0], H^s(\mathbb{R}^d) \cap \partial_{z_1}H^s(\mathbb{R}^d))$ (for $s > 1 + d/2$). If, in addition, $\Delta_{z_\perp}\partial_{z_1}^{-1}A^{\rm in} \in \partial_{z_1}H^{s-3}(\mathbb{R}^d)$, then $\Delta_{z_\perp}\partial_{z_1}^{-2}A \in L^\infty([0,\tau_0], H^{s-3}(\mathbb{R}^d))$ (see [53] or Lemma 3 in [42]).

Theorem 3. *(See [20].) Let* $s \in \mathbb{N}$ *be such that* $s > 1 + \frac{d}{2}$. *Assume that we have a family* $(A_\varepsilon^{\rm in}, \phi_\varepsilon^{\rm in})_{0<\varepsilon<1}$ *such that*

$$\Lambda \equiv \sup_{0<\varepsilon<1} \big\|\big(A_\varepsilon^{\rm in}, \partial_{z_1}\phi_\varepsilon^{\rm in}, \varepsilon\nabla_{z_\perp}\phi_\varepsilon^{\rm in}\big)\big\|_{H^{s+1}(\mathbb{R}^d)} < +\infty.$$

Then, there exist $0 < \varepsilon_0 < 1$, $\tau_0 > 0$ *and* $K > 0$, *depending only on* s *and* Λ, *such that, for* $0 < \varepsilon \leqslant \varepsilon_0$, $T_\varepsilon > \tau_0/\varepsilon^3$ *and there exist two real-valued functions* $A_\varepsilon \in \mathcal{C}([0,\tau_0], H^{s+1}(\mathbb{R}^d))$ *and* $\phi_\varepsilon \in \mathcal{C}([0,\tau_0], \dot{H}^{s+1}(\mathbb{R}^d)) \cap \mathcal{C}([0,\tau_0] \times \mathbb{R}^d)$ *such that* $(A_\varepsilon, \phi_\varepsilon)_{|\tau=0} = (A_\varepsilon^{\rm in}, \phi_\varepsilon^{\rm in})$ *and satisfying*

$$\Psi_\varepsilon(t,x) = \big(1 + \varepsilon^2 A_\varepsilon(\tau,z)\big)\exp\big(i\varepsilon\phi_\varepsilon(\tau,z)\big), \quad \tau = \varepsilon^3 t,\ z_1 \equiv \varepsilon(x_1 - \mathfrak{c}_s t),\ z_\perp \equiv \varepsilon^2 x_\perp \tag{8}$$

with $1 + \varepsilon^2 A_\varepsilon \geqslant \frac{1}{2}$ *and*

$$\sup_{0\leqslant\tau\leqslant\tau_0} \big\{\|A_\varepsilon\|_{H^{s+1}(\mathbb{R}^d)} + \big\|\big(\partial_{z_1}\phi_\varepsilon^{\rm in}, \varepsilon\nabla_{z_\perp}\phi_\varepsilon\big)\big\|_{H^s(\mathbb{R}^d)}\big\} \leqslant K. \tag{9}$$

We assume that there exists a function $A^{\rm in} \in H^{s+1}(\mathbb{R}^d)$ *such that*

$$\big(A_\varepsilon^{\rm in}, \partial_{z_1}\phi_\varepsilon^{\rm in}, \varepsilon\nabla_{z_\perp}\phi_\varepsilon^{\rm in}\big) \to \big(A^{\rm in}, \mathfrak{c}_s A^{\rm in}, 0\big) \quad in\ L^2\big(\mathbb{R}^d\big)$$

and moreover that, if $d \geqslant 2$,

$$\left\| A_\varepsilon^{\rm in} - \frac{\partial_{z_1}\phi_\varepsilon^{\rm in}}{\mathfrak{c}_s} \right\|_{L^2(\mathbb{R}^d)} = \mathcal{O}(\varepsilon) \quad and \quad \big\|\nabla_{z_\perp}\phi_\varepsilon^{\rm in}\big\|_{L^2(\mathbb{R}^d)} = \mathcal{O}(1).$$

Then, we have for $\varepsilon \to 0$ *and every* $\sigma < s + 1$,

$$A_\varepsilon \to \zeta \quad in\ \mathcal{C}\big([0,\tau_0], H^\sigma\big(\mathbb{R}^d\big)\big), \quad and \quad \frac{\partial_{z_1}\phi_\varepsilon^{\rm in}}{\mathfrak{c}_s} \to \zeta \quad in\ \mathcal{C}\big([0,\tau_0], H^{\sigma-1}\big(\mathbb{R}^d\big)\big),$$

where $\zeta \in L^\infty([0,\tau_0], H^{s+1}(\mathbb{R}^d))$ *is the solution of the* (KdV)/(KP-I) *equation with initial datum*[1] $A^{\rm in} \in H^{s+1}(\mathbb{R}^d)$. *Furthermore,*

[1] If $d \geqslant 2$, we actually have $\nabla_{z_\perp}\partial_{z_1}^{-1}A^{\rm in} \in L^2(\mathbb{R}^d)$ and $\nabla_{z_\perp}A^{\rm in} \in H^s(\mathbb{R}^d)$, which is sufficient to guarantee the continuity in time for ζ.

$$\sup_{0\leqslant\tau\leqslant\tau_0}\left\|A_\varepsilon-\frac{\partial_{z_1}\phi_\varepsilon}{\mathfrak{c}_s}\right\|_{L^2(\mathbb{R}^d)}=o(1),\tag{10}$$

and if $d\geqslant 2$,

$$\sup_{0\leqslant\tau\leqslant\tau_0}\left\|A_\varepsilon-\frac{\partial_{z_1}\phi_\varepsilon}{\mathfrak{c}_s}\right\|_{L^2(\mathbb{R}^d)}\leqslant K\varepsilon\quad\textit{and}\quad\sup_{0\leqslant\tau\leqslant\tau_0}\|\nabla_{z_\perp}\phi_\varepsilon\|_{L^2(\mathbb{R}^d)}\leqslant K.\tag{11}$$

Remark 1.1. In [20], Lemma 1, the proof implicitly assumes that the potential function F is nonnegative. This is not a problem for the study of the (KdV) limit since, in the end, we prove that $|\Psi|$ is uniformly close to 1. Therefore, since $F(\varrho)\sim\mathfrak{c}_s^2(\varrho-1)^2/4$ as $\varrho\to 1$, one can modify F away from 1 in order to have "$F\geqslant 0$" and afterwards observe that the solution for the modified nonlinearity is actually a solution for the original one for ε small enough. However, for a correct statement of Lemma 1 in [20], one needs to add that F is nonnegative.

Since the above result give a description of a wave propagating at the speed of sound, it is natural to investigate the behaviour of the travelling waves of (NLS) in the transonic limit, that is for travelling waves of speed $c\simeq\mathfrak{c}_s$, and expect a convergence, up to similar rescalings, to the (KdV)/(KP-I) solitary wave. The (KdV)/(KP-I) equation does have solitary waves provided $\Gamma\neq 0$ (otherwise, (KdV)/(KP-I) is linear), and $1\leqslant d\leqslant 3$ (see [24] when $d\geqslant 2$). For the Gross–Pitaevskii nonlinearity ($f(\varrho)=\varrho-1$), explicit integration of the travelling waves equation can be carried out in dimension $d=1$ (see [52,12]), and this convergence can be checked explicitely. Still for the Gross–Pitaevskii nonlinearity, we refer to [11] for the proof of the transonic limit in the two dimensional case. For a more general nonlinearity f, see [16] for the case $d=1$, using ODE techniques, and [18] for the dimensions $d=2$ and $d=3$.

In Theorem 3, the convergence was obtained through a compactness argument, which does not provide a quantitative error estimate. The purpose of this paper is to provide a convergence result for this (KdV)/(KP-I) asymptotic regime with an error bound comparable to the one obtained in Theorem 2 of [14] for a general nonlinearity f and any dimension $d\geqslant 1$. As a first remark, note that the zero mass assumption $\int_{\mathbb{R}}A(z_1,z_\perp)\,dz_1=0$ for every $z_\perp\in\mathbb{R}^{d-1}$, which allows to define the term $\partial_{z_1}^{-1}A$, is *necessary* in order to prove rigorously a consistency result of the (KP-I) approximation, as explained by D. Lannes in [40]. However, this zero mass assumption is not natural from the physical point of view. This is the reason why D. Lannes and J.-C. Saut have proposed in [42] weakly transverse Boussinesq type systems that are formally equivalent to the (KP) equation but for which no zero mass assumption is needed and for which natural consistency error bounds can be proved. This is the point of view we shall adopt for our problem.

1.2. Comparing to a weakly transverse Boussinesq system

Working in the hydrodynamical variables $(A_\varepsilon,U_\varepsilon=(U_\varepsilon^1,U_\varepsilon^\perp)\equiv\mathfrak{c}_s^{-1}(\partial_{z_1}\phi_\varepsilon,\nabla_{z_\perp}\phi_\varepsilon))$, (3) becomes

$$\begin{cases}\dfrac{1}{\mathfrak{c}_s}\partial_\tau A_\varepsilon-\dfrac{1}{\varepsilon^2}\partial_{z_1}A_\varepsilon+2U_\varepsilon^1\partial_{z_1}A_\varepsilon+2\varepsilon^2U_\varepsilon^\perp\cdot\nabla_{z_\perp}A_\varepsilon+\dfrac{1}{\varepsilon^2}\big(1+\varepsilon^2A_\varepsilon\big)\big(\partial_{z_1}U_\varepsilon^1+\varepsilon^2\nabla_{z_\perp}\cdot U_\varepsilon^\perp\big)=0\\ \dfrac{1}{\mathfrak{c}_s}\partial_\tau U_\varepsilon-\dfrac{1}{\varepsilon^2}\partial_{z_1}U_\varepsilon+2\big(U_\varepsilon^1\partial_{z_1}+\varepsilon^2U_\varepsilon^\perp\cdot\nabla_{z_\perp}\big)U_\varepsilon+\dfrac{1}{\mathfrak{c}_s^2\varepsilon^4}\partial_{z_1}\big[f\big(\big(1+\varepsilon^2A_\varepsilon\big)^2\big)\big]\\ \quad-\dfrac{1}{\mathfrak{c}_s^2}\nabla_z\left(\dfrac{\partial_{z_1}^2A_\varepsilon+\varepsilon^2\Delta_{z_\perp}A_\varepsilon}{1+\varepsilon^2A_\varepsilon}\right)=0.\end{cases}\tag{12}$$

Notice that, when we neglect the quantum pressure, (12) is a symmetrizable hyperbolic system in the variables $(A_\varepsilon,U_\varepsilon^1,\varepsilon U_\varepsilon^\perp)$ (and not $(A_\varepsilon,U_\varepsilon^1,U_\varepsilon^\perp)$ due to the weak transversality), for which the symmetrizers

$$\mathrm{Diag}\left(1,\frac{\mathfrak{c}_s^2}{2f'((1+\varepsilon^2A_\varepsilon)^2)},\dots,\frac{\mathfrak{c}_s^2}{2f'((1+\varepsilon^2A_\varepsilon)^2)}\right)\quad\text{or}\quad\mathrm{Diag}\left(\frac{2}{\mathfrak{c}_s^2}f'\big(\big(1+\varepsilon^2A_\varepsilon\big)^2\big),1,\dots,1\right)\tag{13}$$

can be used. Therefore, it is natural to propose, in the spirit of [42], the following Boussinesq type system

$$\begin{cases} \dfrac{1}{\mathfrak{c}_s}\partial_\tau \mathrm{A}_\varepsilon - \dfrac{1}{\varepsilon^2}\partial_{z_1}\mathrm{A}_\varepsilon + \dfrac{1}{\varepsilon^2}\partial_{z_1}\mathrm{U}_\varepsilon^1 + 2\mathrm{U}_\varepsilon^1\partial_{z_1}\mathrm{A}_\varepsilon + \mathrm{A}_\varepsilon\partial_{z_1}\mathrm{U}_\varepsilon^1 + 2\varepsilon^2\mathrm{U}_\varepsilon^\perp\cdot\nabla_{z_\perp}\mathrm{A}_\varepsilon \\ \quad + \big(1+\varepsilon^2\mathrm{A}_\varepsilon\big)\nabla_{z_\perp}\cdot\mathrm{U}_\varepsilon^\perp = 0 \\ \dfrac{1}{\mathfrak{c}_s}\partial_\tau \mathrm{U}_\varepsilon - \dfrac{1}{\varepsilon^2}\partial_{z_1}\mathrm{U}_\varepsilon + 2\mathrm{U}_\varepsilon^1\partial_{z_1}\mathrm{U}_\varepsilon + 2\varepsilon^2\mathrm{U}_\varepsilon^\perp\cdot\nabla_{z_\perp}\mathrm{U}_\varepsilon + \dfrac{1}{\varepsilon^2}\nabla_z\mathrm{A}_\varepsilon + (\Gamma-5)\mathrm{A}_\varepsilon\nabla_z\mathrm{A}_\varepsilon \\ \quad - \dfrac{1}{\mathfrak{c}_s^2}\partial_{z_1}^3\mathrm{U}_\varepsilon = 0. \end{cases} \tag{$\mathcal{B}_\varepsilon$}$$

Here, we have used the Taylor expansion (4) for the nonlinearity f and the definition of Γ. Notice that in this system, we have replaced A_ε by U_ε in the dispersive terms, which is justified by the fact that, by (5), we expect $A_\varepsilon \approx \mathfrak{c}_s^{-1}\partial_{z_1}\phi_\varepsilon = U_\varepsilon^1$. This allows to have the structure of a symmetrizable hyperbolic system in the variables $(\mathrm{A}_\varepsilon, \mathrm{U}_\varepsilon^1, \varepsilon\mathrm{U}_\varepsilon^\perp)$, since the dispersive term is then a diagonal term with constant coefficients. Indeed, we can use the symmetrizer

$$\Sigma\big(\varepsilon^2\mathrm{A}_\varepsilon\big) \equiv \mathrm{Diag}\bigg(\frac{1+(\Gamma-5)\varepsilon^2\mathrm{A}_\varepsilon}{1+\varepsilon^2\mathrm{A}_\varepsilon}, 1, \dots, 1\bigg),$$

the first one of (13) having the disadvantage of making the dispersive term with nonconstant coefficients. We may observe that if, in $(\mathcal{B}_\varepsilon)$, we replace $\partial_{z_1}^3\mathrm{U}_\varepsilon^1$ and $\partial_{z_1}^3\mathrm{U}_\varepsilon^\perp$ by $\partial_{z_1}^3\mathrm{A}_\varepsilon^1$ and $\partial_{z_1}^3\mathrm{A}_\varepsilon^\perp$ respectively, we no longer have a symmetrizable hyperbolic system, and the local well-posedness of the resulting system would then be a delicate issue, see [47] for a Boussinesq system (see also [54]). In view of this very nice structure of $(\mathcal{B}_\varepsilon)$, we prove the following local well-posedness result.

Proposition 1. *Let $\Lambda > 0$ and $s \in \mathbb{N}$ be such that $s > 3 + \frac{d}{2}$. Assume that $0 < \varepsilon < 1$ and that $(\mathrm{A}_\varepsilon^{\mathrm{in}}, \mathrm{U}_\varepsilon^{\mathrm{in}})$ is an initial datum for $(\mathcal{B}_\varepsilon)$ such that*

$$\big\|\big(\mathrm{A}_\varepsilon^{\mathrm{in}}, \mathrm{U}_\varepsilon^{\mathrm{in},1}, \varepsilon\mathrm{U}_\varepsilon^{\mathrm{in},\perp}\big)\big\|_{H^s(\mathbb{R}^d)} \leqslant \Lambda.$$

Then, there exists $\tau_* > 0$ and K depending only on Λ and s (and not on $\varepsilon \in (0,1)$) such that $(\mathcal{B}_\varepsilon)$ has a unique solution $(A_\varepsilon, \mathrm{U}_\varepsilon^1, \varepsilon\mathrm{U}_\varepsilon^\perp) \in L^\infty([0,\tau_*], H^s(\mathbb{R}^d))$, and this solution satisfies

$$\sup_{0\leqslant\tau\leqslant\tau_*} \big\|\big(A_\varepsilon, \mathrm{U}_\varepsilon^1, \varepsilon\mathrm{U}_\varepsilon^\perp\big)\big\|_{H^s(\mathbb{R}^d)} \leqslant K. \tag{14}$$

Moreover, if $\mathrm{U}_\varepsilon^{\mathrm{in}}$ is a gradient vector field, then for $0 \leqslant \tau \leqslant \tau_*$, $\mathrm{U}_\varepsilon(\tau)$ is also a gradient vector field.

As it is the case in [42], let us emphasize that if $d \geqslant 2$, we do not control *a priori* $\mathrm{U}_\varepsilon^\perp$ but only $\varepsilon\mathrm{U}_\varepsilon^\perp$, due to the anistropy of the scaling. We now stress the link between the system $(\mathcal{B}_\varepsilon)$ and the (KdV) and the (KP-I) equations.

Proposition 2. *Assume $d = 1$, $s \in \mathbb{N}$ such that $s \geqslant 5$ and let $0 < \varepsilon < 1$ and $\Lambda > 0$ be given. Let $(\mathrm{A}_\varepsilon^{\mathrm{in}}, \mathrm{U}_\varepsilon^{\mathrm{in}})$ be such that*

$$\big\|\big(\mathrm{A}_\varepsilon^{\mathrm{in}}, \mathrm{U}_\varepsilon^{\mathrm{in}}\big)\big\|_{H^s(\mathbb{R})} \leqslant \Lambda,$$

and let $(\mathrm{A}_\varepsilon, \mathrm{U}_\varepsilon) \in L^\infty([0,\tau_*], H^s(\mathbb{R}))$ be the solution of $(\mathcal{B}_\varepsilon)$ for the initial datum $(\mathrm{A}_\varepsilon^{\mathrm{in}}, \mathrm{U}_\varepsilon^{\mathrm{in}})$ provided by Proposition 1. Then, for some constant K depending only on Λ, we have

$$\sup_{0\leqslant\tau\leqslant\tau_*} \|\mathrm{A}_\varepsilon - \mathrm{U}_\varepsilon\|_{H^{s-2}(\mathbb{R})} \leqslant K\big(\big\|\mathrm{A}_\varepsilon^{\mathrm{in}} - \mathrm{U}_\varepsilon^{\mathrm{in}}\big\|_{H^{s-2}(\mathbb{R})} + \varepsilon^2\big). \tag{15}$$

Moreover, if $\zeta_\varepsilon \in \mathcal{C}(\mathbb{R}_+, H^s(\mathbb{R}))$ is the solution of the (KdV) equation

$$\frac{2}{\mathfrak{c}_s}\partial_\tau\zeta_\varepsilon + \Gamma\zeta_\varepsilon\partial_z\zeta_\varepsilon - \frac{1}{\mathfrak{c}_s^2}\partial_z^3\zeta_\varepsilon = 0, \qquad (\zeta_\varepsilon)_{|\tau=0} = A_\varepsilon^{\mathrm{in}},$$

then for some constant K depending only on Λ,

$$\sup_{[0,\tau_0]} \|\mathrm{A}_\varepsilon - \zeta_\varepsilon\|_{H^{s-5}(\mathbb{R})} + \sup_{[0,\tau_0]} \|\mathrm{U}_\varepsilon - \zeta_\varepsilon\|_{H^{s-5}(\mathbb{R})} \leqslant K\big(\big\|\mathrm{A}_\varepsilon^{\mathrm{in}} - \mathrm{U}_\varepsilon^{\mathrm{in}}\big\|_{H^{s-2}(\mathbb{R})} + \varepsilon^2\big). \tag{16}$$

Of course, the comparison (16) is meaningless if $\mathrm{A}_\varepsilon^{\mathrm{in}} - \mathrm{U}_\varepsilon^{\mathrm{in}}$ is not small, which has to be related to the constraint (5).

Proposition 3. *Assume $d \geqslant 2$, $s \in \mathbb{N}$ such that $s > 3 + \frac{d}{2}$ and let $0 < \varepsilon < 1$ and $\Lambda > 0$ be given. Let $(A_\varepsilon, \mathrm{U}_\varepsilon^1, \varepsilon \mathrm{U}_\varepsilon^\perp) \in L^\infty([0, \tau_*], H^s(\mathbb{R}^d))$ be, as in Proposition 1, a solution of $(\mathcal{B}_\varepsilon)$ for an initial datum $(\mathrm{A}_\varepsilon^{\mathrm{in}}, \mathrm{U}_\varepsilon^{\mathrm{in}})$ such that $\mathrm{U}_\varepsilon^{\mathrm{in}}$ is a gradient vector field and*

$$\big\| \big(\mathrm{A}_\varepsilon^{\mathrm{in}}, \mathrm{U}_\varepsilon^{\mathrm{in},1}, \varepsilon \mathrm{U}_\varepsilon^{\mathrm{in},\perp}\big) \big\|_{H^s(\mathbb{R}^d)} \leqslant \Lambda.$$

Then, for some constant K depending only on Λ, we have, for $0 \leqslant \tau \leqslant \tau_*$,

$$\begin{cases} \big\| \partial_{z_1}^2 (\mathrm{A}_\varepsilon - \mathrm{U}_\varepsilon^1) \big\|_{H^{s-3}(\mathbb{R}^d)} \leqslant K \big(\big\| \partial_{z_1}^2 (\mathrm{A}_\varepsilon^{\mathrm{in}} - \mathrm{U}_\varepsilon^{\mathrm{in}}) \big\|_{H^{s-3}(\mathbb{R}^d)} + \varepsilon^2 \big) \\ \big\| \partial_{z_1} \big(\mathrm{A}_\varepsilon - \mathrm{U}_\varepsilon^1\big) \big\|_{H^{s-2}(\mathbb{R}^d)} \leqslant K \big(\big\| \partial_{z_1} (\mathrm{A}_\varepsilon^{\mathrm{in}} - \mathrm{U}_\varepsilon^{\mathrm{in}}) \big\|_{H^{s-2}(\mathbb{R}^d)} + \varepsilon \big), \end{cases} \tag{17}$$

as well as

$$\big\| \mathrm{A}_\varepsilon - \mathrm{U}_\varepsilon^1 \big\|_{L^2(\mathbb{R}^d)} + \varepsilon \big\| \mathrm{U}_\varepsilon^\perp \big\|_{L^2(\mathbb{R}^d)} \leqslant K \big(\big\| \mathrm{A}_\varepsilon^{\mathrm{in}} - \mathrm{U}_\varepsilon^{\mathrm{in},1} \big\|_{L^2(\mathbb{R}^d)} + \varepsilon \big\| \mathrm{U}_\varepsilon^{\mathrm{in},\perp} \big\|_{L^2(\mathbb{R}^d)} + \varepsilon \big). \tag{18}$$

Moreover, if we have a family of initial data $(A_\varepsilon^{\mathrm{in}}, \mathrm{U}_\varepsilon^{\mathrm{in},1}, \varepsilon \mathrm{U}_\varepsilon^{\mathrm{in},\perp})_{0<\varepsilon<1}$ such that $\mathrm{U}_\varepsilon^{\mathrm{in}}$ is a gradient vector field

$$\big\| \big(A_\varepsilon^{\mathrm{in}}, \mathrm{U}_\varepsilon^{\mathrm{in}}\big) \big\|_{H^s(\mathbb{R}^d)} \leqslant \Lambda, \qquad \big\| A_\varepsilon^{\mathrm{in}} - \mathrm{U}_\varepsilon^{\mathrm{in},1} \big\|_{L^2(\mathbb{R}^d)} \leqslant \Lambda \varepsilon, \qquad \big\| \mathrm{U}_\varepsilon^{\mathrm{in},\perp} \big\|_{L^2(\mathbb{R}^d)} \leqslant \Lambda$$

and, for some $\zeta^{\mathrm{in}} \in H^s(\mathbb{R}^d)$,

$$\big(A_\varepsilon^{\mathrm{in}}, \mathrm{U}_\varepsilon^{\mathrm{in},1}\big) \to \big(\zeta^{\mathrm{in}}, \zeta^{\mathrm{in}}\big) \quad \text{in } L^2\big(\mathbb{R}^d\big),$$

then we have, for any $0 \leqslant \sigma < s$,

$$A_\varepsilon \to \zeta \quad \text{and} \quad \mathrm{U}_\varepsilon^1 \to \zeta \quad \text{in } \mathcal{C}\big([0, \tau_*], H^\sigma\big(\mathbb{R}^d\big)\big),$$

where $\zeta \in L^\infty([0, \tau_*], H^s(\mathbb{R}^d))$ solves the (KP-I) *equation*

$$\frac{2}{\mathfrak{c}_s} \partial_\tau \zeta + \Gamma \zeta \partial_{z_1} \zeta - \frac{1}{\mathfrak{c}_s^2} \partial_{z_1}^3 \zeta + \Delta_{z_\perp} \partial_{z_1}^{-1} \zeta = 0, \qquad \zeta_{|\tau=0} = \zeta^{\mathrm{in}}.$$

Remark 1. The estimates in (17) are very anisotropic due to the fact that the natural bound is on $\varepsilon \mathrm{U}_\varepsilon^\perp$ and not on $\mathrm{U}_\varepsilon^\perp$, but we can use that $\partial_{z_1} \mathrm{U}_\varepsilon^\perp = \nabla_{z_\perp} \mathrm{U}_\varepsilon^1$ since U_ε is a gradient to improve the bounds. Note that in [42], the convergence of the weakly transverse Boussinesq system to uncoupled (KP) equations is shown (see Theorem 1 there) by a WKB expansion. Here, the wave propagating to the left is trivial. The hypothesis for Theorem 1 in [42] are not exactly the same as in Proposition 3: for instance, we do not impose conditions like $\partial_{z_2}^2 \zeta^{\mathrm{in}} \in \partial_{z_1}^2 H^s(\mathbb{R}^2)$. The proof of Proposition 3 relies on a compactness argument close to [20], and not a WKB expansion.

The link between the Boussinesq system $(\mathcal{B}_\varepsilon)$ and the (KdV) and the (KP-I) equations being clarified, we can state our main result.

Theorem 4. *Let $\Lambda > 0$, $0 < \varepsilon < 1$ and $s \in \mathbb{N}$ be such that $s > 3 + \frac{d}{2}$. Assume that $(A_\varepsilon^{\mathrm{in}}, \phi_\varepsilon^{\mathrm{in}})$ is such that*

$$\big\| \big(A_\varepsilon^{\mathrm{in}}, \partial_{z_1} \phi_\varepsilon^{\mathrm{in}}, \nabla_{z_\perp} \phi_\varepsilon^{\mathrm{in}}\big) \big\|_{H^s(\mathbb{R}^d)} \leqslant \Lambda.$$

Then, there exists $0 < \varepsilon_0 < 1$ and K depending on Λ and s such that, for $0 < \varepsilon < \varepsilon_0$, (NLS) *has a unique solution $\Psi_\varepsilon \in \Psi_\varepsilon^{\mathrm{in}} + \mathcal{C}([0, \tau_0/\varepsilon^3], H^s(\mathbb{R}^d))$, with $\tau_0 \geqslant 1/(K\Lambda)$, that can be written*

$$\Psi_\varepsilon(t, x) = \big(1 + \varepsilon^2 A_\varepsilon(\tau, z)\big) \exp\big(i \varepsilon \phi_\varepsilon(\tau, z)\big), \qquad \tau = \varepsilon^3 t,\ z_1 \equiv \varepsilon(x_1 - \mathfrak{c}_s t),\ z_\perp \equiv \varepsilon^2 x_\perp$$

with $1 + \varepsilon^2 A_\varepsilon \geqslant \frac{1}{2}$ and

$$\|A_\varepsilon\|_{\mathcal{C}([0,\tau_0], H^s(\mathbb{R}^d))} + \|\partial_{z_1} \phi_\varepsilon, \varepsilon \nabla_{z_\perp} \phi_\varepsilon\|_{\mathcal{C}([0,\tau_0], H^{s-1}(\mathbb{R}^d))} \leqslant K.$$

Denoting $(\mathrm{A}_\varepsilon, \mathrm{U}_\varepsilon^1, \mathrm{U}_\varepsilon^\perp) \in L^\infty([0, \tau_*], H^s(\mathbb{R}^d))$ the solution to $(\mathcal{B}_\varepsilon)$ for the initial datum $(A_\varepsilon^{\mathrm{in}}, \mathfrak{c}_s^{-1} \partial_{z_1} \phi_\varepsilon^{\mathrm{in}}, \mathfrak{c}_s^{-1} \nabla_{z_\perp} \phi_\varepsilon^{\mathrm{in}})$, we have, for $0 \leqslant \tau \leqslant \min(\tau_0, \tau_*)$,

$$\left\| \left(A_\varepsilon, \frac{\partial_{z_1}\phi_\varepsilon}{\mathfrak{c}_s}, \frac{\varepsilon\nabla_{z_\perp}\phi_\varepsilon}{\mathfrak{c}_s} \right) - \left(\mathrm{A}_\varepsilon, \mathrm{U}_\varepsilon^1, \varepsilon\mathrm{U}_\varepsilon^\perp \right) \right\|_{H^{s-1}(\mathbb{R}^d)} \leqslant K\varepsilon^2\tau.$$

This result is quite close to Theorem 1 in [14], where the functions $(A_\varepsilon + U_\varepsilon)/2$ and $(A_\varepsilon - U_\varepsilon)/2$ are shown to be ε^2 close to the solutions of two (KdV) equation with appropriate initial data. Here, we compare directly to the Boussinesq system $(\mathcal{B}_\varepsilon)$ via an estimate of the τ-derivative of $(A_\varepsilon, U_\varepsilon)$. The estimates of Proposition 2 and 3 can thus be transposed to $(A_\varepsilon, U_\varepsilon)$, leading in particular to the following corollary.

Corollary 1. *If $d = 1$ and under the assumptions of Theorem 4 with $s \geqslant 5$, we have, for $0 < \varepsilon < \varepsilon_0$ and some constant K depending only on Λ,*

$$\sup_{[0,\min(\tau_0,\tau_*)]} \left\{ \|A_\varepsilon - \zeta_\varepsilon\|_{H^s(\mathbb{R})} + \|U_\varepsilon - \zeta_\varepsilon\|_{H^s(\mathbb{R})} \right\} \leqslant K\left(\left\| A_\varepsilon^{\mathrm{in}} - U_\varepsilon^{\mathrm{in}} \right\|_{H^s(\mathbb{R})} + \varepsilon^2 \right)$$

where $\zeta_\varepsilon \in \mathcal{C}(\mathbb{R}_+, H^s(\mathbb{R}))$ denotes the solution to the (KdV) *equation*

$$\frac{2}{\mathfrak{c}_s}\partial_\tau\zeta_\varepsilon + \Gamma\zeta_\varepsilon\partial_z\zeta_\varepsilon - \frac{1}{\mathfrak{c}_s^2}\partial_z^3\zeta_\varepsilon = 0, \qquad (\zeta_\varepsilon)_{|\tau=0} = A_\varepsilon^{\mathrm{in}}.$$

Proof of Corollary 1. Denoting $(\mathrm{A}_\varepsilon, \mathrm{U}_\varepsilon) \in L^\infty([0,\tau_*], H^s(\mathbb{R}))$ the solution to $(\mathcal{B}_\varepsilon)$, we have

$$\sup_{[0,\tau_0]} \left\| \left(A_\varepsilon, \frac{\partial_z\phi_\varepsilon}{\mathfrak{c}_s} \right) - (\mathrm{A}_\varepsilon, \mathrm{U}_\varepsilon) \right\|_{H^{s-1}(\mathbb{R})} \leqslant K\varepsilon^2\tau \leqslant K\varepsilon^2$$

by Theorem 4, since $s \geqslant 5 > 3 + 1/2$. Moreover, since $s \geqslant 5$, applying Proposition 2, there holds

$$\sup_{[0,\tau_0]} \|\mathrm{A}_\varepsilon - \zeta_\varepsilon\|_{H^{s-5}(\mathbb{R})} + \sup_{[0,\tau_0]} \|\mathrm{U}_\varepsilon - \zeta_\varepsilon\|_{H^{s-5}(\mathbb{R})} \leqslant K\left(\left\| \mathrm{A}_\varepsilon^{\mathrm{in}} - \mathrm{U}_\varepsilon^{\mathrm{in}} \right\|_{H^{s-2}(\mathbb{R})} + \varepsilon^2 \right)$$

where ζ_ε solves (KdV) with initial datum $A_\varepsilon^{\mathrm{in}}$. As a consequence,

$$\begin{aligned} &\sup_{[0,\min(\tau_0,\tau_*)]} \left\{ \|A_\varepsilon - \zeta_\varepsilon\|_{H^{s-5}(\mathbb{R})} + \|U_\varepsilon - \zeta_\varepsilon\|_{H^{s-5}(\mathbb{R})} \right\} \\ &\leqslant K \sup_{[0,\min(\tau_0,\tau_*)]} \left\{ \left\| \left(A_\varepsilon, \frac{\partial_z\phi_\varepsilon}{\mathfrak{c}_s} \right) - (\mathrm{A}_\varepsilon, \mathrm{U}_\varepsilon) \right\|_{H^{s-1}(\mathbb{R})} + \|\mathrm{A}_\varepsilon - \zeta_\varepsilon\|_{H^{s-5}(\mathbb{R})} + \|\mathrm{U}_\varepsilon - \zeta_\varepsilon\|_{H^{s-5}(\mathbb{R})} \right\} \\ &\leqslant K\left(\left\| A_\varepsilon^{\mathrm{in}} - U_\varepsilon^{\mathrm{in}} \right\|_{H^{s-2}(\mathbb{R})} + \varepsilon^2 \right), \end{aligned}$$

as desired. □

Notice that we obtain in this way in dimension $d = 1$ a comparison result with the (KdV) equation with an error $\mathcal{O}(\varepsilon^2)$ as in Theorem 2, [14], with assumptions that are basically the same (note that we do not need the L^2 boundedness of the "$\varepsilon\partial_z^{s+6}A_\varepsilon^{\mathrm{in}}$" derivative). Of course, the convergence by compactness in Theorem 3 holds in a larger space than the one where we prove quantitative error bounds. Our result holds for a general nonlinearity and does not rely on the integrability of the one dimensional Gross–Pitaevskii equation but only on singular hyperbolic systems. However, since we do not benefit of the *a priori* bounds deduced from the integrability (as in [13]), we do not have an exponential bound on the error but work on a bounded interval in τ. Note that from our previous discussion, no reasonable comparison result with the (KP-I) equation itself has to be expected.

1.3. Expanding in powers of ε

One natural way to get error estimates would be to justify an expansion of A_ε and ϕ_ε in powers of ε. These expansions are indeed justified for the WKB asymptotics: see, *e.g.*, [29,31,19]. In the physical literature, this is actually the way the (KP-I) equation is formally derived for (NLS) (see [9,34,45]). We would like to point out that, in view of the fact that the limit is singular, this power expansion is not formally correct in the sense that this requires very strong well-preparedness assumptions on the initial data and this expansion cannot be valid at arbitrary order.

We consider the system (12), where $\partial_{z_1}^{-1} U_\varepsilon^1$ stands for $\mathfrak{c}_s^{-1}\phi_\varepsilon$:

$$\begin{cases} \frac{1}{\mathfrak{c}_s}\partial_\tau A_\varepsilon - \frac{1}{\varepsilon^2}\partial_{z_1} A_\varepsilon + 2U_\varepsilon^1 \partial_{z_1} A_\varepsilon + 2\varepsilon^2 (\nabla_{z_\perp}\partial_{z_1}^{-1} U_\varepsilon^1)\cdot \nabla_{z_\perp} A_\varepsilon + \frac{1}{\varepsilon^2}(1+\varepsilon^2 A_\varepsilon)\big[\partial_{z_1} U_\varepsilon^1 + \varepsilon^2 \Delta_{z_\perp}\partial_{z_1}^{-1} U_\varepsilon^1\big] = 0 \\ \frac{1}{\mathfrak{c}_s}\partial_\tau U_\varepsilon^1 - \frac{1}{\varepsilon^2}\partial_{z_1} U_\varepsilon^1 + 2U_\varepsilon^1 \partial_{z_1} U_\varepsilon^1 + 2\varepsilon^2 (\nabla_{z_\perp}\partial_{z_1}^{-1} U_\varepsilon^1)\cdot \nabla_{z_\perp} U_\varepsilon^1 + \frac{1}{\varepsilon^2}\partial_{z_1} A_\varepsilon \\ \quad + \frac{1}{\mathfrak{c}_s^2 \varepsilon^4}\partial_{z_1}\big[f\big([1+\varepsilon^2 A_\varepsilon]^2\big) - \mathfrak{c}_s^2 \varepsilon^2 A_\varepsilon\big] = \frac{1}{\mathfrak{c}_s^2}\partial_{z_1}\left(\frac{\partial_{z_1}^2 A_\varepsilon + \varepsilon^2 \Delta_{z_\perp} A_\varepsilon}{1+\varepsilon^2 A_\varepsilon}\right). \end{cases}$$

We assume a formal expansion

$$A_\varepsilon = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots, \qquad U_\varepsilon^1 = U_0^1 + \varepsilon U_1^1 + \varepsilon^2 U_2^1 + \cdots,$$

where the functions A_k and U_k^1 are localized, insert this into (12) and collect the terms of the same formal order in ε. We then consider initial data having the same expansions:

$$A_\varepsilon^{\mathrm{in}} = A_0^{\mathrm{in}} + \varepsilon A_1^{\mathrm{in}} + \varepsilon^2 A_2^{\mathrm{in}} + \cdots, \qquad U_\varepsilon^{\mathrm{in},1} = U_0^{\mathrm{in},1} + \varepsilon U_1^{\mathrm{in},1} + \varepsilon^2 U_2^{\mathrm{in},1} + \cdots.$$

At order ε^{-2}, we obtain $U_0^1 = A_0$ (and this is natural in view of (5)). The terms of order ε^{-1} provide

$$U_1^1 = A_1$$

and we point out that this is indeed a constraint of well-preparedness on the terms A_1 and U_1^1 at initial time, since we must have

$$U_1^{\mathrm{in},1} = A_1^{\mathrm{in}},$$

and this condition is not natural, even though in [9], the expansion only involves even powers of ε. We now turn to the terms of order ε^0:

$$\begin{cases} \frac{1}{\mathfrak{c}_s}\partial_\tau U_0^1 + 2U_0^1 \partial_{z_1} U_0^1 - \frac{1}{\mathfrak{c}_s^2}\partial_{z_1}^3 A_0 + (\Gamma - 5) A_0 \partial_{z_1} A_0 + \partial_{z_1} A_2 - \partial_{z_1} U_2^1 = 0 \\ \frac{1}{\mathfrak{c}_s}\partial_\tau A_0 + A_0 \partial_{z_1} U_0^1 + 2U_0^1 \partial_{z_1} A_0 + \Delta_{z_\perp}\partial_{z_1}^{-1} U_0^1 + \partial_{z_1} U_2^1 - \partial_{z_1} A_2 = 0. \end{cases}$$

We can solve this equation in (A_2, U_2^1) if and only if the two equations are compatible, that is if and only if $A_0 = U_0^1$ is a solution of

$$\frac{2}{\mathfrak{c}_s}\partial_\tau A_0 - \frac{1}{\mathfrak{c}_s^2}\partial_{z_1}^3 A_0 + \Gamma A_0 \partial_{z_1} A_0 + \Delta_{z_\perp}\partial_{z_1}^{-1} A_0 = 0,$$

which is (KdV)/(KP-I). Then, using the (KdV)/(KP-I) equation for A_0, we are left with

$$\partial_{z_1} A_2 - \partial_{z_1} U_2^1 = \frac{1}{\mathfrak{c}_s}\partial_\tau A_0 + 3A_0 \partial_{z_1} A_0 + \Delta_{z_\perp}\partial_{z_1}^{-1} A_0 = \frac{1}{2\mathfrak{c}_s^2}\partial_{z_1}^3 A_0 + \left(3 - \frac{\Gamma}{2}\right) A_0 \partial_{z_1} A_0 + \frac{1}{2}\Delta_{z_\perp}\partial_{z_1}^{-1} A_0,$$

that is

$$A_2 = U_2^1 + \frac{1}{2\mathfrak{c}_s^2}\partial_{z_1}^2 A_0 + \frac{1}{4}(6 - \Gamma) A_0^2 + \frac{1}{2}\Delta_{z_\perp}\partial_{z_1}^{-2} A_0. \tag{19}$$

Here again, we obtain a strong constraint on (A_2, U_2^1) at the initial time:

$$A_2^{\mathrm{in}} = U_2^{\mathrm{in},1} + \frac{1}{2\mathfrak{c}_s^2}\partial_{z_1}^2 A_0^{\mathrm{in}} + \frac{1}{4}(6 - \Gamma)\big(A_0^{\mathrm{in}}\big)^2 - \frac{1}{2}\Delta_{z_\perp}\partial_{z_1}^{-2} A_0^{\mathrm{in}},$$

which is not natural since the rigorous results in Theorems 2, 3, 4 do not make such preparedness assumptions on the initial data. Moreover, in dimensions $d \geqslant 2$, the term $\Delta_{z_\perp}\partial_{z_1}^{-2} A_0^{\mathrm{in}}$ is not well defined in general: for instance, if $A_0^{\mathrm{in}} = \partial_{z_1}\{\mathrm{e}^{-z_1^2 - z_2^2}\} = -2z_1 \mathrm{e}^{-z_1^2 - z_2^2}$, then $\partial_{z_2}^2 \partial_{z_1}^{-1} A_0^{\mathrm{in}} = 2(2z_2^2 - 1)\mathrm{e}^{-z_1^2 - z_2^2}$ has no z_1 antiderivative in L^2.

The terms of order ε^1 give

$$\begin{cases} \frac{1}{\mathfrak{c}_s}\partial_\tau U_1^1 - \frac{1}{\mathfrak{c}_s^2}\partial_{z_1}^3 A_1 + (\Gamma-5)A_0\partial_{z_1}A_1 + (\Gamma-5)A_1\partial_{z_1}A_0 + 2U_0^1\partial_{z_1}U_1^1 + 2U_1^1\partial_{z_1}U_0^1 + \partial_{z_1}A_3 - \partial_{z_1}U_3^1 = 0 \\ \frac{1}{\mathfrak{c}_s}\partial_\tau A_1 + A_0\partial_{z_1}U_1^1 + A_1\partial_{z_1}U_0^1 + 2U_0^1\partial_{z_1}A_1 + 2U_1^1\partial_{z_1}A_0 + \Delta_{z_\perp}\partial_{z_1}^{-1}U_1^1 + \partial_{z_1}U_3^1 - \partial_{z_1}A_3 = 0. \end{cases}$$

Here again, we have a compatibility condition between these two equations, which implies that $A_1 = U_1^1$ must verify the (KdV)/(KP-I) equation linearized around A_0

$$\frac{2}{\mathfrak{c}_s}\partial_\tau A_1 + \Gamma A_0\partial_{z_1}A_1 + \Gamma A_1\partial_{z_1}A_0 - \frac{1}{\mathfrak{c}_s^2}\partial_{z_1}^3 A_1 + \Delta_{z_\perp}\partial_{z_1}^{-1}A_1 = 0,$$

and it remains (since $A_0 = U_0^1$ and $A_1 = U_1^1$)

$$\begin{aligned} \partial_{z_1}A_3 - \partial_{z_1}U_3^1 &= \frac{1}{\mathfrak{c}_s}\partial_\tau A_1 + A_0\partial_{z_1}U_1^1 + A_1\partial_{z_1}U_0^1 + 2U_0^1\partial_{z_1}A_1 + 2U_1^1\partial_{z_1}A_0 + \Delta_{z_\perp}\partial_{z_1}^{-1}U_1^1 \\ &= \left(3 - \frac{\Gamma}{2}\right)\partial_{z_1}(A_0A_1) + \frac{1}{2\mathfrak{c}_s^2}\partial_{z_1}^2 A_1 + \frac{1}{2}\Delta_{z_\perp}\partial_{z_1}^{-1}A_1. \end{aligned}$$

In [9], the expansion is assumed even in ε, hence $A_1 = U_1^1 = 0$, and then it is natural to choose $A_3 = U_3^1 = 0$.

For the terms of order ε^2, we have, for some coefficient q_3 coming from the Taylor expansion of $f((1+\alpha)^2)$:

$$\begin{cases} \frac{1}{\mathfrak{c}_s}\partial_\tau U_2^1 - \frac{1}{\mathfrak{c}_s^2}\partial_{z_1}^3 A_2 + \frac{1}{\mathfrak{c}_s^2}\partial_{z_1}\left(A_0\partial_{z_1}^2 A_0\right) - \frac{1}{\mathfrak{c}_s^2}\partial_{z_1}\Delta_{z_\perp}A_0 \\ \quad + (\Gamma-5)\partial_{z_1}(A_2A_0) + (\Gamma-5)A_1\partial_{z_1}A_1 + 3q_3A_0^2\partial_{z_1}A_0 \\ \quad + 2\partial_{z_1}\left(U_0^1U_2^1\right) + 2U_1^1\partial_{z_1}U_1^1 + 2\left(\nabla_{z_\perp}\partial_{z_1}^{-1}U_0^1\right)\cdot\nabla_{z_\perp}U_0^1 + \partial_{z_1}A_4 - \partial_{z_1}U_4^1 = 0 \\ \frac{1}{\mathfrak{c}_s}\partial_\tau A_2 + A_0\partial_{z_1}U_2^1 + A_2\partial_{z_1}U_0^1 + A_1\partial_{z_1}U_1^1 + 2U_0^1\partial_{z_1}A_2 + 2U_2^1\partial_{z_1}A_0 + 2U_1^1\partial_{z_1}A_1 \\ \quad + 2\left(\nabla_{z_\perp}\partial_{z_1}^{-1}U_0^1\right)\cdot\nabla_{z_\perp}A_0 + \Delta_{z_\perp}\partial_{z_1}^{-1}U_2^1 + A_0\Delta_{z_\perp}\partial_{z_1}^{-1}U_0^1 + \partial_{z_1}U_4^1 - \partial_{z_1}A_4 = 0. \end{cases}$$

Compatibility of the second equation with the first one then implies

$$\begin{aligned} &\frac{1}{\mathfrak{c}_s}\partial_\tau\left(A_2 + U_2^1\right) - \frac{1}{\mathfrak{c}_s^2}\partial_{z_1}^3 A_2 + (\Gamma-5)\partial_{z_1}(A_0A_2) + \Gamma A_1\partial_{z_1}A_1 + \partial_{z_1}\left(q_3A_0^3\right) + 2\partial_{z_1}\left(U_0^1A_2\right) \\ &\quad + \frac{1}{\mathfrak{c}_s^2}\partial_{z_1}\left(A_0\partial_{z_1}^2 A_0\right) + \partial_{z_1}\left(2A_0U_2^1\right) + A_0\partial_{z_1}U_2^1 + A_2\partial_{z_1}A_0 + 2A_0\partial_{z_1}A_2 + 2U_2^1\partial_{z_1}A_0 \\ &\quad + 4\left(\nabla_{z_\perp}\partial_{z_1}^{-1}U_0^1\right)\cdot\nabla_{z_\perp}A_0 + \Delta_{z_\perp}\partial_{z_1}^{-1}U_2^1 + A_0\Delta_{z_\perp}\partial_{z_1}^{-1}A_0 - \frac{1}{\mathfrak{c}_s^2}\partial_{z_1}\Delta_{z_\perp}A_0 = 0. \end{aligned} \tag{20}$$

Using the expression (19) of A_2 in terms of U_2^1 and the (KdV)/(KP-I) equation for A_0, we obtain

$$\frac{1}{\mathfrak{c}_s}\partial_\tau\left(U_2^1 - A_2\right) + \frac{1}{4}(6-\Gamma)A_0\Delta_{z_\perp}\partial_{z_1}^{-1}A_0 + \frac{\Gamma}{8}\Delta_{z_\perp}\partial_{z_1}^{-1}\left(A_0^2\right) + \frac{1}{4}\Delta_{z_\perp}^2\partial_{z_1}^{-3}A_0 + \partial_{z_1}\{\ldots\} = 0,$$

where the term $\{\ldots\}$ depends on A_1, U_2^1, A_0. Here again, we observe that the term $\Delta_{z_\perp}^2\partial_{z_1}^{-3}A_0$ is not well defined in general, and that the expression $\Delta_{z_\perp}\partial_{z_1}^{-1}(A_0^2)$ is not in L^2 since $A_0^2 \geqslant 0$ (when A_0 is nontrivial). This means that if $d \geqslant 2$, the expansion is not formally correct up to the cancellation of the terms of order ε^2.

We would like to point out another difficulty when we cancel the terms of order ε^2, and restrict ourselves to the dimension $d = 1$. Under weak assumptions on the nonlinearity, the Cauchy problem for (NLS) is known to be locally well-posed in $\Psi^{\mathrm{in}} + H^1$, see [28]. This implies in particular that when we lift $\Psi = A\mathrm{e}^{i\varphi}$, we must have, by Sobolev imbedding, the existence of the limits $\varphi(t, \pm\infty)$ as well as $\varphi(t, \pm\infty) = \varphi^{\mathrm{in}}(t, \pm\infty)$ for any $t \geqslant 0$. Note that we consider here only solutions with $|\Psi| \approx 1$. As a consequence, the (generalized Riemann, say) integral $\int_{\mathbb{R}} U_\varepsilon\,dz = \int_{\mathbb{R}} U_0\,dz + \varepsilon\int_{\mathbb{R}} U_1\,dz + \varepsilon^2\int_{\mathbb{R}} U_2\,dz + \cdots$ must be independent of τ. Since U_0 (resp. U_1) solves

(KdV) (resp. linearized (KdV)), it is (formally) true that $\int_{\mathbb{R}} U_0\,dz$ (resp. $\int_{\mathbb{R}} U_1\,dz$) is conserved. For U_2, assuming the conservation of $\int_{\mathbb{R}} U_2\,dz$, we deduce from (19) that

$$\partial_\tau \int_{\mathbb{R}} A_2\,dz = \frac{1}{4}(6-\Gamma)\partial_\tau \int_{\mathbb{R}} A_0^2\,dz = 0, \tag{21}$$

since the (KdV) flow conserves the L^2 norm. On the other hand, (20) becomes

$$\begin{aligned}
&\frac{1}{\mathfrak{c}_s}\partial_\tau(A_2+U_2) - \frac{1}{2\mathfrak{c}_s^2}\partial_z^3(A_2+U_2) + \frac{\Gamma-4}{2}\partial_z\big(A_0(A_2+U_2)\big) + 3\partial_z\big(A_0[A_2+U_2]\big)\\
&\quad = \frac{1}{2\mathfrak{c}_s^2}\partial_z^3\left(\frac{1}{2\mathfrak{c}_s^2}\partial_z^2 A_0 + \frac{1}{4}(6-\Gamma)A_0^2\right) + \left(\frac{1}{2\mathfrak{c}_s^2}\partial_z^2 A_0 + \frac{1}{4}(6-\Gamma)A_0^2\right)\partial_z A_0\\
&\qquad - \frac{\Gamma-4}{2}\left(\frac{1}{2\mathfrak{c}_s^2}\partial_z^2 A_0 + \frac{1}{4}(6-\Gamma)A_0^2\right) - \Gamma A_1\partial_z A_1 - \frac{1}{\mathfrak{c}_s^2}\partial_z\big(A_0\partial_z^2 A_0\big) - \partial_z\big(q_3 A_0^3\big),
\end{aligned}$$

and since $2\partial_z^2 A_0\partial_z A_0 = \partial_z[(\partial_z A_0)^2]$ and $3A_0^2\partial_z A_0 = \partial_z[A_0^3]$, all the terms in the right-hand side are z-derivatives except the term involving A_0^2 in the before last line. Therefore, formal z integration of the above equation provides, still assuming $\int_{\mathbb{R}} U_2\,dz$ constant,

$$\frac{1}{\mathfrak{c}_s}\partial_\tau \int_{\mathbb{R}} A_2\,dz = \frac{(\Gamma-4)(\Gamma-6)}{8}\int_{\mathbb{R}} A_0^2\,dz,$$

which contradicts (21) (when A_0 is nontrivial), except in the particular cases $\Gamma = 6$ which happens for the Gross–Pitaevskii nonlinearity since $f'' = 0$ everywhere, or $\Gamma = 4$. This second argument suggest that we may not in general be able to cancel out the terms in ε^2 with an expansion in ε.

We finally point out that in [42], the convergence of the weakly transverse Boussinesq system to the (KP) equation was shown through an expansion in ε similar to the one discussed here, which leads, similarly to (19), to hypothesis like $\partial_{z_2}^2\zeta^{\text{in}} \in \partial_{z_1}^2 H^s(\mathbb{R}^2)$. The results in Theorems 1, 2 and 3 do not rely on the justification of some expansion in ε. Actually, expanding in ε with even powers so that the equations are solved up to the natural error $\mathcal{O}(\varepsilon^2)$ suggest that we may compare the true solution $(A_\varepsilon, U_\varepsilon)$ to the approximate one $(A_0+\varepsilon^2 A_2, U_0+\varepsilon^2 U_2)$ up to an error $\mathcal{O}(\varepsilon^2)$. The condition (19) appears then somehow unnatural because the terms $(\varepsilon^2 A_2, \varepsilon^2 U_2)$ involved are of the order of the error $\mathcal{O}(\varepsilon^2)$.

1.4. Formal derivation of (gKdV)/(gKP-I) equation in the degenerate case $\Gamma = 0$

When

$$\Gamma = 6 + \frac{4}{\mathfrak{c}_s^2}f''(1) = 0,$$

(KdV)/(KP-I) is a linear dispersive equation. In order to see nonlinear effects, it is thus natural to enlarge the size of the data. It turns out that the natural scaling is now

$$\Psi(t,x) = \big(1+\varepsilon A_\varepsilon(\tau,z)\big)\exp\big(i\phi_\varepsilon(\tau,z)\big) \quad \tau = \varepsilon^3 t,\ z_1 \equiv \varepsilon(x_1 - \mathfrak{c}_s t),\ z_\perp \equiv \varepsilon^2 x_\perp. \tag{22}$$

Plugging this into (NLS), we obtain the system

$$\begin{cases}
\partial_\tau A_\varepsilon - \dfrac{\mathfrak{c}_s}{\varepsilon^2}\partial_{z_1}A_\varepsilon + \dfrac{2}{\varepsilon}\partial_{z_1}\phi_\varepsilon\partial_{z_1}A_\varepsilon + 2\varepsilon\nabla_{z_\perp}\phi_\varepsilon\cdot\nabla_{z_\perp}A_\varepsilon + \dfrac{1}{\varepsilon^2}(1+\varepsilon A_\varepsilon)\big(\partial_{z_1}^2\phi_\varepsilon + \varepsilon^2\Delta_{z_\perp}\phi_\varepsilon\big) = 0\\[2ex]
\partial_\tau\phi_\varepsilon - \dfrac{\mathfrak{c}_s}{\varepsilon^2}\partial_{z_1}\phi_\varepsilon + \dfrac{1}{\varepsilon}(\partial_{z_1}\phi_\varepsilon)^2 + \varepsilon|\nabla_{z_\perp}\phi_\varepsilon|^2 + \dfrac{1}{\varepsilon^3}f\big((1+\varepsilon A_\varepsilon)^2\big) - \dfrac{\partial_{z_1}^2 A_\varepsilon + \varepsilon^2\Delta_{z_\perp}A_\varepsilon}{1+\varepsilon A_\varepsilon} = 0.
\end{cases} \tag{23}$$

As $\varepsilon \to 0$ and if $A_\varepsilon \to A$ and $\phi_\varepsilon \to \phi$, we infer as above that at leading order, for both equations

$$\mathfrak{c}_s A = \partial_{z_1}\phi. \tag{24}$$

However, it has to be noticed that (23) has also singular terms of order ε^{-1}. Assuming that f is of class $\mathcal{C}^4$ near $\varrho = 1$ and using the Taylor expansion

$$f\big((1+\alpha)^2\big) = \mathfrak{c}_s^2\alpha + \left(\frac{\mathfrak{c}_s^2}{2} + 2f''(1)\right)\alpha^2 + \left(2f''(1) + \frac{4}{3}f'''(1)\right)\alpha^3 + f_4(\alpha),$$

with $f_4(\alpha) = \mathcal{O}(\alpha^4)$ as $\alpha \to 0$, the formally singular terms in system (23) give

$$\begin{cases} \dfrac{1}{\varepsilon^2}\big(\partial_{z_1}^2\phi_\varepsilon - \mathfrak{c}_s\partial_{z_1}A_\varepsilon\big) + \dfrac{1}{\varepsilon}\big(2\partial_{z_1}\phi_\varepsilon\partial_{z_1}A_\varepsilon + A_\varepsilon\partial_{z_1}^2\phi_\varepsilon\big) = \mathcal{O}(1) \\ \dfrac{\mathfrak{c}_s}{\varepsilon^2}(\mathfrak{c}_s A_\varepsilon - \partial_{z_1}\phi_\varepsilon) + \dfrac{1}{\varepsilon}\big((\partial_{z_1}\phi_\varepsilon)^2 + \big(\mathfrak{c}_s^2/2 + 2f''(1)\big)A_\varepsilon^2\big) = \mathcal{O}(1). \end{cases}$$

We recall that $\Gamma = 0$ if and only if $-2f''(1) = 3\mathfrak{c}_s^2$. Furthermore, since $\mathfrak{c}_s A_\varepsilon = \partial_{z_1}\phi_\varepsilon + \mathcal{O}(\varepsilon)$, we infer for both equations in the above system (formally)

$$\partial_{z_1}\phi_\varepsilon - \mathfrak{c}_s A_\varepsilon = -\frac{3\varepsilon}{2}\mathfrak{c}_s A_\varepsilon^2 + \mathcal{O}\big(\varepsilon^2\big). \tag{25}$$

Adding $\mathfrak{c}_s^{-1}$ times the first equation of (23) to $\mathfrak{c}_s^{-2}\partial_{z_1}$ times the second one, we get (using $-2f''(1) = 3\mathfrak{c}_s^2$),

$$\begin{aligned} &\frac{1}{\mathfrak{c}_s}\partial_\tau\left(A_\varepsilon + \frac{1}{\mathfrak{c}_s}\partial_{z_1}\phi_\varepsilon\right) - \frac{1}{\mathfrak{c}_s^2}\partial_{z_1}\left(\frac{\partial_{z_1}^2 A_\varepsilon + \varepsilon^2\Delta_{z_\perp}A_\varepsilon}{1+\varepsilon A_\varepsilon}\right) + (1+\varepsilon A_\varepsilon)\Delta_{z_\perp}\phi_\varepsilon + \frac{1}{\mathfrak{c}_s^2}\big(6f''(1) + 4f'''(1)\big)A_\varepsilon^2\partial_{z_1}A_\varepsilon \\ &\quad + \frac{1}{\mathfrak{c}_s\varepsilon}\left\{2\partial_{z_1}\phi_\varepsilon\partial_{z_1}A_\varepsilon + A_\varepsilon\partial_{z_1}^2\phi_\varepsilon + \frac{1}{\mathfrak{c}_s}\partial_{z_1}\big[(\partial_{z_1}\phi_\varepsilon)^2\big] - \frac{5\mathfrak{c}_s}{2}\partial_{z_1}\big(A_\varepsilon^2\big)\right\} \\ &\quad = -\frac{2\varepsilon}{\mathfrak{c}_s}\nabla_{z_\perp}\phi_\varepsilon\cdot\nabla_{z_\perp}A_\varepsilon - \frac{\varepsilon}{\mathfrak{c}_s^2}\partial_{z_1}\big[|\nabla_{z_\perp}\phi_\varepsilon|^2\big] - \frac{1}{\mathfrak{c}_s^2\varepsilon^3}\partial_{z_1}\big[f_4(\varepsilon A_\varepsilon)\big]. \end{aligned} \tag{26}$$

We have to pay attention to the second line in (26) due to the factor ε^{-1}. Using (25), the leading (quadratic) order terms cancel out and the second line in (26) is

$$\begin{aligned} &\frac{1}{\mathfrak{c}_s\varepsilon}\left\{2\partial_{z_1}A_\varepsilon\left(\mathfrak{c}_s A_\varepsilon - \frac{3\varepsilon}{2}\mathfrak{c}_s A_\varepsilon^2\right) + A_\varepsilon\partial_{z_1}\left(\mathfrak{c}_s A_\varepsilon - \frac{3\varepsilon}{2}\mathfrak{c}_s A_\varepsilon^2\right) + \frac{1}{\mathfrak{c}_s}\partial_{z_1}\left[\left(\mathfrak{c}_s A_\varepsilon - \frac{3\varepsilon}{2}\mathfrak{c}_s A_\varepsilon^2\right)^2\right] - 5\mathfrak{c}_s A_\varepsilon\partial_{z_1}A_\varepsilon\right\} \\ &\quad + \mathcal{O}(\varepsilon) \\ &\quad = -15A_\varepsilon^2\partial_{z_1}A_\varepsilon + \mathcal{O}(\varepsilon). \end{aligned}$$

As a consequence, passing to the (formal) limit $\varepsilon \to 0$ in (26) yields the modified (KdV)/(KP-I) equation

$$\frac{2}{\mathfrak{c}_s}\partial_\tau A - \frac{1}{\mathfrak{c}_s^2}\partial_{z_1}^3 A + \Gamma' A^2\partial_{z_1}A + \Delta_{z_\perp}\partial_{z_1}^{-1}A = 0, \tag{mKdV)/(mKP-I}$$

where the coefficient

$$\Gamma' \equiv \frac{4f'''(1)}{\mathfrak{c}_s^2} - 24$$

involves a third order derivative of f at 1. The nature of (mKdV)/(mKP-I) strongly depends on the sign of Γ': it is defocussing for $\Gamma' > 0$ (without solitary waves) but focusing when $\Gamma' < 0$ (with solitary waves). Indeed, in dimension $d = 1$, we have two solitons of speed $-1/(2\mathfrak{c}_s)$

$$\mathrm{w}_\pm(z) \equiv \pm\frac{\sqrt{-6/(\Gamma'\mathfrak{c}_s^2)}}{\cosh(z)}$$

(recall that (mKdV)/(mKP-I) is odd in A: if A is a solution, so is $-A$), and if $d \geqslant 2$, we have existence of (at least two) nontrivial solitary waves to (mKP-I) if and only if $d = 2$ and $\Gamma' < 0$ (*cf.* [24]).

We can clearly go further and derive more generally (gKdV)/(gKP-I) equations from (NLS) when suitable coefficients like Γ and Γ' vanish. More precisely, for some given $m \in \mathbb{N}$, assume that f is of class $\mathcal{C}^{m+3}$ near $\varrho = 1$ and that we have

$$\frac{f^{(j)}(1)}{(j+1)!} = (-1)^{j+1}\frac{\mathfrak{c}_s^2}{4} \quad \text{for } 1 \leqslant j < m+2, \tag{27}$$

the equality for $j=1$ being always true by definition of $\mathfrak{c}_s$, namely $\mathfrak{c}_s = \sqrt{2f'(1)}$. If $m=1$, this requires $f''(1) = -3\mathfrak{c}_s^2/2$, which holds true if $\Gamma = 0$. Consider now (f is supposed smooth enough) the Taylor expansion of $f((1+\alpha)^2)$ near the origin:

$$\frac{1}{\mathfrak{c}_s^2} f\big((1+\alpha)^2\big) = \sum_{k=1}^{m+1} q_k \alpha^k + q_{m+2}\alpha^{m+2} + \mathcal{O}\big(\alpha^{m+3}\big) = \sum_{k=1}^{m+2}\Bigg(\sum_{\substack{0\leqslant \ell\leqslant j\\ j+\ell=k}} \frac{f^{(j)}(1)2^{\ell-j}}{\ell!(j-\ell)!}\Bigg)\alpha^k + \mathcal{O}\big(\alpha^{m+3}\big).$$

The ansatz (2) and (22) are then changed for

$$\Psi(t,x) = \big(1+\varepsilon^{\frac{2}{m+1}}A_\varepsilon(\tau,z)\big)\exp\big(i\varepsilon^{\frac{1-m}{1+m}}\phi_\varepsilon(\tau,z)\big) \quad \tau=\varepsilon^3 t,\ z_1\equiv\varepsilon(x_1-\mathfrak{c}_s t),\ z_\perp\equiv\varepsilon^2 x_\perp. \tag{28}$$

Inserting this into (NLS) yields, with $U_\varepsilon \equiv \mathfrak{c}_s^{-1}\nabla_z\phi_\varepsilon$,

$$\begin{cases} \dfrac{1}{\mathfrak{c}_s}\partial_\tau A_\varepsilon - \dfrac{1}{\varepsilon^2}\partial_{z_1}A_\varepsilon + 2\varepsilon^{\frac{2}{m+1}-2}U_\varepsilon^1\partial_{z_1}A_\varepsilon + 2\varepsilon^{\frac{2}{m+1}}U_\varepsilon^\perp\cdot\nabla_{z_\perp}A_\varepsilon \\ \quad + \dfrac{1}{\varepsilon^2}\big(1+\varepsilon^{\frac{2}{m+1}}A_\varepsilon\big)\big(\partial_{z_1}U_\varepsilon^1 + \varepsilon^2\nabla_{z_\perp}\cdot U_\varepsilon^\perp\big) = 0 \\ \dfrac{1}{\mathfrak{c}_s}\partial_\tau U_\varepsilon^1 - \dfrac{1}{\varepsilon^2}\partial_{z_1}U_\varepsilon^1 + 2\varepsilon^{\frac{2}{m+1}-2}U_\varepsilon^1\partial_{z_1}U_\varepsilon^1 + 2\varepsilon^{\frac{2}{m+1}}U_\varepsilon^\perp\cdot\nabla_{z_\perp}U_\varepsilon^1 + \displaystyle\sum_{k=1}^{m+1} kq_k\varepsilon^{\frac{2(k-1)}{m+1}-2}A_\varepsilon^{k-1}\partial_{z_1}A_\varepsilon \\ \quad + (m+2)q_{m+2}A_\varepsilon^{m+1}\partial_{z_1}A_\varepsilon + \mathcal{O}\big(\varepsilon^{\frac{2}{m+1}}\big) - \dfrac{1}{\mathfrak{c}_s^2}\partial_{z_1}\Bigg(\dfrac{\partial_{z_1}^2 A_\varepsilon + \varepsilon^2\Delta_{z_\perp}A_\varepsilon}{1+\varepsilon^{\frac{2}{m+1}}A_\varepsilon}\Bigg) = 0. \end{cases} \tag{29}$$

Comparing the singular terms in both equations, we obtain

$$\begin{cases} \partial_{z_1}U_\varepsilon^1 - \partial_{z_1}A_\varepsilon + \varepsilon^{\frac{2}{m+1}}\big(2U_\varepsilon^1\partial_{z_1}A_\varepsilon + A_\varepsilon\partial_{z_1}U_\varepsilon^1\big) = \mathcal{O}\big(\varepsilon^2\big) \\ \partial_{z_1}A_\varepsilon - \partial_{z_1}U_\varepsilon^1 + 2\varepsilon^{\frac{2}{m+1}}U_\varepsilon^1\partial_{z_1}U_\varepsilon^1 + \displaystyle\sum_{k=2}^{m+1} kq_k\varepsilon^{\frac{2(k-1)}{m+1}}A_\varepsilon^{k-1}\partial_{z_1}A_\varepsilon = \mathcal{O}\big(\varepsilon^2\big). \end{cases}$$

For $m=0$, we recover the constraint (5), and for $m=1$, we obtain (25). When $m\geqslant 1$, this system is also a single constraint. Indeed, letting $\epsilon\equiv\varepsilon^{2/(m+1)}$, so that $\epsilon^{m+1}=\varepsilon^2$, we shall see that the two equations in the above system formally reduce to the single constraint

$$U_\varepsilon^1 = A_\varepsilon - \frac{3}{2}\epsilon A_\varepsilon^2 + 2\epsilon^2 A_\varepsilon^3 - \frac{5}{2}\epsilon^3 A_\varepsilon^4 + \cdots + (-1)^m\frac{m+2}{2}\epsilon^m A_\varepsilon^{m+1} + \mathcal{O}\big(\epsilon^{m+1}\big). \tag{30}$$

For the first equation, this follows immediately by induction on m. Formally integrating in z_1, the second one can be written

$$A_\varepsilon - U_\varepsilon^1 + \epsilon\big(U_\varepsilon^1\big)^2 + \sum_{k=2}^{m+1} q_k\epsilon^{k-1}A_\varepsilon^k = \mathcal{O}\big(\epsilon^{m+1}\big).$$

From (30) and after a little algebra, this is equivalent to

$$\sum_{k=2}^{m+1}(-1)^k\frac{k+1}{2}\epsilon^{k-1}A_\varepsilon^k + \sum_{k=2}^{m+1}(-1)^k\frac{(k+1)(k-1)(k+6)}{24}\epsilon^{k-1}A_\varepsilon^k + \sum_{k=2}^{m+1}q_k\epsilon^{k-1}A_\varepsilon^k = \mathcal{O}\big(\epsilon^{m+1}\big),$$

that is to

$$\forall 2\leqslant k\leqslant m+1, \quad q_k = (-1)^{k-1}\frac{(k+1)(k+2)(k+3)}{24}. \tag{31}$$

The relation (31) is actually verified when the $f^{(j)}(1)$'s verify (27), as can be seen by noticing that then, for $\varrho \to 1$,

$$f(\varrho) = \sum_{j=1}^{m+1} \frac{f^{(j)}(1)}{j!}(\varrho-1)^j + \mathcal{O}\big((\varrho-1)^{m+2}\big) = \sum_{j=1}^{m+1} \frac{\mathfrak{c}_s^2}{4}(-1)^{j+1}(j+1)(\varrho-1)^j + \mathcal{O}\big((\varrho-1)^{m+2}\big)$$
$$= \frac{\mathfrak{c}_s^2}{4}\sum_{j=1}^{+\infty}(-1)^{j+1}(j+1)(\varrho-1)^j + \mathcal{O}\big((\varrho-1)^{m+2}\big) = \frac{\mathfrak{c}_s^2}{4}\left(1-\frac{1}{\varrho^2}\right) + \mathcal{O}\big((\varrho-1)^{m+2}\big),$$

thus, for $\alpha \to 0$,

$$\frac{1}{\mathfrak{c}_s^2} f\big((1+\alpha)^2\big) = \frac{1}{4}\left(1-\frac{1}{(1+\alpha)^4}\right) + \mathcal{O}\big(\alpha^{m+2}\big) = \sum_{k=1}^{m+1}(-1)^{k-1}\frac{(k+1)(k+2)(k+3)}{24}\alpha^k + \mathcal{O}\big(\alpha^{m+2}\big).$$

Note that when (30) is satisfied, we have

$$\epsilon\big(2U_\varepsilon^1\partial_{z_1}A_\varepsilon + A_\varepsilon\partial_{z_1}U_\varepsilon^1\big) + 2\epsilon U_\varepsilon^1\partial_{z_1}U_\varepsilon^1 + \sum_{k=2}^{m+1} kq_k\epsilon^{k-1}A_\varepsilon^{k-1}\partial_{z_1}A_\varepsilon$$
$$= (-1)^m\frac{\epsilon^{m+1}}{24}(m+2)(m+3)(m+4)(m+5)A_\varepsilon^{m+1}\partial_{z_1}A_\varepsilon + \mathcal{O}\big(\epsilon^{m+2}\big). \tag{32}$$

Adding now the two equations of (29) and using (32), we infer

$$\frac{1}{\mathfrak{c}_s}\partial_\tau\big(A_\varepsilon + U_\varepsilon^1\big) - \frac{1}{\mathfrak{c}_s^2}\partial_{z_1}\left(\frac{\partial_{z_1}^2 A_\varepsilon + \varepsilon^2\Delta_{z_\perp}A_\varepsilon}{1+\epsilon A_\varepsilon}\right) + (1+\epsilon A_\varepsilon)\nabla_{z_\perp}\cdot U_\varepsilon^\perp$$
$$+ (m+2)q_{m+2}A_\varepsilon^{m+1}\partial_{z_1}A_\varepsilon + \frac{(-1)^m}{24}(m+2)(m+3)(m+4)(m+5)A_\varepsilon^{m+1}\partial_{z_1}A_\varepsilon$$
$$= -2\epsilon U_\varepsilon^\perp\cdot\nabla_{z_\perp}U_\varepsilon^1 - 2\epsilon U_\varepsilon^\perp\cdot\nabla_{z_\perp}A_\varepsilon + \mathcal{O}(\epsilon), \tag{33}$$

where the $\mathcal{O}(\epsilon)$ contains the remainder in the Taylor expansion and the contribution coming from (32). The formal limit is then the (gKdV)/(gKP-I) equation

$$\frac{2}{\mathfrak{c}_s}\partial_\tau A + \Gamma^{(m)}A^{m+1}\partial_{z_1}A + \Delta_{z_\perp}\partial_{z_1}^{-1}A - \frac{1}{\mathfrak{c}_s^2}\partial_{z_1}^3 A = 0, \tag{gKdV)/(gKP-I}$$

where the coefficient $\Gamma^{(m)}$ involves a derivative of order $f^{(m+2)}$ of f at $\rho = 1$ and is defined by

$$\Gamma^{(m)} \equiv (m+2)q_{m+2} + \frac{(-1)^m}{24}(m+2)(m+3)(m+4)(m+5),$$

and clearly $\Gamma^{(0)} = \Gamma$ if $m = 0$ and $\Gamma^{(1)} = \Gamma'$ if $m = 1$. It is also clear that $\Gamma^{(m)}$ vanishes if and only if $q_{m+2} = (-1)^{m+1}\frac{(m+3)(m+4)(m+5)}{24}$, which is (31) for $k = m+2$.

Remark 2. As we have seen during the computation, the nonlinearity given by

$$f(\varrho) = \frac{\mathfrak{c}_s^2}{4}\left(1-\frac{1}{\varrho^2}\right),$$

at least locally near $\varrho = 1$, is extremely specific. Indeed, *all* the coefficients $\Gamma^{(m)}$, $m \in \mathbb{N}\cup\{0\}$ vanish for this nonlinearity, in view of the fact that $f^{(j)}(1) = (-1)^{j+1}(j+1)!\frac{\mathfrak{c}_s^2}{4}$ for any $j \in \mathbb{N}_0$.

Remark 3. If one prefers to express A_ε in terms of U_ε^1 in (30), one obtains

$$A_\varepsilon = \sum_{k=1}^{m+1}\frac{1\cdot 3\cdot\cdots\cdot(2k-1)}{k!}\epsilon^{k-1}\big[U_\varepsilon^1\big]^k + \mathcal{O}\big(\epsilon^{m+1}\big). \tag{34}$$

Indeed, (30) provides (formally)

$$\epsilon U_\varepsilon^1 = \sum_{j=0}^{+\infty} (-1)^j \frac{j+2}{2} \epsilon^{j+1} A_\varepsilon^{j+1} + \mathcal{O}\big(\epsilon^{m+1}\big) = \frac{1}{2}\partial_\epsilon \left(\sum_{j=0}^{+\infty} (-1)^j \epsilon^{j+2} A_\varepsilon^{j+1} \right) + \mathcal{O}\big(\epsilon^{m+1}\big)$$
$$= \frac{1}{2}\partial_\epsilon \left(\frac{\epsilon^2 A_\varepsilon}{1+\epsilon A_\varepsilon} \right) + \mathcal{O}\big(\epsilon^{m+1}\big) = \frac{1}{2}\left(1 - \frac{1}{(1+\epsilon A_\varepsilon)^2} \right) + \mathcal{O}\big(\epsilon^{m+1}\big),$$

hence $1+\epsilon A_\varepsilon = (1-2\epsilon U_\varepsilon^1)^{-1/2} + \mathcal{O}(\epsilon^{m+1})$ and the result follows by Taylor expansion.

We would like to conclude this section with a discussion on the free wave regime studied in [10]. This wave regime holds for an initial datum for (NLS) of the type

$$\Psi^{\rm in}(x) = \big(1+\varepsilon A_\varepsilon^{\rm in}(z)\big)\exp\big(i\phi_\varepsilon^{\rm in}(z)\big), \quad z \equiv \varepsilon x \tag{35}$$

and relies on the ansatz

$$\Psi(t,x) = \big(1+\varepsilon A_\varepsilon(\mathfrak{t},z)\big)\exp\big(i\phi_\varepsilon(\mathfrak{t},z)\big), \quad \mathfrak{t} = \varepsilon t,\ z \equiv \varepsilon x. \tag{36}$$

The main result in [10] is the following.

Theorem 5. *(See [10].) Let $\Lambda > 0$ and $s \in \mathbb{R}$ be such that $s > 1 + \frac{d}{2}$. We consider an initial datum for the Gross–Pitaevskii equation*

$$i\frac{\partial \Psi}{\partial t} + \Delta\Psi = \Psi\big(|\Psi|^2 - 1\big) \tag{GP}$$

of the type $\Psi_\varepsilon^{\rm in}(x) = (1+\varepsilon A_\varepsilon^{\rm in}(z))\exp(i\phi_\varepsilon^{\rm in}(z))$, $z = \varepsilon x$, with

$$\big\|A_\varepsilon^{\rm in}\big\|_{H^{s+1}(\mathbb{R}^d)} + \big\|\nabla_z \phi_\varepsilon^{\rm in}\big\|_{H^s(\mathbb{R}^d)} \leqslant \Lambda.$$

Then, there exists a positive constant $K_0 = K_0(s,d)$ such that if $K_0\varepsilon\Lambda \leqslant 1$, then (GP) has a unique solution $\Psi_\varepsilon \in \Psi_\varepsilon^{\rm in} + \mathcal{C}([0, 1/(K_0\varepsilon\Lambda)], H^{s+1}(\mathbb{R}^d, \mathbb{C})$ with initial datum $\Psi_\varepsilon^{\rm in}$, which can be written under the form (36) with

$$\sup_{0\leqslant \mathfrak{t}\leqslant 1/(K_0\varepsilon\Lambda)} \big\|A_\varepsilon(\mathfrak{t})\big\|_{H^{s+1}(\mathbb{R}^d)} + \big\|\nabla_z\phi_\varepsilon(\mathfrak{t})\big\|_{H^s(\mathbb{R}^d)} \leqslant K_0\Lambda \quad \text{and} \quad \frac{1}{2} \leqslant \rho = 1+\varepsilon A_\varepsilon \leqslant 2. \tag{37}$$

Furthermore, if $(\mathfrak{a}_\varepsilon, \mathfrak{u}_\varepsilon)$ denotes the solution to the free wave equation

$$\begin{cases} \partial_{\mathfrak{t}}\mathfrak{a}_\varepsilon + 2\nabla_z\cdot\mathfrak{u}_\varepsilon = 0 \\ \partial_{\mathfrak{t}}\mathfrak{u}_\varepsilon + \frac{1}{2}\nabla_z\mathfrak{a}_\varepsilon = 0 \end{cases} \tag{38}$$

with intial datum $(A_\varepsilon^{\rm in}, \nabla_z\phi_\varepsilon^{\rm in})$, then, for $0 \leqslant \mathfrak{t} \leqslant 1/(K_0\varepsilon\Lambda)$, there holds

$$\big\|(A_\varepsilon, U_\varepsilon)(\mathfrak{t}) - (\mathfrak{a}_\varepsilon, \mathfrak{u}_\varepsilon)(\mathfrak{t})\big\|_{H^{s-2}(\mathbb{R}^d)\times H^{s-2}(\mathbb{R}^d)} \leqslant K_0\varepsilon\mathfrak{t}\big(\Lambda^2 + \varepsilon\Lambda\big). \tag{39}$$

This underlines that the free wave regime is a good approximation for large $\mathfrak{t}$, namely $t \ll \varepsilon^{-1}$. Actually, $\mathfrak{t} \approx 1$ ($t \simeq \varepsilon^{-1}$) is the time scale for the Euler regime, and since we linearize around a constant state, we expect that the asymptotics hold for large t. We may refer to, *e.g.*, [17] for a survey on the different long wave regimes for (NLS) (Euler regime, wave regime, ...). In the case $d = 1$, $m = 1$, the initial datum for the (mKdV) regime is also of the type (35). However, due to the cancellation of Γ and the nonlinear preparedness assumption (25) of the data, we formally obtain solutions on a much larger time interval $\tau \simeq 1$, that is $t \approx \varepsilon^{-3}$ or $\mathfrak{t} \approx \varepsilon^{-2}$.

Remark 4. It seems that actually, in Theorem 5, the norm $\|A_\varepsilon^{\rm in}\|_{H^{s+1}(\mathbb{R}^d)}$ needs to be replaced by $\|A_\varepsilon^{\rm in}\|_{H^s(\mathbb{R}^d)} + \varepsilon\|A_\varepsilon^{\rm in}\|_{H^{s+1}(\mathbb{R}^d)}$, and similarly in (37). Indeed, in Proposition 1 in [10], we see that "z" is controled in H^s, but the imaginary part of z is $2\frac{\nabla\rho}{\rho}$, with $\rho = 1+\varepsilon A_\varepsilon$, so that only $\varepsilon\|A_\varepsilon^{\rm in}\|_{H^{s+1}(\mathbb{R}^d)}$ is involved and not just $\|A_\varepsilon\|_{H^{s+1}(\mathbb{R}^d)}$. This means that the right-hand side of (39) should presumably be replaced by $K_0\varepsilon\mathfrak{t}(\Lambda^2+\Lambda)$.

1.5. Some rigorous justification of the (gKdV)/(gKP-I) equation

We present here our rigorous convergence result to the (gKdV)/(gKP-I) equation, but, as we shall see, it does not hold on the scale $\tau = \varepsilon^3 t \approx 1$.

Determining the right time scale. When one wants to justify the (gKdV)/(gKP-I) asymptotic regime, the main difficulty is the presence, in systems (23) and (29), of singular terms with nonconstant coefficient. In comparison with the justification of the (KdV)/(KP-I) limit, where we prove first the Sobolev bounds and then the error estimate involving the preparedness assumption, the difficulty for proving the (gKdV)/(gKP-I) limit on the natural time scale $\tau = \varepsilon^3 t$ is to break down a vicious circle: the Sobolev bounds depend on the preparation of the data, which itself depends on the Sobolev bounds. Despite our efforts, we have not been able to solve this problem, even working in a space of analytic functions. Another aspect which appears for this problem on the time scale $\tau = \varepsilon^3 t \approx 1$ is that we always have to expand much further than the expected natural order. For instance, the constraint (30), namely, considering $d = 1$ for simplicity,

$$U_\varepsilon = A_\varepsilon - \frac{3}{2}\epsilon A_\varepsilon^2 + 2\epsilon^2 A_\varepsilon^3 - \frac{5}{2}\epsilon^3 A_\varepsilon^4 + \dots + (-1)^m \frac{m+2}{2}\epsilon^m A_\varepsilon^{m+1} + \mathcal{O}(\epsilon^{m+1}),$$

requires to expand $(A_\varepsilon, U_\varepsilon)$ up to $\mathcal{O}(\epsilon^{m+1})$. However, this induces in the equations a consistency error only $\mathcal{O}(1)$ due to the singular term in $1/\varepsilon^2 = 1/\epsilon^{m+1}$. Hence we may hope to prove only $(A_\varepsilon, U_\varepsilon) - (A_0, U_0) - \epsilon(A_1, U_1) - \dots - \epsilon^m(A_m, U_m) = \mathcal{O}(1)$, which is useless. Furthermore, we do not have any equation for the evolution of (A_1, U_1), $(A_2, U_2), \dots$ and there is clearly no uniqueness when solving (30). Expanding $(A_\varepsilon, U_\varepsilon)$ up to $\mathcal{O}(\epsilon^{m+r})$ for some $r \geqslant 1$ provides a consistency error $\mathcal{O}(\epsilon^{r-1})$, which is not sufficient for proving that (30) remains true, except for $r \geqslant m+2$. When $m = 1$, hence $\epsilon = \varepsilon$, this means that we have to expand $(A_\varepsilon, U_\varepsilon)$ up to $\mathcal{O}(\varepsilon^4)$ instead of the natural $\mathcal{O}(\varepsilon^2)$. This is the same mechanism which shows that the Sobolev bounds at one order require an expansion of the data at a much larger order.

As a consequence, it is natural to work on a smaller time scale. In view of the result of [10] given in Theorem 5, the wave time scale seems natural. Notice that for an initial datum of the form

$$\Psi^{\rm in}(x) = \big(1 + \epsilon A_\varepsilon^{\rm in}(z)\big) \exp\left(i\frac{\epsilon}{\varepsilon}\phi_\varepsilon^{\rm in}(z)\right), \quad z \equiv \varepsilon x,$$

where the small parameter $\varepsilon^2 \ll \epsilon \ll 1$ may be different from ε (compare with (38)), the free wave regime holds for $t \ll (\varepsilon\epsilon)^{-1}$. We thus introduce the time scale $\theta = \varepsilon\epsilon t = \epsilon$t, that is we replace (22) by

$$\Psi(t,x) = \big(1 + \epsilon A_\varepsilon(\theta, z)\big) \exp\left(i\frac{\epsilon}{\varepsilon}\phi_\varepsilon(\theta, z)\right), \quad \theta = \varepsilon\epsilon t,\ z_1 \equiv \varepsilon(x_1 - \mathfrak{c}_s t),\ z_\perp \equiv \varepsilon^2 x_\perp, \tag{40}$$

which changes (3) for a system with $U_\varepsilon \equiv \mathfrak{c}_s^{-1}\nabla_z\phi_\varepsilon$ where the singular terms have constant coefficients:

$$\begin{cases} \partial_\theta A_\varepsilon - \dfrac{\mathfrak{c}_s}{\epsilon}\partial_{z_1}A_\varepsilon + 2U_\varepsilon^1\partial_{z_1}A_\varepsilon + 2\varepsilon^2 U_\varepsilon^\perp \cdot \nabla_{z_\perp}A_\varepsilon + \dfrac{1}{\epsilon}(1+\epsilon A_\varepsilon)\big(\partial_{z_1}U_\varepsilon^1 + \varepsilon^2\nabla_{z_\perp}\cdot U_\varepsilon^\perp\big) = 0 \\ \partial_\theta U_\varepsilon - \dfrac{\mathfrak{c}_s}{\epsilon}\partial_{z_1}U_\varepsilon + 2U_\varepsilon^1\partial_{z_1}U_\varepsilon + 2\varepsilon^2 U_\varepsilon^\perp \cdot \nabla_{z_\perp}U_\varepsilon + \dfrac{1}{\epsilon^2}\nabla_z\big[f\big((1+\epsilon A_\varepsilon)^2\big)\big] \\ \quad = \dfrac{\varepsilon^2}{\epsilon}\nabla_z\left[\dfrac{\partial_{z_1}^2 A_\varepsilon + \varepsilon^2\Delta_{z_\perp}A_\varepsilon}{1+\epsilon A_\varepsilon}\right]. \end{cases} \tag{41}$$

The free wave regime studied in [10] then holds for $\theta \ll 1$.

Approximation of the right-going wave by the Burgers equation for $\theta \approx 1$. Let us investigate what can be shown concerning an expansion in ε for the one dimensional situation with only one wave propagating to the right. We then try to expand further in $\epsilon = \varepsilon$ the result in [10] with the assumption that the wave going to the left is negligible. This leads us to consider the one dimensional system, where $\theta = \varepsilon^2 t$ and $U_\varepsilon = \mathfrak{c}_s^{-1}\partial_z\phi_\varepsilon$,

$$\begin{cases} \frac{1}{\mathfrak{c}_s}\partial_\theta A_\varepsilon - \frac{1}{\varepsilon}\partial_z A_\varepsilon + 2U_\varepsilon \partial_z A_\varepsilon + \frac{1}{\varepsilon}(1+\varepsilon A_\varepsilon)\partial_z U_\varepsilon = 0 \\ \frac{1}{\mathfrak{c}_s}\partial_\theta U_\varepsilon - \frac{1}{\varepsilon}\partial_z U_\varepsilon + 2U_\varepsilon \partial_z U_\varepsilon + \frac{1}{\mathfrak{c}_s^2\varepsilon^2}\partial_z\big(f\big([1+\varepsilon A_\varepsilon]^2\big)\big) = \varepsilon\partial_z\Big(\frac{\partial_z^2 A_\varepsilon}{1+\varepsilon A_\varepsilon}\Big). \end{cases} \tag{42}$$

When plugging a formal expansion in ε for $A_\varepsilon = A_0 + \varepsilon A_1 + \cdots$ and $U_\varepsilon = U_0 + \varepsilon U_1 + \cdots$ into (42) and arguing as in Section 1.3, we find that $A_0 = U_0$ verifies the (inviscid) Burgers equation (sometimes, it is also called the Hopf equation)

$$\frac{2}{\mathfrak{c}_s}\partial_\theta \mathfrak{a} + \Gamma \mathfrak{a}\partial_z \mathfrak{a} = 0 \tag{43}$$

under the additional hypothesis that

$$U_1^{\mathrm{in}} - A_1^{\mathrm{in}} = \frac{\Gamma - 6}{4}\big[A_0^{\mathrm{in}}\big]^2,$$

so that the relation $U_1 - A_1 = \frac{\Gamma-6}{4}A_0^2$ holds true for positive times. Here is a rigorous result in this direction, without this last extra assumption (as (19) was not necessary for proving the convergence to the (KdV)/(KP-I) equation but required by the expansion in ε). We emphasize that we focus on the right-going wave.

Proposition 4. *Assume $d = 1$, $\Lambda > 0$ and $s \in \mathbb{N}$ be such that $s \geqslant 3$. We consider an initial datum $(A_\varepsilon^{\mathrm{in}}, U_\varepsilon^{\mathrm{in}})$ for (42) verifying*

$$\big\|A_\varepsilon^{\mathrm{in}}\big\| \leqslant \Lambda \quad \text{and} \quad \big\|U_\varepsilon^{\mathrm{in}}\big\| \leqslant \Lambda. \tag{44}$$

Then, there exist $\theta_* > 0$ and a positive constant $\varepsilon_0 = \varepsilon_0(\Lambda, s)$ such that if $0 < \varepsilon \leqslant \varepsilon_0$, then (42) has a unique solution $(A_\varepsilon, U_\varepsilon) \in \mathcal{C}([0,\theta_*], H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}))$ with initial datum $(A_\varepsilon^{\mathrm{in}}, U_\varepsilon^{\mathrm{in}})$, verifying

$$\sup_{0\leqslant\theta\leqslant\theta_*} \big\|A_\varepsilon(\theta)\big\|_{H^s(\mathbb{R})} + \big\|\partial_z\phi_\varepsilon(\theta)\big\|_{H^{s-1}(\mathbb{R})} \leqslant K_0\Lambda \quad \text{and} \quad \frac{1}{2} \leqslant \rho = 1 + \varepsilon A_\varepsilon \leqslant 2. \tag{45}$$

Furthermore, if $\mathfrak{a}_\varepsilon \in \mathcal{C}([0,\theta_0], H^s(\mathbb{R}))$ denotes the solution to the (inviscid) Burgers equation

$$\frac{2}{\mathfrak{c}_s}\partial_\theta \mathfrak{a} + \Gamma \mathfrak{a}\partial_z \mathfrak{a} = 0$$

with intial datum $A_\varepsilon^{\mathrm{in}}$, then, for $0 \leqslant \theta \leqslant \min(\theta_0, \theta_*)$, there holds

$$\big\|A_\varepsilon(\theta) - \mathfrak{a}_\varepsilon(\theta)\big\|_{H^{s-3}(\mathbb{R})} + \big\|U_\varepsilon(\theta) - \mathfrak{a}_\varepsilon(\theta)\big\|_{H^{s-3}(\mathbb{R})} \leqslant K\big(\big\|A_\varepsilon^{\mathrm{in}} - U_\varepsilon^{\mathrm{in}}\big\|_{H^{s-1}(\mathbb{R})} + \varepsilon\theta\big).$$

This result provides an expansion for A_ε and U_ε up to $\mathcal{O}(\varepsilon)$ uniformly for $0 \leqslant \theta \leqslant \min(\theta_0, \theta_*) \approx 1$, whereas the result in Theorem 5, [10] takes into account left and right-going waves but is restricted to $0 \leqslant \theta \ll 1$. If $\Gamma \neq 0$, the approximation $\mathfrak{a}_\varepsilon$ has a nontrivial dynamics on the time scale $\theta \approx 1$. We shall now investigate what happens when $\Gamma = 0$, on the time scale θ. More precisely, we shall work up to $\theta \lesssim |\ln\varepsilon|$.

Justification of the (gKdV)/(gKP-I) equation for large $\boldsymbol{\theta}$. Here, we make the assumption (27) for some $m \in \mathbb{N}$, and recall that $\epsilon = \varepsilon^{2/(m+1)}$. Since we shall work for $\theta \lesssim |\ln\varepsilon|$, there holds $\tau = \epsilon^m\theta \lesssim \epsilon^m|\ln\varepsilon|$, so that the solution $\zeta(\tau)$ to (gKdV)/(gKP-I) has moved from $\approx \epsilon^m|\ln\varepsilon| \ll 1$ from its initial value. As a consequence, any error estimate between A_ε and $\zeta(\tau)$ for $\theta \lesssim |\ln\varepsilon|$ is meaningful only if the error is $\ll \epsilon^m|\ln\varepsilon|$. Since we shall justify an expansion in ϵ, this will force us to solve the equations up to an error $\mathcal{O}(\epsilon^{m+1})$. Proceeding in this way, we shall then prove a Gronwall estimate which roughly reads

$$\begin{aligned} &\big\|A_\varepsilon(\theta) - \big(A_0 + \epsilon A_1 + \cdots + \epsilon^{m+1}A_{m+1}\big)(\tau)\big\| + \big\|U_\varepsilon(\theta) - \big(U_0 + \epsilon U_1 + \cdots + \epsilon^{m+1}U_{m+1}\big)(\tau)\big\| \\ &\quad \lesssim \epsilon^{m+1}\mathrm{e}^{C_0\theta}, \end{aligned} \tag{46}$$

where $A_0 = U_0 = \zeta(\tau)$. For $\theta \lesssim |\ln\varepsilon| \approx |\ln\epsilon|$, the right-hand side remains small. Clearly, in this expansion, the terms $\epsilon^{m+1}A_{m+1}$ and $\epsilon^{m+1}U_{m+1}$ are useless in view of the error $\geqslant \epsilon^{m+1}$, but they are necessary in order to have a consistency error in $\mathcal{O}(\epsilon^{m+1})$. This leads to constraints such as (19) or $U_1^{\rm in} - A_1^{\rm in} = \frac{\Gamma-6}{4}[A_0^{\rm in}]^2$ for the Burgers equation. We have seen that for (A_1, U_1), (A_2, U_2), ..., we have no evolution equation on the time scale τ (but they are (formally) stationary on the time scale θ). Therefore, it may seem strange to justify an expansion which seems up to $\mathcal{O}(\epsilon^{m+1})$ without knowing the true dynamics of (A_1, U_1), (A_2, U_2), ... However, this is not unconsistent, since for $\theta \lesssim |\ln\epsilon|$, the term ϵU_1 for instance, has moved from its initial condition of at most $\epsilon \times \tau = \epsilon^{m+1}\theta$, which is much *smaller* than $\epsilon^{m+1}\mathrm{e}^{C_0\theta}$ when θ is large.

Our result is based on an expansion in ϵ, thus we shall have the above mentioned constraints on the initial data, such as $\Delta_{z_\perp}\partial_{z_1}^{-2}\zeta^{\rm in} \in H^{s+1}(\mathbb{R}^d)$, although we believe that they are actually not necessary. In view of the form of (46), note that a distinction has to be made between the case $m = 1$ and the case $m \geqslant 2$. Indeed, we have seen that we wish to have $\|A_\varepsilon(\theta) - \zeta(\tau)\| = \|U_\varepsilon(\theta) - \zeta(\tau)\| = o(\epsilon^m|\ln\varepsilon|)$, and *a priori*, we infer from (46) that $\|A_\varepsilon(\theta) - \zeta(\tau)\| = \|U_\varepsilon(\theta) - \zeta(\tau)\| \approx \epsilon$. If $m = 1$, it is true that $\epsilon = o(\epsilon^m|\ln\varepsilon|)$, but if $m \geqslant 2$, this is no longer the case, which means that we cannot compare both A_ε and U_ε to ζ in a significant way. Indeed, in view of (30), that is

$$U_\varepsilon - \left\{A_\varepsilon - \frac{3}{2}\epsilon A_\varepsilon^2 + 2\epsilon^2 A_\varepsilon^3 - \frac{5}{2}\epsilon^3 A_\varepsilon^4 + \cdots + (-1)^m \frac{m+2}{2}\epsilon^m A_\varepsilon^{m+1}\right\} = \mathcal{O}(\epsilon^{m+1}),$$

we cannot have at the same time $A_1 = 0$ and $U_1 = 0$. In the case $m \geqslant 2$, we shall privilege the comparison of ζ to the amplitude A_ε and then impose $A_1 = A_2 = \cdots = A_{m-1} = 0$, which in turn implies, via (30), a strong constraint on the expansion of U_ε at the initial time, both for $U_\varepsilon^{\rm in,1}$ and for $U_\varepsilon^{\rm in,\perp} = \nabla_{z_\perp}\partial_{z_1}^{-1}U_\varepsilon^{\rm in,1}$ (since U_ε is a gradient vector field). This is the reason why we shall present two results. The first one (Theorem 6 below) in one space dimension and where we want to compare the amplitude A_ε to ζ, which requires $A_1 = A_2 = \cdots = A_{m-1} = 0$, in particular at the initial time. The second one (Theorem 7 below) in space dimension $d \geqslant 2$, and where we compare the first component of the gradient vector field U_ε to ζ, which requires $U_1^1 = U_2^1 = \cdots = U_{m-1}^1 = 0$. Of course when $d = 1$, one could make a statement where we compare U_ε to ζ (with $U_1^1 = U_2^1 = \cdots = U_{m-1}^1 = 0$). However, in dimension $d \geqslant 2$, since U_ε is a gradient vector field, this imposes some constraints in the expansion in ϵ for U_ε^1 and $U_\varepsilon^\perp$, which prevents the comparison between A_ε and ζ (since we must have $A_1 = A_2 = \cdots = A_{m-1} = 0$), at least when $m \geqslant 2$. We may now state our main results for this section.

Theorem 6. *We assume $d = 1$. Let $\Lambda > 0$, $s, m \in \mathbb{N}$ such that $s \geqslant 2$ and (27) holds. We fix $\zeta^{\rm in} \in H^{s+5}(\mathbb{R})$ and denote $\zeta \in \mathcal{C}([0, \tau_*], H^{s+5}(\mathbb{R}))$ the solution to the* (gKdV) *equation*

$$\frac{2}{\mathfrak{c}_s}\partial_\tau\zeta + \Gamma^{(m)}\zeta^{m+1}\partial_z\zeta - \frac{1}{\mathfrak{c}_s^2}\partial_z^3\zeta = 0$$

for the initial datum $\zeta^{\rm in}$. We fix $A_m^{\rm in} \in H^{s+5}(\mathbb{R})$ and consider an initial datum $(A_\varepsilon^{\rm in}, U_\varepsilon^{\rm in} = \partial_z\phi_\varepsilon^{\rm in})$ for (41) satisfying

$$\|A_\varepsilon^{\rm in} - \zeta^{\rm in} - \epsilon^m A_m^{\rm in}\|_{H^s(\mathbb{R})} \leqslant \Lambda\varepsilon^2 = \Lambda\epsilon^{m+1},$$

and

$$\left\|U_\varepsilon^{\rm in} - \left\{A_\varepsilon^{\rm in} - \frac{3}{2}\epsilon\big[A_\varepsilon^{\rm in}\big]^2 + 2\epsilon^2\big[A_\varepsilon^{\rm in}\big]^3 - \frac{5}{2}\epsilon^3\big[A_\varepsilon^{\rm in}\big]^4 + \cdots + (-1)^m\frac{m+2}{2}\epsilon^m\big[A_\varepsilon^{\rm in}\big]^{m+1}\right\}\right\|_{H^s(\mathbb{R})} \leqslant \Lambda\varepsilon^2 = \Lambda\epsilon^{m+1}.$$

Then, there exist two (small) positive constants μ and $\varepsilon_0 > 0$, depending only on s, Λ and the functions $\zeta^{\rm in}$ and $A_m^{\rm in}$ such that (41) has a unique solution $(A_\varepsilon, U_\varepsilon) \in \mathcal{C}([0, \mu|\ln\epsilon|], H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}))$ if $0 < \varepsilon \leqslant \varepsilon_0$. Moreover, there exists a positive constant C, depending only on s, Λ and the functions $\zeta^{\rm in}$ and $A_m^{\rm in}$ such that, for $\theta \in [0, \mu|\ln\epsilon|]$, we have

$$\|A_\varepsilon(\theta) - \zeta(\epsilon^m\theta)\|_{H^s(\mathbb{R})} \leqslant C\big(\epsilon^m\|A_m^{\rm in}\|_{H^s(\mathbb{R})} + \epsilon^{m+1}\mathrm{e}^{\theta/(2\mu)}\big) \leqslant C\epsilon^m$$

and

$$\left\|U_\varepsilon(\theta) - \left\{A_\varepsilon - \frac{3}{2}\epsilon A_\varepsilon^2 + 2\epsilon^2 A_\varepsilon^3 - \frac{5}{2}\epsilon^3 A_\varepsilon^4 + \cdots + (-1)^m\frac{m+2}{2}\epsilon^m A_\varepsilon^{m+1}\right\}(\theta)\right\|_{H^{s-1}(\mathbb{R})} \leqslant C\epsilon^{m+1}\mathrm{e}^{\theta/(2\mu)} \leqslant C\epsilon^{m+\frac{1}{2}}.$$

We recall that the assumption $\|A_\varepsilon^{\text{in}} - \zeta^{\text{in}} - \epsilon^m A_m^{\text{in}}\|_{H^s(\mathbb{R})} \leqslant \Lambda\varepsilon^2 = \Lambda\epsilon^{m+1}$ corresponds to the hypothesis $A_1 = A_2 = \cdots = A_{m-1} = 0$ at $\theta = 0$. Our second result holds in arbitrary dimension $d \geqslant 1$. Since we privilege the vector field U_ε, we no longer compute U_ε^1 from A_ε by (30) but compute A_ε from U_ε^1 by (34).

Theorem 7. *We assume $d \geqslant 1$. Let $\Lambda > 0$, $s, m \in \mathbb{N}$ such that $s > 1 + \frac{d}{2}$ and (27) holds. We fix $\zeta^{\text{in}} \in H^{s+5}(\mathbb{R}^d)$ and assume moreover, if $d \geqslant 2$, that*

$$\zeta^{\text{in}} \in \partial_{z_1} H^{s+5}(\mathbb{R}^d) \quad \text{and} \quad \Delta_{z_\perp}\partial_{z_1}^{-1}\zeta^{\text{in}} \in \partial_{z_1} H^{s+2}(\mathbb{R}^d).$$

We then denote $\zeta \in \mathcal{C}([0, \tau_*], H^s(\mathbb{R}^d))$ the solution to the (gKdV)/(gKP-I) *equation*

$$\frac{2}{\mathfrak{c}_s}\partial_\tau\zeta + \Gamma^{(m)}\zeta^{m+1}\partial_{z_1}\zeta + \Delta_{z_\perp}\partial_{z_1}^{-1}\zeta - \frac{1}{\mathfrak{c}_s^2}\partial_{z_1}^3\zeta = 0$$

for the initial datum ζ^{in}. We consider an initial datum $(A_\varepsilon^{\text{in}}, U_\varepsilon^{\text{in}} = \nabla_z\phi_\varepsilon^{\text{in}})$ for (41) satisfying

$$U_\varepsilon^{\text{in}} = \nabla_z\partial_{z_1}^{-1}\zeta^{\text{in}} \quad \text{and} \quad \left\| A_\varepsilon^{\text{in}} - \sum_{k=1}^{m+1}\frac{1\cdot 3\cdot\cdots\cdot(2k-1)}{k!}\epsilon^{k-1}\big[U_\varepsilon^1\big]^k \right\|_{H^s(\mathbb{R}^d)} \leqslant \Lambda\epsilon^{m+1} = \Lambda\varepsilon^2.$$

Then, there exist two (small) positive constants μ and ε_0, depending only on s, d, Λ and the function ζ^{in} such that (41) has a unique solution $(A_\varepsilon, U_\varepsilon) \in \mathcal{C}([0, \mu|\ln\epsilon|], H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d))$. Moreover, there exists a positive constant C, depending only on s, d, Λ and the function ζ^{in} such that, for $\theta \in [0, \mu|\ln\epsilon|]$, we have

$$\big\|U_\varepsilon^1(\theta) - \zeta\big(\epsilon^m\theta\big)\big\|_{H^{s-1}(\mathbb{R}^d)} \leqslant C\epsilon^{m+1}\mathrm{e}^{\theta/(2\mu)} \leqslant C\epsilon^{m+\frac{1}{2}}, \qquad \big\|\varepsilon U_\varepsilon^\perp(\theta)\big\|_{H^{s-1}(\mathbb{R}^d)} \leqslant C\epsilon^{m+1}\mathrm{e}^{\theta/(2\mu)} \leqslant C\epsilon^{m+\frac{1}{2}},$$

and

$$\left\| A_\varepsilon(\theta) - \sum_{k=1}^{m+1}\frac{1\cdot 3\cdot\cdots\cdot(2k-1)}{k!}\epsilon^{k-1}\big[U_\varepsilon^1(\theta)\big]^k \right\|_{H^s(\mathbb{R}^d)} \leqslant C\epsilon^{m+1}\mathrm{e}^{\theta/(2\mu)} \leqslant C\epsilon^{m+\frac{1}{2}}.$$

As an illustration for these two theorems, take $m = 1$, $U^{\text{in}} = \nabla_z\partial_{z_1}^{-1}\zeta^{\text{in}}$, and $A^{\text{in}} = \zeta^{\text{in}} + \frac{3}{2}[\zeta^{\text{in}}]^2$. Then, we prove that $A_\varepsilon(\theta)$ and $U_\varepsilon^1(\theta)$ are equal to $\zeta(\varepsilon\theta) + \mathcal{O}(\varepsilon)$ uniformly for $0 \leqslant \theta \leqslant \mu|\ln\varepsilon|$, whereas ζ has moved from its initial condition about $\varepsilon|\ln\varepsilon| \gg \varepsilon$. Theorems 6 and 7 then provide a justification of the (gKdV)/(gKP-I) limit on the time scale $t \lesssim (\varepsilon\epsilon)^{-1}|\ln\epsilon| \approx \varepsilon^{-1-\frac{2}{m+1}}|\ln\varepsilon|$, which is much smaller than the expected one $t \lesssim \varepsilon^{-3}$ (recall $m \geqslant 1$), but much larger than the natural one for the free wave regime $t \ll (\varepsilon\epsilon)^{-1}$ with both left and right-going waves (*cf.* Theorem 5 due to [10]), or the time scale $t \approx \varepsilon^{-2}$ which is suitable for a right-going wave approximated by the Burgers equation (see Proposition 4).

In [22], T. Colin and D. Lannes justify the Davey–Stewartson approximation for WKB initial data in hyperbolic systems. Their situation bears some common feature with our one: the transport equation (analogous to the free wave equation for us) governs formally the dynamics on the time scale say $t \simeq 1/\delta$, and the diffractive (formal) approximation holds on the time scale $t \simeq 1/\delta^2$, where δ is some samall parameter. However, the rigorous justification of the Davey–Stewartson approximation in [22] is for times $t \lesssim |\ln\delta|/\delta$, which is here again much smaller than the diffractive scale $t \simeq 1/\delta^2$, but much larger than the transport scale $t \ll \delta^{-1}$. However, in [22], this is the occurrence of resonances which prevent the approximation to hold up to times of order $t \simeq 1/\delta^2$, whereas in our situation, this is the occurrence of nonlinear singular terms. It is then not completely clear on which time scale the (gKdV)/(gKP-I) approximation is valid. We shall study this problem numerically in some forthcoming work.

Similarly to the (KdV)/(KP-I) limit, we may wonder what is known for the (gKdV)/(gKP-I) asymptotic limit for the travelling waves. Concerning the one dimensional problem, we refer to [16], where the ODE argument still works for the (gKdV) limit as soon as the (gKdV) equation has solitary waves, that is the nonlinearity is even, or focusing and odd. In particular, when $\Gamma = 0 > \Gamma'$, this gives rise to two branches of solutions in the transonic limit. In higher dimension, note that the (gKP-I) equation which is not (KP-I) (that is with nonlinearity which is not quadratic) has travelling wave only if $d = 2$ and the nonlinearity is either cubic focusing or quartic (see [24]). In [21], we have investigated numerically the existence and properties of the travelling waves for (NLS) in dimension two. In the focusing case $\Gamma = 0 > \Gamma'$ for (mKP-I), we have also obtained, as in [16], two branches of solutions in the transonic limit. So far, we do not know any mathematical result concerning this convergence to (mKP-I) for the travelling waves.

The main ingredient in the proofs for the above results is to use the trick of E. Grenier [31]. The idea is to write the wave function Ψ solution to (NLS) under the form

$$\Psi = a\exp(i\varphi),$$

where φ is real-valued but a is *complex-valued*, which is a modified Madelung transform where amplitude and phase are no longer the true ones. Then, we do not split (NLS) separating real and imaginary parts, which would lead to the first system in (1), but decide instead to solve

$$\begin{cases} \partial_t a + 2\nabla\phi\cdot\nabla a + a\Delta\varphi = i\Delta a \\ \partial_t\varphi + |\nabla\varphi|^2 + f(|a|^2) = 0. \end{cases}$$

The point is that *if* (a,φ) solves this system, *then* $\Psi = a\exp(i\varphi)$ solves (NLS). The advantage of this system is that it is a symmetrizable hyperbolic system (if $f' > 0$, which will be the case here) with a skew-adjoint, constant coefficient, perturbation for which existence or comparison results can be easily derived.

1.6. Derivation of the (mKdV)/(mKP-I) equation from the Landau–Lifshitz model

In the Landau–Lifshitz model for planar ferromagnets in the case of an easy-plane anisotropy, the spin density $\mathfrak{m} = \mathfrak{m}(t,x) = (\mathfrak{m}_1,\mathfrak{m}_2,\mathfrak{m}_3) \in \mathbb{S}^2$, $t\in\mathbb{R}_+$, $x\in\mathbb{R}^d$, obeys (see [39,37,38,45]) the equation

$$\frac{\partial\mathfrak{m}}{\partial t} = \mathfrak{m}\times(\Delta\mathfrak{m} - \mathfrak{m}_3\vec{e}_3), \quad \vec{e}_3 \equiv (0,0,1). \tag{LL}$$

The physical dimensions are $d = 1$, 2 or 3. The Landau–Lifshitz equation (LL) formally conserves the energy

$$\int_{\mathbb{R}^d} |\nabla\mathfrak{m}|^2 + \mathfrak{m}_3^2\,dx.$$

Concerning the local well-posedness of (LL), we shall use the following result.

Theorem 8. *Let $s\in\mathbb{N}$ with $s > +\frac{d}{2}$. If $\mathfrak{m}^{\mathrm{in}} \in \mathcal{C}(\mathbb{R}^d,\mathbb{S}^2)$ verifies $\nabla\mathfrak{m}^{\mathrm{in}} \in H^s(\mathbb{R}^d,(\mathbb{R}^d)^3)$, then there exists $t_* = \frac{1}{C(s,d)\|\nabla\mathfrak{m}^{\mathrm{in}}\|_{H^s}} > 0$ such that* (LL) *has a unique solution $\mathfrak{m}\in L^\infty([0,t_*],\mathbb{S}^2)$ with $\nabla\mathfrak{m}\in L^\infty([0,t_*],H^s((\mathbb{R}^d)^3))$.*

The proof of Theorem 8 is omitted, since it follows from the arguments in [50] (the extra term $\mathfrak{m}_3\vec{e}_3$ is harmless), or in [26], where the heat flow into the manifold $\mathbb{S}^2$ is used, which would lead for (LL) to the parabolic regularization:

$$\frac{\partial\mathfrak{m}^\nu}{\partial t} = \nu\big(\Delta\mathfrak{m}^\nu + \big|\nabla\mathfrak{m}^\nu\big|^2\mathfrak{m}^\nu - \mathfrak{m}_3^\nu\vec{e}_3\big) + \mathfrak{m}^\nu\times\big(\Delta\mathfrak{m}^\nu - \mathfrak{m}_3^\nu\vec{e}_3\big),$$

and then letting $\nu\to 0$.

The Eq. (LL) may be recast as a nonlinear Schrödinger type equation by using the stereographic projection

$$\Psi \equiv \frac{\mathfrak{m}_1 + i\mathfrak{m}_2}{1+\mathfrak{m}_3},$$

which is valid for $\mathfrak{m}_3 \neq -1$. This transforms (LL) into the nonlinear Schrödinger type equation

$$i\frac{\partial\Psi}{\partial t} + \Delta\Psi + \frac{1-|\Psi|^2}{1+|\Psi|^2}\Psi = \frac{2\bar{\Psi}}{1+|\Psi|^2}\left(\sum_{j=1}^d(\partial_j\Psi)^2\right), \tag{47}$$

which also possesses the gauge invariance, but which is quasilinear and not semilinear as (NLS). We may also find the hydrodynamical form by using the Madelung transform $\Psi = A\mathrm{e}^{i\varphi}$, provided Ψ does not vanish, which yields

$$\begin{cases} \partial_t A + 2\dfrac{1-A^2}{1+A^2}(\nabla\varphi)\cdot\nabla A + A\Delta\varphi = 0 \\ \partial_t\varphi + \dfrac{1-A^2}{1+A^2}|\nabla\varphi|^2 + \dfrac{A^2-1}{A^2+1} - \dfrac{\Delta A}{A} + \dfrac{2|\nabla A|^2}{1+A^2} = 0 \end{cases} \tag{48}$$

or, in variables ($\rho \equiv A^2$, $U \equiv \nabla_x \varphi$),

$$\begin{cases} \partial_t \rho + 2\dfrac{1-\rho}{1+\rho} U \cdot \nabla \rho + 2\rho \nabla \cdot U = 0 \\ \partial_t U + \nabla\left(\dfrac{1-\rho}{1+\rho}|U|^2\right) + \nabla\left(\dfrac{\rho-1}{\rho+1}\right) - \nabla\left(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) + \nabla\left(\dfrac{|\nabla\rho|^2}{2\rho(1+\rho)}\right) = 0. \end{cases} \tag{49}$$

Notice that here, the speed of sound is equal to $2\frac{d}{d\rho}(\frac{\rho-1}{\rho+1})_{|\rho=1} = 1$, and that the associated Euler type system (in the long wave regime) is different from the usual one. The result below will no longer rely on the trick of E. Grenier, where we allow the amplitude to be complex-valued, thus we shall work with the true hydrodynamical variables ($\rho = A^2, U = \nabla_x \varphi$). In order to put forward the (mKdV)/(mKP-I) limit, we follow [45] (although this work was related to the question of travelling waves), and make the long wave ansatz

$$\Psi(t,x) = \sqrt{1+\varepsilon A_\varepsilon(\tau,z)}\exp\big(i\phi_\varepsilon(\tau,z)\big) \quad \tau = \varepsilon^3 t,\ z_1 \equiv \varepsilon(x_1 - t),\ z_\perp \equiv \varepsilon^2 x_\perp, \tag{50}$$

which is actually similar to the one used in Section 1.4 when $\Gamma = 0$. We plug (50) in (47) and deduce as above the system

$$\begin{cases} \partial_\tau A_\varepsilon - \dfrac{1}{\varepsilon^2}\partial_{z_1} A_\varepsilon - \dfrac{2A_\varepsilon}{2+\varepsilon A_\varepsilon}\big[\partial_{z_1}\phi_\varepsilon \partial_{z_1} A_\varepsilon + \varepsilon^2 \nabla_{z_\perp}\phi_\varepsilon \cdot \nabla_{z_\perp} A_\varepsilon\big] + \dfrac{2}{\varepsilon^2}(1+\varepsilon A_\varepsilon)\big(\partial_{z_1}^2\phi_\varepsilon + \varepsilon^2\Delta_{z_\perp}\phi_\varepsilon\big) = 0 \\ \partial_\tau \phi_\varepsilon - \dfrac{1}{\varepsilon^2}\partial_{z_1}\phi_\varepsilon - \dfrac{A_\varepsilon}{2+\varepsilon A_\varepsilon}\big[(\partial_{z_1}\phi_\varepsilon)^2 + \varepsilon^2|\nabla_{z_\perp}\phi_\varepsilon|^2\big] + \dfrac{1}{\varepsilon^2}\cdot\dfrac{A_\varepsilon}{2+\varepsilon A_\varepsilon} \\ \quad - \dfrac{\partial_{z_1}^2\sqrt{1+\varepsilon A_\varepsilon} + \varepsilon^2\Delta_{z_\perp}\sqrt{1+\varepsilon A_\varepsilon}}{\varepsilon\sqrt{1+\varepsilon A_\varepsilon}} + \dfrac{\varepsilon}{2(1+\varepsilon A_\varepsilon)(2+\varepsilon A_\varepsilon)}\big[(\partial_{z_1}A_\varepsilon)^2 + \varepsilon^2|\nabla_{z_\perp}A_\varepsilon|^2\big] = 0. \end{cases} \tag{51}$$

The singular terms in ε^{-2} are

$$-\frac{1}{\varepsilon^2}\partial_{z_1}A_\varepsilon + \frac{2}{\varepsilon^2}\partial_{z_1}^2\phi_\varepsilon \quad \text{and} \quad -\frac{1}{\varepsilon^2},\partial_{z_1}\phi_\varepsilon + \frac{1}{2\varepsilon^2}A_\varepsilon,$$

which gives as before the constraint $A = 2\partial_{z_1}\phi$ (comparing to the case of the (NLS) equation, since $\sqrt{1+\varepsilon A_\varepsilon} = 1 + \varepsilon A_\varepsilon/2 + \mathcal{O}(\varepsilon^2)$, there is an extra factor 2 to the formula "$\mathfrak{c}_s A = \partial_{z_1}\phi$"). As in subsection 1.4, the formally singular terms in (51) reduce, for both equations, to the single preparedness condition:

$$2\partial_{z_1}\phi_\varepsilon - A_\varepsilon = -\frac{\varepsilon}{2}A_\varepsilon^2 + \mathcal{O}\big(\varepsilon^2\big) \quad \text{or} \quad A_\varepsilon = 2\partial_{z_1}\phi_\varepsilon + 2\varepsilon(\partial_{z_1}\phi_\varepsilon)^2 + \mathcal{O}\big(\varepsilon^2\big). \tag{52}$$

Noticing that $\frac{\alpha}{2+\alpha} = \frac{\alpha}{2} - \frac{\alpha^2}{4} + \frac{\alpha^3}{8} + \mathcal{O}_{\alpha\to 0}(\alpha^4)$, we add here again the first equation of (51) to $2\partial_{z_1}$ times the second one and get

$$\begin{aligned} &\partial_\tau(A_\varepsilon + 2\partial_{z_1}\phi_\varepsilon) - 2\partial_{z_1}\left(\frac{\partial_{z_1}^2\sqrt{1+\varepsilon A_\varepsilon} + \varepsilon^2\Delta_{z_\perp}\sqrt{1+\varepsilon A_\varepsilon}}{\varepsilon\sqrt{1+\varepsilon A_\varepsilon}}\right) + 2(1+\varepsilon A_\varepsilon)\Delta_{z_\perp}\phi_\varepsilon \\ &\quad - \partial_{z_1}\big(\big(A_\varepsilon + F_1(\varepsilon A_\varepsilon)\big)(\partial_{z_1}\phi_\varepsilon)^2\big) - \big(A_\varepsilon + F_2(\varepsilon A_\varepsilon)\big)\partial_{z_1}\phi_\varepsilon\partial_{z_1}A_\varepsilon \\ &\quad + \frac{1}{\varepsilon}\big\{2A_\varepsilon\partial_{z_1}^2\phi_\varepsilon - A_\varepsilon\partial_{z_1}A_\varepsilon\big\} + \frac{3}{4}A_\varepsilon^2\partial_{z_1}A_\varepsilon \\ &= \frac{2\varepsilon^2 A_\varepsilon}{2+\varepsilon A_\varepsilon}\nabla_{z_\perp}\phi_\varepsilon \cdot \nabla_{z_\perp}A_\varepsilon - \frac{1}{\varepsilon^3}\partial_{z_1}\big[f_4(\varepsilon A_\varepsilon)\big] \\ &\quad - \varepsilon\partial_{z_1}\left\{\frac{(\partial_{z_1}A_\varepsilon)^2 + \varepsilon^2|\nabla_{z_\perp}A_\varepsilon|^2}{(1+\varepsilon A_\varepsilon)(2+\varepsilon A_\varepsilon)}\right\} + \varepsilon^2\partial_{z_1}\left\{\frac{A_\varepsilon}{2+\varepsilon A_\varepsilon}|\nabla_{z_\perp}\phi_\varepsilon|^2\right\}. \end{aligned} \tag{53}$$

Here, we have $f_4(\alpha) = \mathcal{O}(\alpha^4)$ and $F_1(\alpha)$, $F_2(\alpha) = \mathcal{O}(\alpha)$ as $\alpha \to 0$. As in the previous subsection, in the second line of (53), the formally singular term $\{2A_\varepsilon\partial_{z_1}^2\phi_\varepsilon - A_\varepsilon\partial_{z_1}A_\varepsilon\}/\varepsilon$ becomes, in view of (52),

$$-A_\varepsilon^2\partial_{z_1}A_\varepsilon,$$

hence (53) implies, on the formal level, if $A_\varepsilon \to A$ and $\phi_\varepsilon \to \phi$, with $A = \partial_{z_1}\phi$, the convergence to the (mKdV)/(mKP-I) focusing equation

$$2\partial_\tau A - \partial_{z_1}^3 A - \frac{3}{2}A^2\partial_{z_1} A + \Delta_{z_\perp}\partial_{z_1}^{-1} A = 0.$$

For a slightly different model, where the Maxwell equation is taken into account, H. Leblond in [43] also derives (formally) an asymptotic regime given by the (mKP) equation. In the work [30] by P. Germain and F. Rousset, the (KdV)/(KP-I) asymptotic regime is studied starting from the Schrödinger map problem into a manifold in a general geometrical framework, which includes the (LL) equation as a particular case. Their result proves the convergence to a geometrical (KdV)/(KP-I) equation in a scaling comparable to (2) and includes as a particular case the (NLS) equation, that is the results presented in Section 1.1. It turns out that for (LL), this would lead to the linear Airy equation (for the phase φ such that $\mathfrak{m} = \mathrm{e}^{i\varphi} \in \mathbb{S}^1 \subset \mathbb{S}^2$) on the time scale $\tau \approx 1$. The method of proof is different since the target is a general manifold, whereas our analysis of (LL) relies on the stereographic projection.

Concerning (LL), we shall prove the following justification of the (mKdV)/(mKP-I) asymptotic regime. We give a statement close to the one in Theorem 7, but here again, in dimension $d = 1$, one could write down the result where we compare the amplitude A_ε to ζ, allowing an expansion of $A_\varepsilon^{\mathrm{in}}$ up to $\mathcal{O}(\varepsilon^2)$, similar to Theorem 6. Note that we work here in the variables $\theta = \varepsilon^2 t$ and $z = (z_1, z_\perp) = (\varepsilon x_1, \varepsilon^2 x_\perp)$, so that (51) with $U_\varepsilon \equiv \nabla_z \phi_\varepsilon$ is changed for

$$\begin{cases} \partial_\theta A_\varepsilon - \dfrac{1}{\varepsilon}\partial_{z_1} A_\varepsilon - \dfrac{2\varepsilon A_\varepsilon}{2+\varepsilon A_\varepsilon}\big[U_\varepsilon^1 \partial_{z_1} A_\varepsilon + \varepsilon^2 U_\varepsilon^\perp \cdot \nabla_{z_\perp} A_\varepsilon\big] + \dfrac{2}{\varepsilon}(1+\varepsilon A_\varepsilon)\big(\partial_{z_1} U_\varepsilon^1 + \varepsilon^2 \nabla_{z_\perp}\cdot U_\varepsilon^\perp\big) = 0 \\ \partial_\theta U_\varepsilon - \dfrac{1}{\varepsilon}\partial_{z_1} U_\varepsilon - \nabla_z\Big(\dfrac{\varepsilon A_\varepsilon}{2+\varepsilon A_\varepsilon}\big[\big[U_\varepsilon^1\big]^2 + \varepsilon^2\big|U_\varepsilon^\perp\big|^2\big]\Big) + \dfrac{1}{\varepsilon}\nabla_z\Big(\dfrac{A_\varepsilon}{2+\varepsilon A_\varepsilon}\Big) \\ \quad - \nabla_z\Big(\dfrac{\partial_{z_1}^2\sqrt{1+\varepsilon A_\varepsilon} + \varepsilon^2\Delta_{z_\perp}\sqrt{1+\varepsilon A_\varepsilon}}{\sqrt{1+\varepsilon A_\varepsilon}}\Big) + \varepsilon^2\nabla_z\Big(\dfrac{(\partial_{z_1}A_\varepsilon)^2 + \varepsilon^2|\nabla_{z_\perp}A_\varepsilon|^2}{(1+\varepsilon A_\varepsilon)(2+\varepsilon A_\varepsilon)}\Big) = 0. \end{cases} \tag{54}$$

Theorem 9. *Let $\Lambda > 0$ and $s \in \mathbb{N}$ be such that $s > 1 + \frac{d}{2}$. We fix $\zeta^{\mathrm{in}} \in H^{s+6}(\mathbb{R}^d)$ and assume moreover, if $d \geqslant 2$, that*

$$\zeta^{\mathrm{in}} \in \partial_{z_1} H^{s+6}\big(\mathbb{R}^d\big) \quad \text{and} \quad \Delta_{z_\perp}\partial_{z_1}^{-1}\zeta^{\mathrm{in}} \in \partial_{z_1} H^{s+3}\big(\mathbb{R}^d\big).$$

We then denote $\zeta \in \mathcal{C}([0,\tau_*], H^{s+6}(\mathbb{R}^d))$ the solution to the (mKdV)/(mKP-I) *equation*

$$2\partial_\tau \zeta - \partial_{z_1}^3\zeta - \frac{3}{2}\zeta^2\partial_{z_1}\zeta + \Delta_{z_\perp}\partial_{z_1}^{-1}\zeta = 0$$

for the initial datum ζ^{in}. We consider an initial datum $(A_\varepsilon^{\mathrm{in}}, \nabla_z\phi^{\mathrm{in}})$ for (54) such that

$$\nabla_z\phi^{\mathrm{in}} = \frac{1}{2}\nabla_z\partial_{z_1}^{-1}\zeta^{\mathrm{in}} \quad \text{and} \quad \Big\|A_\varepsilon^{\mathrm{in}} - \zeta^{\mathrm{in}} - \frac{\varepsilon}{2}\big[\zeta^{\mathrm{in}}\big]^2\Big\|_{H^s(\mathbb{R}^d)} + \varepsilon\Big\|A_\varepsilon^{\mathrm{in}} - \zeta^{\mathrm{in}} - \frac{\varepsilon}{2}\big[\zeta^{\mathrm{in}}\big]^2\Big\|_{H^{s+1}(\mathbb{R}^d)} \leqslant \Lambda\varepsilon^2.$$

Then, there exists two (small) positive constants ε_0 and μ, depending only on s, d, Λ and the function ζ^{in} such that, if $0 < \varepsilon < \varepsilon_0$, (54) has a unique solution $(A_\varepsilon, U_\varepsilon) \in \mathcal{C}([0, \mu|\ln\varepsilon|], H^s(\mathbb{R}^d) \times H^{s+1}(\mathbb{R}^d))$. Moreover, there exists a positive constant C, depending only on s, d, Λ and the function ζ^{in} such that, for $\theta \in [0, \mu|\ln\varepsilon|]$, we have

$$\big\|2U_\varepsilon^1(\theta) - \zeta(\varepsilon\theta)\big\|_{H^s(\mathbb{R}^d)} + \big\|2\varepsilon U_\varepsilon^\perp(\theta) - \varepsilon\nabla_{z_\perp}\zeta(\varepsilon\theta)\big\|_{H^s(\mathbb{R}^d)} \leqslant C\varepsilon^2\mathrm{e}^{\theta/(2\mu)} \leqslant C\varepsilon^{\frac{3}{2}}$$

and

$$\Big\|A_\varepsilon(\theta) - \zeta(\varepsilon\theta) - \frac{\varepsilon}{2}\zeta^2(\varepsilon\theta)\Big\|_{H^s(\mathbb{R}^d)} + \varepsilon\Big\|A_\varepsilon(\theta) - \zeta(\varepsilon\theta) - \frac{\varepsilon}{2}\zeta^2(\varepsilon\theta)\Big\|_{H^{s+1}(\mathbb{R}^d)} \leqslant C\varepsilon^2\mathrm{e}^{\theta/(2\mu)} \leqslant C\varepsilon^{\frac{3}{2}}$$

so that in particular

$$\big\|A_\varepsilon(\theta) - \zeta(\varepsilon\theta)\big\|_{H^s(\mathbb{R}^d)} \leqslant C\varepsilon.$$

In connection with this result, an analogous convergence from (LL) to (mKdV)/(mKP-I) holds for the travelling waves. For the one dimensional case, this follows from explicit integration (see [44,25]): for $0 \leqslant c < 1$, the only travelling wave $\mathfrak{m}(t,x) = \mathfrak{m}_c(x - ct)$ to (LL) is given by

$$\mathfrak{m}_c(x) = \left(\frac{c}{\cosh(x\sqrt{1-c^2})}, \tanh\left(x\sqrt{1-c^2}\right), \pm\frac{\sqrt{1-c^2}}{\cosh(x\sqrt{1-c^2})}\right),$$

up to the natural symmetries of the problem: rotation around the x_3 axis and translation. From this explicit formula we have, for instance, with $\varepsilon = \sqrt{1-c^2}$, and U_c given by the stereographic projection

$$U_c = \frac{\mathfrak{m}_{c,1} + i\mathfrak{m}_{c,2}}{1+\mathfrak{m}_{c,3}},$$

the relation (recall $z = \varepsilon x$)

$$|U_c|^2(x) - 1 = \frac{1-\mathfrak{m}_{c,3}(x)}{1+\mathfrak{m}_{c,3}(x)} - 1 = \mp 2\frac{\frac{\sqrt{1-c^2}}{\cosh(x\sqrt{1-c^2})}}{1+\frac{\sqrt{1-c^2}}{\cosh(x\sqrt{1-c^2})}} = \mp 2\frac{\frac{\varepsilon}{\cosh(\varepsilon x)}}{1+\frac{\varepsilon}{\cosh(\varepsilon x)}} = \mp 2\frac{\frac{\varepsilon}{\cosh(z)}}{1+\frac{\varepsilon}{\cosh(z)}}.$$

This shows clearly that

$$\varepsilon A_\varepsilon(z) = |U_c|^2(x) - 1 = \frac{\pm 2\varepsilon}{\cosh(z)} + \mathcal{O}\left(\varepsilon^2\right),$$

where $\frac{\pm 2}{\cosh(z)}$ is the (mKdV) solitary wave (of speed $-1/2$). In the two dimensional situation, the numerical simulations and formal computations in [45], similar to those above, suggest the convergence to the (mKP-I) ground state in the transonic limit.

Concerning the associated wave regime, where we remove the space translation and work on the shorter time scale $t \approx \varepsilon^{-2}$, let us quote two papers. The first one is due to J. Shatah and C. Zeng [49], where the strong convergence to the wave map equation

$$\partial_{\mathrm{t}}^2\mathfrak{m} = \Delta_z\mathfrak{m} + |\nabla_z\mathfrak{m}|^2\mathfrak{m}, \tag{55}$$

with $\mathfrak{m} \in \mathbb{S}^1 \subset \mathbb{S}^2$ the equator, is shown. Actually, a more general result is proven, which corresponds for (LL) to the particular case of the target manifold $\mathbb{S}^2$ and $B_k = 0$ for all $1 \leqslant k \leqslant d$. Of course, once we have lifted the $\mathbb{S}^1$-valued map $\mathfrak{m} = \mathrm{e}^{i\varphi}$, the wave map equation (55) reduces to the free wave equation

$$\partial_{\mathrm{t}}^2\varphi = \Delta_z\varphi.$$

The result of [49] is proved for the time scale $t = \varepsilon^{-1}\mathrm{t} \approx \varepsilon^{-1}$, *i.e.* t of order one. Comparing with the result in [10], where the convergence is proved for $t \ll \varepsilon^{-2}$, that is $\mathrm{t} \ll \varepsilon^{-1}$, this is a smaller time scale, and this is in particular due to the fact that when $B_k = B_k(\mathfrak{m})$ is nonzero, the term $B_k(\mathfrak{m})\mathfrak{m}$ in Eq. (SM) in [49] prevents in general from having existence of smooth solutions for large times.[2] On the other hand, A. Capella, C. Melcher and F. Otto in [15] provide a weak convergence result to a wave map type equation (see [15] for a precise statement) for a model similar to (LL) (but also including dissipation and the stray-field coming from Maxwell equations). Their result also holds on the time scale $t = \varepsilon^{-1}\mathrm{t} \approx \varepsilon^{-1}$, for weak convergences and locally in space. Finally, the results in [49] and [15] do not provide error bounds. Our last result is about the free wave regime associated with (LL). In order to state it, we have to work in the variables $(\mathrm{t}, z) = (\varepsilon t, \varepsilon x)$, and write the solution Ψ of (47) under the form given by

$$\Psi(t,x) = \sqrt{1+\varepsilon A_\varepsilon(\mathrm{t},z)}\exp\left(i\phi_\varepsilon(\mathrm{t},z)\right), \quad \mathrm{t} = \varepsilon t, \ z = \varepsilon x,$$

so that, denoting $U_\varepsilon \equiv \nabla_z\phi_\varepsilon$, (47) becomes

[2] It seems that there is a small mistake in the statement of the Theorem (convergence) in [49, p. 302]. Indeed, from the formulas on p. 310, it is not always true that "$G''(p_*)\zeta = 0$" at the initial time, therefore at $t = 0$, we do not have "$\partial_t p_*(0) = \iota B_k\partial_k u(0)$" (which would mean for (55) $\partial_{\mathrm{t}}\mathfrak{m} = 0$ at $t = 0$) but $\partial_t p_*(0) = \iota B_k\partial_k u(0) + \iota\lim_{\epsilon\to 0}[\epsilon^{-1}G(u_\epsilon(0))]$. Furthermore, it is not clear that the convergences in [49] are strong in $H^\ell(\mathbb{R}^d)$ since they follow from a compactness argument.

$$\begin{cases} \partial_t A_\varepsilon + 2\nabla_z \cdot U_\varepsilon = \dfrac{2\varepsilon^2 A_\varepsilon}{2+\varepsilon A_\varepsilon} U_\varepsilon \cdot \nabla_z A_\varepsilon - 2\varepsilon A_\varepsilon \nabla_z \cdot U_\varepsilon \\ \partial_t U_\varepsilon + \dfrac{1}{2}\nabla_z A_\varepsilon = -\varepsilon^2 \nabla_z \left(\dfrac{A_\varepsilon}{2+\varepsilon A_\varepsilon} |U_\varepsilon|^2 \right) + \varepsilon^3 \nabla_z \left(\dfrac{|\nabla_z A_\varepsilon|^2}{2(1+\varepsilon A_\varepsilon)(2+\varepsilon A_\varepsilon)} \right) \\ \qquad\qquad + \varepsilon \nabla_z \left(\dfrac{A_\varepsilon^2}{2(2+\varepsilon A_\varepsilon)} \right) + \varepsilon \nabla_z \left(\dfrac{\Delta_z \sqrt{1+\varepsilon A_\varepsilon}}{\sqrt{1+\varepsilon A_\varepsilon}} \right). \end{cases} \tag{56}$$

Theorem 10. *Let $\Lambda > 0$ and $s \in \mathbb{N}$ be such that $s > 5 + \frac{d}{2}$. We consider an initial datum for (56) of the type $(A_\varepsilon^{\mathrm{in}}, \nabla_z \phi_\varepsilon^{\mathrm{in}}) \in H^{s+1}(\mathbb{R}^d) \times H^s(\mathbb{R}^d, \mathbb{R}^d)$, with*

$$\left\| A_\varepsilon^{\mathrm{in}} \right\|_{H^s(\mathbb{R}^d)} + \varepsilon \left\| A_\varepsilon^{\mathrm{in}} \right\|_{H^{s+1}(\mathbb{R}^d)} + \left\| \nabla_z \phi_\varepsilon^{\mathrm{in}} \right\|_{H^s(\mathbb{R}^d)} \leqslant \Lambda.$$

Then, there exists a positive constant $K_0 = K_0(s,d)$ such that if $K_0 \varepsilon \Lambda \leqslant 1$, then (56) has a unique solution $(A_\varepsilon, U_\varepsilon) \in \mathcal{C}([0, 1/(K_0\varepsilon\Lambda)], H^{s+1}(\mathbb{R}^d) \times H^s(\mathbb{R}^d, \mathbb{R}^d))$ with initial datum $(A_\varepsilon^{\mathrm{in}}, \nabla_z \phi_\varepsilon^{\mathrm{in}})$, and it verifies

$$\sup_{0 \leqslant \mathrm{t} \leqslant 1/(K_0\varepsilon\Lambda)} \left[\left\| A_\varepsilon(\mathrm{t}) \right\|_{H^s(\mathbb{R}^d)} + \varepsilon \left\| A_\varepsilon(\mathrm{t}) \right\|_{H^{s+1}(\mathbb{R}^d)} + \left\| \nabla_z \phi_\varepsilon(\mathrm{t}) \right\|_{H^s(\mathbb{R}^d)} \right] \leqslant K_0 \Lambda \tag{57}$$

and, for $0 \leqslant \mathrm{t} \leqslant 1/(K_0\varepsilon\Lambda)$, $x \in \mathbb{R}^d$,

$$\frac{1}{2} \leqslant \rho(\mathrm{t}, x) = 1 + \varepsilon A_\varepsilon(\mathrm{t}, x) \leqslant 2.$$

Furthermore, if $(\mathfrak{a}_\varepsilon, \mathfrak{u}_\varepsilon)$ denotes the solution to the free wave equation

$$\begin{cases} \partial_t \mathfrak{a}_\varepsilon + 2\nabla_z \cdot \mathfrak{u}_\varepsilon = 0 \\ \partial_t \mathfrak{u}_\varepsilon + \dfrac{1}{2} \nabla_z \mathfrak{a}_\varepsilon = 0 \end{cases} \tag{58}$$

with intial datum $(A_\varepsilon^{\mathrm{in}}, \nabla_z \phi_\varepsilon^{\mathrm{in}})$, then, for $0 \leqslant \mathrm{t} \leqslant 1/(K_0\varepsilon\Lambda)$, there holds

$$\left\| (A_\varepsilon, U_\varepsilon)(\mathrm{t}) - (\mathfrak{a}_\varepsilon, \mathfrak{u}_\varepsilon)(\mathrm{t}) \right\|_{H^{s-2}(\mathbb{R}^d) \times H^{s-2}(\mathbb{R}^d)} \leqslant K_0 \varepsilon \mathrm{t} \left(\Lambda + \Lambda^2 \right).$$

We emphasize that [49] prove uniform Sobolev bounds in this regime for t of order one, whereas here, we obtain these uniform bounds for the much larger time scale $\mathrm{t} \leqslant 1/(K_0\varepsilon\Lambda)$. Moreover, we prove a comparison result with strong convergences. The main ingredient in the proof of Theorems 9 and 10 is to use an extended formulation and an augmented system as for the analysis in [8] of the Cauchy problem for the Euler–Korteweg system. This approach was also used in [10] for the free wave regime. An alternative to the well-posedness result in Theorem 8 would be to rely on this extended formulation as in [8]. In comparison with the results for (NLS) that we prove using the trick of E. Grenier ([31]), for the latter approach, the formulation (49) is more appropriate. We mention that one could use the extended formulation for the analysis of (NLS), for instance for the (gKdV)/(gKP-I) limit (Theorem 6), but we have privileged the approach of E. Grenier in view of the simplicity of the structure of hyperbolic symmetrizable system perturbed by a skew-adjoint, constant coefficient, perturbation. The differences in the statements for both approaches only rely on the loss of derivatives for the uniform Sobolev bounds. On the other hand, it is plausible that one may improve the uniform Sobolev bounds (57) to larger time scales, using the dispersive properties of the equation, as it is done in [10]. We have not tackled this question here. Finally, let us mention that since we are in a situation analogous to the case $\Gamma = 0$ for (NLS), the result associated to what we prove in Proposition 4 would be here simply a comparison of A_ε and U_ε to the solution of the trivial "Burgers" equation $\partial_\theta \mathfrak{a} = 0$.

2. Properties of the Boussinesq system and comparison result

We shall use the fact that for $s > d/2$, $H^s(\mathbb{R}^d)$ is an algebra, and that

$$\|fg\|_{H^s(\mathbb{R}^d)} \leqslant C_1 \|f\|_{H^s(\mathbb{R}^d)} \|g\|_{H^s(\mathbb{R}^d)}.$$

Moreover, we have the tame estimates (see, *e.g.*, [51])

$$\left\| \partial_z^\alpha (fg) - f \partial_z^\alpha g \right\|_{L^2(\mathbb{R}^d)} \leqslant C_k \left(\|f\|_{H^k} \|g\|_{L^\infty(\mathbb{R}^d)} + \|\nabla_z f\|_{L^\infty(\mathbb{R}^d)} \|g\|_{H^{k-1}(\mathbb{R}^d)} \right), \quad |\alpha| \leqslant k. \tag{59}$$

2.1. Proof of Proposition 1: local well-posedness of the Boussinesq system ($\mathcal{B}_\varepsilon$)

The proof of Proposition 1 is very close to the proof of Theorem 4 in [20], and thus will be only sketched. We set $\mathcal{Y} = (\mathcal{Y}^0, \mathcal{Y}^1, \mathcal{Y}^\perp)^t \equiv (A_\varepsilon, U^1_\varepsilon, \varepsilon U^\perp_\varepsilon)^t \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-1} = \mathbb{R}^{1+d}$, $\nabla^\varepsilon = (\partial_{z_1}, \varepsilon\nabla_{z_\perp})$, and write the system ($\mathcal{B}_\varepsilon$) under the abstract form:

$$\frac{1}{\mathfrak{c}_s}\partial_\tau \mathcal{Y} + \frac{1}{\varepsilon^2} H\big(\varepsilon^2\mathcal{Y}, \nabla^\varepsilon\big)\mathcal{Y} = L\big(\nabla^\varepsilon\big)\mathcal{Y}, \tag{60}$$

where $L(\nabla^\varepsilon)$ is the constant coefficients third order differential operator

$$L(\nabla_\varepsilon) \equiv \frac{1}{\mathfrak{c}_s^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \partial^3_{z_1} & 0 \\ 0 & 0 & \partial^3_{z_1} \end{pmatrix},$$

and $H(\varepsilon^2\mathcal{Y}, \nabla^\varepsilon)$ is a first order hyperbolic operator

$$H\big(\varepsilon^2\mathcal{Y}, \nabla^\varepsilon\big) = \sum_{j=1}^d H^k\big(\varepsilon^2\mathcal{Y}\big)\nabla^\varepsilon_j,$$

with symbol

$$H\big(\varepsilon^2\mathcal{Y}, \xi\big) = \sum_{j=1}^d H^j\big(\varepsilon^2\mathcal{Y}\big)\xi_j$$
$$= \begin{pmatrix} (-\xi_1 + 2\varepsilon^2\mathcal{Y}^1\xi_1 + 2\varepsilon^2\mathcal{Y}^\perp\cdot\xi_\perp) & (1+\varepsilon^2\mathcal{Y}^0)\xi_1 & (1+\varepsilon^2\mathcal{Y}^0)\xi^t_\perp \\ (1+(\Gamma-5)\varepsilon^2\mathcal{Y}^0)\xi_1 & -\xi_1 + 2\varepsilon^2\mathcal{Y}^1\xi_1 + 2\varepsilon^2\mathcal{Y}^\perp\cdot\xi_\perp & 0 \\ (1+(\Gamma-5)\varepsilon^2\mathcal{Y}^0)\xi_\perp & 0 & (-\xi_1 + 2\varepsilon^2\mathcal{Y}^1\xi_1 + 2\varepsilon^2\mathcal{Y}^\perp\cdot\xi_\perp)\mathrm{I}_{d-1} \end{pmatrix}.$$

We may symmetrize this system by using, as we have said,

$$\Sigma\big(\varepsilon^2\mathcal{Y}\big) = \mathrm{Diag}\bigg(\frac{1+(\Gamma-5)\varepsilon^2\mathcal{Y}^0}{1+\varepsilon^2\mathcal{Y}^0}, 1, \ldots, 1\bigg).$$

Indeed, we have

$$\Sigma\big(\varepsilon^2\mathcal{Y}\big)L\big(\nabla^\varepsilon\big) = L\big(\nabla^\varepsilon\big) = \frac{1}{\mathfrak{c}_s^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \partial^3_{z_1} & 0 \\ 0 & 0 & \partial^3_{z_1} \end{pmatrix}$$

which is a skew symmetric operator, and the matrix

$$\Sigma\big(\varepsilon^2\mathcal{Y}\big)H\big(\varepsilon^2\mathcal{Y}, \xi\big) = \begin{pmatrix} * & (1+(\Gamma-5)\varepsilon^2\mathcal{Y}^0)\xi_1 & (1+(\Gamma-5)\varepsilon^2\mathcal{Y}^0)\xi^t \\ (1+(\Gamma-5)\varepsilon^2\mathcal{Y}^0)\xi_1 & * & 0 \\ (1+(\Gamma-5)\varepsilon^2\mathcal{Y}^0)\xi_\perp & 0 & *\mathrm{I}_{d-1} \end{pmatrix}$$

(where the coefficients $*$ are nonrelevant) is symmetric for every $\xi \in \mathbb{R}^d$ and, by an integration by parts,

$$\forall W \in H^1\big(\mathbb{R}^d\big), \quad \big|\big\langle \Sigma\big(\varepsilon^2\mathcal{Y}\big)H\big(\varepsilon^2\mathcal{Y}, \nabla^\varepsilon\big)W, W\big\rangle_{L^2}\big| \leqslant K\varepsilon^2\|\nabla\mathcal{Y}\|_{L^\infty}\|W\|^2_{L^2}.$$

Therefore, the local in time existence and uniqueness for smooth solutions $\mathcal{Y} \in L^\infty([0, T^\varepsilon), H^s)$, with $s > 3 + d/2$ for this type of system is classical. In order to prove that $T^\varepsilon \geqslant \tau_*$, where $\tau_* > 0$ is independent of $0 < \varepsilon < 1$, we follow readily [20] (these are classical arguments, see *e.g.* [48]), which gives the estimate

$$\big\|\mathcal{Y}(\tau)\big\|^2_{H^s} \leqslant C\Bigg(\big\|\mathcal{Y}^{\mathrm{in}}\big\|^2_{H^s} + \int_0^\tau \big[\varepsilon^2\big\|\partial_{\bar\tau}\mathcal{Y}(\bar\tau)\big\|_{L^\infty} + \big\|\mathcal{Y}(\bar\tau)\big\|_{W^{1,\infty}}\big]\big\|\mathcal{Y}(\bar\tau)\big\|^2_{H^s}\,d\bar\tau\Bigg).$$

We use the equation and the Sobolev imbedding to estimate the bracket:

$$\varepsilon^2 \|\partial_{\bar\tau}\mathcal{Y}(\bar\tau)\|_{L^\infty} + \|\mathcal{Y}(\bar\tau)\|_{W^{1,\infty}} \leqslant C\|\mathcal{Y}(\bar\tau)\|_{W^{3,\infty}} \leqslant C\|\mathcal{Y}(\bar\tau)\|_{H^s},$$

provided $s > 3 + d/2$ (due to the third order derivative, we loose one more derivative than in [20]). Therefore,

$$\|\mathcal{Y}(\tau)\|_{H^s}^2 \leqslant C\left(\|\mathcal{Y}^{\mathrm{in}}\|_{H^s}^2 + \int_0^\tau \|\mathcal{Y}(\bar\tau)\|_{H^s}^3 \, d\bar\tau\right),$$

and the result then follows easily. We shall repeatedly use this structure of hyperbolic system with a constant coefficient dispersive term with a symmetrizer which leaves invariant this dispersive term to prove either existence/uniqueness of solution either comparison results. The fact that U_ε remains a gradient if it is a gradient initially comes immediately from the structure of the equation. The proof is complete. □

2.2. *Proof of Theorem 4*

The first point is to compare (12) and $(\mathcal{B}_\varepsilon)$, and the main difference between these two systems is that we have changed $\partial_{z_1} A_\varepsilon$ for $\partial_{z_1} U_\varepsilon^1$. By estimating the time derivative of $(A_\varepsilon, U_\varepsilon^1, \varepsilon U_\varepsilon^\perp)$, we shall derive the following estimate.

Lemma 1. *There exists some constant K, depending only on Λ, such that for $0 \leqslant \tau \leqslant \tau_0$, we have*

$$\begin{cases} (d=1) \quad \|\partial_z(A_\varepsilon - U_\varepsilon)\|_{H^{s-2}} \leqslant K\left(\|\partial_z(A_\varepsilon^{\mathrm{in}} - U_\varepsilon^{\mathrm{in}})\|_{H^{s-2}} + \varepsilon^2\right) \\ (d\geqslant 2) \quad \|\partial_{z_1}(A_\varepsilon - U_\varepsilon^1)\|_{H^{s-2}} \leqslant K\left(\|\partial_{z_1}(A_\varepsilon^{\mathrm{in}} - U_\varepsilon^{1,\mathrm{in}})\|_{H^{s-2}} + \varepsilon\right) \\ (d\geqslant 2) \quad \|\partial_{z_1}^2(A_\varepsilon - U_\varepsilon^1)\|_{H^{s-3}} \leqslant K\left(\|\partial_{z_1}^2(A_\varepsilon^{\mathrm{in}} - U_\varepsilon^{1,\mathrm{in}})\|_{H^{s-3}} + \varepsilon^2\right). \end{cases}$$

Proof. We recall that in [20], the solution Ψ_ε was constructed using the trick of E. Grenier [31]. We first solve the system where a_ε is complex-valued, u_ε^1 and $u_\varepsilon^\perp$ real-valued and where $\langle\cdot,\cdot\rangle$ denotes the real scalar product in $\mathbb{C}$:

$$\begin{cases} \frac{1}{\mathfrak{c}_s}\partial_\tau a_\varepsilon - \frac{1}{\varepsilon^2}\partial_{z_1} a_\varepsilon + \frac{1}{\varepsilon^2}\partial_{z_1} u_\varepsilon^1 + 2u_\varepsilon^1\partial_{z_1} a_\varepsilon + a_\varepsilon \partial_{z_1} u_\varepsilon^1 + 2\varepsilon^2 u_\varepsilon^\perp\cdot\nabla_{z_\perp} a_\varepsilon + (1+\varepsilon^2 a_\varepsilon)\nabla_{z_\perp}\cdot u_\varepsilon^\perp \\ \quad = \frac{i}{\varepsilon\mathfrak{c}_s}(\partial_{z_1}^2 a_\varepsilon + \varepsilon^2\Delta_{z_\perp} a_\varepsilon) \\ \frac{1}{\mathfrak{c}_s}\partial_\tau u_\varepsilon^1 - \frac{1}{\varepsilon^2}\partial_{z_1} u_\varepsilon^1 + 2u_\varepsilon^1\partial_{z_1} u_\varepsilon^1 + 2\varepsilon^2 u_\varepsilon^\perp\cdot\nabla_{z_\perp} u_\varepsilon^1 + \frac{2f'(|1+\varepsilon^2 a_\varepsilon|^2)}{\varepsilon^2\mathfrak{c}_s^2}\langle 1+\varepsilon^2 a_\varepsilon, \partial_{z_1} a_\varepsilon\rangle = 0 \\ \frac{1}{\mathfrak{c}_s}\partial_\tau u_\varepsilon^\perp - \frac{1}{\varepsilon^2}\partial_{z_1} u_\varepsilon^\perp + 2u_\varepsilon^1\partial_{z_1} u_\varepsilon^\perp + 2\varepsilon^2 u_\varepsilon^\perp\cdot\nabla_{z_\perp} u_\varepsilon^\perp + \frac{2f'(|1+\varepsilon^2 a_\varepsilon|^2)}{\varepsilon^2\mathfrak{c}_s^2}\langle 1+\varepsilon^2 a_\varepsilon, \nabla_{z_\perp} a_\varepsilon\rangle = 0, \end{cases} \tag{61}$$

with the initial conditions $(a_\varepsilon, u_\varepsilon)_{|\tau=0} = (A_\varepsilon^{\mathrm{in}}, U_\varepsilon^{\mathrm{in}}) \in \mathbb{R}\times\mathbb{R}^d \subset \mathbb{C}\times\mathbb{R}^d$. Then, following [3], we define the phase function Θ_ε by the formula

$$\Theta_\varepsilon(\tau, z) \equiv \phi_\varepsilon^{\mathrm{in}}(z) - \int_0^\tau \left[|u_\varepsilon^1|^2 + \varepsilon^2|u_\varepsilon^\perp|^2 + f(|1+\varepsilon^2 a_\varepsilon|^2)\right](\bar\tau, z)\, d\bar\tau,$$

and then check that $u_\varepsilon = \nabla_z \Theta_\varepsilon$ and that the function

$$\Psi_\varepsilon(t,x) \equiv (1+\varepsilon^2 a_\varepsilon(\tau,z))\exp(i\varepsilon\Theta_\varepsilon(\tau,z)), \quad \tau = \varepsilon^3 t,\ z_1 = \varepsilon(x_1 - \mathfrak{c}_s t),\ z_\perp = \varepsilon^2 x_\perp$$

is indeed a solution of (NLS). This has been achieved (see the proof of Theorem 4 in [20]) for $s > 2 + d/2$ on some time interval $[0, \tau_0]$, where $\tau_0 > 0$ is independent of $0 < \varepsilon < 1$, and with the uniform bounds:

$$\|(a_\varepsilon, u_\varepsilon^1, \varepsilon u_\varepsilon^\perp)\|_{H^s} \leqslant K. \tag{62}$$

Notice that the use of this strategy is possible if we work with the variables $(A, \nabla\phi)$, but not with the variables $(\rho = |A|^2, \nabla\phi)$, since the main interest of (61) is that the dispersive term in the right-hand side of the first equation has constant coefficient, and this is no longer the case with the density $\rho = |A|^2$.

The bound (62) leads to (9) after the change of variables from $(a_\varepsilon, u_\varepsilon)$ to $(A_\varepsilon, U_\varepsilon)$, namely

$$A_\varepsilon \equiv \frac{|1+\varepsilon^2 a_\varepsilon| - 1}{\varepsilon^2}, \qquad U_\varepsilon \equiv u_\varepsilon - \frac{i\varepsilon}{\mathfrak{c}_s}\left(\frac{\nabla a_\varepsilon}{1+\varepsilon^2 a_\varepsilon} - \frac{\nabla A_\varepsilon}{1+\varepsilon^2 A_\varepsilon}\right), \tag{63}$$

which looses one derivative in U_ε (notice also that U_ε is indeed real-valued since $|1+\varepsilon^2 a_\varepsilon| = 1+\varepsilon^2 A_\varepsilon$). We then let

$$\left(\dot a_\varepsilon, \dot U^1_\varepsilon, \varepsilon \dot U^\perp_\varepsilon\right) \equiv \partial_\tau\left(a_\varepsilon, u^1_\varepsilon, \varepsilon u^\perp_\varepsilon\right)$$

and apply ∂_τ to (61) to obtain

$$\begin{cases} \dfrac{1}{\mathfrak{c}_s}\partial_\tau \dot a_\varepsilon - \dfrac{1}{\varepsilon^2}\partial_{z_1}\dot a_\varepsilon + \dfrac{1}{\varepsilon^2}\partial_{z_1}\dot u^1_\varepsilon + 2u^1_\varepsilon \partial_{z_1}\dot a_\varepsilon + a_\varepsilon \partial_{z_1}\dot u^1_\varepsilon + 2\varepsilon^2 u^\perp_\varepsilon\cdot\nabla_{z_\perp}\dot a_\varepsilon + \left(1+\varepsilon^2 a_\varepsilon\right)\nabla_{z_\perp}\cdot \dot u^\perp_\varepsilon \\ \quad = \dfrac{i}{\varepsilon\mathfrak{c}_s}\left(\partial^2_{z_1}\dot a^1_\varepsilon + \varepsilon^2\Delta_{z_\perp}\dot a_\varepsilon\right) + \mathcal{C}_\varepsilon \\ \dfrac{1}{\mathfrak{c}_s}\partial_\tau \dot u^1_\varepsilon - \dfrac{1}{\varepsilon^2}\partial_{z_1}\dot u^1_\varepsilon + 2u^1_\varepsilon\partial_{z_1}\dot u^1_\varepsilon + 2\varepsilon^2 u^\perp_\varepsilon\cdot\nabla_{z_\perp}\dot u^1_\varepsilon + \dfrac{2f'(|1+\varepsilon^2 a_\varepsilon|^2)}{\varepsilon^2\mathfrak{c}_s^2}\left\langle 1+\varepsilon^2 a_\varepsilon, \partial_{z_1}\dot a_\varepsilon\right\rangle = \mathcal{C}^1_\varepsilon \\ \dfrac{1}{\mathfrak{c}_s}\partial_\tau \dot u^\perp_\varepsilon - \dfrac{1}{\varepsilon^2}\partial_{z_1}\dot u^\perp_\varepsilon + 2u^1_\varepsilon\partial_{z_1}\dot u^\perp_\varepsilon + 2\varepsilon^2 u^\perp_\varepsilon\cdot\nabla_{z_\perp}\dot u^\perp_\varepsilon + \dfrac{2f'(|1+\varepsilon^2 a_\varepsilon|^2)}{\varepsilon^2\mathfrak{c}_s^2}\left\langle 1+\varepsilon^2 a_\varepsilon, \nabla_{z_\perp}\dot a_\varepsilon\right\rangle = \mathcal{C}^\perp_\varepsilon. \end{cases} \tag{64}$$

Here, the commutators $\mathcal{C}_\varepsilon$, $\mathcal{C}^1_\varepsilon$ and $\mathcal{C}^\perp_\varepsilon$ are defined by

$$\mathcal{C}_\varepsilon \equiv -2\dot u^1_\varepsilon\partial_{z_1}a_\varepsilon - \dot a_\varepsilon\partial_{z_1}u^1_\varepsilon - 2\varepsilon^2\dot u^\perp_\varepsilon\cdot\nabla_{z_\perp}a_\varepsilon - \varepsilon^2\dot a_\varepsilon\nabla_{z_\perp}\cdot u^\perp_\varepsilon,$$

$$\mathcal{C}^1_\varepsilon \equiv -2\dot u^1_\varepsilon\partial_{z_1}u^1_\varepsilon - 2\varepsilon^2\dot u^\perp_\varepsilon\cdot\nabla_{z_\perp}u^1_\varepsilon - \left\langle \partial_\tau\left[\frac{2f'(|1+\varepsilon^2 a_\varepsilon|^2)}{\varepsilon^2\mathfrak{c}_s^2}\left(1+\varepsilon^2 a_\varepsilon\right)\right], \partial_{z_1}a_\varepsilon\right\rangle,$$

$$\mathcal{C}^\perp_\varepsilon \equiv -2\dot u^1_\varepsilon\partial_{z_1}u^\perp_\varepsilon - 2\varepsilon^2\dot u^\perp_\varepsilon\cdot\nabla_{z_\perp}u^\perp_\varepsilon - \left\langle \partial_\tau\left[\frac{2f'(|1+\varepsilon^2 a_\varepsilon|^2)}{\varepsilon^2\mathfrak{c}_s^2}\left(1+\varepsilon^2 a_\varepsilon\right)\right]\nabla_{z_\perp}a_\varepsilon\right\rangle.$$

As for the Boussinesq system, denoting $\Upsilon = (\Upsilon^0, \Upsilon^1, \Upsilon^\perp)^t \equiv (\dot a_\varepsilon, \dot u^1_\varepsilon, \varepsilon\dot u^\perp_\varepsilon)^t$ allows to write (64) under the form of a hyperbolic system with smooth coefficients $\mathcal{X} = (\mathcal{X}^0, \mathcal{X}^1_\varepsilon, \varepsilon\mathcal{X}^\perp_\varepsilon)^t \equiv (a_\varepsilon, u^1_\varepsilon, \varepsilon u^\perp_\varepsilon)^t \in L^\infty([0,\tau_*], H^s)$

$$\frac{1}{\mathfrak{c}_s}\partial_\tau\Upsilon + \frac{1}{\varepsilon^2}\mathcal{H}\left(\varepsilon^2\mathcal{X}, \nabla^\varepsilon\right)\Upsilon = \frac{1}{\varepsilon}\mathcal{L}\left(\nabla^\varepsilon\right)\Upsilon + \mathcal{S}_\varepsilon(\Upsilon), \tag{65}$$

with

$$\mathcal{L}(\nabla^\varepsilon) \equiv \frac{i}{\mathfrak{c}_s}\begin{pmatrix} \partial^2_{z_1} + \varepsilon^2\Delta_{z_\perp} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{H}\left(\varepsilon^2\mathcal{X}, \nabla^\varepsilon\right) = \sum_{j=1}^d \mathcal{H}^j\left(\varepsilon^2\mathcal{X}\right)\nabla^\varepsilon_j,$$

where

$$\mathcal{H}\left(\varepsilon^2\mathcal{X}, \xi\right) = \sum_{j=1}^d \mathcal{H}^j\left(\varepsilon^2\mathcal{X}\right)\xi_j$$
$$= \begin{pmatrix} (-\xi_1 + 2\varepsilon^2\mathcal{X}^1\xi_1 + 2\varepsilon^2\mathcal{X}^\perp\cdot\xi_\perp) & (1+\varepsilon^2\mathcal{X}^0)\xi_1 & (1+\varepsilon^2\mathcal{X}^0)\xi^t_\perp \\ \frac{2}{\mathfrak{c}_s^2}f'(|1+\varepsilon^2\mathcal{X}^0|^2)\xi_1 & -\xi_1 + 2\varepsilon^2\mathcal{X}^1\xi_1 + 2\varepsilon^2\mathcal{X}^\perp\cdot\xi_\perp & 0 \\ \frac{2}{\mathfrak{c}_s^2}f'(|1+\varepsilon^2\mathcal{X}^0|^2)\xi_\perp & 0 & (-\xi_1 + 2\varepsilon^2\mathcal{X}^1\xi_1 + 2\varepsilon^2\mathcal{X}^\perp\cdot\xi_\perp)\mathrm{I}_{d-1} \end{pmatrix}.$$

Moreover, the source term $\mathcal{S}_\varepsilon$ given by the commutators $\mathcal{C}_\varepsilon$, $\mathcal{C}^1_\varepsilon$ and $\mathcal{C}^\perp_\varepsilon$ enjoys the estimate, for $0 \leqslant \tau \leqslant \tau_*$,

$$\left\|\mathcal{S}_\varepsilon(\Upsilon)\right\|_{H^{s-2}} \leqslant K\|\Upsilon\|_{H^{s-2}} \tag{66}$$

using (62) and that H^σ is an algebra with $\sigma = s-2 > d/2$. Since $s-2 > d/2$, the local well-posedness of the linear system (65) in H^{s-2} is standard, and we indeed have $(\dot a_\varepsilon, \dot U^1_\varepsilon, \varepsilon\dot U^\perp_\varepsilon)_{|\tau=0} \in H^{s-2}$ (from (61)). Hence, it remains to show that the maximal solution is defined on a time interval $[0, \tau^*_\varepsilon]$ such that $\tau^*_\varepsilon \geqslant \tau_0$ for ε small. The symmetrizer

$$\mathfrak{S}_\varepsilon(\varepsilon^2\mathcal{X}^0) \equiv \mathrm{Diag}\left(1_{\mathbb{C}}, \frac{\mathfrak{c}_s^2}{2f'(|1+\varepsilon^2\mathcal{X}^0|^2)}, \dots, \frac{\mathfrak{c}_s^2}{2f'(|1+\varepsilon^2\mathcal{X}^0|^2)}\right)$$

is well adapted (see [20]), since it keeps the dispersive term with constant coefficients, and is such that the matrix $\mathfrak{S}_\varepsilon\mathcal{H}(\varepsilon^2\mathcal{X},\xi)$ is symmetric for every $\xi\in\mathbb{R}^d$. By applying ∂_z^α with $\alpha\in\mathbb{N}^d$ such that $|\alpha|\leqslant s-2$, we infer as in [20], using the tame estimate (59), that for $0\leqslant\tau<\min(\tau_\varepsilon^*,\tau_*)$,

$$\frac{d}{d\tau}\big(\big(\mathfrak{S}_\varepsilon(\varepsilon^2\mathcal{X}^0)\partial_z^\alpha\Upsilon,\partial_z^\alpha\Upsilon\big)_{L^2}\big) \leqslant C\big(\varepsilon^2\|\partial_\tau\mathcal{X}^0\|_{L^\infty}+\|\mathcal{X}\|_{W^{1,\infty}}\big)\|\Upsilon\|_{H^{s-2}}^2 + C\|\mathcal{S}_\varepsilon(\Upsilon)\|_{H^{s-2}}\|\Upsilon\|_{H^{s-2}}.$$

Using (66), the Sobolev imbedding and the uniform bounds (62), we deduce

$$\frac{d}{d\tau}\big(\big(\mathfrak{S}_\varepsilon(\varepsilon^2\mathcal{X}^0)\partial_z^\alpha\Upsilon,\partial_z^\alpha\Upsilon\big)_{L^2}\big) \leqslant K\|\Upsilon(\tau)\|_{H^{s-2}}^2,$$

where K depends only on Λ, hence by the Gronwall lemma, it comes $\tau_\varepsilon^*>\tau_*$ and

$$\sup_{0\leqslant\tau\leqslant\tau_*}\|\Upsilon(\tau)\|_{H^{s-2}} \leqslant K\|\Upsilon^{\mathrm{in}}\|_{H^{s-2}}. \tag{67}$$

We recall that we wish to bound $\partial_{z_1}(A_\varepsilon-U_\varepsilon^1)$ or $\partial_{z_1}^2(A_\varepsilon-U_\varepsilon^1)$. By (63) and the uniform bounds (62), we have

$$A_\varepsilon = \frac{|1+\varepsilon^2 a_\varepsilon|-1}{\varepsilon^2} = \mathrm{Re}(a_\varepsilon)+\mathcal{O}_{H^s}(\varepsilon^2),$$

and then

$$\begin{aligned}
\partial_{z_1}(A_\varepsilon-U_\varepsilon^1) &= \partial_{z_1}\left(\mathrm{Re}(a_\varepsilon)-u_\varepsilon^1+\frac{i\varepsilon}{\mathfrak{c}_s}\left[\frac{\partial_{z_1}a_\varepsilon}{1+\varepsilon^2 a_\varepsilon}-\frac{\partial_{z_1}A_\varepsilon}{1+\varepsilon^2 A_\varepsilon}\right]\right)+\mathcal{O}_{H^{s-1}}(\varepsilon^2)\\
&= \partial_{z_1}\left(\mathrm{Re}(a_\varepsilon)-u_\varepsilon^1+\frac{i\varepsilon}{\mathfrak{c}_s}[\partial_{z_1}a_\varepsilon-\partial_{z_1}A_\varepsilon]\right)+\mathcal{O}_{H^{s-1}}(\varepsilon^2)\\
&= \partial_{z_1}\left(\mathrm{Re}(a_\varepsilon)-u_\varepsilon^1-\frac{\varepsilon}{\mathfrak{c}_s}\partial_{z_1}\mathrm{Im}(a_\varepsilon)\right)+\mathcal{O}_{H^{s-1}}(\varepsilon^2)\\
&= \mathrm{Re}\left\{\partial_{z_1}(a_\varepsilon-u_\varepsilon^1)+\frac{i\varepsilon}{\mathfrak{c}_s}[\partial_{z_1}^2+\varepsilon^2\Delta_{z_\perp}]a_\varepsilon\right\}+\mathcal{O}_{H^{s-1}}(\varepsilon^2) && (68)\\
&= \mathrm{Re}\left\{\frac{\varepsilon^2}{\mathfrak{c}_s}\partial_\tau a_\varepsilon\right\}+\varepsilon^2\nabla_{z_\perp}\cdot u_\varepsilon^\perp+\mathcal{O}_{H^{s-1}}(\varepsilon^2). && (69)
\end{aligned}$$

For (69), we use the uniform bounds (9), and in particular the uniform bound on $\varepsilon u_\varepsilon^\perp$. This is also the reason why we have singled out (if $d\geqslant 2$), the term $\nabla_{z_\perp}\cdot u_\varepsilon^\perp$. Let us assume first $d=1$, so that the term $\nabla_{z_\perp}\cdot u_\varepsilon^\perp$ disappears. Then, since at the initial time, $a_\varepsilon=A_\varepsilon$ is real-valued, we infer in particular from (68) that

$$\big[\partial_z(A_\varepsilon-U_\varepsilon)\big]_{|\tau=0} = \left[\frac{\varepsilon^2}{\mathfrak{c}_s}\partial_\tau a_\varepsilon\right]_{|\tau=0}+\mathcal{O}_{H^{s-1}}(\varepsilon^2).$$

Furthermore, we deduce easily from (61) that

$$\left\{\frac{\varepsilon^2}{\mathfrak{c}_s}\partial_\tau u_\varepsilon\right\}_{|\tau=0} = \big[\partial_z(U_\varepsilon-A_\varepsilon)\big]_{|\tau=0}+\mathcal{O}_{H^{s-1}}(\varepsilon^2) = \partial_z\big(U_\varepsilon^{\mathrm{in}}-A_\varepsilon^{\mathrm{in}}\big)+\mathcal{O}_{H^{s-1}}(\varepsilon^2),$$

so that

$$\|\varepsilon^2\Upsilon^{\mathrm{in}}\|_{H^{s-2}} \leqslant K\varepsilon^2+K\|\partial_z(U_\varepsilon^{\mathrm{in}}-A_\varepsilon^{\mathrm{in}})\|_{H^{s-2}}.$$

Then, taking the H^{s-2} norm in (69) and using (67), we deduce that for $0\leqslant\tau\leqslant\tau_*$ and some constant K depending only on Λ,

$$\begin{aligned}
\|\partial_z(A_\varepsilon-U_\varepsilon)\|_{H^{s-2}} &\leqslant K\big(\|\varepsilon^2\partial_\tau a_\varepsilon\|_{H^{s-2}}+\varepsilon^2\big)\\
&\leqslant K\big(\|\varepsilon^2\Upsilon^{\mathrm{in}}\|_{H^{s-2}}+\varepsilon^2\big)\\
&\leqslant K\big(\|\partial_z(U_\varepsilon^{\mathrm{in}}-A_\varepsilon^{\mathrm{in}})\|_{H^{s-2}}+\varepsilon^2\big).
\end{aligned}$$

This finishes the proof of the one dimensional case. When $d \geqslant 2$, the point is that we do not control $u_\varepsilon^\perp$ but only $\varepsilon u_\varepsilon^\perp$. Nevertheless, if $d \geqslant 2$, the same argument shows the second statement in Lemma 1. For the third statement, we may use that u_ε is a gradient, hence $\partial_{z_1} u_\varepsilon^\perp = \nabla_{z_\perp} u_\varepsilon^1$. Therefore, it is natural to apply ∂_{z_1} to (68) and infer

$$\begin{aligned}
\big[\partial_{z_1}^2 (A_\varepsilon - U_\varepsilon)\big]_{|\tau=0} &= \partial_{z_1}\left[\frac{\varepsilon^2}{\mathfrak{c}_s}\partial_\tau a_\varepsilon\right]_{|\tau=0} + \varepsilon^2 \partial_{z_1}\nabla_{z_\perp}\cdot u_\varepsilon^\perp + \mathcal{O}_{H^{s-2}}\big(\varepsilon^2\big) \\
&= \partial_{z_1}\left[\frac{\varepsilon^2}{\mathfrak{c}_s}\partial_\tau a_\varepsilon\right]_{|\tau=0} + \varepsilon^2 \Delta_{z_\perp} u_\varepsilon^1 + \mathcal{O}_{H^{s-2}}\big(\varepsilon^2\big) \\
&= \partial_{z_1}\left[\frac{\varepsilon^2}{\mathfrak{c}_s}\partial_\tau a_\varepsilon\right]_{|\tau=0} + \mathcal{O}_{H^{s-2}}\big(\varepsilon^2\big).
\end{aligned}$$

Hence, we deduce in a similar way

$$\big\|\partial_{z_1}^2 (A_\varepsilon - U_\varepsilon)\big\|_{H^{s-3}} \leqslant K \big\|\partial_{z_1}^2\big(A_\varepsilon^{\text{in}} - U_\varepsilon^{\text{in}}\big)\big\|_{H^{s-3}} + K\varepsilon^2,$$

and this finishes the proof of Lemma 1. □

We now turn to the proof of Theorem 4. We recall that $(A_\varepsilon, U_\varepsilon^1, U_\varepsilon^\perp)$ solves (12), that is

$$\begin{cases}
\dfrac{1}{\mathfrak{c}_s}\partial_\tau A_\varepsilon - \dfrac{1}{\varepsilon^2}\partial_{z_1} A_\varepsilon + 2U_\varepsilon^1 \partial_{z_1} A_\varepsilon + 2\varepsilon^2 U_\varepsilon^\perp\cdot\nabla_{z_\perp} A_\varepsilon + \dfrac{1}{\varepsilon^2}\big(1+\varepsilon^2 A_\varepsilon\big)\big(\partial_{z_1} U_\varepsilon^1 + \varepsilon^2 \nabla_{z_\perp}\cdot U_\varepsilon^\perp\big) = 0 \\
\dfrac{1}{\mathfrak{c}_s}\partial_\tau U_\varepsilon^1 - \dfrac{1}{\varepsilon^2}\partial_{z_1} U_\varepsilon^1 + 2\big(U_\varepsilon^1\partial_{z_1} + \varepsilon^2 U_\varepsilon^\perp\cdot\nabla_{z_\perp}\big)U_\varepsilon^1 + \dfrac{1}{\mathfrak{c}_s^2\varepsilon^4}\partial_{z_1}\big[f\big(\big(1+\varepsilon^2 A_\varepsilon\big)^2\big)\big] \\
\quad - \dfrac{1}{\mathfrak{c}_s^2}\partial_{z_1}\left(\dfrac{\partial_{z_1}^2 A_\varepsilon + \varepsilon^2\Delta_{z_\perp} A_\varepsilon}{1+\varepsilon^2 A_\varepsilon}\right) = 0 \\
\dfrac{1}{\mathfrak{c}_s}\partial_\tau U_\varepsilon^\perp - \dfrac{1}{\varepsilon^2}\partial_{z_1} U_\varepsilon^\perp + 2\big(U_\varepsilon^1\partial_{z_1} + \varepsilon^2 U_\varepsilon^\perp\cdot\nabla_{z_\perp}\big)U_\varepsilon^\perp + \dfrac{1}{\mathfrak{c}_s^2\varepsilon^4}\nabla_{z_\perp}\big[f\big(\big(1+\varepsilon^2 A_\varepsilon\big)^2\big)\big] \\
\quad - \dfrac{1}{\mathfrak{c}_s^2}\nabla_{z_\perp}\left(\dfrac{\partial_{z_1}^2 A_\varepsilon + \varepsilon^2\Delta_{z_\perp} A_\varepsilon}{1+\varepsilon^2 A_\varepsilon}\right) = 0.
\end{cases}$$

Using the Taylor expansion (4) and Lemma 1, we shall deduce that $(A_\varepsilon, U_\varepsilon^1, U_\varepsilon^\perp)^t$ solves the Boussinesq system $\mathcal{B}_\varepsilon$ up to an error $\mathcal{O}(\varepsilon^2)$. More precisely, we have, with the notations of subsection 2.1 and $Y \equiv (A_\varepsilon, U_\varepsilon^1, \varepsilon U_\varepsilon^\perp)^t$,

$$\frac{1}{\mathfrak{c}_s}\partial_\tau Y + \frac{1}{\varepsilon^2} H\big(\varepsilon^2 Y, \nabla^\varepsilon\big) Y = L\big(\nabla^\varepsilon\big) Y + \frac{1}{\mathfrak{c}_s^2}\mathrm{Err}_\varepsilon,$$

where, using once again that $\partial_{z_1} U_\varepsilon^\perp = \nabla_{z_\perp} U_\varepsilon^1$,

$$\mathrm{Err}_\varepsilon \equiv \begin{pmatrix} 0 \\ -\frac{1}{\mathfrak{c}_s^2\varepsilon^4}\partial_{z_1}(f_3(\varepsilon^2 A_\varepsilon)) \\ -\frac{1}{\mathfrak{c}_s^2\varepsilon^4}\nabla_{z_\perp}(f_3(\varepsilon^2 A_\varepsilon)) \end{pmatrix} + \begin{pmatrix} 0 \\ \partial_{z_1}^3 (A_\varepsilon - U_\varepsilon^1) - \partial_{z_1}(\frac{\varepsilon^2 A_\varepsilon}{1+\varepsilon^2 A_\varepsilon}\partial_{z_1}^2 A_\varepsilon) + \varepsilon^2\partial_{z_1}(\frac{\Delta_{z_\perp} A_\varepsilon}{1+\varepsilon^2 A_\varepsilon}) \\ \partial_{z_1}^2\nabla_{z_\perp}(A_\varepsilon - U_\varepsilon^1) - \nabla_{z_\perp}(\frac{\varepsilon^2 A_\varepsilon}{1+\varepsilon^2 A_\varepsilon}\partial_{z_1}^2 A_\varepsilon) + \varepsilon^2\nabla_{z_\perp}(\frac{\Delta_{z_\perp} A_\varepsilon}{1+\varepsilon^2 A_\varepsilon}) \end{pmatrix}.$$

From the uniform bounds (9) and Lemma 1 to control the terms $\partial_{z_1}^3 (A_\varepsilon - U_\varepsilon^1)$ and $\partial_{z_1}^2 \nabla_{z_\perp}(A_\varepsilon - U_\varepsilon^1)$, we infer that Err_ε verifies, for some constant K depending only on Λ,

$$\sup_{0\leqslant\tau\leqslant\tau_0} \|\mathrm{Err}_\varepsilon\|_{H^{s-3}} \leqslant K\big(\big\|A_\varepsilon^{\text{in}} - U_\varepsilon^{1,\text{in}}\big\|_{H^s} + \varepsilon^2\big).$$

The error estimate follows then easily since the unperturbed system is symmetrizable in variables $(A_\varepsilon, U_\varepsilon^1, \varepsilon U_\varepsilon^\perp)$ as in the proof of Proposition 1. □

2.3. Proof of Propositions 2 and 3

Proof of Proposition 2. Notice first that arguing as in Lemma 1, namely estimating the time derivative of $(\mathrm{A}_\varepsilon, \mathrm{U}_\varepsilon)$ yields the estimate

$$\big\|\partial_z(\mathrm{A}_\varepsilon - \mathrm{U}_\varepsilon)\big\|_{H^{s-3}(\mathbb{R})} \leqslant K\big(\big\|\partial_z\big(\mathrm{A}_\varepsilon^{\mathrm{in}} - \mathrm{U}_\varepsilon^{\mathrm{in}}\big)\big\|_{H^{s-3}(\mathbb{R})} + \varepsilon^2\big). \tag{70}$$

The exponent is now $s-3$ instead of $s-2$ since the dispersive term is here of third order. In order to prove (15), it remains then to show the L^2 estimate, and we shall argue as in [13] (proof of Proposition 4 there). From the two equations of $(\mathcal{B}_\varepsilon)$, we obtain

$$\frac{1}{\mathfrak{c}_s}\partial_\tau(\mathrm{A}_\varepsilon - \mathrm{U}_\varepsilon) - \frac{2}{\varepsilon^2}\partial_z(\mathrm{A}_\varepsilon - \mathrm{U}_\varepsilon) + 2\mathrm{U}_\varepsilon\partial_z\mathrm{A}_\varepsilon + \mathrm{A}_\varepsilon\partial_z\mathrm{U}_\varepsilon - 2\mathrm{U}_\varepsilon\partial_z\mathrm{U}_\varepsilon - (\Gamma-5)\mathrm{A}_\varepsilon\partial_z\mathrm{A}_\varepsilon + \frac{1}{\mathfrak{c}_s^2}\partial_z^3\mathrm{U}_\varepsilon = 0.$$

We then define $\mathrm{W}_\varepsilon \equiv \mathrm{A}_\varepsilon - \mathrm{U}_\varepsilon$ and write the equation for W_ε under the form

$$\frac{2}{\mathfrak{c}_s}\partial_\tau\mathrm{W}_\varepsilon - \frac{2}{\varepsilon^2}\partial_z\mathrm{W}_\varepsilon + (2\mathrm{U}_\varepsilon - \mathrm{A}_\varepsilon)\partial_z\mathrm{W}_\varepsilon = (\Gamma-6)\mathrm{A}_\varepsilon\partial_z\mathrm{A}_\varepsilon - \frac{1}{\mathfrak{c}_s^2}\partial_z^3\mathrm{U}_\varepsilon. \tag{71}$$

It then follows from integration by parts that

$$\frac{2}{\mathfrak{c}_s}\frac{d}{d\tau}\int_{\mathbb{R}}\mathrm{W}_\varepsilon^2\,dz = \int_{\mathbb{R}}\partial_z(2\mathrm{U}_\varepsilon - \mathrm{A}_\varepsilon)\mathrm{W}_\varepsilon^2\,dz - (\Gamma-6)\int_{\mathbb{R}}\mathrm{A}_\varepsilon^2\partial_z\mathrm{W}_\varepsilon\,dz + \frac{2}{\mathfrak{c}_s^2}\int_{\mathbb{R}}\partial_z\mathrm{W}_\varepsilon\partial_z^2\mathrm{U}_\varepsilon\,dz.$$

We now integrate in time and use the uniform bound (62) to infer

$$\frac{2}{\mathfrak{c}_s}\int_{\mathbb{R}}\mathrm{W}_\varepsilon^2\,dz \leqslant \frac{2}{\mathfrak{c}_s}\|\mathrm{A}_\varepsilon^{\mathrm{in}} - \mathrm{U}_\varepsilon^{\mathrm{in}}\|_{L^2}^2 + K\int_0^\tau\int_{\mathbb{R}}\mathrm{W}_\varepsilon^2(\bar\tau)\,dz\,d\bar\tau + \int_0^\tau\int_{\mathbb{R}}\partial_z\mathrm{W}_\varepsilon\left\{\frac{2}{\mathfrak{c}_s^2}\partial_z^2\mathrm{U}_\varepsilon - (\Gamma-6)\mathrm{A}_\varepsilon^2\right\}dz\,d\bar\tau.$$

We now express $\frac{2}{\varepsilon^2}\partial_z\mathrm{W}_\varepsilon$ from (71) and obtain

$$\begin{aligned}
&\frac{2}{\mathfrak{c}_s}\int_{\mathbb{R}}\mathrm{W}_\varepsilon^2(\tau)\,dz \\
&\quad\leqslant \frac{2}{\mathfrak{c}_s}\big\|\mathrm{A}_\varepsilon^{\mathrm{in}} - \mathrm{U}_\varepsilon^{\mathrm{in}}\big\|_{L^2}^2 + K\int_0^\tau\int_{\mathbb{R}}\mathrm{W}_\varepsilon^2(\bar\tau)\,dz\,d\bar\tau \\
&\qquad+ \frac{\varepsilon^2}{\mathfrak{c}_s}\int_0^\tau\int_{\mathbb{R}}\partial_\tau\mathrm{W}_\varepsilon\left\{\frac{2}{\mathfrak{c}_s^2}\partial_z^2\mathrm{U}_\varepsilon - (\Gamma-6)\mathrm{A}_\varepsilon^2\right\}dz\,d\bar\tau \\
&\qquad+ \frac{\varepsilon^2}{2}\int_0^\tau\int_{\mathbb{R}}\left\{(2\mathrm{U}_\varepsilon - \mathrm{A}_\varepsilon)\partial_z\mathrm{W}_\varepsilon - (\Gamma-6)\mathrm{A}_\varepsilon\partial_z\mathrm{A}_\varepsilon + \frac{1}{\mathfrak{c}_s^2}\partial_z^3\mathrm{U}_\varepsilon\right\}\left\{\frac{2}{\mathfrak{c}_s^2}\partial_z^2\mathrm{U}_\varepsilon - (\Gamma-6)\mathrm{A}_\varepsilon^2\right\}dz\,d\bar\tau.
\end{aligned} \tag{72}$$

For the second line in (72), we integrate by parts in time:

$$\begin{aligned}
&\frac{\varepsilon^2}{\mathfrak{c}_s}\int_0^\tau\int_{\mathbb{R}}\partial_\tau\mathrm{W}_\varepsilon\left\{\frac{2}{\mathfrak{c}_s^2}\partial_z^2\mathrm{U}_\varepsilon - (\Gamma-6)\mathrm{A}_\varepsilon^2\right\}dz\,d\bar\tau \\
&\quad= -\frac{\varepsilon^2}{\mathfrak{c}_s}\int_0^\tau\int_{\mathbb{R}}\mathrm{W}_\varepsilon\partial_\tau\left\{\frac{2}{\mathfrak{c}_s^2}\partial_z^2\mathrm{U}_\varepsilon - (\Gamma-6)\mathrm{A}_\varepsilon^2\right\}dz\,d\bar\tau + \frac{\varepsilon^2}{\mathfrak{c}_s}\int_{\mathbb{R}}\mathrm{W}_\varepsilon(\tau)\left\{\frac{2}{\mathfrak{c}_s^2}\partial_z^2\mathrm{U}_\varepsilon(\tau) - (\Gamma-6)\mathrm{A}_\varepsilon^2(\tau)\right\}dz \\
&\qquad- \frac{\varepsilon^2}{\mathfrak{c}_s}\int_{\mathbb{R}}\mathrm{W}_\varepsilon(0)\left\{\frac{2}{\mathfrak{c}_s^2}\partial_z^2\mathrm{U}_\varepsilon(0) - (\Gamma-6)\mathrm{A}_\varepsilon^2(0)\right\}dz.
\end{aligned}$$

The last and before last terms are easily estimated by $K\varepsilon^2\|\mathrm{A}_\varepsilon^{\mathrm{in}} - \mathrm{U}_\varepsilon^{\mathrm{in}}\|_{L^2}$ and $K\varepsilon^2\|\mathrm{W}_\varepsilon\|_{L^2}$ respectively. Moreover, from $(\mathcal{B}_\varepsilon)$, we have

$$\frac{\varepsilon^2}{\mathfrak{c}_s}\partial_\tau\partial_z^2\mathrm{U}_\varepsilon(\tau) = -\partial_z^3\mathrm{W}_\varepsilon + \mathcal{O}_{L^2}\big(\varepsilon^2\big)$$

since $s \geqslant 5$, thus, for $0 \leqslant \tau \leqslant \tau_0$,

$$-\frac{\varepsilon^2}{\mathfrak{c}_s}\int_0^\tau\int_{\mathbb{R}} \mathrm{W}_\varepsilon\partial_\tau\left\{\frac{2}{\mathfrak{c}_s^2}\partial_z^2\mathrm{U}_\varepsilon\right\}dz\,d\bar\tau \leqslant \frac{2}{\mathfrak{c}_s^2}\int_0^\tau\int_{\mathbb{R}} \mathrm{W}_\varepsilon\partial_z^3\mathrm{W}_\varepsilon\,dz + K\varepsilon^2\int_0^\tau \|\mathrm{W}_\varepsilon\|_{L^2}\,d\bar\tau \leqslant 0 + K\varepsilon^4 + \int_0^\tau \|\mathrm{W}_\varepsilon\|_{L^2}^2\,d\bar\tau.$$

Similarly, using the inequality $2ab \leqslant a^2 + b^2$,

$$\frac{\varepsilon^2}{\mathfrak{c}_s}(\Gamma - 6)\int_{\mathbb{R}} \mathrm{W}_\varepsilon\partial_\tau\big[\mathrm{A}_\varepsilon^2\big]dz \leqslant 2(\Gamma - 6)\int_{\mathbb{R}} \mathrm{A}_\varepsilon\mathrm{W}_\varepsilon\partial_z\mathrm{W}_\varepsilon\,dz + K\varepsilon^2\|\mathrm{W}_\varepsilon\|_{L^2} \leqslant K\|\mathrm{W}_\varepsilon\|_{L^2}^2 + K\varepsilon^4$$

by integration by parts for the last integral. For the third line in (72), the uniform bound (62) gives, for $0 \leqslant \tau \leqslant \tau_0$,

$$\frac{\varepsilon^2}{2}\int_0^\tau\int_{\mathbb{R}} (2\mathrm{U}_\varepsilon - \mathrm{A}_\varepsilon)\partial_z\mathrm{W}_\varepsilon\left\{\frac{1}{\mathfrak{c}_s^2}\partial_z^2\mathrm{U}_\varepsilon - (\Gamma - 6)\mathrm{A}_\varepsilon^2\right\}dz\,d\bar\tau \leqslant K\varepsilon^2\int_0^\tau \|\mathrm{W}_\varepsilon\|_{L^2}\,d\bar\tau \leqslant K\varepsilon^4 + \int_0^\tau \|\mathrm{W}_\varepsilon\|_{L^2}^2\,d\bar\tau.$$

In addition,

$$\int_{\mathbb{R}}\left\{-(\Gamma - 6)\mathrm{A}_\varepsilon\partial_z\mathrm{A}_\varepsilon + \frac{1}{\mathfrak{c}_s^2}\partial_z^3\mathrm{U}_\varepsilon\right\}\left\{\frac{2}{\mathfrak{c}_s^2}\partial_z^2\mathrm{U}_\varepsilon - (\Gamma - 6)\mathrm{A}_\varepsilon^2\right\}dz = \int_{\mathbb{R}} \partial_z\left\{\left[\frac{1}{\mathfrak{c}_s^2}\partial_z^2\mathrm{U}_\varepsilon - \frac{\Gamma - 6}{2}\mathrm{A}_\varepsilon^2\right]^2\right\}dz = 0.$$

Consequently, (72) implies

$$\big\|\mathrm{W}_\varepsilon(\tau)\big\|_{L^2}^2 \leqslant \big\|\mathrm{A}_\varepsilon^{\mathrm{in}} - \mathrm{U}_\varepsilon^{\mathrm{in}}\big\|_{L^2}^2 + K\int_0^\tau \big\|\mathrm{W}_\varepsilon(\bar\tau)\big\|_{L^2}^2\,d\bar\tau + K\varepsilon^4,$$

and (15) then follows from the Gronwall lemma.

Defining

$$\mathfrak{Z}_\varepsilon \equiv \frac{\mathrm{A}_\varepsilon + \mathrm{U}_\varepsilon}{2}$$

and summing the two equations of $(\mathcal{B}_\varepsilon)$, we obtain

$$\frac{2}{\mathfrak{c}_s}\partial_\tau\mathfrak{Z}_\varepsilon + \Gamma\mathfrak{Z}_\varepsilon\partial_z\mathfrak{Z}_\varepsilon - \frac{1}{\mathfrak{c}_s^2}\partial_z^3\mathfrak{Z}_\varepsilon = -(\Gamma - 8)\mathrm{W}_\varepsilon\partial_z\mathfrak{Z}_\varepsilon - (\Gamma - 6)\mathfrak{Z}_\varepsilon\partial_z\mathrm{W}_\varepsilon - (\Gamma - 6)\mathrm{W}_\varepsilon\partial_z\mathrm{W}_\varepsilon - \frac{1}{\mathfrak{c}_s^2}\partial_z^3\mathrm{W}_\varepsilon. \tag{73}$$

As a consequence, by a crude estimate of the right-hand side of (73) and using (70), we have

$$\left\|\frac{2}{\mathfrak{c}_s}\partial_\tau\mathfrak{Z}_\varepsilon + \Gamma\mathfrak{Z}_\varepsilon\partial_z\mathfrak{Z}_\varepsilon - \frac{1}{\mathfrak{c}_s^2}\partial_z^3\mathfrak{Z}_\varepsilon\right\|_{H^{s-5}} \leqslant K\big(\big\|\mathrm{A}_\varepsilon^{\mathrm{in}} - \mathrm{U}_\varepsilon^{\mathrm{in}}\big\|_{H^{s-2}} + \varepsilon^2\big). \tag{74}$$

By very standard estimates involving (59) (since $\partial_z\mathfrak{Z}_\varepsilon \in L^\infty([0,\tau_0], L^\infty)$), we deduce that for $0 \leqslant \tau \leqslant \tau_0$,

$$\sup_{[0,\tau_0]} \|\mathfrak{Z}_\varepsilon - \tilde\zeta_\varepsilon\|_{H^{s-5}} \leqslant K\big(\big\|\mathrm{A}_\varepsilon^{\mathrm{in}} - \mathrm{U}_\varepsilon^{\mathrm{in}}\big\|_{H^{s-2}} + \varepsilon^2\big), \tag{75}$$

where $\tilde\zeta_\varepsilon$ is the solution of the initial value problem

$$\frac{2}{\mathfrak{c}_s}\partial_\tau\tilde\zeta_\varepsilon + \Gamma\tilde\zeta_\varepsilon\partial_z\tilde\zeta_\varepsilon - \frac{1}{\mathfrak{c}_s^2}\partial_z^3\tilde\zeta_\varepsilon = 0, \qquad (\tilde\zeta_\varepsilon)_{|\tau=0} = (\mathfrak{Z}_\varepsilon)_{|\tau=0} = \frac{\mathrm{A}_\varepsilon^{\mathrm{in}} + \mathrm{U}_\varepsilon^{\mathrm{in}}}{2}.$$

Since ζ_ε is the solution of the initial value problem

$$\frac{2}{\mathfrak{c}_s}\partial_\tau \zeta_\varepsilon + \Gamma \zeta_\varepsilon \partial_z \zeta_\varepsilon - \frac{1}{\mathfrak{c}_s^2}\partial_z^3 \zeta_\varepsilon = 0, \qquad (\zeta_\varepsilon)_{|\tau=0} = A_\varepsilon^{\mathrm{in}},$$

it follows that

$$\sup_{[0,\tau_0]} \|\zeta_\varepsilon - \tilde{\zeta}_\varepsilon\|_{H^{s-5}} \leqslant C(\tau_0) \big\| A_\varepsilon^{\mathrm{in}} - \mathrm{U}_\varepsilon^{\mathrm{in}} \big\|_{H^{s-2}},$$

hence

$$\sup_{[0,\tau_0]} \|\mathrm{A}_\varepsilon - \zeta_\varepsilon\|_{H^{s-5}} + \sup_{[0,\tau_0]} \|\mathrm{U}_\varepsilon - \zeta_\varepsilon\|_{H^{s-5}} \leqslant K\big(\big\|\mathrm{A}_\varepsilon^{\mathrm{in}} - \mathrm{U}_\varepsilon^{\mathrm{in}}\big\|_{H^{s-2}} + \varepsilon^2\big), \tag{76}$$

as wished. □

Proof of Proposition 3 ($d \geqslant 2$). A first approach for proving Proposition 3 is to use the arguments in [42], for some initial data satisfying some preparedness assumptions (Assumption 1, p. 2866 in [42]). We shall give another argument following the lines of the proof of Theorem 3, but we give some details since the preparation hypothesis $\mathrm{A}_\varepsilon \simeq U_\varepsilon^1$ is slightly different and since we used in [20] the conservation of the energy and momentum for (NLS).

As a first step, note that the estimates (17) can be shown as in the proof of Lemma 1. Furthermore, using (14), we infer from $(\mathcal{B}_\varepsilon)$

$$\begin{aligned} &\frac{1}{\mathfrak{c}_s}\partial_\tau\big(\mathrm{A}_\varepsilon - \mathrm{U}_\varepsilon^1\big) - \frac{2}{\varepsilon^2}\partial_{z_1}\big(\mathrm{A}_\varepsilon - \mathrm{U}_\varepsilon^1\big) + \big(2\mathrm{U}_\varepsilon^1 - \mathrm{A}_\varepsilon\big)\partial_{z_1}\big(\mathrm{A}_\varepsilon - \mathrm{U}_\varepsilon^1\big) + (\Gamma - 6)\mathrm{A}_\varepsilon \partial_{z_1}\mathrm{A}_\varepsilon \\ &\quad + \nabla_{z_\perp}\cdot \mathrm{U}_\varepsilon^\perp + \frac{1}{\mathfrak{c}_s^2}\partial_{z_1}^3 \mathrm{U}_\varepsilon^1 = \mathcal{O}_{L^2}(\varepsilon), \end{aligned} \tag{77}$$

so that, integrating by parts,

$$\begin{aligned} &\frac{1}{2\mathfrak{c}_s}\frac{d}{d\tau}\int_{\mathbb{R}^d} \big(\mathrm{A}_\varepsilon - \mathrm{U}_\varepsilon^1\big)^2\,dz - \int_{\mathbb{R}^d} \mathrm{U}_\varepsilon^\perp \cdot \nabla_{z_\perp}\big(\mathrm{A}_\varepsilon - \mathrm{U}_\varepsilon^1\big)\,dz \\ &\quad \leqslant \big(\big\|\partial_{z_1}\big[2\mathrm{U}_\varepsilon^1 - \mathrm{A}_\varepsilon\big]\big\|_{L^\infty} + 1\big)\big\|\mathrm{A}_\varepsilon - \mathrm{U}_\varepsilon^1\big\|_{L^2}^2 + \int_{\mathbb{R}^d}\bigg\{\frac{\Gamma - 6}{2}\mathrm{A}_\varepsilon^2 + \frac{1}{\mathfrak{c}_s^2}\partial_{z_1}^2 \mathrm{U}_\varepsilon^1\bigg\}\partial_{z_1}\big(\mathrm{A}_\varepsilon - \mathrm{U}_\varepsilon^1\big)\,dz + K\varepsilon^2. \end{aligned}$$

As for the proof of Proposition 2, we report $\frac{2}{\varepsilon^2}\partial_{z_1}(\mathrm{A}_\varepsilon - \mathrm{U}_\varepsilon^1)$ from (77) and integrate in time to get, by (14),

$$\begin{aligned} &\frac{2}{\mathfrak{c}_s}\big\|\big(\mathrm{A}_\varepsilon - \mathrm{U}_\varepsilon^1\big)(\tau)\big\|_{L^2}^2 - 4\int_0^\tau \int_{\mathbb{R}^d} \mathrm{U}_\varepsilon^\perp \cdot \nabla_{z_\perp}\big(\mathrm{A}_\varepsilon - \mathrm{U}_\varepsilon^1\big)\,dz\,d\bar{\tau} \\ &\quad \leqslant K\big\|\mathrm{A}_\varepsilon^{\mathrm{in}} - \mathrm{U}_\varepsilon^{\mathrm{in},1}\big\|_{L^2}^2 + K\varepsilon^2 + K\int_0^\tau \big\|\big(\mathrm{A}_\varepsilon - \mathrm{U}_\varepsilon^1\big)(\bar{\tau})\big\|_{L^2}^2\,d\bar{\tau} \\ &\qquad - \frac{\varepsilon^2}{\mathfrak{c}_s}\int_0^\tau\int_{\mathbb{R}^d}\big(\mathrm{A}_\varepsilon - \mathrm{U}_\varepsilon^1\big)\partial_\tau\bigg\{\frac{2}{\mathfrak{c}_s^2}\partial_{z_1}^2 \mathrm{U}_\varepsilon^1 - (\Gamma - 6)\mathrm{A}_\varepsilon^2\bigg\}\,dz\,d\bar{\tau}. \end{aligned}$$

Combining $(\mathcal{B}_\varepsilon)$ with (62), we have here again

$$\frac{\varepsilon^2}{\mathfrak{c}_s}\partial_\tau \partial_{z_1}^2 \mathrm{U}_\varepsilon^1 = -\partial_{z_1}^3\big(\mathrm{A}_\varepsilon - \mathrm{U}_\varepsilon^1\big) + \mathcal{O}_{L^2}(\varepsilon)$$

since $\varepsilon \mathrm{U}_\varepsilon^\perp$ is uniformly bounded in H^s. Furthermore, for the term involving $\partial_\tau[\mathrm{A}_\varepsilon^2]$, comparing with the case $d = 1$, we have the extra term

$$2(\Gamma-6)\frac{\varepsilon^2}{\mathfrak{c}_s}\int_0^\tau\int_{\mathbb{R}^d} \mathrm{A}_\varepsilon\big(\mathrm{A}_\varepsilon-\mathrm{U}_\varepsilon^1\big)\big[\nabla_{z_\perp}\cdot\mathrm{U}_\varepsilon^\perp+\mathcal{O}_{L^2}(\varepsilon)\big]\,d\bar\tau\leqslant K\varepsilon\big\|\mathrm{A}_\varepsilon-\mathrm{U}_\varepsilon^1\big\|_{L^2}\leqslant K\big\|\mathrm{A}_\varepsilon-\mathrm{U}_\varepsilon^1\big\|_{L^2}^2+K\varepsilon^2,$$

since $\varepsilon\mathrm{U}_\varepsilon^\perp$ is uniformly bounded in H^s. The loss ε instead of ε^2 seems unavoidable since we do not have cancellations with other terms (even those in the right-hand side of (77)). This leads to

$$\big\|\big(\mathrm{A}_\varepsilon-\mathrm{U}_\varepsilon^1\big)(\tau)\big\|_{L^2}^2-2\mathfrak{c}_s\int_0^\tau\int_{\mathbb{R}^d}\mathrm{U}_\varepsilon^\perp\cdot\nabla_{z_\perp}\big(\mathrm{A}_\varepsilon-\mathrm{U}_\varepsilon^1\big)\,dz\,d\bar\tau\leqslant K\big\|\mathrm{A}_\varepsilon^{\mathrm{in}}-\mathrm{U}_\varepsilon^{\mathrm{in},1}\big\|_{L^2}^2+K\varepsilon^2 + K\int_0^\tau\big\|\big(\mathrm{A}_\varepsilon-\mathrm{U}_\varepsilon^1\big)(\bar\tau)\big\|_{L^2}^2\,d\bar\tau. \tag{78}$$

On the other hand, since $\partial_{z_1}\mathrm{U}_\varepsilon^\perp=\nabla_{z_\perp}\mathrm{U}_\varepsilon^1$, we deduce from $(\mathcal{B}_\varepsilon)$

$$\frac{1}{2\mathfrak{c}_s}\frac{d}{d\tau}\int_{\mathbb{R}^d}\varepsilon^2\big|\mathrm{U}_\varepsilon^\perp\big|^2\,dz+\int_{\mathbb{R}^d}\mathrm{U}_\varepsilon^\perp\cdot\nabla_{z_\perp}\big(\mathrm{A}_\varepsilon-\mathrm{U}_\varepsilon^1\big)\,dz\leqslant\varepsilon^2\big\|\partial_{z_1}\mathrm{U}_\varepsilon^1+\varepsilon^2\nabla_{z_\perp}\mathrm{U}_\varepsilon^\perp\big\|_{L^\infty}\big\|\mathrm{U}_\varepsilon^\perp\big\|_{L^2}^2+K\varepsilon^2 \leqslant K\varepsilon^2\big\|\mathrm{U}_\varepsilon^\perp\big\|_{L^2}^2+K\varepsilon^2, \tag{79}$$

thus

$$\big\|\varepsilon\mathrm{U}_\varepsilon^\perp(\tau)\big\|_{L^2}^2+2\mathfrak{c}_s\int_0^\tau\int_{\mathbb{R}^d}\mathrm{U}_\varepsilon^\perp\cdot\nabla_{z_\perp}\big(\mathrm{A}_\varepsilon-\mathrm{U}_\varepsilon^1\big)\,dz\,d\bar\tau\leqslant\big\|\varepsilon\mathrm{U}_\varepsilon^{\mathrm{in},\perp}\big\|_{L^2}^2+K\int_0^\tau\big\|\varepsilon\mathrm{U}_\varepsilon^\perp\big\|_{L^2}^2\,d\bar\tau+K\varepsilon^2. \tag{80}$$

Consequently, in view of the cancellation of the integrals in the left-hand sides of (78) and (80),

$$\big\|\big(\mathrm{A}_\varepsilon-\mathrm{U}_\varepsilon^1\big)(\tau)\big\|_{L^2}^2+\big\|\varepsilon\mathrm{U}_\varepsilon^\perp(\tau)\big\|_{L^2}^2 \leqslant\big\|\big(\mathrm{A}_\varepsilon^{\mathrm{in}}-\mathrm{U}_\varepsilon^{\mathrm{in},1}\big)\big\|_{L^2}^2+\big\|\varepsilon\mathrm{U}_\varepsilon^{\mathrm{in},\perp}\big\|_{L^2}^2+K\int_0^\tau\big\|\big(\mathrm{A}_\varepsilon-\mathrm{U}_\varepsilon^1\big)(\bar\tau)\big\|_{L^2}^2+\big\|\varepsilon\mathrm{U}_\varepsilon^\perp\big\|_{L^2}^2\,d\bar\tau+K\varepsilon^2$$

hence, by the Gronwall lemma,

$$\sup_{0\leqslant\tau\leqslant\tau_0}\big\{\big\|\mathrm{A}_\varepsilon-\mathrm{U}_\varepsilon^1\big\|_{L^2}+\varepsilon\big\|\mathrm{U}_\varepsilon^\perp\big\|_{L^2}\big\}\leqslant K\big(\big\|\mathrm{A}_\varepsilon^{\mathrm{in}}-\mathrm{U}_\varepsilon^{\mathrm{in},1}\big\|_{L^2}+\varepsilon\big\|\mathrm{U}_\varepsilon^{\mathrm{in},\perp}\big\|_{L^2}+\varepsilon\big),$$

as wished for (18).

Finally, using (14) and (17), we deduce from $(\mathcal{B}_\varepsilon)$ that

$$\frac{1}{\mathfrak{c}_s}\partial_\tau\mathrm{A}_\varepsilon-\frac{1}{\varepsilon^2}\partial_{z_1}\mathrm{A}_\varepsilon+\frac{1}{\varepsilon^2}\partial_{z_1}\mathrm{U}_\varepsilon^1+3\mathrm{A}_\varepsilon\partial_{z_1}\mathrm{A}_\varepsilon+\nabla_{z_\perp}\cdot\mathrm{U}_\varepsilon^\perp=\mathcal{O}_{L^2}\big(\varepsilon+\big\|\partial_{z_1}\big(\mathrm{A}_\varepsilon^{\mathrm{in}}-\mathrm{U}_\varepsilon^{\mathrm{in},1}\big)\big\|_{H^{s-2}}\big)$$

and

$$\frac{1}{\mathfrak{c}_s}\partial_\tau\mathrm{U}_\varepsilon^1-\frac{1}{\varepsilon^2}\partial_{z_1}\mathrm{U}_\varepsilon^1+\frac{1}{\varepsilon^2}\partial_{z_1}\mathrm{A}_\varepsilon+(\Gamma-3)\mathrm{U}_\varepsilon^1\partial_{z_1}\mathrm{U}_\varepsilon^1-\frac{1}{\mathfrak{c}_s^2}\partial_{z_1}^3\mathrm{U}_\varepsilon^1=\mathcal{O}_{L^2}\big(\varepsilon+\big\|\partial_{z_1}\big(\mathrm{A}_\varepsilon^{\mathrm{in}}-\mathrm{U}_\varepsilon^{\mathrm{in},1}\big)\big\|_{H^{s-2}}\big),$$

hence

$$\left|\frac{1}{2\mathfrak{c}_s}\frac{d}{d\tau}\int_{\mathbb{R}^d}\mathrm{A}_\varepsilon^2+\big[\mathrm{U}_\varepsilon^1\big]^2\,dz-\int_{\mathbb{R}^d}\mathrm{U}_\varepsilon^\perp\cdot\nabla_{z_\perp}\mathrm{A}_\varepsilon\,dz\right|\leqslant K\big(\varepsilon^2+\big\|\partial_{z_1}\big(\mathrm{A}_\varepsilon^{\mathrm{in}}-\mathrm{U}_\varepsilon^{\mathrm{in},1}\big)\big\|_{H^{s-2}}^2\big).$$

Inserting (18) into (79) and since $\nabla_{z_\perp}\mathrm{U}_\varepsilon^1=\partial_{z_1}\mathrm{U}_\varepsilon^\perp$, we infer

$$\left|\frac{1}{2\mathfrak{c}_s}\frac{d}{d\tau}\int_{\mathbb{R}^d}\varepsilon^2\big|\mathrm{U}_\varepsilon^\perp\big|^2\,dz+\int_{\mathbb{R}^d}\mathrm{U}_\varepsilon^\perp\cdot\nabla_{z_\perp}\mathrm{A}_\varepsilon\,dz\right|\leqslant\left|\int_{\mathbb{R}^d}\mathrm{U}_\varepsilon^\perp\cdot\big(\partial_{z_1}\mathrm{U}_\varepsilon^\perp\big)\,dz\right|+K\varepsilon^2\big\|\mathrm{U}_\varepsilon^\perp\big\|_{L^2}^2+K\varepsilon^2 \leqslant 0+K\big\|\mathrm{A}_\varepsilon^{\mathrm{in}}-\mathrm{U}_\varepsilon^{\mathrm{in},1}\big\|_{L^2}^2+K\varepsilon^2\big\|\mathrm{U}_\varepsilon^{\mathrm{in},\perp}\big\|_{L^2}^2+K\varepsilon^2,$$

and then

$$\left|\frac{1}{2\mathfrak{c}_s}\frac{d}{d\tau}\int\limits_{\mathbb{R}^d} \mathrm{A}_\varepsilon^2+\big[\mathrm{U}_\varepsilon^1\big]^2+\varepsilon^2\big|\mathrm{U}_\varepsilon^\perp\big|^2\,dz\right|\leqslant K\big\|\mathrm{A}_\varepsilon^{\mathrm{in}}-\mathrm{U}_\varepsilon^{\mathrm{in},1}\big\|_{L^2}^2+K\varepsilon^2\big\|\mathrm{U}_\varepsilon^{\mathrm{in},\perp}\big\|_{L^2}^2+K\varepsilon^2.$$

Combining this with (18), this allows to show the almost conservation law, for $0\leqslant\tau\leqslant\tau_*$,

$$\int\limits_{\mathbb{R}^d} 2\mathrm{A}_\varepsilon^2+\varepsilon^2\big|\mathrm{U}_\varepsilon^\perp\big|^2\,dz=\int\limits_{\mathbb{R}^d} 2\big[\mathrm{A}_\varepsilon^{\mathrm{in}}\big]^2+\varepsilon^2\big|\mathrm{U}_\varepsilon^{\mathrm{in},\perp}\big|^2\,dz+\mathcal{O}\big(\big\|\mathrm{A}_\varepsilon^{\mathrm{in}}-\mathrm{U}_\varepsilon^{\mathrm{in},1}\big\|_{L^2(\mathbb{R}^d)}^2+\varepsilon^2\big\|\mathrm{U}_\varepsilon^{\mathrm{in},\perp}\big\|_{L^2(\mathbb{R}^d)}^2+\varepsilon^2\big),\tag{81}$$

and a similar statement can be made with $\int_{\mathbb{R}^d} 2[\mathrm{U}_\varepsilon^1]^2+\varepsilon^2|\mathrm{U}_\varepsilon^\perp|^2\,dz$.

At this stage, we note that we have all the ingredients needed for the proof of Theorem 6 in [20]. Indeed, in [20], we used the conservation of the energy and the momentum to show that

$$\mathrm{A}_\varepsilon-\mathrm{U}_\varepsilon^1\to 0\quad\text{and}\quad \varepsilon\mathrm{U}_\varepsilon^{\mathrm{in},\perp}\to 0\quad\text{in } L^2,$$

but here, the estimate (18) ensures these convergences. Furthermore, the uniform L^2 bound on $\mathrm{U}_\varepsilon^\perp$ comes directly from (18) and the assumptions in Proposition 3 and provide (see [20]) compactness in time. Then, the almost conservation law (81) guarantees that there is no loss of L^2 norm in the compactness argument. This finishes the proof. □

3. Proof of Proposition 4

The proof of Proposition 4 turns out to be quite similar to the justification of the (KdV) limit. Indeed, we use once again the trick of E. Grenier and first solve the one dimensional system, with a_ε complex-valued,

$$\begin{cases}\dfrac{1}{\mathfrak{c}_s}\partial_\theta a_\varepsilon-\dfrac{1}{\varepsilon}\partial_z a_\varepsilon+2u_\varepsilon\partial_z a_\varepsilon+\dfrac{1}{\varepsilon}(1+\varepsilon a_\varepsilon)\partial_z a_\varepsilon=\dfrac{i}{\mathfrak{c}_s}\partial_z^2 a_\varepsilon\\ \dfrac{1}{\mathfrak{c}_s}\partial_\theta u_\varepsilon-\dfrac{1}{\varepsilon}\partial_z u_\varepsilon+2u_\varepsilon\partial_z u_\varepsilon+\dfrac{1}{\mathfrak{c}_s^2\varepsilon^2}\partial_z\big(f\big(|1+\varepsilon a_\varepsilon|^2\big)\big)=0.\end{cases}\tag{82}$$

Following [20] (proof of Theorem 4 there) or the proof of Proposition 1, we see that there exists $\theta_*>0$ and $\varepsilon_0>0$ such that, if $0<\varepsilon\leqslant\varepsilon_0$, there exists a unique solution $(a_\varepsilon,u_\varepsilon)\in L^\infty([0,\theta_*],H^s(\mathbb{R}))$ to (82). Moreover, for some absolute constant K_0, there holds the uniform bound

$$\sup_{0\leqslant\theta\leqslant\theta_*}\big\|a_\varepsilon(\theta)\big\|_{H^s(\mathbb{R})}+\big\|u_\varepsilon(\theta)\big\|_{H^s(\mathbb{R})}\leqslant K_0\Lambda\quad\text{and}\quad\frac{1}{2}\leqslant|1+\varepsilon a_\varepsilon|\leqslant 2.$$

As in the proof of Lemma 1, we may show that

$$\big\|\partial_z\big[a_\varepsilon(\theta)-u_\varepsilon(\theta)\big]\big\|_{H^{s-2}(\mathbb{R})}\leqslant K\big(\big\|\partial_z\big[a_\varepsilon^{\mathrm{in}}-u_\varepsilon^{\mathrm{in}}\big]\big\|_{H^{s-2}(\mathbb{R})}+\varepsilon\big).$$

Via the formula (63), this yields (45) and

$$\big\|\partial_z\big[A_\varepsilon(\theta)-U_\varepsilon(\theta)\big]\big\|_{H^{s-2}(\mathbb{R})}\leqslant K\big(\big\|\partial_z\big[A_\varepsilon^{\mathrm{in}}-U_\varepsilon^{\mathrm{in}}\big]\big\|_{H^{s-2}(\mathbb{R})}+\varepsilon\big).$$

Combining the two equations in (42), we deduce

$$\begin{aligned}&\frac{1}{\mathfrak{c}_s}\partial_\theta(A_\varepsilon-U_\varepsilon)-\frac{2}{\varepsilon}\partial_z(A_\varepsilon-U_\varepsilon)+(2U_\varepsilon-A_\varepsilon)\partial_z(A_\varepsilon-U_\varepsilon)-(\Gamma-5)A_\varepsilon\partial_z A_\varepsilon\\&\quad=-\frac{\varepsilon}{\mathfrak{c}_s}\partial_z\left(\frac{\partial_z^2 A_\varepsilon}{1+\varepsilon A_\varepsilon}\right)+\frac{1}{\mathfrak{c}_s^2\varepsilon^2}\partial_z\big(f_3(\varepsilon A_\varepsilon)\big),\end{aligned}$$

where $f_3(\alpha)=\mathcal{O}_{\alpha\to 0}(\alpha^3)$ is given by the Taylor expansion (4) of f. In particular, since $s\geqslant 3$, we infer

$$\frac{1}{\mathfrak{c}_s}\partial_\theta(A_\varepsilon-U_\varepsilon)-\frac{2}{\varepsilon}\partial_z(A_\varepsilon-U_\varepsilon)+(2U_\varepsilon-A_\varepsilon)\partial_z(A_\varepsilon-U_\varepsilon)-(\Gamma-5)A_\varepsilon\partial_z A_\varepsilon=\mathcal{O}_{L^2}(\varepsilon),$$

uniformly for $0\leqslant\theta\leqslant\theta_*$. The L^2 estimate for $A_\varepsilon-U_\varepsilon$ can then be derived by following the lines of the proof of Proposition 2, using the fact that $A_\varepsilon\partial_z A_\varepsilon=\partial_z(A_\varepsilon^2/2)$. Once the estimate

$$\sup_{0\leqslant\theta\leqslant\theta_*}\big\|A_\varepsilon(\theta)-U_\varepsilon(\theta)\big\|_{H^{s-1}(\mathbb{R})}\leqslant K\big(\big\|A_\varepsilon^{\mathrm{in}}-U_\varepsilon^{\mathrm{in}}\big\|_{H^{s-1}(\mathbb{R})}+\varepsilon\big)$$

is shown, we have

$$\frac{1}{\mathfrak{c}_s}\partial_\theta(A_\varepsilon+U_\varepsilon)+2U_\varepsilon\partial_z(A_\varepsilon+U_\varepsilon)+A_\varepsilon\partial_zU_\varepsilon+(\Gamma-5)A_\varepsilon\partial_zA_\varepsilon=\mathcal{O}_{H^{s-3}}(\varepsilon),$$

or

$$\frac{1}{\mathfrak{c}_s}\partial_\theta(A_\varepsilon+U_\varepsilon)+\Gamma(A_\varepsilon+U_\varepsilon)\partial_z(A_\varepsilon+U_\varepsilon)=\mathcal{O}_{H^{s-3}}(\varepsilon).$$

The result follows then from a classical comparison argument involving (59) similar to the proof of Proposition 2 (see Section 2.3).

4. Justification of the (gKdV)/(gKP-I) limit as a large time asymptotics for the free wave regime

This section is devoted to the proofs of Theorems 6 and 7. We wish to solve the (NLS) equation by using the trick of E. Grenier, that is to solve the system (61) written in our scaling with $\theta=\varepsilon\epsilon t$ and where a_ε is complex-valued and u_ε real-valued:

$$\begin{cases}\dfrac{1}{\mathfrak{c}_s}\partial_\theta a_\varepsilon-\dfrac{1}{\epsilon}\partial_{z_1}a_\varepsilon+2u_\varepsilon^1\partial_{z_1}a_\varepsilon+2\epsilon^{m+1}u_\varepsilon^\perp\cdot\nabla_{z_\perp}a_\varepsilon+\dfrac{1}{\epsilon}(1+\epsilon a_\varepsilon)\big(\partial_{z_1}u_\varepsilon^1+\epsilon^{m+1}\nabla_{z_\perp}\cdot u_\varepsilon^\perp\big)\\ \quad=\dfrac{i\varepsilon}{\epsilon\mathfrak{c}_s}\big(\partial_{z_1}^2a_\varepsilon+\epsilon^{m+1}\Delta_{z_\perp}a_\varepsilon\big)\\ \dfrac{1}{\mathfrak{c}_s}\partial_\theta u_\varepsilon-\dfrac{1}{\epsilon}\partial_{z_1}u_\varepsilon+2u_\varepsilon^1\partial_{z_1}u_\varepsilon+2\varepsilon^2u_\varepsilon^\perp\cdot\nabla_{z_\perp}u_\varepsilon+\dfrac{2f'(|1+\epsilon a_\varepsilon|^2)}{\epsilon\mathfrak{c}_s^2}\langle 1+\epsilon a_\varepsilon,\nabla_z a_\varepsilon\rangle=0.\end{cases}\tag{83}$$

The initial data $(a_\varepsilon^{\mathrm{in}},u_\varepsilon^{\mathrm{in}})$ will be chosen appropriately later on, so that the natural relation

$$\frac{\nabla\Psi^{\mathrm{in}}}{\Psi^{\mathrm{in}}}=\frac{\epsilon\nabla_z a^{\mathrm{in}}}{1+\epsilon a^{\mathrm{in}}}+i\mathfrak{c}_s\frac{\epsilon}{\varepsilon}u_\varepsilon^{\mathrm{in}}\tag{84}$$

holds true (since we have $\Psi=(1+\epsilon a_\varepsilon)\mathrm{e}^{i\epsilon\varphi_\varepsilon/\varepsilon}$). The proof is divided in two steps. In the first one, we construct an approximate solution, and then prove an error estimate.

4.1. Construction of an approximate solution

In view of the coefficient ε/ϵ in front of the dispersive term in (83) and since we expect a_ε real-valued at leading order, it is natural to look for the approximate solution with an expansion of the form:

$$\begin{cases}a_\varepsilon^{\mathrm{app}}=\mathfrak{a}+i\varepsilon\mathfrak{b}=\big(\mathfrak{a}_0+\epsilon\mathfrak{a}_1+\epsilon^2\mathfrak{a}_2+\cdots+\epsilon^m\mathfrak{a}_m+\cdots\big)+i\varepsilon\big(\mathfrak{b}_0+\epsilon\mathfrak{b}_1+\epsilon^2\mathfrak{b}_2+\cdots+\epsilon^m\mathfrak{b}_m+\cdots\big)\\ u_\varepsilon^{\mathrm{app}}=\mathfrak{u}_0+\epsilon\mathfrak{u}_1+\epsilon^2\mathfrak{u}_2+\cdots+\epsilon^m\mathfrak{u}_m+\cdots,\end{cases}$$

where the functions $\mathfrak{u}_k$, $\mathfrak{a}_k$ and $\mathfrak{b}_k$ depend on the variables (z,τ) and are real-valued. We may try to construct this approximate solution by cancellation of the powers of ϵ in (83) until we have solved the equations up to an $\mathcal{O}(\epsilon^{m+1})$ error. However, we have to pay attention to some point for the imaginary part $\mathfrak{b}$. Indeed, if we assume $\mathfrak{b}=\mathfrak{b}(\tau)$ only and since $\partial_\theta=\varepsilon^2/\epsilon\partial_\tau=\epsilon^m\partial_\tau$, the imaginary part of the first equation in (83) then reduces to

$$\begin{aligned}0&=\varepsilon\epsilon^m\frac{1}{\mathfrak{c}_s}\partial_\tau\mathfrak{b}-\frac{\varepsilon}{\epsilon}\partial_{z_1}\mathfrak{b}+2\varepsilon u_\varepsilon^{\mathrm{app},1}\partial_{z_1}\mathfrak{b}+\varepsilon\mathfrak{b}\partial_{z_1}u_\varepsilon^{\mathrm{app},1}-\frac{\varepsilon}{\epsilon\mathfrak{c}_s}\partial_{z_1}^2\mathfrak{a}+\mathcal{O}\big(\epsilon^{m+1}\big)\\&=-\frac{\varepsilon}{\epsilon}\left(\partial_{z_1}\mathfrak{b}+\frac{1}{\mathfrak{c}_s}\partial_{z_1}^2\mathfrak{a}-2\epsilon u_\varepsilon^{\mathrm{app},1}\partial_{z_1}\mathfrak{b}-\epsilon\mathfrak{b}\partial_{z_1}u_\varepsilon^{\mathrm{app},1}\right)+\mathcal{O}\big(\epsilon^{m+1}\big).\end{aligned}\tag{85}$$

From the fact that $\varepsilon/\epsilon=\epsilon^{\frac{m-1}{2}}\gg\epsilon^{m+1}$ (for every $m\geqslant 0$), it is thus necessary to choose

$$\mathfrak{b}_0=-\frac{1}{\mathfrak{c}_s}\partial_{z_1}\mathfrak{a}_0.\tag{86}$$

However, to solve at next orders, we need to solve an ode in z_1 and not a time dependent problem. For instance, for $\mathfrak{b}_1$, this becomes

$$\mathfrak{c}_s\partial_{z_1}\mathfrak{b}_1+\partial^2_{z_1}\mathfrak{a}_1=-2u^1_0\partial^2_{z_1}\mathfrak{a}_0-\partial_{z_1}\mathfrak{a}_0\partial_{z_1}u^1_0=-2\partial_{z_1}[\mathfrak{a}_0\partial_{z_1}\mathfrak{a}_0]+[\partial_{z_1}\mathfrak{a}_0]^2,$$

since we shall have $u^1_0=\mathfrak{a}_0$. Clearly, this problem cannot be solved with $\mathfrak{b}_1$ in some H^s space exactly, since the source term does not have zero integral in z_1. To remedy this problem, we shall roughly speaking let $\mathfrak{b}_1,\mathfrak{b}_2,\dots$ depend on θ.

Let us define the consistency errors

$$-\mathcal{R}_a\equiv\frac{1}{\mathfrak{c}_s}\frac{\partial a^{\rm app}_\varepsilon}{\partial\theta}-\frac{1}{\epsilon}\partial_{z_1}a^{\rm app}_\varepsilon+2u^{\rm app,1}_\varepsilon\partial_{z_1}a^{\rm app}_\varepsilon+2\epsilon^{m+1}u^{\rm app,\perp}_\varepsilon\cdot\nabla_{z_\perp}a^{\rm app}_\varepsilon+\frac{1}{\epsilon}\big(1+\epsilon a^{\rm app}_\varepsilon\big)\big(\partial_{z_1}u^{\rm app,1}_\varepsilon+\epsilon^{m+1}\nabla_{z_\perp}\cdot u^{\rm app,\perp}_\varepsilon\big)-\frac{i\varepsilon}{\mathfrak{c}_s\epsilon}\big(\partial^2_{z_1}a^{\rm app}_\varepsilon+\epsilon^{m+1}\Delta_{z_\perp}a^{\rm app}_\varepsilon\big),\tag{87}$$

and

$$-\mathcal{R}_u\equiv\frac{1}{\mathfrak{c}_s}\partial_\theta u^{\rm app}_\varepsilon-\frac{1}{\epsilon}\partial_{z_1}u^{\rm app}_\varepsilon+2u^{\rm app,1}_\varepsilon\partial_{z_1}u^{\rm app}_\varepsilon+2\varepsilon^2u^{\rm app,\perp}_\varepsilon\cdot\nabla_{z_\perp}u^{\rm app}_\varepsilon+\frac{2f'(|1+\epsilon a^{\rm app}_\varepsilon|^2)}{\epsilon\mathfrak{c}_s^2}\big\langle1+\epsilon a^{\rm app}_\varepsilon,\nabla_z a^{\rm app}_\varepsilon\big\rangle.\tag{88}$$

The next lemma provides the construction of an approximate solution $(a^{\rm app}_\varepsilon,u^{\rm app}_\varepsilon)$ for the one dimensional case (Theorem 6). The changes required for Theorem 7 will be given next.

Lemma 2. *Assume $d=1$. Under the assumptions of Theorem 6, there exist initial data $(a^{\rm in}_\varepsilon,u^{\rm in}_\varepsilon)$ satisfying (84) and an approximate solution $(a^{\rm app}_\varepsilon,u^{\rm app}_\varepsilon)$ such that we have*

$$\|\mathcal{R}_a\|_{H^s}+\|\mathcal{R}_u\|_{H^s}\leqslant C\epsilon^{m+1}$$

as well as

$$\big\|a^{\rm in}_\varepsilon-a^{\rm app}_\varepsilon(\theta=0)\big\|_{H^s}+\big\|u^{\rm in}_\varepsilon-u^{\rm app}_\varepsilon(\theta=0)\big\|_{H^s}\leqslant C\epsilon^{m+1}.\tag{89}$$

Proof. The proof is divided in 4 steps.

Step 1: Definition of the approximate solution. We set

$$\mathfrak{a}_0=\mathfrak{u}_0\equiv\zeta(\tau)\in\mathcal{C}\big([0,\tau_*],H^{s+5}\big),\qquad\mathfrak{a}_1=\dots=\mathfrak{a}_{m-1}\equiv0$$

(if $m=1$, the second condition is void), so that $\mathfrak{a}=\mathfrak{a}_0+\mathcal{O}(\epsilon^m)$. We have seen in Section 1.3 that in order to cancel out the terms of order $\epsilon^{-1},\epsilon^0,\dots,\epsilon^{m-1}$, then (30) must hold true, that is

$$\mathfrak{u}=\mathfrak{a}-\frac{3}{2}\epsilon\mathfrak{a}^2+2\epsilon^2\mathfrak{a}^3-\frac{5}{2}\epsilon^3\mathfrak{a}^4+\dots+(-1)^m\frac{m+2}{2}\epsilon^m\mathfrak{a}^{m+1}+\mathcal{O}\big(\epsilon^{m+1}\big).$$

Therefore, we also set

$$\forall1\leqslant k\leqslant m-1,\quad\mathfrak{u}_k\equiv(-1)^k\frac{k+2}{2}\zeta^{k+1}(\tau)\in\mathcal{C}\big([0,\tau_*],H^{s+5}\big),$$

fix

$$\mathfrak{a}_m(\tau,z)\equiv A^{\rm in}_m(z)\in H^{s+5},\qquad\mathfrak{u}_m(\tau,z)\equiv A^{\rm in}_m(z)+(-1)^m\frac{m+2}{2}\zeta^{m+1}(\tau,z)\in H^{s+5}$$

and choose $\mathfrak{u}_{m+1}\equiv0$ and

$$\mathfrak{a}_{m+1}\equiv-\frac{1}{2\mathfrak{c}_s^2}\partial^2_{z_1}\zeta-\frac{1}{2}\left(\sum_{k=0}^m\mathfrak{u}_k(\tau)\mathfrak{u}_{m-k}(\tau)\right)-\frac{1}{2}\left(\left[q_{m+2}+(-1)^m\frac{m+3}{2}\right]\zeta^{m+2}(\tau)-2\zeta(\tau)\mathfrak{a}_m(\tau)\right)\in L^\infty\big([0,\tau_*],H^{s+2}\big),$$

which is some (arbitrary) solution to what will be an analogue of (19). Concerning the imaginary part $\mathfrak{b}$, we recall (see (85)) that we wish to solve

$$0 = \frac{\varepsilon}{\mathfrak{c}_s}\partial_\theta \mathfrak{b} - \frac{\varepsilon}{\epsilon}\partial_z \mathfrak{b} + 2\varepsilon u_\varepsilon^{\mathrm{app}}\partial_z \mathfrak{b} + \varepsilon \mathfrak{b}\partial_z u_\varepsilon^{\mathrm{app}} - \frac{\varepsilon}{\epsilon \mathfrak{c}_s}\partial_z^2 \mathfrak{a} + \mathcal{O}\big(\epsilon^{m+1}\big)$$

up to $\mathcal{O}(\epsilon^{m+1})$, and since $\frac{\varepsilon}{\epsilon} = \epsilon^{\frac{m-1}{2}}$, this requires to solve

$$\frac{\epsilon}{\mathfrak{c}_s}\partial_\theta \mathfrak{b} - \partial_z \mathfrak{b} + \frac{1}{\mathfrak{c}_s}\partial_z^2 \mathfrak{a} - 2\epsilon u_\varepsilon^{\mathrm{app}}\partial_z \mathfrak{b} - \epsilon \mathfrak{b}\partial_z u_\varepsilon^{\mathrm{app}} = 0$$

up to $\mathcal{O}(\epsilon^{\frac{m+3}{2}})$. For that purpose, we first define (*cf.* (86))

$$\underline{\mathfrak{b}} \equiv -\frac{1}{\mathfrak{c}_s}\partial_z \mathfrak{a}(\tau) \in \mathcal{C}\big([0,\tau_*], H^{s+1}\big),$$

and we omit the dependency on ε to simplify the notations. Next, we set $\underline{u} \equiv u_\varepsilon^{\mathrm{app}}$ (here again, it depends on ε) and define the function $\tilde{\mathfrak{b}}_\varepsilon = \tilde{\mathfrak{b}}_\varepsilon(\theta)$ as the solution of the high speed transport equation

$$\frac{1}{\mathfrak{c}_s}\partial_\theta \tilde{\mathfrak{b}}_\varepsilon - \frac{1}{\epsilon}\partial_z \tilde{\mathfrak{b}}_\varepsilon + 2\underline{u}\partial_z \tilde{\mathfrak{b}}_\varepsilon + \tilde{\mathfrak{b}}_\varepsilon \partial_z \underline{u} = \frac{G}{\mathfrak{c}_s} \equiv -2\underline{u}\partial_z \underline{\mathfrak{b}} - \underline{\mathfrak{b}}\partial_z \underline{u}, \quad \tilde{\mathfrak{b}}_\varepsilon(\theta = 0) = 0, \tag{90}$$

and finally set

$$\mathfrak{b} \equiv \underline{\mathfrak{b}} + \tilde{\mathfrak{b}}_\varepsilon.$$

We shall prove that $\tilde{\mathfrak{b}}_\varepsilon$ is rather small.

Step 2: Sobolev estimates for $\tilde{\mathfrak{b}}_\varepsilon$. The basic idea is to consider the simplified one dimensional problem, where the source term is independent of θ (the source term in (90) depends on $\tau = \varepsilon^m\theta$):

$$\partial_\theta \beta - \frac{1}{\epsilon}\partial_z \beta = g(z), \qquad \beta(\theta = 0) = 0,$$

with solution given by the method of characteristics:

$$\beta(\theta, z) = \int_0^\theta g\left(z + \frac{\theta - \bar{\theta}}{\epsilon}\right) d\bar{\theta}.$$

From this formula, it comes $\beta(\theta, z) = \epsilon \int_z^{z-\theta/\epsilon} g$, which shows that β is small in L^∞ if $g \in L^1$ and that $\partial_z \beta$ is small in L^∞ simply assuming $g \in L^\infty$. We shall follow the same type of compuations for (90). For the extra term $\underline{\mathfrak{b}}\partial_z \underline{u}$, we shall use that $\partial_z \underline{u}$ has a bounded antiderivative (even though $\partial_z \underline{u} \notin L^1_{z_1}(\mathbb{R})$). We use the method of characteristics and introduce the solution Z (we omit the dependency on ϵ) to the problem

$$\partial_\theta Z(\theta, y) = -\frac{\mathfrak{c}_s}{\epsilon} + 2\mathfrak{c}_s \underline{u}\big(\epsilon^m\theta, Z(\theta, y)\big), \qquad Z(\theta = 0, y) = y.$$

Since $\underline{u}$ is uniformly Lipschitz continuous in z for $\theta \in [0, \epsilon^{-1}]$, the flow Z is well defined for $\theta \in [0, \epsilon^{-1}]$ and verifies, for some constant $C \geqslant 1$ independent of $\epsilon \leqslant 1$ and $\theta \in [0, \epsilon^{-1}]$,

$$\left| Z(\theta, y) - y + \frac{\theta}{\epsilon} \right| \leqslant C. \tag{91}$$

We now consider ϵ small enough so that $2\epsilon \|\underline{u}\|_{L^\infty([0,\tau_*]\times\mathbb{R})} \leqslant 1/2$. Applying the method of characteristics, we see that $\tilde{\mathfrak{b}}_\varepsilon$ satisfies, for every $y \in \mathbb{R}$ and $\theta \in [0, |\ln \epsilon|]$,

$$\frac{d}{d\theta}\Big(\tilde{\mathfrak{b}}_\varepsilon\big(\theta, Z(\theta, y)\big)\Big) + \mathfrak{c}_s \tilde{\mathfrak{b}}_\varepsilon\big(\theta, Z(\theta, y)\big)\partial_z \underline{u}\big(\epsilon^m\theta, Z(\theta, y)\big) = G\big(\epsilon^m\theta, Z(\theta, y)\big). \tag{92}$$

As a consequence, by Duhamel's formula,

$$\tilde{\mathfrak{b}}_\varepsilon\big(\theta, Z(\theta, y)\big) = \int_0^\theta \exp\left(\mathfrak{c}_s \int_{\bar\theta}^{\theta} \partial_{z_1}\underline{u}\big(\epsilon^m\theta', Z\big(\theta', y\big)\big)\, d\theta'\right) G\big(\epsilon^m\bar\theta, Z(\bar\theta, y)\big)\, d\bar\theta. \tag{93}$$

Let us now estimate the integral in the exponential in (93) by writing and using the change of variables $\mathrm{y} = Z(\theta', y)$, or $\theta' = \theta'_y(\mathrm{y})$

$$\begin{aligned}
\int_{\bar\theta}^{\theta} \partial_z\underline{u}\big(\epsilon^m\theta', Z\big(\theta', y\big)\big)\, d\theta' &= \int_{\bar\theta}^{\theta}\left\{\partial_z\underline{u}^{\mathrm{in}}\big(Z\big(\theta', y\big)\big) + \int_0^{\epsilon^m\theta'} \partial_\tau\partial_z\underline{u}\big(\underline{\theta}, Z\big(\theta', y\big)\big)\, d\underline{\theta}\right\} d\theta' \\
&= \int_{Z(\theta,y)}^{Z(\bar\theta,y)}\left\{\partial_z\underline{u}^{\mathrm{in}}(\mathrm{y}) + \int_0^{\epsilon^m\theta'_y(\mathrm{y})} \partial_\tau\partial_z\underline{u}(\underline{\theta}, \mathrm{y})\, d\underline{\theta}\right\} \frac{\epsilon\, d\mathrm{y}}{1 - 2\epsilon\underline{u}(\theta'_y(\mathrm{y}), \mathrm{y})} \\
&= \int_{Z(\theta,y)}^{Z(\bar\theta,y)} \epsilon\partial_z\underline{u}^{\mathrm{in}}(\mathrm{y})\, d\mathrm{y} + \int_{Z(\theta,y)}^{Z(\bar\theta,y)}\int_0^{\epsilon^m\theta'_y(\mathrm{y})} \epsilon\partial_\tau\partial_z\underline{u}(\underline{\theta}, \mathrm{y})\, d\underline{\theta}\, d\mathrm{y} \\
&\quad + \int_{Z(\theta,y)}^{Z(\bar\theta,y)}\left\{\partial_z\underline{u}^{\mathrm{in}}(\mathrm{y}) + \int_0^{\epsilon^m\theta'_y(\mathrm{y})} \partial_\tau\partial_z\underline{u}(\underline{\theta}, \mathrm{y})\, d\underline{\theta}\right\} \frac{2\epsilon^2\underline{u}(\theta'_y(\mathrm{y}), \mathrm{y})\, d\mathrm{y}}{1 - 2\epsilon\underline{u}(\theta'_y(\mathrm{y}), \mathrm{y})}.
\end{aligned}$$

We infer by Cauchy–Schwarz that the third integral is $\leqslant C\epsilon^2$. Moreover, by direct computation, the first one is equal to

$$\epsilon\big\{\underline{u}^{\mathrm{in}}\big(Z(\bar\theta, y)\big) - \underline{u}^{\mathrm{in}}\big(Z(\theta, y)\big)\big\},$$

and since $|Z(\theta, y) - Z(\bar\theta, y)| \leqslant C|\theta - \bar\theta|/\epsilon$, we infer by Cauchy–Schwarz that the second integral is

$$C\epsilon \times \sqrt{\theta/\epsilon} \times \epsilon^m|\ln\epsilon| \leqslant C|\ln\epsilon|^{3/2}\epsilon^{m+1/2},$$

uniformly in $0 \leqslant \bar\theta \leqslant \theta \leqslant |\ln\epsilon|$, y. As a consequence,

$$\int_{\bar\theta}^{\theta} \partial_z\underline{u}\big(\epsilon^m\theta', Z\big(\theta', y\big)\big)\, d\theta' = \epsilon\big\{\underline{u}^{\mathrm{in}}\big(Z(\bar\theta, y)\big) - \underline{u}^{\mathrm{in}}\big(Z(\theta, y)\big)\big\} + \mathcal{O}\big(\epsilon^2\big), \tag{94}$$

and using once again the change of variables $\mathrm{y} = Z(\bar\theta, y)$, it then follows that

$$\begin{aligned}
\big|\tilde{\mathfrak{b}}_\varepsilon\big(\theta, Z(\theta, y)\big)\big| &\leqslant C\int_0^\theta |G|\big(\epsilon^m\bar\theta, Z(\bar\theta, y)\big)\, d\bar\theta \\
&\leqslant C\int_0^\theta \big|G^{\mathrm{in}}\big|\big(Z(\bar\theta, y)\big)\, d\bar\theta + C\int_0^\theta\int_0^{\epsilon^m\theta'} |\partial_\tau G|\big(\underline{\theta}, Z(\bar\theta, y)\big)\, d\underline{\theta}\, d\bar\theta \\
&\leqslant C\int_{Z(\theta,y)}^{y} \big|G^{\mathrm{in}}\big|(\mathrm{y}) \frac{\epsilon\, d\mathrm{y}}{1 - 2\epsilon u_\varepsilon^{\mathrm{app},1}(\theta'_y(\mathrm{z}), \mathrm{y})} \\
&\quad + C\int_{Z(\theta,y)}^{y}\int_0^{\epsilon^m|\ln\epsilon|} |\partial_\tau G|(\underline{\theta}, \mathrm{y})\, d\underline{\theta} \frac{\epsilon\, d\mathrm{y}}{1 - 2\epsilon u_\varepsilon^{\mathrm{app},1}(\theta'_y(\mathrm{y}), \mathrm{y})}.
\end{aligned}$$

We now fix $z \in \mathbb{R}$ and let $y = Z(\theta, \cdot)^{-1}(z)$ in the above formula to deduce

$$\begin{aligned}\left|\tilde{\mathfrak{b}}_\varepsilon(\theta, z)\right| &\leqslant C\epsilon \int_z^{Z(\theta,\cdot)^{-1}(z)} \left|G^{\mathrm{in}}\right|(\mathrm{y})\, d\mathrm{y} + C\epsilon \int_z^{Z(\theta,\cdot)^{-1}(z)} \int_0^{\epsilon^m|\ln\epsilon|} |\partial_\tau G|(\underline{\theta}, \mathrm{y})\, d\underline{\theta}\, d\mathrm{y} \\ &\leqslant C\epsilon\left|G^{\mathrm{in}}\right| \star \mathbf{1}_{[-\theta/\epsilon - C, 0]}(z) + C\epsilon \int_0^{\epsilon^m|\ln\epsilon|} |\partial_\tau G|(\underline{\theta}, \cdot) \star \mathbf{1}_{[-\theta/\epsilon - C, 0]}(z)\, d\underline{\theta}.\end{aligned}$$

Here, we have used (91) for the last inequality, which gives that $z = Z(\theta, y) = y - \frac{\theta}{\epsilon} + \mathcal{O}(1)$ uniformly in (y, θ). Classical convolution estimates then yield, if $0 \leqslant \theta \leqslant |\ln\epsilon|$,

$$\epsilon\left\|\left|G^{\mathrm{in}}\right| \star \mathbf{1}_{[-\theta/\epsilon - C, 0]}\right\|_{L^2} \leqslant \epsilon\left\|G^{\mathrm{in}}\right\|_{L^1} \|\mathbf{1}_{[-\theta/\epsilon - C, 0]}\|_{L^2} \leqslant C\epsilon\sqrt{\frac{\theta}{\epsilon} + C} \leqslant C\sqrt{\epsilon|\ln\epsilon|}$$

and

$$\epsilon\left\|\left|G^{\mathrm{in}}\right| \star \mathbf{1}_{[-\theta/\epsilon - C, 0]}\right\|_{L^\infty} \leqslant \epsilon\left\|G^{\mathrm{in}}\right\|_{L^1} \|\mathbf{1}_{[-\theta/\epsilon - C, 0]}\|_{L^\infty} \leqslant C\epsilon.$$

Note that when $\int_{\mathbb{R}} G \neq 0$, that is when G is not the z-derivative of some localized function, it does seem possible to improve very much the L^2 bound (see however [41] for refined estimates for secular growth). Arguing similarly for the other term (which is actually smaller in view of the $\underline{\theta}$-integration), we arrive at

$$\sup_{0\leqslant\theta\leqslant|\ln\epsilon|} \left\|\tilde{\mathfrak{b}}_\varepsilon(\theta)\right\|_{L^2} \leqslant C\sqrt{\epsilon|\ln\epsilon|} \quad \text{and} \quad \sup_{0\leqslant\theta\leqslant|\ln\epsilon|} \left\|\tilde{\mathfrak{b}}_\varepsilon(\theta)\right\|_{L^\infty} \leqslant C\epsilon.$$

Let us now estimate the derivatives of $\tilde{\mathfrak{b}}_\varepsilon$. As explained at the beginning of this step, they enjoy a better behaviour. Applying ∂_z to (90) yields

$$\frac{1}{\mathfrak{c}_s}\partial_\theta\partial_z\tilde{\mathfrak{b}}_\varepsilon - \frac{1}{\epsilon}\partial_z\partial_z\tilde{\mathfrak{b}}_\varepsilon + 2\underline{u}\partial_z\partial_z\tilde{\mathfrak{b}}_\varepsilon + 3\partial_z\tilde{\mathfrak{b}}_\varepsilon\partial_z\underline{u} = \frac{\partial_z G}{\mathfrak{c}_s} - 2\tilde{\mathfrak{b}}_\varepsilon\partial_z^2\underline{u}, \quad \partial_z\tilde{\mathfrak{b}}_\varepsilon(\theta = 0) = 0,$$

which has a structure similar to (90). Arguing as for (93), we deduce

$$\begin{aligned}\partial_z\tilde{\mathfrak{b}}_\varepsilon(\theta, z) = &\int_0^\theta \exp\left(3\mathfrak{c}_s \int_{\bar{\theta}}^\theta \partial_z\underline{u}\big(\epsilon^m\theta', Z\big(\theta', Z(\theta, \cdot)^{-1}(z)\big)\big)\, d\theta'\right)\partial_z G\big(\epsilon^m\bar{\theta}, Z\big(\bar{\theta}, Z(\theta, \cdot)^{-1}(z)\big)\big)\, d\bar{\theta} \\ &- 2\mathfrak{c}_s \int_0^\theta \exp\left(3\mathfrak{c}_s \int_{\bar{\theta}}^\theta \partial_z\underline{u}\big(\epsilon^m\theta', Z\big(\theta', Z(\theta, \cdot)^{-1}(z)\big)\big)\, d\theta'\right) \\ &\times \tilde{\mathfrak{b}}_\varepsilon\big(\bar{\theta}, Z\big(\bar{\theta}, Z(\theta, \cdot)^{-1}(z)\big)\big)\partial_z^2\underline{u}\big(\epsilon^m\bar{\theta}, Z\big(\bar{\theta}, Z(\theta, \cdot)^{-1}(z)\big)\big)\, d\bar{\theta} \\ = &\, I + II.\end{aligned}$$

By (94), we see that the exponential is equal to $1 + \mathcal{O}(\epsilon)$ uniformly for $0 \leqslant \bar{\theta} \leqslant \theta \leqslant |\ln\epsilon|$ and z. To estimate II, we bound $\tilde{\mathfrak{b}}_\varepsilon$ by $\mathcal{O}(\epsilon)$ in L^∞ and use once again the change of variable $\mathrm{y} = Z(\bar{\theta}, Z(\theta, \cdot)^{-1}(z))$, which gains a factor ϵ, and the convolution estimate to infer

$$\sup_{0\leqslant\theta\leqslant|\ln\epsilon|} \|II\|_{L^2} \leqslant C\epsilon^2\sqrt{|\ln\epsilon|/\epsilon} \leqslant C\epsilon.$$

For I, similarly, we get

$$\sup_{0\leqslant\theta\leqslant|\ln\epsilon|} \left\| I - \int_0^\theta \partial_{z_1} G^{\mathrm{in}}\big(Z\big(\bar{\theta}, Z(\theta, \cdot)^{-1}(z_1)\big), z_\perp\big)\, d\bar{\theta}\right\|_{L^2} \leqslant C\epsilon.$$

Making the change of variable $\mathrm{y} = Z(\bar{\theta}, Z(\theta, \cdot)^{-1}(z_1))$, we obtain

$$\int_0^\theta \partial_z G^{\mathrm{in}}\big(Z\big(\bar\theta, Z(\theta,\cdot)^{-1}(z)\big)\big)\,d\bar\theta = \int_{Z(\theta,\cdot)^{-1}(z)}^{z} \partial_z G^{\mathrm{in}}(\mathrm{y})\frac{\epsilon\,d\mathrm{y}}{1-2\epsilon\underline{u}(\theta'_{Z(\theta,\cdot)^{-1}(z)}(\mathrm{y}),\mathrm{y})}$$
$$= \int_{Z(\theta,\cdot)^{-1}(z)}^{z} \partial_z G^{\mathrm{in}}(\mathrm{y},z_\perp)\epsilon\,d\mathrm{y} + \mathcal{O}_{L^\infty([0,|\ln\epsilon|],L^2)}\big(\epsilon^2\sqrt{|\ln\epsilon|/\epsilon}\big),$$

using one more time the convolution estimate. The first integral is explicitely computed (now, we have a z-derivative): $\epsilon\{G^{\mathrm{in}}(z) - G^{\mathrm{in}}(Z(\theta,\cdot)^{-1}(z))\}$. Gathering these estimates, we conclude

$$\sup_{0\leqslant\theta\leqslant|\ln\epsilon|}\big\|\partial_z\tilde{\mathfrak{b}}_\varepsilon(\theta)\big\|_{L^2}\leqslant C\epsilon.$$

In a similar way, we derive

$$\sup_{0\leqslant\theta\leqslant|\ln\epsilon|}\big\|\partial_z\tilde{\mathfrak{b}}_\varepsilon(\theta)\big\|_{H^s}\leqslant C\epsilon.$$

To summarize, we have proved that $\mathfrak{b}$ verifies, for $0\leqslant\theta\leqslant|\ln\epsilon|$,

$$\|\mathfrak{b}\|_{H^{s+1}}\leqslant C,\qquad \big\|\partial_z\mathfrak{b}-\partial_z\underline{\mathfrak{b}}\big(\tau=\epsilon^m\theta\big)\big\|_{H^s}\leqslant C\varepsilon$$

and

$$\frac{\varepsilon}{\mathfrak{c}_s}\partial_\theta\mathfrak{b} - \frac{\varepsilon}{\epsilon}\partial_z\mathfrak{b} + 2\varepsilon u_\varepsilon^{\mathrm{app}}\partial_z\mathfrak{b} + \varepsilon\mathfrak{b}\partial_z u_\varepsilon^{\mathrm{app}} - \frac{\varepsilon}{\epsilon\mathfrak{c}_s}\partial_z^2\mathfrak{a} = 0.$$

Step 3: Choice of the initial data for (83) and error estimate. We recall that when we use the trick of E. Grenier, the initial data for (83) and (41) must verify (*cf.* (63))

$$A_\varepsilon^{\mathrm{in}} \equiv \frac{|1+\epsilon a_\varepsilon^{\mathrm{in}}|-1}{\epsilon},\qquad U_\varepsilon^{\mathrm{in}} \equiv u_\varepsilon^{\mathrm{in}} - \frac{i\varepsilon}{\mathfrak{c}_s}\left(\frac{\partial_z a_\varepsilon^{\mathrm{in}}}{1+\epsilon a_\varepsilon^{\mathrm{in}}} - \frac{\partial_z A_\varepsilon^{\mathrm{in}}}{1+\epsilon A_\varepsilon^{\mathrm{in}}}\right). \tag{95}$$

First, we have $|1+\epsilon a_\varepsilon^{\mathrm{app}}|^2 = |1+\epsilon\mathfrak{a}+i\varepsilon\epsilon\mathfrak{b}|^2 = (1+\epsilon\mathfrak{a})^2+\epsilon^2\varepsilon^2\mathfrak{b}^2 = (1+\epsilon\mathfrak{a})^2+\mathcal{O}_{H^s}(\epsilon^{m+3}[1+\sqrt{\epsilon|\ln\epsilon|}])$ (uniformly for $0\leqslant\theta\leqslant|\ln\epsilon|$) by the estimates in Step 1. We may then define, for $0\leqslant\theta\leqslant|\ln\epsilon|$, a real-valued quantity $\underline{a}_\varepsilon = \mathcal{O}(\epsilon^{m+2})$ such that, defining

$$a_\varepsilon^{\mathrm{in}} \equiv A_\varepsilon^{\mathrm{in}} + i\varepsilon\mathfrak{b}(\theta=0) + \underline{a}_\varepsilon = A_\varepsilon^{\mathrm{in}} - i\frac{\varepsilon}{\mathfrak{c}_s}\partial_{z_1}\zeta^{\mathrm{in}} + \underline{a}_\varepsilon,$$

the first equality in (95) is verified. We then define $u_\varepsilon^{\mathrm{in}}$ through the second equality in (95). We now give estimates for the error between $(a_\varepsilon^{\mathrm{in}}, u_\varepsilon^{\mathrm{in}})$ and $(a_\varepsilon^{\mathrm{app,in}}, u_\varepsilon^{\mathrm{app,in}})$. By construction, we have

$$\begin{aligned} a_\varepsilon^{\mathrm{in}} - a_\varepsilon^{\mathrm{app,in}} &= \big[A_\varepsilon^{\mathrm{in}} + i\varepsilon\mathfrak{b}(\theta=0) + \underline{a}_\varepsilon\big] - \big[\mathfrak{a}(\theta=0) + i\varepsilon\mathfrak{b}(\theta=0)\big] \\ &= A_\varepsilon^{\mathrm{in}} - \big(\zeta^{\mathrm{in}} + \epsilon^m A_m^{\mathrm{in}} + \epsilon^{m+1}\mathfrak{a}_{m+1}(\theta=0)\big) + \mathcal{O}_{H^s}\big(\epsilon^{m+2}\big) = \mathcal{O}_{H^s}\big(\epsilon^{m+1}\big). \end{aligned}$$

Consequently,

$$\frac{i\varepsilon}{\mathfrak{c}_s}\left(\frac{\partial_z a_\varepsilon^{\mathrm{in}}}{1+\epsilon a_\varepsilon^{\mathrm{in}}} - \frac{\partial_z A_\varepsilon^{\mathrm{in}}}{1+\epsilon A_\varepsilon^{\mathrm{in}}}\right) = \frac{i\varepsilon}{\mathfrak{c}_s\epsilon}\partial_z\left[\log_{\mathbb{C}}\left(\frac{1+\epsilon a_\varepsilon^{\mathrm{in}}}{1+\epsilon A_\varepsilon^{\mathrm{in}}}\right)\right] = \mathcal{O}_{H^s}\big(\varepsilon\epsilon^{m+2}\big) = \mathcal{O}_{H^s}\big(\epsilon^{m+3}\big),$$

thus $U_\varepsilon^{\mathrm{in}} - u_\varepsilon^{\mathrm{in}} = \mathcal{O}_{H^s}(\epsilon^{m+3})$, and this implies $u_\varepsilon^{\mathrm{in}} - u_\varepsilon^{\mathrm{app}}(\theta=0) = \mathcal{O}_{H^s}(\epsilon^{m+2})$. As a consequence, we have constructed initial data $(a_\varepsilon^{\mathrm{in}}, u_\varepsilon^{\mathrm{in}})$ verifying (95) as well as (89).

Step 4: Error estimate for the residuals. From the estimates of Step 1, we have $\mathfrak{b} = \mathfrak{b}_0 + \mathcal{O}_{H^{s+1}}(\sqrt{\epsilon\theta})$. This is then just for $\mathfrak{b}$ that the expansion in ϵ is not completely rigorous in the sense that we do not claim that $\tilde{\mathfrak{b}}_\varepsilon$ is of order ϵ in H^{s+1}. The term $\mathfrak{b}$ appears in the nonlinearity $f(|1+\epsilon a_\varepsilon^{\mathrm{app}}|^2)$, but since we have already seen in Step 3 that $|1+\epsilon a_\varepsilon|^2 = (1+\epsilon\mathfrak{a})^2 + \mathcal{O}_{H^s}(\epsilon^{m+3})$, the expansion in ϵ is actually true. For the imaginary part of $\mathcal{R}_a$, we have

$$-\mathrm{Im}(\mathcal{R}_a) = \varepsilon\left\{\frac{1}{\mathfrak{c}_s}\frac{\partial \mathfrak{b}}{\partial\theta} - \frac{1}{\epsilon}\partial_z\mathfrak{b} + 2u_\varepsilon^{\mathrm{app}}\partial_z\mathfrak{b} + \mathfrak{b}\partial_z u_\varepsilon^{\mathrm{app}} - \frac{1}{\mathfrak{c}_s\epsilon}\partial_z^2\mathfrak{a}\right\}$$
$$= \mathcal{O}_{H^s}\left(\epsilon^{m+1}\right) + \varepsilon\left\{\frac{1}{\mathfrak{c}_s}\frac{\partial \mathfrak{b}}{\partial\theta} - \frac{1}{\epsilon}\partial_z\mathfrak{b} + 2u_\varepsilon^{\mathrm{app}}\partial_z\mathfrak{b} + \mathfrak{b}\partial_z u_\varepsilon^{\mathrm{app}} - \frac{1}{\mathfrak{c}_s\epsilon}\partial_z^2\mathfrak{a}\right\},$$

and by construction of $\mathfrak{b}$ (see Step 1), we precisely get $-\mathrm{Im}(\mathcal{R}_a) = \mathcal{O}_{H^s}(\epsilon^{m+1})$. We turn finally to the real part of $\mathcal{R}_a$, and since $\mathfrak{b}$ only appears in the last term with ε/ϵ in front of, we obtain

$$-\mathrm{Re}(\mathcal{R}_a) = \frac{1}{\mathfrak{c}_s}\frac{\partial\mathfrak{a}}{\partial\theta} - \frac{1}{\epsilon}\partial_z\mathfrak{a} + 2u_\varepsilon^{\mathrm{app}}\partial_z\mathfrak{a} + \frac{1}{\epsilon}(1+\epsilon\mathfrak{a})\partial_z u_\varepsilon^{\mathrm{app}} + \frac{\epsilon^m}{\mathfrak{c}_s}\partial_z^2\underline{\mathfrak{b}} + \frac{\epsilon^m}{\mathfrak{c}_s}\partial_z^2\tilde{\mathfrak{b}}_\varepsilon + \mathcal{O}_{H^s}\left(\epsilon^{m+1}\right).$$

From the estimate of Step 1, we have $\partial_{z_1}^2\tilde{\mathfrak{b}}_\varepsilon = \mathcal{O}_{H^s}(\epsilon)$, and by construction, $\mathfrak{c}_s\underline{\mathfrak{b}} = -\partial_z\mathfrak{a} = -\partial_z\mathfrak{a}_0 + \mathcal{O}_{H^s}(\epsilon)$. Since the expansion in ϵ is now correct, we know that $\mathrm{Re}(\mathcal{R}_a)$ and $\mathcal{R}_u$ are of order $\mathcal{O}(\epsilon^{m-1})$ by construction of the terms $\mathfrak{a}_k, \mathfrak{u}_k, 0\leqslant k\leqslant m$. Let us now inspect the terms of order ϵ^m in $-\mathrm{Re}(\mathcal{R}_a)$ and $-\mathcal{R}_u$ respectively:

$$\begin{cases} \dfrac{1}{\mathfrak{c}_s}\dfrac{\partial\zeta}{\partial\tau} - \partial_z\mathfrak{a}_{m+1} + 2\mathfrak{u}_m\partial_z\zeta + 2\zeta\partial_z\mathfrak{a}_m + \partial_z\mathfrak{u}_{m+1} + \mathfrak{a}_m\partial_z\zeta + \zeta\partial_z\mathfrak{u}_m - \dfrac{1}{\mathfrak{c}_s^2}\partial_z^3\zeta \\ \dfrac{1}{\mathfrak{c}_s}\partial_\tau\zeta - \partial_z\mathfrak{u}_{m+1} + \partial_z\left(\displaystyle\sum_{k=0}^m \mathfrak{u}_k\mathfrak{u}_{m-k}\right) + \partial_z\mathfrak{a}_{m+1} + \partial_z\left(q_{m+2}\zeta^{m+2} - 5\zeta\mathfrak{a}_m\right), \end{cases}$$

as can be seen from the computations in Section 1.4 (we keep the same notations). These two quantities vanish if and only if their sum and difference vanish, that is

$$\begin{cases} \dfrac{2}{\mathfrak{c}_s}\dfrac{\partial\zeta}{\partial\tau} + 2\mathfrak{u}_m\partial_z\zeta + 2\zeta\partial_z\mathfrak{a}_m + \mathfrak{a}_m\partial_z\zeta + \zeta\partial_z\mathfrak{u}_m + \partial_z\left(\displaystyle\sum_{k=0}^m \mathfrak{u}_k\mathfrak{u}_{m-k}\right) + \partial_z\left(q_{m+2}\zeta^{m+2} - 5\zeta\mathfrak{a}_m\right) - \dfrac{1}{\mathfrak{c}_s^2}\partial_z^3\zeta \\ 2\partial_z(\mathfrak{u}_{m+1} - \mathfrak{a}_{m+1}) = 2\mathfrak{u}_m\partial_z\zeta + 2\zeta\partial_z\mathfrak{a}_m + \mathfrak{a}_m\partial_z\zeta + \zeta\partial_z\mathfrak{u}_m + \partial_z\left(\displaystyle\sum_{k=0}^m \mathfrak{u}_k\mathfrak{u}_{m-k}\right) \\ \quad + \partial_z\left(q_{m+2}\zeta^{m+2} - 5\zeta\mathfrak{a}_m\right) + \dfrac{1}{\mathfrak{c}_s^2}\partial_z^3\zeta. \end{cases}$$

Once we have reported the expressions of the $\mathfrak{u}_k$'s, the first equation is precisely the (gKdV) equation. Since by construction $\mathfrak{u}_m = \mathfrak{a}_m + (-1)^m(m+2)\zeta^{m+1}/2$, we see that the right-hand side of the second equation becomes

$$\partial_z\left(\sum_{k=0}^m \mathfrak{u}_k\mathfrak{u}_{m-k}\right) + \partial_z\left(\left[q_{m+2} + (-1)^m\frac{m+3}{2}\right]\zeta^{m+2} - 2\zeta\mathfrak{a}_m\right) + \frac{1}{\mathfrak{c}_s^2}\partial_z^3\zeta,$$

which is indeed a z-derivative. By our (arbitrary) choice for $\mathfrak{u}_{m+1}$ and $\mathfrak{a}_{m+1}$, we get the conclusion. Note that the fact that we can integrate in z the last equation is actually not linked to the precise choice for $(\mathfrak{a}_m, \mathfrak{u}_m)$. The proof of Lemma 2 is complete. □

Lemma 3. *Assume $d\geqslant 1$. Under the assumptions of Theorem 7, there exist initial data $(a_\varepsilon^{\mathrm{in}}, u_\varepsilon^{\mathrm{in}})$ satisfying (84) and an approximate solution $(a_\varepsilon^{\mathrm{app}}, u_\varepsilon^{\mathrm{app}})$ such that we have*

$$\|\mathcal{R}_a\|_{H^s} + \left\|\left(\mathcal{R}_u^1, \varepsilon\mathcal{R}_u^\perp\right)\right\|_{H^s} \leqslant C\epsilon^{m+1}$$

as well as

$$\left\|a_\varepsilon^{\mathrm{in}} - a_\varepsilon^{\mathrm{app}}(\theta=0)\right\|_{H^s} + \left\|u_\varepsilon^{\mathrm{in},1} - u_\varepsilon^{\mathrm{app},1}(\theta=0)\right\|_{H^s} \leqslant C\epsilon^{m+1}. \tag{96}$$

Proof. We shall only point out the few differences with the proof of Lemma 2.

Step 1: Definition of the approximate solution. We set

$$\mathfrak{u}_0 \equiv \nabla_z\partial_{z_1}^{-1}\zeta(\tau) \in \mathcal{C}\left([0,\tau_*], H^{s+4}\right), \qquad \mathfrak{u}_1 = \cdots = \mathfrak{u}_{m+1} \equiv 0,$$

(if $m=1$, the second condition is void), so that $\mathfrak{u}^1=\zeta(\tau)+\mathcal{O}(\epsilon^m)$. For the amplitude, the relation (34) imposes to choose

$$\forall 0\leqslant k\leqslant m,\quad \mathfrak{a}_k\equiv\frac{1\cdot 3\cdot\dots\cdot(2k+1)}{(k+1)!}\zeta^{k+1}(\tau)\in\mathcal{C}\big([0,\tau_*],H^{s+5}\cap\partial_{z_1}H^{s+5}\big),$$

and for $\mathfrak{a}_{m+1}$, we fix (arbitrarily)

$$\mathfrak{a}_{m+1}\equiv-\frac{1}{2\mathfrak{c}_s^2}\partial_{z_1}^2\zeta(\tau)-\frac{1}{2}\left(\left[q_{m+2}+(-1)^m\frac{m+3}{2}\right]\zeta^{m+2}(\tau)-2\zeta\,\mathfrak{a}_m\right)-\frac{1}{2}\Delta_{z_\perp}\partial_{z_1}^{-2}\zeta(\tau)\in L^\infty\big([0,\tau_*],H^{s+2}\big)$$

(by the result in [53] or Lemma 3 in [42]). Note that the sum $\sum_{k=0}^m\mathfrak{u}_k\mathfrak{u}_{m-k}$ now vanishes for our choice of the $\mathfrak{u}_k$'s. Concerning the imaginary part, as for Lemma 2, we choose

$$\mathfrak{b}\equiv\underline{\mathfrak{b}}+\tilde{\mathfrak{b}}_\varepsilon,$$

where

$$\underline{\mathfrak{b}}\equiv-\frac{1}{\mathfrak{c}_s}\partial_{z_1}\mathfrak{a}(\tau)\in\mathcal{C}\big([0,\tau_*],H^{s+1}\big)$$

and the function $\tilde{\mathfrak{b}}_\varepsilon=\tilde{\mathfrak{b}}_\varepsilon(\theta)$ is the solution of the high speed transport equation

$$\frac{1}{\mathfrak{c}_s}\partial_\theta\tilde{\mathfrak{b}}_\varepsilon-\frac{1}{\epsilon}\partial_{z_1}\tilde{\mathfrak{b}}_\varepsilon+2u_\varepsilon^{\mathrm{app},1}\partial_{z_1}\tilde{\mathfrak{b}}_\varepsilon+\tilde{\mathfrak{b}}_\varepsilon\partial_{z_1}u_\varepsilon^{\mathrm{app},1}=\frac{G}{\mathfrak{c}_s}\equiv-2u_\varepsilon^{\mathrm{app},1}\partial_{z_1}\underline{\mathfrak{b}}-\underline{\mathfrak{b}}\partial_{z_1}u_\varepsilon^{\mathrm{app},1},\qquad\tilde{\mathfrak{b}}_\varepsilon(\theta=0)=0.\tag{97}$$

Step 2: Sobolev estimates for $\tilde{\mathfrak{b}}_\varepsilon$. Observing that the high speed transport equation (97) only involves the z_1 coordinate, we deduce as in the proof of Lemma 2 that $\tilde{\mathfrak{b}}_\varepsilon$ verifies first

$$\sup_{0\leqslant\theta\leqslant|\ln\epsilon|}\big\|\tilde{\mathfrak{b}}_\varepsilon(\theta)\big\|_{L^2}\leqslant C\sqrt{\epsilon|\ln\epsilon|}\quad\text{and}\quad\sup_{0\leqslant\theta\leqslant|\ln\epsilon|}\big\|\tilde{\mathfrak{b}}_\varepsilon(\theta)\big\|_{L^\infty}\leqslant C\epsilon,$$

hence for any $\alpha\in\mathbb{N}_0^{d-1}$ with $|\alpha|\leqslant s+1$

$$\sup_{0\leqslant\theta\leqslant|\ln\epsilon|}\big\|\partial_{z_\perp}^\alpha\tilde{\mathfrak{b}}_\varepsilon(\theta)\big\|_{L^2}\leqslant C\sqrt{\epsilon|\ln\epsilon|}\quad\text{and}\quad\sup_{0\leqslant\theta\leqslant|\ln\epsilon|}\big\|\partial_{z_\perp}^\alpha\tilde{\mathfrak{b}}_\varepsilon(\theta)\big\|_{L^\infty}\leqslant C\epsilon.$$

As before, the z_1-derivative is shown here again to have a better behaviour:

$$\sup_{0\leqslant\theta\leqslant|\ln\epsilon|}\big\|\partial_{z_1}\tilde{\mathfrak{b}}_\varepsilon(\theta)\big\|_{H^s}\leqslant C\epsilon.$$

Therefore, $\mathfrak{b}$ verifies, for $0\leqslant\theta\leqslant|\ln\epsilon|$,

$$\|\mathfrak{b}\|_{H^s}\leqslant C,\qquad\big\|\partial_{z_1}\mathfrak{b}-\partial_{z_1}\underline{\mathfrak{b}}\big(\tau=\epsilon^m\theta\big)\big\|_{H^s}\leqslant C\varepsilon$$

and

$$\frac{\varepsilon}{\mathfrak{c}_s}\partial_\theta\mathfrak{b}-\frac{\varepsilon}{\epsilon}\partial_{z_1}\mathfrak{b}+2\varepsilon u_\varepsilon^{\mathrm{app},1}\partial_{z_1}\mathfrak{b}+\varepsilon\mathfrak{b}\partial_{z_1}u_\varepsilon^{\mathrm{app},1}-\frac{\varepsilon}{\epsilon\mathfrak{c}_s}\partial_{z_1}^2\mathfrak{a}=0.$$

Step 3: Choice of the initial data for (83) and error estimate and *Step 4: Error estimate for the residuals*. They are very similar to Step 3 and Step 4 in the proof of Lemma 2, taking into account the transverse variable, thus we omit the proof.

4.2. Error estimate

We look for an exact solution of the modified Madelung system (83) under the form

$$(a_\varepsilon,u_\varepsilon)=\big(a_\varepsilon^{\mathrm{app}},u_\varepsilon^{\mathrm{app}}\big)+(\mathcal{A}_\varepsilon,\mathcal{U}_\varepsilon).$$

Since the system (83) is symmetrizable and the dispersive term has constant coefficient and is skew-adjoint, the error estimate, for $|\alpha|\leqslant s$,

$$\frac{d}{d\theta}\big(\big(\mathfrak{S}_\varepsilon\big(\epsilon\big(a_\varepsilon^{\rm app}+\mathcal{A}_\varepsilon\big)\big)\partial_z^\alpha\Upsilon,\partial_z^\alpha\Upsilon\big)_{L^2}\big)$$
$$\leqslant C\big(\epsilon\big\|\partial_\theta\big(a_\varepsilon^{\rm app}+\mathcal{A}_\varepsilon\big)\big\|_{L^\infty}+\big\|a_\varepsilon^{\rm app}+\mathcal{A}_\varepsilon\big\|_{W^{1,\infty}}+1\big)\|\Upsilon\|_{H^s}^2+C\epsilon^{2(m+1)},$$

with $\Upsilon=(\mathcal{A}_\varepsilon,\mathcal{U}_\varepsilon^1,\varepsilon\mathcal{U}_\varepsilon^\perp)$, follows immediately. Recall that at time $\theta=0$, Υ is $\mathcal{O}(\epsilon^{m+1})$, even though we have included the terms of order ϵ^{m+1} in the approximate solution. We denote by $\theta_\varepsilon\in(0,|\ln\epsilon|)$ the maximal time for which $\|\epsilon\partial_\theta\Upsilon\|_{L^\infty}+\|\mathcal{A}_\varepsilon\|_{W^{1,\infty}}\leqslant 1$. Then, we infer from the Gronwall inequality that for $0\leqslant\theta\leqslant\theta_\varepsilon$,

$$\big\|\Upsilon(\theta)\big\|_{H^s}^2\leqslant\big\{\big\|\Upsilon(\theta=0)\big\|_{H^s}^2+\epsilon^{2(m+1)}\big\}\mathrm{e}^{2C\theta}\leqslant C\epsilon^{2(m+1)}\mathrm{e}^{2C\theta},$$

where C is a constant depending only on s, d, Λ and the function ζ. This guarantees that $\theta_\varepsilon\leqslant\mu|\ln\epsilon|$ for some small constant $0<\mu<1/C$ depending only on s, d, Λ and the function ζ and provided ε is sufficiently small. We finally use the formula (63) to infer that for $\theta_\varepsilon\leqslant\mu|\ln\epsilon|$,

$$\big\|A_\varepsilon-\mathrm{Re}\big(a_\varepsilon^{\rm app}\big)\big\|_{H^s}+\big\|U_\varepsilon^1-u_\varepsilon^{\mathrm{app},1}\big\|_{H^{s-1}}\leqslant C\epsilon^{m+1}\mathrm{e}^{\frac{\theta}{2\mu}}.$$

This completes the proofs of Theorems 6 and 7.

5. Justification of the wave and the (mKdV)/(mKP-I) limit for the Landau–Lifshitz equation

5.1. Proof of the free wave limit for the Landau–Lifshitz equation

In order to prove the Sobolev bounds (57) on the suitable time interval, we shall not proceed as in [50] and [49]. Indeed, they apply ∂_t to the equation, and obtain a wave equation of the form

$$\partial_t^2\mathfrak{m}+\Delta^2\mathfrak{m}=\cdots.$$

Using the scales $\mathrm{t}=\varepsilon t$ and $z=\varepsilon x$, this becomes

$$\partial_{\mathrm{t}}^2\mathfrak{m}+\varepsilon^2\Delta^2\mathfrak{m}=\cdots,$$

for which the natural high order functional is

$$\sum_{\substack{\alpha\in\mathbb{N}_0^d\\|\alpha|\leqslant s}}\int_{\mathbb{R}^d}\big|\partial_{\mathrm{t}}\partial_z^\alpha\mathfrak{m}\big|^2+\varepsilon^2\big|\Delta\partial_z^\alpha\mathfrak{m}\big|^2\,dz.$$

This functional controls $\partial_{\mathrm{t}}\mathfrak{m}$ in H^s. Taking the cross product of the equation with $\mathfrak{m}$, we infer that

$$\mathfrak{m}\times\partial_t\mathfrak{m}=-\varepsilon\Delta\mathfrak{m}-\varepsilon|\nabla\mathfrak{m}|^2\mathfrak{m}+\frac{\mathfrak{m}_3}{\varepsilon}\vec{e}_3-\varepsilon\left(\frac{\mathfrak{m}_3}{\varepsilon}\right)^2\mathfrak{m},$$

hence the functional controls $\frac{\mathfrak{m}_3}{\varepsilon}$ in H^s, but only $\varepsilon\Delta\mathfrak{m}$ in H^s and not $\Delta\mathfrak{m}$, which should be on the same level.

From (49), we deduce that the gradient vector field

$$V\equiv\frac{\nabla\rho}{2\rho}=\frac{1}{2}\nabla\ln(\rho)$$

satisfies

$$\partial_t V+2\nabla\left(\frac{1-\rho}{1+\rho}U\cdot V\right)+\Delta U=0,$$

since $\nabla(\nabla\cdot U)=\nabla(\Delta\varphi)=\Delta U$. Moreover,

$$\Delta V=\frac{1}{2}\Delta\nabla\ln(\rho)=\nabla\Delta\ln(\sqrt{\rho})=\nabla\left(\nabla\cdot\left[\frac{\nabla\sqrt{\rho}}{\sqrt{\rho}}\right]\right)=\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}-\frac{|\nabla\sqrt{\rho}|^2}{\rho}\right)=\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}-\frac{|\nabla\rho|^2}{4\rho^2}\right),$$

thus

$$\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) = \Delta V + \nabla\big(|V|^2\big),$$

and we may rewrite the equation for U under the form

$$\partial_t U + \nabla\left(\frac{1-\rho}{1+\rho}|U|^2\right) + \nabla\left(\frac{\rho-1}{\rho+1}\right) - \Delta V = \nabla\big(|V|^2\big) - \nabla\left(\frac{2\rho|V|^2}{1+\rho}\right) = \nabla\left(\frac{1-\rho}{1+\rho}|V|^2\right).$$

Consequently, the complex-valued gradient vector field

$$Z \equiv U - iV$$

verifies

$$\partial_t Z - i\Delta Z + \nabla\left(\frac{1-\rho}{1+\rho}(Z\cdot Z)\right) + \nabla\left(\frac{\rho-1}{\rho+1}\right) = 0,$$

where we have set, for $Z, \tilde{Z} \in \mathbb{C}^d$, $Z\cdot\tilde{Z} = \sum_{j=1}^d Z^j \tilde{Z}^j \in \mathbb{C}$. Therefore, we have obtained the augmented system

$$\begin{cases} \partial_t \rho + 2\dfrac{1-\rho}{1+\rho} U\cdot\nabla\rho + 2\rho\nabla\cdot U = 0 \\ \nabla\rho = 2\rho V \\ \partial_t Z - i\Delta Z + \nabla\left(\dfrac{1-\rho}{1+\rho}(Z\cdot Z)\right) + \nabla\left(\dfrac{\rho-1}{\rho+1}\right) = 0. \end{cases} \tag{98}$$

On the other hand, from the stereographic projection, we have

$$\mathfrak{m} = \left(\frac{2\,\mathrm{Re}(\Psi)}{1+|\Psi|^2}, \frac{2\,\mathrm{Im}(\Psi)}{1+|\Psi|^2}, \frac{1-|\Psi|^2}{1+|\Psi|^2}\right),$$

so that the energy has the expression

$$\begin{aligned} E(\mathfrak{m}) &= \int_{\mathbb{R}^d} 4\left|\nabla\left(\frac{\Psi}{1+|\Psi|^2}\right)\right|^2 + 4\left|\nabla\left(\frac{1}{1+|\Psi|^2}\right)\right|^2 + \left(\frac{1-|\Psi|^2}{1+|\Psi|^2}\right)^2 dx \\ &= \int_{\mathbb{R}^d} \frac{4\rho}{(1+\rho)^2}|\nabla\varphi|^2 + \frac{4}{\rho(1+\rho)^2}|\nabla\rho|^2 + \left(\frac{1-\rho}{1+\rho}\right)^2 dx \\ &= \int_{\mathbb{R}^d} \frac{4\rho}{(1+\rho)^2}|Z|^2 + \left(\frac{1-\rho}{1+\rho}\right)^2 dx. \end{aligned}$$

We now use the scaled variables $\theta = \varepsilon^2 t$ and $z = \varepsilon x$, which transform (98) and the energy into

$$\begin{cases} \partial_\theta \rho + 2\dfrac{1-\rho}{1+\rho} U_\varepsilon\cdot\nabla_z\rho + 2\rho\nabla_z\cdot U_\varepsilon = 0 \\ \nabla_z\rho = 2\rho V_\varepsilon \\ \partial_\theta Z_\varepsilon - i\Delta Z_\varepsilon + \nabla_z\left(\dfrac{1-\rho}{1+\rho}(Z_\varepsilon\cdot Z_\varepsilon)\right) + \dfrac{1}{\varepsilon^2}\nabla_z\left(\dfrac{\rho-1}{\rho+1}\right) = 0. \end{cases} \tag{99}$$

and

$$E(\mathfrak{m}) = \varepsilon^{2-d}\int_{\mathbb{R}^d} \frac{4\rho}{(1+\rho)^2}|Z_\varepsilon|^2 + \frac{1}{\varepsilon^2}\left(\frac{1-\rho}{1+\rho}\right)^2 dz = \varepsilon^{2-d}E_\varepsilon(\Psi),$$

with

$$V_\varepsilon \equiv \frac{V}{\varepsilon}, \qquad U_\varepsilon \equiv \frac{U}{\varepsilon}, \qquad Z_\varepsilon \equiv \frac{Z}{\varepsilon}, \qquad \rho = 1 + \varepsilon a.$$

Note that V_ε is of order ε. By Theorem 8, we have local in time well-posedness for the system (99), say for $0 \leqslant \theta \leqslant \theta_\varepsilon$. We define $\bar{\theta}_\varepsilon \in (0, \theta_\varepsilon]$ to be the maximal time for which, for any $0 \leqslant \theta \leqslant \bar{\theta}_\varepsilon$,

$$\frac{1}{2} \leqslant \big|\Psi(\theta, \cdot)\big| \leqslant 2. \tag{100}$$

Note that the conservation of energy already provides, for $0 \leqslant \theta \leqslant \bar{\theta}_\varepsilon$,

$$\begin{aligned}\frac{1}{K_0}\left(\big\|Z_\varepsilon(\theta)\big\|_{L^2}^2 + \left\|\frac{\rho(\theta)-1}{\varepsilon}\right\|_{L^2}^2\right) &\leqslant \int_{\mathbb{R}^d} \frac{4\rho}{(1+\rho)^2}|Z_\varepsilon|^2 + \frac{1}{\varepsilon^2}\left(\frac{1-\rho}{1+\rho}\right)^2 dz \\ &= \int_{\mathbb{R}^d} \frac{4\rho^{\mathrm{in}}}{(1+\rho^{\mathrm{in}})^2}|Z_\varepsilon^{\mathrm{in}}|^2 + \frac{1}{\varepsilon^2}\left(\frac{1-\rho^{\mathrm{in}}}{1+\rho^{\mathrm{in}}}\right)^2 dz \\ &\leqslant K_0\left(\big\|Z_\varepsilon^{\mathrm{in}}\big\|_{L^2}^2 + \left\|\frac{\rho^{\mathrm{in}}-1}{\varepsilon}\right\|_{L^2}^2\right)\end{aligned}$$

where the constant K_0 is absolute. As we shall see, the expression of the energy in variables (ρ, Z_ε) suggests a good candidate for a high order functional, since the weights play the role of a suitable symmetrizer.

Proposition 5. *Let $s > 1 + d/2$. There exists $C = C(s,d)$, depending only on s and d, such that, for any $\alpha \in \mathbb{N}_0^d$ with $0 < |\alpha| \leqslant s$, there holds*

$$\begin{aligned}&\frac{d}{d\theta}\int_{\mathbb{R}^d} \frac{4\rho}{(1+\rho)^2}\big|\partial_z^\alpha Z_\varepsilon\big|^2 + \frac{4}{\varepsilon^2(1+\rho)^4}\big[\partial_z^\alpha \rho\big]^2 dz \\ &\quad\leqslant C(s,d)\left(\left\|\frac{\rho-1}{\varepsilon}\right\|_{H^s}^2 + \big\|Z_\varepsilon\big\|_{H^s}^2\right)\left(1 + \varepsilon\left\|\frac{\rho-1}{\varepsilon}\right\|_{H^s}^2 + \varepsilon\|Z_\varepsilon\|_{H^s}^2\right).\end{aligned}$$

Remark 5. The nonlinear effect is rather weak in view of the factor ε in front of. This is related to the fact that the system (49) has a remarkable symmetry property. Indeed, in the regime we are considering, where $\rho = 1 + \varepsilon a$ this system is somehow close to

$$\begin{cases}\partial_\theta a_\varepsilon + \dfrac{2}{\varepsilon}(1+\varepsilon a_\varepsilon)\nabla\cdot u_\varepsilon = \mathcal{O}\big(\varepsilon^2\big) \\ \partial_\theta u_\varepsilon + \dfrac{1}{\varepsilon}\nabla\left(\dfrac{a_\varepsilon}{2+\varepsilon a_\varepsilon}\right) = \partial_\theta u_\varepsilon + \dfrac{1}{2\varepsilon}(1+\varepsilon a_\varepsilon)\nabla a_\varepsilon + \mathcal{O}(\varepsilon) = \mathcal{O}\big(\varepsilon^2\big),\end{cases}$$

which can be symmetrized by using the constant coefficient symmetrizer $\left(\begin{smallmatrix}1 & 0\\ 0 & 4\end{smallmatrix}\right)$.

Proof. Let $\alpha \in \mathbb{N}_0^d$ be such that $0 < |\alpha| \leqslant s$. As a first step, we compute

$$\frac{d}{d\theta}\int_{\mathbb{R}^d} \frac{4\rho}{(1+\rho)^2}\big|\partial_z^\alpha Z_\varepsilon\big|^2 dz = \int_{\mathbb{R}^d} \frac{8\rho}{(1+\rho)^2}\big\langle\partial_z^\alpha Z_\varepsilon, \partial_\theta\partial_z^\alpha Z_\varepsilon\big\rangle dz + \int_{\mathbb{R}^d} \frac{4(1-\rho)}{(1+\rho)^3}\big|\partial_z^\alpha Z_\varepsilon\big|^2 \partial_\theta \rho\, dz. \tag{101}$$

Applying ∂_z^α to the third equation in (99) and reporting yields

$$\begin{aligned}&\int_{\mathbb{R}^d} \frac{8\rho}{(1+\rho)^2}\big\langle\partial_z^\alpha Z_\varepsilon, \partial_\theta\partial_z^\alpha Z_\varepsilon\big\rangle dz \\ &\quad= \int_{\mathbb{R}^d} \frac{8\rho}{(1+\rho)^2}\big\langle\partial_z^\alpha Z_\varepsilon, i\Delta\partial_z^\alpha Z_\varepsilon\big\rangle dz - \int_{\mathbb{R}^d} \frac{8\rho}{(1+\rho)^2}\left\langle\partial_z^\alpha Z_\varepsilon, \partial_z^\alpha \nabla_z\left(\frac{1-\rho}{1+\rho}(Z_\varepsilon\cdot Z_\varepsilon)\right)\right\rangle dz \\ &\qquad - \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} \frac{8\rho}{(1+\rho)^2}\left\langle\partial_z^\alpha Z_\varepsilon, \partial_z^\alpha \nabla_z\left(\frac{\rho-1}{\rho+1}\right)\right\rangle dz.\end{aligned} \tag{102}$$

We integrate by parts the first integral, using that $\langle \partial_j \partial_z^\alpha Z_\varepsilon, i\partial_j \partial_z^\alpha Z_\varepsilon\rangle = 0$ pointwise for any $1 \leqslant j \leqslant d$:

$$\int_{\mathbb{R}^d} \frac{8\rho}{(1+\rho)^2}\langle \partial_z^\alpha Z_\varepsilon, i\Delta\partial_z^\alpha Z_\varepsilon\rangle\,dz = -\int_{\mathbb{R}^d} \frac{8(1-\rho)}{(1+\rho)^3}(\nabla_z\rho)\cdot\langle \partial_z^\alpha Z_\varepsilon, i\nabla_z\partial_z^\alpha Z_\varepsilon\rangle\,dz$$
$$= -16\int_{\mathbb{R}^d} \frac{\rho(1-\rho)}{(1+\rho)^3}\langle \partial_z^\alpha Z_\varepsilon, iV_\varepsilon\cdot\nabla_z\partial_z^\alpha Z_\varepsilon\rangle\,dz. \tag{103}$$

Using (59) and Cauchy–Schwarz, we also have

$$-\int_{\mathbb{R}^d} \frac{8\rho}{(1+\rho)^2}\left\langle \partial_z^\alpha Z_\varepsilon, \partial_z^\alpha\nabla_z\left(\frac{1-\rho}{1+\rho}(Z_\varepsilon\cdot Z_\varepsilon)\right)\right\rangle dz$$
$$\leqslant -\int_{\mathbb{R}^d} \frac{8\rho(1-\rho)}{(1+\rho)^3}\langle \partial_z^\alpha Z_\varepsilon, \partial_z^\alpha\nabla_z(Z_\varepsilon\cdot Z_\varepsilon)\rangle\,dz$$
$$+ C(s,d)\left(\left\|\frac{1-\rho}{1+\rho}\right\|_{H^{s+1}}\|Z_\varepsilon\cdot Z_\varepsilon\|_{L^\infty} + \left\|\frac{1-\rho}{1+\rho}\right\|_{L^\infty}\|Z_\varepsilon\cdot Z_\varepsilon\|_{H^s}\right)$$
$$\leqslant -\int_{\mathbb{R}^d} \frac{8\rho(1-\rho)}{(1+\rho)^3}\langle \partial_z^\alpha Z_\varepsilon, \partial_z^\alpha\nabla_z(Z_\varepsilon\cdot Z_\varepsilon)\rangle\,dz + C(s,d)\big(1+\varepsilon\sqrt{E_\varepsilon(\Psi)} + \|Z_\varepsilon\|_{H^s}\big)\|Z_\varepsilon\|_{H^s}^2. \tag{104}$$

Here, we have used that H^s is an algebra and that $\nabla_z\rho = 2\rho V_\varepsilon$. Using once again (59), we deduce

$$-\int_{\mathbb{R}^d} \frac{8\rho(1-\rho)}{(1+\rho)^3}\langle \partial_z^\alpha Z_\varepsilon, \partial_z^\alpha\nabla_z(Z_\varepsilon\cdot Z_\varepsilon)\rangle\,dz \leqslant -\int_{\mathbb{R}^d} \frac{16\rho(1-\rho)}{(1+\rho)^3}\langle \partial_z^\alpha Z_\varepsilon, Z_\varepsilon\cdot\partial_z^\alpha\nabla_z Z_\varepsilon\rangle\,dz + C(s,d)\|Z_\varepsilon\|_{H^s}^3.$$

Furthermore, by integration by parts, we infer

$$-\int_{\mathbb{R}^d} \frac{16\rho(1-\rho)}{(1+\rho)^3}\langle \partial_z^\alpha Z_\varepsilon, Z_\varepsilon\cdot\partial_z^\alpha\nabla_z Z_\varepsilon\rangle\,dz$$
$$= -\int_{\mathbb{R}^d} \frac{16\rho(1-\rho)}{(1+\rho)^3}\langle \partial_z^\alpha Z_\varepsilon, \mathrm{Re}(Z_\varepsilon)\cdot\nabla_z\partial_z^\alpha Z_\varepsilon\rangle\,dz - \int_{\mathbb{R}^d} \frac{16\rho(1-\rho)}{(1+\rho)^3}\langle \partial_z^\alpha Z_\varepsilon, i\,\mathrm{Im}(Z_\varepsilon)\cdot\nabla_z\partial_z^\alpha Z_\varepsilon\rangle\,dz$$
$$= 8\int_{\mathbb{R}^d} |\partial_z^\alpha Z_\varepsilon|^2\nabla_z\cdot\left(\frac{\rho(1-\rho)}{(1+\rho)^3}U_\varepsilon\right)dz + \int_{\mathbb{R}^d} \frac{16\rho(1-\rho)}{(1+\rho)^3}\langle \partial_z^\alpha Z_\varepsilon, iV_\varepsilon\cdot\nabla_z\partial_z^\alpha Z_\varepsilon\rangle\,dz.$$

Notice that the last integral is exactly the opposite of the right-hand side of (103) (this is due to the weight $4\rho/(1+\rho)^2$ for the $\partial_z^\alpha Z_\varepsilon$ part) and that the before last integral is, by Sobolev imbedding ($s > 1 + d/2$),

$$\leqslant C(s,d)\|Z_\varepsilon\|_{H^s}^2\big(\|\nabla_z\cdot U_\varepsilon\|_{L^\infty}\|\rho-1\|_{L^\infty} + \|U_\varepsilon\|_{L^\infty}\|\nabla_z\rho\|_{L^\infty}\big) \leqslant C(s,d)\varepsilon\|Z_\varepsilon\|_{H^s}^3.$$

Therefore, reporting these estimates into (103) and (102) provides

$$\int_{\mathbb{R}^d} \frac{8\rho}{(1+\rho)^2}\langle \partial_z^\alpha Z_\varepsilon, \partial_\theta\partial_z^\alpha Z_\varepsilon\rangle\,dz$$
$$\leqslant C(s,d)\big(1+\varepsilon\|Z_\varepsilon\|_{H^s}\big)\|Z_\varepsilon\|_{H^s}^2 - \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} \frac{8\rho}{(1+\rho)^2}\left\langle \partial_z^\alpha Z_\varepsilon, \partial_z^\alpha\nabla_z\left(\frac{\rho-1}{\rho+1}\right)\right\rangle dz. \tag{105}$$

Inserting (105) into (101) gives

$$\frac{d}{d\theta}\int_{\mathbb{R}^d} \frac{4\rho}{(1+\rho)^2}|\partial_z^\alpha Z_\varepsilon|^2\,dz$$

$$\leqslant C(s,d)\big(1+\varepsilon\|Z_\varepsilon\|_{H^s}\big)\|Z_\varepsilon\|_{H^s}^2 + \int_{\mathbb{R}^d} 4\big|\partial_z^\alpha Z_\varepsilon\big|^2 \left\{ \frac{1-\rho}{(1+\rho)^3}\partial_\theta \rho + 2\nabla_z \cdot \left(\frac{\rho(1-\rho)}{(1+\rho)^3} U_\varepsilon \right) \right\} dz$$

$$- \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} \frac{8\rho}{(1+\rho)^2} \left\langle \partial_z^\alpha Z_\varepsilon, \partial_z^\alpha \nabla_z \left(\frac{\rho-1}{\rho+1} \right) \right\rangle dz$$

$$\leqslant C(s,d)\big(1+\varepsilon\|Z_\varepsilon\|_{H^s}\big)\|Z_\varepsilon\|_{H^s}^2 - \int_{\mathbb{R}^d} |\partial_z^\alpha Z_\varepsilon|^2 \frac{16\rho}{(1+\rho)^4} U_\varepsilon \cdot \nabla_z \rho \, dz$$

$$- \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} \frac{8\rho}{(1+\rho)^2} \left\langle \partial_z^\alpha Z_\varepsilon, \partial_z^\alpha \nabla_z \left(\frac{\rho-1}{\rho+1} \right) \right\rangle dz,$$

where we have used the first equation in (99) for the last inequality. By Sobolev imbedding, we have $|U_\varepsilon \cdot \nabla_z \rho| \leqslant C\rho \|Z_\varepsilon\|_{L^\infty} \|\nabla_z \rho\|_{L^\infty} \leqslant C\varepsilon \|Z_\varepsilon\|_{H^s} \|(\rho-1)/\varepsilon\|_{H^s} \leqslant C\varepsilon(\|Z_\varepsilon\|_{H^s}^2 + \|(\rho-1)/\varepsilon\|_{H^s}^2)$, hence

$$\frac{d}{d\theta}\int_{\mathbb{R}^d} \frac{4\rho}{(1+\rho)^2}\big|\partial_z^\alpha Z_\varepsilon\big|^2 dz \leqslant C(s,d)\big(1+\varepsilon\|Z_\varepsilon\|_{H^s}^2 + \varepsilon\big\|(\rho-1)/\varepsilon\big\|_{H^s}^2\big)\|Z_\varepsilon\|_{H^s}^2$$
$$- \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} \frac{8\rho}{(1+\rho)^2} \left\langle \partial_z^\alpha Z_\varepsilon, \partial_z^\alpha \nabla_z \left(\frac{\rho-1}{\rho+1} \right) \right\rangle dz. \tag{106}$$

It remains to study the last integral in (106):

$$-\frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} \frac{8\rho}{(1+\rho)^2} \left\langle \partial_z^\alpha Z_\varepsilon, \partial_z^\alpha \nabla_z \left(\frac{\rho-1}{\rho+1} \right) \right\rangle dz = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} \frac{16\rho}{(1+\rho)^2} \left\langle \partial_z^\alpha U_\varepsilon, \partial_z^\alpha \left(\frac{\nabla_z \rho}{(1+\rho)^2} \right) \right\rangle dz.$$

Thanks to a new use of (59), there holds

$$\left\| \partial_z^\alpha \left(\frac{\nabla_z \rho}{(1+\rho)^2} \right) - \frac{\nabla_z \partial_z^\alpha \rho}{(1+\rho)^2} \right\|_{L^2}$$
$$= \left\| \partial_z^\alpha \left(\left[\frac{1}{(1+\rho)^2} - \frac{1}{4} \right] \nabla_z \rho \right) - \left[\frac{1}{(1+\rho)^2} - \frac{1}{4} \right] \nabla_z \partial_z^\alpha \rho \right\|_{L^2}$$
$$\leqslant C(s,d)\left(\left\| \frac{1}{(1+\rho)^2} - \frac{1}{4} \right\|_{H^s} \|\nabla_z \rho\|_{L^\infty} + \left\| \nabla_z \left[\frac{1}{(1+\rho)^2} - \frac{1}{4} \right] \right\|_{L^\infty} \|\nabla_z \rho\|_{H^{s-1}} \right)$$
$$\leqslant C(s,d)\varepsilon^2 \left\| \frac{\rho-1}{\varepsilon} \right\|_{H^s}^2,$$

where we have used the Sobolev imbedding for the last inequality, since $s > 1 + d/2$. Thus, by Cauchy–Schwarz,

$$-\frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} \frac{8\rho}{(1+\rho)^2} \left\langle \partial_z^\alpha Z_\varepsilon, \partial_z^\alpha \nabla_z \left(\frac{\rho-1}{\rho+1} \right) \right\rangle dz \leqslant -\frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} \frac{16\rho}{(1+\rho)^4} \big\langle \partial_z^\alpha U_\varepsilon, \partial_z^\alpha \nabla_z \rho \big\rangle dz + C(s,d)\left\| \frac{\rho-1}{\varepsilon} \right\|_{H^s}^2$$
$$\leqslant \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} \frac{16\rho}{(1+\rho)^4} \big\langle \partial_z^\alpha (\nabla_z \cdot U_\varepsilon), \partial_z^\alpha \rho \big\rangle dz + C(s,d)\left\| \frac{\rho-1}{\varepsilon} \right\|_{H^s}^2,$$

after integrating by parts. Inserting this into (106) yields

$$\frac{d}{d\theta}\int_{\mathbb{R}^d} \frac{4\rho}{(1+\rho)^2}\big|\partial_z^\alpha Z_\varepsilon\big|^2 dz \leqslant C(s,d)\left(1+\varepsilon\|Z_\varepsilon\|_{H^s}^2 + \varepsilon\left\| \frac{\rho-1}{\varepsilon} \right\|_{H^s}^2\right)\|Z_\varepsilon\|_{H^s}^2$$
$$+ C(s,d)\left\| \frac{\rho-1}{\varepsilon} \right\|_{H^s}^2 + \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} \frac{16\rho}{(1+\rho)^4} \big\langle \partial_z^\alpha (\nabla_z \cdot U_\varepsilon), \partial_z^\alpha \rho \big\rangle dz. \tag{107}$$

Now, observe that, by (99),

$$\partial_\theta\left(\frac{4}{\varepsilon^2(1+\rho)^4}\big[\partial_z^\alpha\rho\big]^2\right)+\frac{16}{\varepsilon^2(1+\rho)^4}\left\langle\partial_z^\alpha\rho,\partial_z^\alpha\left(\frac{1-\rho}{1+\rho}U_\varepsilon\cdot\nabla_z\rho+\rho\nabla_z\cdot U_\varepsilon\right)\right\rangle$$
$$+\frac{16}{\varepsilon^2(1+\rho)^5}\left(\frac{1-\rho}{1+\rho}U_\varepsilon\cdot\nabla_z\rho+\rho\nabla_z\cdot U_\varepsilon\right)\big[\partial_z^\alpha\rho\big]^2=0.$$

Integrating and using (59), we obtain

$$\frac{d}{d\theta}\int_{\mathbb{R}^d}\frac{4}{\varepsilon^2(1+\rho)^4}\big[\partial_z^\alpha\rho\big]^2\,dz+\int_{\mathbb{R}^d}\frac{16}{\varepsilon^2(1+\rho)^4}\left\langle\partial_z^\alpha\rho,\frac{1-\rho}{1+\rho}U_\varepsilon\cdot\nabla_z\partial_z^\alpha\rho\right\rangle dz+\int_{\mathbb{R}^d}\frac{16\rho}{\varepsilon^2(1+\rho)^4}\langle\partial_z^\alpha\rho,\nabla_z\cdot\partial_z^\alpha U_\varepsilon\rangle\,dz$$
$$\leqslant C(s,d)\left(\left\|\frac{\rho-1}{\varepsilon}\right\|_{H^s}^2+\|U_\varepsilon\|_{H^s}^2\right).$$

We integrate by parts:

$$\int_{\mathbb{R}^d}\frac{16}{\varepsilon^2(1+\rho)^4}\left\langle\partial_z^\alpha\rho,\frac{1-\rho}{1+\rho}U_\varepsilon\cdot\nabla_z\partial_z^\alpha\rho\right\rangle dz=\int_{\mathbb{R}^d}\frac{8(1-\rho)}{\varepsilon^2(1+\rho)^5}U_\varepsilon\cdot\nabla_z\big(\big[\partial_z^\alpha\rho\big]^2\big)\,dz$$
$$=-\frac{8}{\varepsilon^2}\int_{\mathbb{R}^d}\big[\partial_z^\alpha\rho\big]^2\nabla_z\cdot\left(\frac{(1-\rho)}{(1+\rho)^5}U_\varepsilon\right)dz.$$

Therefore,

$$\frac{d}{d\theta}\int_{\mathbb{R}^d}\frac{4}{\varepsilon^2(1+\rho)^4}\big[\partial_z^\alpha\rho\big]^2\,dz\leqslant-\int_{\mathbb{R}^d}\frac{16\rho}{\varepsilon^2(1+\rho)^4}\langle\partial_z^\alpha\rho,\nabla_z\cdot\partial_z^\alpha U_\varepsilon\rangle\,dz$$
$$+C(s,d)\left(\left\|\frac{\rho-1}{\varepsilon}\right\|_{H^s}^2+\|Z_\varepsilon\|_{H^s}^2\right).\tag{108}$$

Combining (107) and (108), we see that the bad (singular) terms cancel out (due to the choice of the weights) and infer

$$\frac{d}{d\theta}\int_{\mathbb{R}^d}\frac{4\rho}{(1+\rho)^2}\big|\partial_z^\alpha Z_\varepsilon\big|^2+\frac{4}{\varepsilon^2(1+\rho)^4}\big[\partial_z^\alpha\rho\big]^2\,dz$$
$$\leqslant C(s,d)\left(\left\|\frac{\rho-1}{\varepsilon}\right\|_{H^s}^2+\|Z_\varepsilon\|_{H^s}^2\right)\left(1+\varepsilon\left\|\frac{\rho-1}{\varepsilon}\right\|_{H^s}^2+\varepsilon\|Z_\varepsilon\|_{H^s}^2\right).$$

The proof of Proposition 5 is complete. □

Proof of Theorem 10. The uniform bounds (57) for $0\leqslant\theta\leqslant\theta_*$, where $\theta_*>0$ does not depend on ε, come directly from Proposition 5 and arguing as in [10], Section 4. For the comparison result with the free wave equation, we need to estimate the right-hand side of (56) in H^{s-2}. Let us observe that for the (GP) equation, (56) becomes

$$\begin{cases}\partial_t A_\varepsilon+2\nabla_z\cdot U_\varepsilon=-2\nabla_z\cdot(A_\varepsilon U_\varepsilon)\\ \partial_t U_\varepsilon+\dfrac{1}{2}\nabla_z A_\varepsilon=-\varepsilon\nabla_z\left(|U_\varepsilon|^2+\dfrac{\Delta_z\sqrt{1+\varepsilon A_\varepsilon}}{\sqrt{1+\varepsilon A_\varepsilon}}\right).\end{cases}$$

The H^{s-2} estimate in [10], Section 4 then follows noticing that $\|\nabla_z\cdot G\|_{H^{s-2}}\leqslant K\|G\|_{H^{s-1}}$ and $\|\nabla_z g\|_{H^{s-2}}\leqslant K\|g\|_{H^{s-1}}$ for any $s\in\mathbb{R}$, as can be seen using Fourier transform (K depends only on s and d). For the equation for U_ε in (56), we may use this fact since the right-hand side is a gradient and get

$$\left\|\partial_t U_\varepsilon+\frac{1}{2}\nabla_z A_\varepsilon\right\|_{H^{s-2}}$$
$$\leqslant K(s,d)\left\|-\frac{\varepsilon^2A_\varepsilon|U_\varepsilon|^2}{2+\varepsilon A_\varepsilon}+\frac{\varepsilon^3|\nabla_z A_\varepsilon|^2}{2(1+\varepsilon A_\varepsilon)(2+\varepsilon A_\varepsilon)}+\frac{\varepsilon A_\varepsilon^2}{2(2+\varepsilon A_\varepsilon)}+\frac{\varepsilon\Delta_z\sqrt{1+\varepsilon A_\varepsilon}}{\sqrt{1+\varepsilon A_\varepsilon}}\right\|_{H^{s-1}}.$$

Since $s > 1 + d/2$, H^{s-1} is an algebra and the uniform bounds (57) imply

$$\left\| \partial_t U_\varepsilon + \frac{1}{2}\nabla_z A_\varepsilon \right\|_{H^{s-2}} \leqslant K(s,d)\big(\varepsilon^2 \Lambda^3 + \varepsilon^2 \Lambda^2 + \varepsilon \Lambda\big) \leqslant K(s,d)\varepsilon\big(\Lambda + \Lambda^2\big),$$

since $K_0(s,d)\varepsilon\Lambda \leqslant 1$. For the equation for A_ε in (56), we no longer have a source term in divergence form. We then modify the argument by invoking the fact that $H^{s-2} \cap L^\infty$ is an algebra (see, for instance, [51]) as soon as $s - 2 \geqslant 0$. Here, $s > 1 + d/2 \geqslant 3/2$ and s is an integer, thus $s \geqslant 2$. As a consequence,

$$\|A_\varepsilon \nabla_z \cdot U_\varepsilon\|_{H^{s-2}} \leqslant K \|A_\varepsilon\|_{H^{s-2}\cap L^\infty} \|\nabla_z \cdot U_\varepsilon\|_{H^{s-2}\cap L^\infty} \leqslant K\Lambda^2$$

using (57) and the Sobolev imbedding $\nabla_z \cdot U_\varepsilon \in H^{s-1} \hookrightarrow L^\infty$ (since $s - 1 > d/2$). Similarly, one has

$$\left\| \frac{A_\varepsilon}{2 + \varepsilon A_\varepsilon} U_\varepsilon \cdot \nabla_z A_\varepsilon \right\|_{H^{s-2}} \leqslant K\Lambda^3,$$

which yields, using once again that $K_0(s,d)\varepsilon\Lambda \leqslant 1$,

$$\|\partial_t A_\varepsilon + 2\nabla_z \cdot U_\varepsilon\|_{H^{s-2}} \leqslant K(s,d)\varepsilon\Lambda^2.$$

Once one has these estimates, the comparison result with the free wave equation (58) can be shown exactly as in [10] Section 4, thus we skip the details.

5.2. *Proof of the (mKdV)/(mKP-I) limit for the Landau–Lifshitz equation*

As for the proof of Theorems 6 and 7, the proof is divided into two steps.

Step 1: Construction of an approximate solution. This time, the expansion in ε is done on the system (51) (instead of what we did with the Madelung system (83) for the (NLS) equation). To construct an approximate solution $(A_\varepsilon^{\mathrm{app}}, U_\varepsilon^{\mathrm{app}}) = (A_0, U_0) + \varepsilon(A_1, U_1) + \varepsilon^2(A_2, U_2)$, the formal computation is very similar to the one in Section 4.1, since the quasilinear terms in (51) do not perturb the leading order terms, thus we skip it. However, since it is important that the vector field Z is a gradient, we shall impose that $U_\varepsilon^{\mathrm{app}}$ is a gradient. We thus choose $A_0(\tau) \equiv \zeta(\tau)$, $2U_0(\tau) \equiv \nabla_z \partial_{z_1}^{-1}\zeta(\tau)$, $U_1 = U_2 = 0$, $A_1(\tau) \equiv A_0^2(\tau)/2 = \zeta^2(\tau)/2$, so that $A_1(\tau) - 2U_1^1(\tau) = A_0^2(\tau)/2 = \zeta^2(\tau)/2$, and finally

$$A_2(\tau) \equiv \frac{1}{4}\zeta^3 + \frac{1}{2}\partial_{z_1}^2 \zeta(\tau) + \Delta_{z_\perp}\partial_{z_1}^{-2}\zeta \in L^\infty\big([0,\tau_*], H^{s+3}\big).$$

The approximate solution then verifies, uniformly for $0 \leqslant \theta \leqslant \tau_*/\varepsilon$,

$$\begin{cases} \partial_\theta A_\varepsilon^{\mathrm{app}} - \dfrac{1}{\varepsilon}\partial_{z_1} A_\varepsilon^{\mathrm{app}} - \dfrac{2\varepsilon A_\varepsilon^{\mathrm{app}}}{2 + \varepsilon A_\varepsilon^{\mathrm{app}}}\big[U_\varepsilon^{\mathrm{app},1}\partial_{z_1} A_\varepsilon^{\mathrm{app}} + \varepsilon^2 U_\varepsilon^{\mathrm{app},\perp} \cdot \nabla_{z_\perp} A_\varepsilon^{\mathrm{app}}\big] \\ \quad + \dfrac{2}{\varepsilon}\big(1 + \varepsilon A_\varepsilon^{\mathrm{app}}\big)\big(\partial_{z_1} U_\varepsilon^{\mathrm{app},1} + \varepsilon^2 \nabla_{z_\perp} \cdot U_\varepsilon^{\mathrm{app},\perp}\big) = \mathcal{O}_{H^{s+1}}\big(\varepsilon^2\big) \\ \partial_\theta U_\varepsilon^{\mathrm{app}} - \dfrac{1}{\varepsilon}\partial_{z_1} U_\varepsilon^{\mathrm{app}} - \nabla_z\Big(\dfrac{\varepsilon A_\varepsilon^{\mathrm{app}}}{2 + \varepsilon A_\varepsilon^{\mathrm{app}}}\big[\big[U_\varepsilon^{\mathrm{app},1}\big]^2 + \varepsilon^2\big|U_\varepsilon^{\mathrm{app},\perp}\big|^2\big]\Big) + \dfrac{1}{\varepsilon}\nabla_z\Big(\dfrac{A_\varepsilon^{\mathrm{app}}}{2 + \varepsilon A_\varepsilon^{\mathrm{app}}}\Big) \\ \quad - \nabla_z\Big(\dfrac{\partial_{z_1}^2\sqrt{1 + \varepsilon A_\varepsilon^{\mathrm{app}}} + \varepsilon^2 \Delta_{z_\perp}\sqrt{1 + \varepsilon A_\varepsilon^{\mathrm{app}}}}{\sqrt{1 + \varepsilon A_\varepsilon^{\mathrm{app}}}}\Big) + \varepsilon^2 \nabla_z\Big(\dfrac{(\partial_{z_1} A_\varepsilon^{\mathrm{app}})^2 + \varepsilon^2|\nabla_{z_\perp} A_\varepsilon^{\mathrm{app}}|^2}{(1 + \varepsilon A_\varepsilon^{\mathrm{app}})(2 + \varepsilon A_\varepsilon^{\mathrm{app}})}\Big) = \mathcal{O}_{H^s}\big(\varepsilon^2\big). \end{cases}$$

Moreover, we have

$$\big\|A_\varepsilon^{\mathrm{in}} - A_\varepsilon^{\mathrm{app}}(\theta = 0)\big\|_{H^{s+3}} + \big\|U_\varepsilon^{\mathrm{in}} - U_\varepsilon^{\mathrm{app}}(\theta = 0)\big\|_{H^{s+3}} \leqslant C\varepsilon^2.$$

As a consequence, denoting $\rho_\varepsilon^{\mathrm{app}} \equiv 1 + \varepsilon A_\varepsilon^{\mathrm{app}}$,

$$Z_\varepsilon^{\mathrm{app}} \equiv \big(U_\varepsilon^{\mathrm{app},1}, \varepsilon U_\varepsilon^{\mathrm{app},\perp}\big) - \frac{i}{2\rho_\varepsilon^{\mathrm{app}}}\big(\partial_{z_1}\rho_\varepsilon^{\mathrm{app}}, \varepsilon\nabla_{z_\perp}\rho_\varepsilon^{\mathrm{app}}\big),$$

we infer

$$\begin{cases} \partial_\theta \rho_\varepsilon^{\mathrm{app}} - \dfrac{1}{\varepsilon}\partial_{z_1}\rho_\varepsilon^{\mathrm{app}} + 2\dfrac{1-\rho_\varepsilon^{\mathrm{app}}}{1+\rho_\varepsilon^{\mathrm{app}}}\mathrm{Re}\big(Z_\varepsilon^{\mathrm{app}}\big)\cdot\nabla^\varepsilon\rho_\varepsilon^{\mathrm{app}} + 2\rho_\varepsilon^{\mathrm{app}}\nabla^\varepsilon\cdot\mathrm{Re}\big(Z_\varepsilon^{\mathrm{app}}\big) = \mathcal{O}_{H^{s+1}}\big(\varepsilon^5\big) \\ \nabla^\varepsilon\rho_\varepsilon^{\mathrm{app}} = 2\rho_\varepsilon^{\mathrm{app}}\mathrm{Im}\big(Z_\varepsilon^{\mathrm{app}}\big) \\ \partial_\theta Z_\varepsilon^{\mathrm{app}} - \dfrac{1}{\varepsilon}\partial_{z_1}Z_\varepsilon^{\mathrm{app}} - i\Delta^\varepsilon Z_\varepsilon^{\mathrm{app}} + \nabla^\varepsilon\left(\dfrac{1-\rho_\varepsilon^{\mathrm{app}}}{1+\rho_\varepsilon^{\mathrm{app}}}\big(Z_\varepsilon^{\mathrm{app}}\cdot Z_\varepsilon^{\mathrm{app}}\big)\right) + \dfrac{1}{\varepsilon^2}\nabla^\varepsilon\left(\dfrac{\rho_\varepsilon^{\mathrm{app}}-1}{\rho_\varepsilon^{\mathrm{app}}+1}\right) = \mathcal{O}_{H^s}\big(\varepsilon^2\big), \end{cases} \tag{109}$$

where $\nabla^\varepsilon \equiv {}^t(\partial_{z_1}, \varepsilon\nabla_{z_\perp})$ and $\Delta^\varepsilon \equiv [\nabla^\varepsilon]^2 = \partial_{z_1}^2 + \varepsilon^2\Delta_{z_\perp}$. In addition, $\varepsilon^{-1}\|\rho_\varepsilon^{\mathrm{in}} - \rho_\varepsilon^{\mathrm{app}}(\theta=0)\|_{H^{s+3}} + \|Z_\varepsilon^{\mathrm{in}} - Z_\varepsilon^{\mathrm{app}}(\theta=0)\|_{H^{s+3}} \leqslant C\varepsilon^2$.

Step 2: Nonlinear stability. Let $(A_\varepsilon = (\rho_\varepsilon - 1)/\varepsilon, U_\varepsilon)$ solve (54) (for which we know local well-posedness). We set

$$\tilde\rho_\varepsilon \equiv \frac{\rho_\varepsilon}{\rho_\varepsilon^{\mathrm{app}}}, \qquad \tilde Z_\varepsilon \equiv \big(U_\varepsilon^1, \varepsilon U_\varepsilon^\perp\big) - \frac{i}{2\rho_\varepsilon}(\partial_{z_1}\rho_\varepsilon, \varepsilon\nabla_{z_\perp}\rho_\varepsilon) - Z_\varepsilon^{\mathrm{app}},$$

so that there holds

$$\begin{cases} \partial_\theta\tilde\rho_\varepsilon - \dfrac{1}{\varepsilon}\partial_{z_1}\tilde\rho_\varepsilon + 2\dfrac{1-\rho_\varepsilon}{1+\rho_\varepsilon}\mathrm{Re}(Z_\varepsilon)\cdot\nabla^\varepsilon\tilde\rho_\varepsilon + 2\dfrac{\tilde\rho_\varepsilon}{\rho_\varepsilon^{\mathrm{app}}}\dfrac{1-\rho_\varepsilon^{\mathrm{app}}}{1+\rho_\varepsilon^{\mathrm{app}}}\mathrm{Re}\big(Z_\varepsilon^{\mathrm{app}}\big)\cdot\nabla^\varepsilon\rho_\varepsilon^{\mathrm{app}} \\ \quad -2\dfrac{\tilde\rho_\varepsilon}{\rho_\varepsilon^{\mathrm{app}}}\dfrac{1-\rho_\varepsilon^{\mathrm{app}}}{1+\rho_\varepsilon^{\mathrm{app}}}\mathrm{Re}\big(Z_\varepsilon^{\mathrm{app}}\big)\cdot\nabla^\varepsilon\rho_\varepsilon^{\mathrm{app}} + 2\tilde\rho_\varepsilon\nabla^\varepsilon\cdot\mathrm{Re}(\tilde Z_\varepsilon) = \mathcal{O}_{H^{s+1}}\big(\varepsilon^5\big) \\ \dfrac{\nabla^\varepsilon\rho_\varepsilon}{\rho_\varepsilon} = \dfrac{\nabla^\varepsilon\rho_\varepsilon^{\mathrm{app}}}{\rho_\varepsilon^{\mathrm{app}}} + \dfrac{\nabla^\varepsilon\tilde\rho_\varepsilon}{\tilde\rho_\varepsilon} = 2\,\mathrm{Im}\big(Z_\varepsilon^{\mathrm{app}}\big) + 2\,\mathrm{Im}(\tilde Z_\varepsilon) \\ \partial_\theta\tilde Z_\varepsilon - \dfrac{1}{\varepsilon}\partial_{z_1}\tilde Z_\varepsilon - i\Delta^\varepsilon\tilde Z_\varepsilon + \nabla^\varepsilon\left(\dfrac{1-\rho_\varepsilon}{1+\rho_\varepsilon}\big(2\tilde Z_\varepsilon\cdot Z_\varepsilon^{\mathrm{app}} + \tilde Z_\varepsilon\cdot\tilde Z_\varepsilon\big)\right) + \nabla^\varepsilon\left(\left[\dfrac{1-\rho_\varepsilon}{1+\rho_\varepsilon} - \dfrac{\rho_\varepsilon^{\mathrm{app}}-1}{\rho_\varepsilon^{\mathrm{app}}+1}\right]Z_\varepsilon^{\mathrm{app}}\cdot Z_\varepsilon^{\mathrm{app}}\right) \\ \quad +\dfrac{1}{\varepsilon^2}\nabla^\varepsilon\left(\dfrac{\rho_\varepsilon-1}{\rho_\varepsilon+1} - \dfrac{\rho_\varepsilon^{\mathrm{app}}-1}{\rho_\varepsilon^{\mathrm{app}}+1}\right) = \mathcal{O}_{H^s}\big(\varepsilon^2\big). \end{cases} \tag{110}$$

For the initial data, we have by construction

$$\varepsilon^{-1}\big\|\tilde\rho_\varepsilon(\theta=0)\big\|_{H^{s+3}} + \big\|\tilde Z_\varepsilon(\theta=0)\big\|_{H^{s+3}} \leqslant C\varepsilon^2.$$

We define here again $0 < \bar\theta_\varepsilon \leqslant |\ln\varepsilon|$ to be the maximal time for which

$$\sup_{0\leqslant\theta\leqslant\bar\theta_\varepsilon}\big\|\tilde\rho_\varepsilon(\theta) - 1\big\|_{H^s} \leqslant \varepsilon,$$

so that

$$\begin{aligned}\big\|\rho_\varepsilon(\theta) - 1\big\|_{H^s} &= \big\|\rho_\varepsilon^{\mathrm{app}}(\theta)\tilde\rho_\varepsilon(\theta) - 1\big\|_{H^s} \leqslant K\big\|\rho_\varepsilon^{\mathrm{app}}(\theta)\big\|_{H^s}\big\|\tilde\rho_\varepsilon(\theta) - 1\big\|_{H^s} + K\big\|\tilde\rho_\varepsilon(\theta)\big\|_{H^s}\big\|\rho_\varepsilon^{\mathrm{app}}(\theta) - 1\big\|_{H^s} \\ &\leqslant C\varepsilon\end{aligned}$$

for $0 \leqslant \theta \leqslant \bar\theta_\varepsilon$. Paralleling the proof of Proposition 5, we shall now prove the following result, where the weight for the potential part has an extra $\rho_\varepsilon^{\mathrm{app}}$ compared to the weight in Proposition 5.

Proposition 6. *If $s > 1 + d/2$, there exists C, depending only on Λ, s and d, such that, for any $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leqslant s$ and any $0 \leqslant \theta \leqslant \bar\theta_\varepsilon$, there holds*

$$\frac{d}{d\theta}\int_{\mathbb{R}^d}\frac{4\rho_\varepsilon}{(1+\rho_\varepsilon)^2}\big|\partial_z^\alpha\tilde Z_\varepsilon\big|^2 + \frac{4\rho_\varepsilon^{\mathrm{app}}}{\varepsilon^2(1+\rho_\varepsilon)^4}\big[\partial_z^\alpha(\tilde\rho_\varepsilon - 1)\big]^2\,dz \leqslant C\left(\varepsilon^4 + \left\|\frac{\tilde\rho_\varepsilon - 1}{\varepsilon}\right\|_{H^s}^2 + \|\tilde Z_\varepsilon\|_{H^s}^2\right). \tag{111}$$

Proof. We shall assume $\alpha \neq 0$, the case $\alpha = 0$ could be treated similarly, or using the conservation of the energy combined with the conservation of $\int_{\mathbb{R}^d} \zeta^2\,dz$. The computations are very close to those for Proposition 5, thus we shall only emphasize on the differences. Let us observe that the before last term in the equation for $\tilde{Z}_\varepsilon$ is easily estimated in H^s, in view of the equality $2\mathrm{Im}(\tilde{Z}_\varepsilon) = \frac{\nabla^\varepsilon \tilde{\rho}_\varepsilon}{\tilde{\rho}_\varepsilon}$:

$$\left\| \nabla^\varepsilon \left(\left[\frac{1-\rho_\varepsilon}{1+\rho_\varepsilon} - \frac{\rho_\varepsilon^{\mathrm{app}} - 1}{\rho_\varepsilon^{\mathrm{app}} + 1} \right] Z_\varepsilon^{\mathrm{app}} \cdot Z_\varepsilon^{\mathrm{app}} \right) \right\|_{H^s} = \left\| \nabla^\varepsilon \left(\frac{2\rho_\varepsilon^{\mathrm{app}}(1-\tilde{\rho}_\varepsilon)}{(1+\rho_\varepsilon)(\rho_\varepsilon^{\mathrm{app}}+1)} Z_\varepsilon^{\mathrm{app}} \cdot Z_\varepsilon^{\mathrm{app}} \right) \right\|_{H^s}$$
$$\leqslant C\big(\|\tilde{\rho}_\varepsilon - 1\|_{H^s} + \|\tilde{Z}_\varepsilon\|_{H^s} \big).$$

Similarly to (102) and (103), one has

$$\int_{\mathbb{R}^d} \frac{8\rho_\varepsilon}{(1+\rho_\varepsilon)^2} \langle \partial_z^\alpha \tilde{Z}_\varepsilon, \partial_\theta \partial_z^\alpha \tilde{Z}_\varepsilon \rangle\,dz \leqslant \int_{\mathbb{R}^d} \frac{8\rho_\varepsilon(1-\rho_\varepsilon)}{(1+\rho_\varepsilon)^3} \partial_{z_1}\rho_\varepsilon \big|\partial_z^\alpha \tilde{Z}_\varepsilon\big|^2\,dz$$
$$- 16 \int_{\mathbb{R}^d} \frac{\rho_\varepsilon(1-\rho_\varepsilon)}{(1+\rho_\varepsilon)^3} \langle \partial_z^\alpha \tilde{Z}_\varepsilon, i\big(\mathrm{Im}(Z_\varepsilon^{\mathrm{app}}) + \mathrm{Im}(\tilde{Z}_\varepsilon)\big) \cdot \nabla^\varepsilon \partial_z^\alpha \tilde{Z}_\varepsilon \rangle\,dz$$
$$- \int_{\mathbb{R}^d} \frac{8\rho}{(1+\rho)^2} \left\langle \partial_z^\alpha \tilde{Z}_\varepsilon, \partial_z^\alpha \nabla^\varepsilon \left(\frac{1-\rho_\varepsilon}{1+\rho_\varepsilon} \big(2\tilde{Z}_\varepsilon \cdot Z_\varepsilon^{\mathrm{app}} + \tilde{Z}_\varepsilon \cdot \tilde{Z}_\varepsilon\big) \right) \right\rangle dz$$
$$- \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} \frac{8\rho}{(1+\rho)^2} \left\langle \partial_z^\alpha \tilde{Z}_\varepsilon, \partial_z^\alpha \nabla^\varepsilon \left(\frac{\rho_\varepsilon - 1}{\rho_\varepsilon + 1} - \frac{\rho_\varepsilon^{\mathrm{app}} - 1}{\rho_\varepsilon^{\mathrm{app}} + 1} \right) \right\rangle dz$$
$$+ C\varepsilon^2 \|\tilde{Z}_\varepsilon\|_{H^s} + C\left(\left\| \frac{\tilde{\rho}_\varepsilon - 1}{\varepsilon} \right\|_{H^s}^2 + \|\tilde{Z}_\varepsilon\|_{H^s}^2 \right),$$

using an integration by parts for the transport term. The first term is $\leqslant C\varepsilon \|\tilde{Z}_\varepsilon\|_{H^s}^2$, and the before last is $\leqslant C\varepsilon^4 + C\|\tilde{Z}_\varepsilon\|_{H^s}^2$. Concerning the third term, arguing as for (104) yields

$$- \int_{\mathbb{R}^d} \frac{8\rho_\varepsilon}{(1+\rho_\varepsilon)^2} \left\langle \partial_z^\alpha \tilde{Z}_\varepsilon, \partial_z^\alpha \nabla^\varepsilon \left(\frac{1-\rho_\varepsilon}{1+\rho_\varepsilon} \big(2\tilde{Z}_\varepsilon \cdot Z_\varepsilon^{\mathrm{app}} + \tilde{Z}_\varepsilon \cdot \tilde{Z}_\varepsilon\big) \right) \right\rangle dz$$
$$\leqslant - \int_{\mathbb{R}^d} \frac{16\rho_\varepsilon(1-\rho_\varepsilon)}{(1+\rho_\varepsilon)^3} \langle \partial_z^\alpha \tilde{Z}_\varepsilon, Z_\varepsilon^{\mathrm{app}} \cdot \partial_z^\alpha \nabla^\varepsilon \tilde{Z}_\varepsilon + \tilde{Z}_\varepsilon \cdot \partial_z^\alpha \nabla^\varepsilon \tilde{Z}_\varepsilon \rangle\,dz + C\big(\|\tilde{\rho}_\varepsilon - 1\|_{H^s}^2 + \|\tilde{Z}_\varepsilon\|_{H^s}^2\big)$$
$$\leqslant - \int_{\mathbb{R}^d} \frac{16\rho_\varepsilon(1-\rho_\varepsilon)}{(1+\rho_\varepsilon)^3} \langle \partial_z^\alpha \tilde{Z}_\varepsilon, \mathrm{Re}\big(Z_\varepsilon^{\mathrm{app}} + \tilde{Z}_\varepsilon\big) \cdot \partial_z^\alpha \nabla^\varepsilon \tilde{Z}_\varepsilon \rangle\,dz$$
$$+ \int_{\mathbb{R}^d} \frac{16\rho_\varepsilon(1-\rho_\varepsilon)}{(1+\rho_\varepsilon)^3} \langle \partial_z^\alpha \tilde{Z}_\varepsilon, i\mathrm{Im}\big(Z_\varepsilon^{\mathrm{app}} + \tilde{Z}_\varepsilon\big) \cdot \partial_z^\alpha \nabla^\varepsilon \tilde{Z}_\varepsilon \rangle\,dz + C\big(\|\tilde{\rho}_\varepsilon - 1\|_{H^s}^2 + \|\tilde{Z}_\varepsilon\|_{H^s}^2\big)$$
$$\leqslant \int_{\mathbb{R}^d} \frac{16\rho_\varepsilon(1-\rho_\varepsilon)}{(1+\rho_\varepsilon)^3} \langle \partial_z^\alpha \tilde{Z}_\varepsilon, i\mathrm{Im}\big(Z_\varepsilon^{\mathrm{app}} + \tilde{Z}_\varepsilon\big) \cdot \partial_z^\alpha \nabla^\varepsilon \tilde{Z}_\varepsilon \rangle\,dz + C\big(\|\tilde{\rho}_\varepsilon - 1\|_{H^s}^2 + \|\tilde{Z}_\varepsilon\|_{H^s}^2\big),$$

where we use that $2\langle \partial_z^\alpha \tilde{Z}_\varepsilon, \mathrm{Re}(Z_\varepsilon^{\mathrm{app}} + \tilde{Z}_\varepsilon) \cdot \partial_z^\alpha \nabla^\varepsilon \tilde{Z}_\varepsilon \rangle = \mathrm{Re}(Z_\varepsilon^{\mathrm{app}} + \tilde{Z}_\varepsilon) \cdot \nabla^\varepsilon |\partial_z^\alpha \tilde{Z}_\varepsilon|^2$ and an integration by parts to bound the first integral. Since here again the terms involving $\mathrm{Im}(Z_\varepsilon^{\mathrm{app}} + \tilde{Z}_\varepsilon)$ cancel out, we deduce

$$\int_{\mathbb{R}^d} \frac{8\rho_\varepsilon}{(1+\rho_\varepsilon)^2} \langle \partial_z^\alpha \tilde{Z}_\varepsilon, \partial_\theta \partial_z^\alpha \tilde{Z}_\varepsilon \rangle\,dz$$
$$\leqslant - \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} \frac{8\rho_\varepsilon}{(1+\rho_\varepsilon)^2} \left\langle \partial_z^\alpha \tilde{Z}_\varepsilon, \partial_z^\alpha \nabla^\varepsilon \left(\frac{\rho_\varepsilon - 1}{\rho_\varepsilon + 1} - \frac{\rho_\varepsilon^{\mathrm{app}} - 1}{\rho_\varepsilon^{\mathrm{app}} + 1} \right) \right\rangle dz + C\left(\varepsilon^4 + \left\| \frac{\tilde{\rho}_\varepsilon - 1}{\varepsilon} \right\|_{H^s}^2 + \|\tilde{Z}_\varepsilon\|_{H^s}^2 \right). \tag{112}$$

Since, in view of the transport equation on ρ_ε,

$$\partial_\theta\left(\frac{8\rho_\varepsilon}{(1+\rho_\varepsilon)^2}\right)=\frac{8}{(1+\rho_\varepsilon)^3}\times\frac{1-\rho_\varepsilon}{\varepsilon}\times\varepsilon\partial_\theta\rho_\varepsilon$$

is uniformly bounded by some absolute constant K_0 for $0\leqslant\theta\leqslant\bar{\theta}_\varepsilon$, we deduce from (112)

$$\frac{d}{d\theta}\int_{\mathbb{R}^d}\frac{4\rho_\varepsilon}{(1+\rho_\varepsilon)^2}\langle\partial_z^\alpha\tilde{Z}_\varepsilon,\partial_z^\alpha\tilde{Z}_\varepsilon\rangle\,dz$$
$$\leqslant-\frac{1}{\varepsilon^2}\int_{\mathbb{R}^d}\frac{8\rho_\varepsilon}{(1+\rho_\varepsilon)^2}\left\langle\partial_z^\alpha\mathrm{Re}(\tilde{Z}_\varepsilon),\partial_z^\alpha\nabla^\varepsilon\left(\frac{2\rho_\varepsilon^{\mathrm{app}}(\tilde{\rho}_\varepsilon-1)}{(\rho_\varepsilon+1)(\rho_\varepsilon^{\mathrm{app}}+1)}\right)\right\rangle dz+C\left(\varepsilon^4+\left\|\frac{\tilde{\rho}_\varepsilon-1}{\varepsilon}\right\|_{H^s}^2+\|\tilde{Z}_\varepsilon\|_{H^s}^2\right).$$

Here, we see that the term in bracket in the integral is slightly different from what we had in (106). Thanks to a new use of (59), we then get, as for (107), keeping aside the terms where we put $\partial_z^\alpha\nabla^\varepsilon$ on each one of the factors in $\frac{\rho_\varepsilon^{\mathrm{app}}}{\rho_\varepsilon^{\mathrm{app}}+1}\times(\tilde{\rho}_\varepsilon-1)\times\frac{1}{(\rho_\varepsilon+1)}$,

$$\frac{d}{d\theta}\int_{\mathbb{R}^d}\frac{4\rho_\varepsilon}{(1+\rho_\varepsilon)^2}\langle\partial_z^\alpha\tilde{Z}_\varepsilon,\partial_z^\alpha\tilde{Z}_\varepsilon\rangle\,dz$$
$$\leqslant-\frac{1}{\varepsilon^2}\int_{\mathbb{R}^d}\frac{16\rho_\varepsilon\rho_\varepsilon^{\mathrm{app}}}{(1+\rho_\varepsilon)^3(\rho_\varepsilon^{\mathrm{app}}+1)}\langle\partial_z^\alpha\mathrm{Re}(\tilde{Z}_\varepsilon),\partial_z^\alpha\nabla^\varepsilon\tilde{\rho}_\varepsilon\rangle\,dz$$
$$+\frac{1}{\varepsilon^2}\int_{\mathbb{R}^d}\frac{16\rho_\varepsilon\rho_\varepsilon^{\mathrm{app}}(\tilde{\rho}_\varepsilon-1)}{(1+\rho_\varepsilon)^2(\rho_\varepsilon^{\mathrm{app}}+1)}\left\langle\partial_z^\alpha\mathrm{Re}(\tilde{Z}_\varepsilon),\partial_z^\alpha\left(\frac{\nabla^\varepsilon\rho_\varepsilon}{(1+\rho_\varepsilon)^2}\right)\right\rangle dz$$
$$-\frac{1}{\varepsilon^2}\int_{\mathbb{R}^d}\frac{16\rho_\varepsilon(\tilde{\rho}_\varepsilon-1)}{(1+\rho_\varepsilon)^3}\left\langle\partial_z^\alpha\mathrm{Re}(\tilde{Z}_\varepsilon),\partial_z^\alpha\nabla^\varepsilon\left(\frac{\rho_\varepsilon^{\mathrm{app}}}{\rho_\varepsilon^{\mathrm{app}}+1}\right)\right\rangle dz+C\left(\varepsilon^4+\left\|\frac{\tilde{\rho}_\varepsilon-1}{\varepsilon}\right\|_{H^s}^2+\|\tilde{Z}_\varepsilon\|_{H^s}^2\right).$$

The last integral is easily estimated by $C\|\frac{\tilde{\rho}_\varepsilon-1}{\varepsilon}\|_{H^s}^2+\|\tilde{Z}_\varepsilon\|_{H^s}^2$, since $\rho_\varepsilon^{\mathrm{app}}-1=\mathcal{O}_{H^{s+1}}(\varepsilon)$ and $\|\tilde{\rho}_\varepsilon-1\|_{L^\infty}\leqslant C\|\tilde{\rho}_\varepsilon-1\|_{H^s}$ by Sobolev imbedding. In the second integral, we replace $\nabla^\varepsilon\rho_\varepsilon=\rho_\varepsilon^{\mathrm{app}}\nabla^\varepsilon\tilde{\rho}_\varepsilon-\tilde{\rho}_\varepsilon\nabla^\varepsilon\rho_\varepsilon^{\mathrm{app}}=\rho_\varepsilon^{\mathrm{app}}\nabla^\varepsilon\tilde{\rho}_\varepsilon-\tilde{\rho}_\varepsilon\mathcal{O}_{H^s}(\varepsilon)$ and infer from (59) that it is

$$\leqslant\frac{1}{\varepsilon^2}\int_{\mathbb{R}^d}\frac{16\rho_\varepsilon[\rho_\varepsilon^{\mathrm{app}}]^2(\tilde{\rho}_\varepsilon-1)}{(1+\rho_\varepsilon)^4(\rho_\varepsilon^{\mathrm{app}}+1)}\langle\partial_z^\alpha\mathrm{Re}(\tilde{Z}_\varepsilon),\partial_z^\alpha\nabla^\varepsilon\tilde{\rho}_\varepsilon\rangle\,dz+C\left(\varepsilon^4+\left\|\frac{\tilde{\rho}_\varepsilon-1}{\varepsilon}\right\|_{H^s}^2+\|\tilde{Z}_\varepsilon\|_{H^s}^2\right).$$

Consequently, by using another integration by parts,

$$\frac{d}{d\theta}\int_{\mathbb{R}^d}\frac{4\rho_\varepsilon}{(1+\rho_\varepsilon)^2}\langle\partial_z^\alpha\tilde{Z}_\varepsilon,\partial_z^\alpha\tilde{Z}_\varepsilon\rangle\,dz\leqslant C\left(\varepsilon^4+\left\|\frac{\tilde{\rho}_\varepsilon-1}{\varepsilon}\right\|_{H^s}^2+\|\tilde{Z}_\varepsilon\|_{H^s}^2\right)$$
$$+\frac{16}{\varepsilon^2}\int_{\mathbb{R}^d}\frac{\rho_\varepsilon\rho_\varepsilon^{\mathrm{app}}}{(1+\rho_\varepsilon)^4}\langle\partial_z^\alpha\nabla^\varepsilon\cdot\mathrm{Re}(\tilde{Z}_\varepsilon),\partial_z^\alpha\tilde{\rho}_\varepsilon\rangle\,dz.\tag{113}$$

Now, observe that, by (109),

$$\partial_\theta\left(\frac{4\rho_\varepsilon^{\mathrm{app}}}{\varepsilon^2(1+\rho_\varepsilon)^4}\left[\partial_z^\alpha\tilde{\rho}_\varepsilon\right]^2\right)+\frac{16\rho_\varepsilon^{\mathrm{app}}\partial_\theta\rho_\varepsilon}{\varepsilon^2(1+\rho_\varepsilon)^5}\left[\partial_z^\alpha\tilde{\rho}_\varepsilon\right]^2-\frac{4\partial_\theta\rho_\varepsilon^{\mathrm{app}}}{\varepsilon^2(1+\rho_\varepsilon)^4}\left[\partial_z^\alpha\tilde{\rho}_\varepsilon\right]^2$$
$$+\frac{16\rho_\varepsilon^{\mathrm{app}}}{\varepsilon^2(1+\rho_\varepsilon)^4}\left\langle\partial_z^\alpha\tilde{\rho}_\varepsilon,\partial_z^\alpha\left(-\frac{1}{2\varepsilon}\partial_{z_1}\tilde{\rho}_\varepsilon+\frac{1-\rho_\varepsilon}{1+\rho_\varepsilon}\mathrm{Re}(Z_\varepsilon)\cdot\nabla^\varepsilon\tilde{\rho}_\varepsilon+\tilde{\rho}_\varepsilon\nabla^\varepsilon\cdot\mathrm{Re}(\tilde{Z}_\varepsilon)\right.\right.$$
$$\left.\left.+\frac{\tilde{\rho}_\varepsilon}{\rho_\varepsilon^{\mathrm{app}}}\left[\frac{1-\rho_\varepsilon}{1+\rho_\varepsilon}\mathrm{Re}(Z_\varepsilon)-\frac{1-\rho_\varepsilon^{\mathrm{app}}}{1+\rho_\varepsilon^{\mathrm{app}}}\mathrm{Re}\big(Z_\varepsilon^{\mathrm{app}}\big)\right]\cdot\nabla^\varepsilon\rho_\varepsilon^{\mathrm{app}}+\mathcal{O}_{H^{s+1}}\big(\varepsilon^5\big)\right)\right\rangle=0.\tag{114}$$

For $0 \leqslant \theta \leqslant \bar{\theta}_\varepsilon$, we have

$$\|\partial_\theta \rho_\varepsilon\|_{L^\infty} = \left\| -\frac{1}{\varepsilon}\partial_{z_1}\rho_\varepsilon + \frac{1-\rho_\varepsilon}{1+\rho_\varepsilon} U_\varepsilon \cdot \nabla^\varepsilon \rho_\varepsilon + \rho_\varepsilon \nabla^\varepsilon \cdot U_\varepsilon \right\|_{L^\infty} \leqslant C,$$

and there also holds $\|\partial_\theta \rho_\varepsilon^{\mathrm{app}}\|_{L^\infty} = \mathcal{O}(\varepsilon)$ uniformly for $\theta \leqslant \tau_*/\varepsilon$. Furthermore, by Cauchy–Schwarz,

$$\left\| \frac{1}{\varepsilon^2} \langle \partial_z^\alpha \tilde{\rho}_\varepsilon, \partial_z^\alpha (\mathcal{O}_{H^{s+1}}(\varepsilon^5)) \rangle \right\|_{L^1} \leqslant C\varepsilon^4 \left\| \frac{\tilde{\rho}_\varepsilon - 1}{\varepsilon} \right\|_{H^s} \leqslant C \left\| \frac{\tilde{\rho}_\varepsilon - 1}{\varepsilon} \right\|_{H^s}^2 + C\varepsilon^8$$

and

$$\left\| \frac{1}{\varepsilon^2} \left\langle \partial_z^\alpha \tilde{\rho}_\varepsilon, \partial_z^\alpha \left\{ \frac{\tilde{\rho}_\varepsilon}{\rho_\varepsilon^{\mathrm{app}}} \left[\frac{1-\rho_\varepsilon}{1+\rho_\varepsilon} \mathrm{Re}(Z_\varepsilon) - \frac{1-\rho_\varepsilon^{\mathrm{app}}}{1+\rho_\varepsilon^{\mathrm{app}}} \mathrm{Re}(Z_\varepsilon^{\mathrm{app}}) \right] \cdot \nabla^\varepsilon \rho_\varepsilon^{\mathrm{app}} \right\} \right\rangle \right\|_{L^1}$$
$$\leqslant C\varepsilon \left\| \frac{\tilde{\rho}_\varepsilon - 1}{\varepsilon} \right\|_{H^s}^2 + C\varepsilon \|\mathrm{Re}(\tilde{Z}_\varepsilon)\|_{H^s}^2.$$

Integrating (114) in $z \in \mathbb{R}^d$, integrating by parts for the singular transport term and using (59), we then obtain

$$\frac{d}{d\theta} \int_{\mathbb{R}^d} \frac{4\rho_\varepsilon^{\mathrm{app}}}{\varepsilon^2(1+\rho_\varepsilon)^4} [\partial_z^\alpha \tilde{\rho}_\varepsilon]^2 \, dz + \int_{\mathbb{R}^d} \frac{4}{\varepsilon^2} \partial_{z_1} \left(\frac{\rho_\varepsilon^{\mathrm{app}}}{(1+\rho_\varepsilon)^4} \right) [\partial_z^\alpha \tilde{\rho}_\varepsilon]^2 \, dz$$
$$+ \int_{\mathbb{R}^d} \frac{16\rho_\varepsilon^{\mathrm{app}}(1-\rho_\varepsilon)}{\varepsilon^2(1+\rho_\varepsilon)^5} \langle \partial_z^\alpha \tilde{\rho}_\varepsilon, \mathrm{Re}(Z_\varepsilon) \cdot \nabla^\varepsilon \partial_z^\alpha \tilde{\rho}_\varepsilon \rangle \, dz + \int_{\mathbb{R}^d} \frac{16\rho_\varepsilon^{\mathrm{app}} \tilde{\rho}_\varepsilon}{\varepsilon^2(1+\rho_\varepsilon)^4} \langle \partial_z^\alpha \tilde{\rho}_\varepsilon, \nabla^\varepsilon \cdot \partial_z^\alpha \mathrm{Re}(\tilde{Z}_\varepsilon) \rangle \, dz$$
$$\leqslant C \left(\varepsilon^4 + \left\| \frac{\tilde{\rho}_\varepsilon - 1}{\varepsilon} \right\|_{H^s}^2 + \|\mathrm{Re}(\tilde{Z}_\varepsilon)\|_{H^s}^2 \right).$$

The first integral has absolute value $\leqslant C\|(\tilde{\rho}_\varepsilon - 1)/\varepsilon\|_{H^s}^2$ since $\|\partial_{z_1} \rho_\varepsilon^{\mathrm{app}}\|_{L^\infty} + \|\partial_{z_1}\rho_\varepsilon\|_{L^\infty} \leqslant C\varepsilon$ for $0 \leqslant \theta \leqslant \bar{\theta}_\varepsilon$. For the second integral, we integrate by parts:

$$\int_{\mathbb{R}^d} \frac{16\rho_\varepsilon^{\mathrm{app}}(1-\rho_\varepsilon)}{\varepsilon^2(1+\rho_\varepsilon)^5} \langle \partial_z^\alpha \tilde{\rho}_\varepsilon, \mathrm{Re}(Z_\varepsilon) \cdot \nabla^\varepsilon \partial_z^\alpha \tilde{\rho}_\varepsilon \rangle \, dz = -\frac{8}{\varepsilon^2} \int_{\mathbb{R}^d} [\partial_z^\alpha \tilde{\rho}_\varepsilon]^2 \nabla^\varepsilon \cdot \left(\rho_\varepsilon^{\mathrm{app}} \frac{(1-\rho_\varepsilon)}{(1+\rho_\varepsilon)^5} \mathrm{Re}(Z_\varepsilon) \right) dz$$
$$\geqslant -C\varepsilon \left\| \frac{\tilde{\rho}_\varepsilon - 1}{\varepsilon} \right\|_{H^s}^2.$$

Therefore, by another integration by parts,

$$\frac{d}{d\theta} \int_{\mathbb{R}^d} \frac{4\rho_\varepsilon^{\mathrm{app}}}{\varepsilon^2(1+\rho_\varepsilon)^4} [\partial_z^\alpha \tilde{\rho}_\varepsilon]^2 \, dz \leqslant \int_{\mathbb{R}^d} \frac{16\rho_\varepsilon^{\mathrm{app}} \tilde{\rho}_\varepsilon}{\varepsilon^2(1+\rho_\varepsilon)^4} \langle \partial_z^\alpha \nabla^\varepsilon \tilde{\rho}_\varepsilon, \partial_z^\alpha \mathrm{Re}(\tilde{Z}_\varepsilon) \rangle \, dz$$
$$+ C \left(\varepsilon^4 + \left\| \frac{\tilde{\rho}_\varepsilon - 1}{\varepsilon} \right\|_{H^s}^2 + \|\tilde{Z}_\varepsilon\|_{H^s}^2 \right). \tag{115}$$

Combining (113) and (115) provides, in view of the cancellation of the bad singular terms,

$$\frac{d}{d\theta} \int_{\mathbb{R}^d} \frac{4\rho_\varepsilon}{(1+\rho_\varepsilon)^2} |\partial_z^\alpha \tilde{Z}_\varepsilon|^2 + \frac{4\rho_\varepsilon^{\mathrm{app}}}{\varepsilon^2(1+\rho_\varepsilon)^4} [\partial_z^\alpha \tilde{\rho}_\varepsilon]^2 \, dz \leqslant C \left(\varepsilon^4 + \left\| \frac{\tilde{\rho}_\varepsilon - 1}{\varepsilon} \right\|_{H^s}^2 + \|\tilde{Z}_\varepsilon\|_{H^s}^2 \right),$$

which is the desired inequality. □

Since at $\theta = 0$, $\tilde{\rho}_\varepsilon = 1 + \mathcal{O}_{H^s}(\varepsilon^3)$ and $\tilde{Z}_\varepsilon = \mathcal{O}_{H^s}(\varepsilon^2)$, (111) and the Gronwall inequality implies, for $0 \leqslant \theta \leqslant \bar{\theta}_\varepsilon$,

$$\left\| \frac{\tilde{\rho}_\varepsilon(\theta) - 1}{\varepsilon} \right\|_{H^s}^2 + \|\tilde{Z}_\varepsilon(\theta)\|_{H^s}^2 \leqslant C\varepsilon^4 \mathrm{e}^{C\theta}.$$

This proves that if $\mu < 1/(2C)$ and $\varepsilon \leqslant \varepsilon_0(\mu, C)$ is sufficiently small, then $\bar{\theta}_\varepsilon > \mu|\ln\varepsilon|$. The end of the proof of Theorem 9 then follows the lines of Section 4.2 thus we omit it. To compare A_ε and $A_\varepsilon^{\mathrm{app}}$, we write

$$A_\varepsilon = \frac{\rho_\varepsilon - 1}{\varepsilon} = \frac{\rho_\varepsilon^{\mathrm{app}}\tilde{\rho}_\varepsilon - 1}{\varepsilon} = \frac{\rho_\varepsilon^{\mathrm{app}} - 1}{\varepsilon} + \rho_\varepsilon^{\mathrm{app}}\frac{\tilde{\rho}_\varepsilon - 1}{\varepsilon} = \zeta(\varepsilon\theta) + \frac{\varepsilon}{2}\zeta^2(\varepsilon\theta) + \mathcal{O}_{H^s}\big(\varepsilon^2 \mathrm{e}^{\theta/\mu}\big),$$

as wished.

Acknowledgements

The support of the ANR *ArDyPitEq* is gratefully acknowledged. I would like to thank Frédéric Rousset warmly for useful discussions, for having drawn my attention on paper [49] and finally for many unsuccessful joint attempts to try to get the Sobolev bounds for the (gKdV)/(gKP-I) asymptotic regime on the time scale $\tau \approx 1$.

References

[1] M. Abid, C. Huepe, S. Metens, C. Nore, C.T. Pham, L.S. Tuckerman, M.E. Brachet, Gross–Pitaevskii dynamics of Bose–Einstein condensates and superfluid turbulence, Fluid Dyn. Res. 33 (5–6) (2003) 509–544.
[2] N. Akhmediev, A. Ankiewicz, R. Grimshaw, Hamiltonian-versus-energy diagrams in soliton theory, Phys. Rev. E 59 (5) (1999) 6088–6096.
[3] T. Alazard, R. Carles, Supercritical geometric optics for nonlinear Schrödinger equations, Arch. Ration. Mech. Anal. 94 (1) (2009) 315–347.
[4] V. Banica, E. Miot, Global existence and collisions for certain configurations of nearly parallel vortex filaments, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 29 (2012) 813–832.
[5] I. Barashenkov, E. Panova, Stability and evolution of the quiescent and travelling solitonic bubbles, Physica D 69 (1–2) (1993) 114–134.
[6] W. Ben Youssef, T. Colin, Rigorous derivation of Korteweg–de Vries-type systems from a general class of nonlinear hyperbolic systems, M2AN Math. Model. Numer. Anal. 34 (4) (2000) 873–911.
[7] W. Ben Youssef, D. Lannes, The long wave limit for a general class of 2D quasilinear hyperbolic problems, Commun. Partial Differ. Equ. 27 (5–6) (2002) 979–1020.
[8] S. Benzoni-Gavage, R. Danchin, S. Descombes, On the well-posedness of the Euler–Korteweg model in several space dimensions, Indiana Univ. Math. J. 56 (2007) 1499–1579.
[9] N. Berloff, P. Roberts, Motions in a Bose condensate: X. New results on stability of axisymmetric solitary waves of the Gross–Pitaevskii equation, J. Phys. A, Math. Gen. 37 (2004) 11333–11351.
[10] F. Béthuel, R. Danchin, D. Smets, On the linear wave regime of the Gross–Pitaevskii equation, J. Anal. Math. 110 (2010) 297–338.
[11] F. Béthuel, P. Gravejat, J.-C. Saut, On the KP-I transonic limit of two-dimensional Gross–Pitaevskii travelling waves, Dyn. Partial Differ. Equ. 5 (3) (2008) 241–280.
[12] F. Béthuel, P. Gravejat, J.-C. Saut, Existence and properties of travelling waves for the Gross–Pitaevskii equation, in: Stationary and Time Dependent Gross–Pitaevskii Equations, in: Contemp. Math., vol. 473, Amer. Math. Soc., Providence, RI, 2008, pp. 55–103.
[13] F. Béthuel, P. Gravejat, J.-C. Saut, D. Smets, On the Korteweg–de Vries long-wave transonic approximation of the Gross–Pitaevskii equation I, Int. Math. Res. Not. 14 (2009) 2700–2748.
[14] F. Béthuel, P. Gravejat, J.-C. Saut, D. Smets, On the Korteweg–de Vries long-wave approximation of the Gross–Pitaevskii equation II, Commun. Partial Differ. Equ. 35 (1) (2010) 113–164.
[15] A. Capella, C. Melcher, F. Otto, Wave-type dynamics in ferromagnetic thin films and the motion of Néel walls, Nonlinearity 20 (11) (2007) 2519–2537.
[16] D. Chiron, Travelling waves for the nonlinear Schrödinger equation with general nonlinearity in dimension one, Nonlinearity 25 (2012) 813–850.
[17] D. Chiron, Three long-wave asymptotic regimes for the nonlinear Schrödinger Equation, in: M. Novaga, G. Orlandi (Eds.), Singularities in Nonlinear Evolution Phenomena and Applications, in: CRM Series, Scuola Normale Superiore, Pisa, 2009, pp. 107–138.
[18] D. Chiron, M. Mariş, Rarefaction pulses for the nonlinear Schrödinger equation in the transonic limit, Commun. Math. Phys. (2013), in press.
[19] D. Chiron, F. Rousset, Geometric optics and boundary layers for nonlinear Schrödinger equations, Commun. Math. Phys. 288 (2) (2009) 503–546.
[20] D. Chiron, F. Rousset, The KdV/KP-I limit of the nonlinear Schrödinger equation, SIAM J. Math. Anal. 42 (1) (2010) 64–96.
[21] D. Chiron, C. Scheid, Travelling waves for the nonlinear Schrödinger equation with general nonlinearity in dimension two, preprint.
[22] T. Colin, D. Lannes, Justification of and long-wave correction to Davey–Stewartson systems from quadratic hyperbolic systems, Discrete Contin. Dyn. Syst. 11 (1) (2004) 83–100.
[23] T. Colin, A. Soyeur, Some singular limits for evolutionary Ginzburg–Landau equations, Asymptot. Anal. 13 (1996) 361–372.
[24] A. de Bouard, J.-C. Saut, Solitary waves of generalized Kadomtsev–Petviashvili equations, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 14 (2) (1997) 211–236.
[25] A. de Laire, Travelling waves for the Landau–Lifshitz equation: nonexistence of small energy solutions and asymptotic behaviour at infinity, preprint.
[26] W. Ding, Y. Wang, Local Schrödinger flow into Kähler manifolds, Sci. China Ser. A 44 (11) (2001) 1446–1464.
[27] C. Gallo, Schrödinger group on Zhidkov spaces, Adv. Differ. Equ. 9 (5–6) (2004) 509–538.

[28] C. Gallo, The Cauchy problem for defocusing nonlinear Schrödinger equations with non-vanishing initial data at infinity, Commun. Partial Differ. Equ. 33 (4–6) (2008) 729–771.

[29] P. Gérard, Remarques sur l'analyse semi-classique de l'équation de Schrödinger non linéaire, in: Séminaire sur les Equations aux Dérivées Partielles, Exp. No. XIII, Ecole Polytechnique, Palaiseau, 1992, 1993, 13 pp.

[30] P. Germain, F. Rousset, Long wave limits for Schrödinger maps, preprint.

[31] E. Grenier, Semiclassical limit of the nonlinear Schrödinger equation in small time, Proc. Am. Math. Soc. 126 (2) (1998) 523–530.

[32] R. Iório, W. Nunes, On equations of KP-type, Proc. R. Soc. Edinb. A 128 (1998) 725–743.

[33] Y. Kivshar, D. Anderson, M. Lisak, Modulational instabilities and dark solitons in a generalized nonlinear Schrödinger equation, Phys. Scr. 47 (1993) 679–681.

[34] Y.S. Kivshar, B. Luther-Davies, Dark optical solitons: physics and applications, Phys. Rep. 298 (1998) 81–197.

[35] Y. Kivshar, D. Pelinovsky, Self-focusing and transverse instabilities of solitary waves, Phys. Rep. 331 (2000) 117–195.

[36] E.B. Kolomeisky, T.J. Newman, J.P. Straley, X. Qi, Low-dimensional bose liquids: beyond the Gross–Pitaevskii approximation, Phys. Rev. Lett. 85 (2000) 1146–1149.

[37] S. Komineas, N. Papanicolaou, Topology and dynamics in ferromagnetic media, Physica D 99 (1) (1996) 81–107.

[38] S. Komineas, N. Papanicolaou, Vortex dynamics in two-dimensional antiferromagnets, Nonlinearity 11 (2) (1998) 265–290.

[39] L. Landau, E. Lifshitz, On the theory of the dispersion of magnetic permeability in ferromagnetic bodies, Phys. Z. Sowjetunion 8 (1935) 153–169.

[40] D. Lannes, Consistency of the KP approximation, in: Dynamical systems and differential equations, Wilmington, NC, 2002 Discrete Contin. Dyn. Syst. (suppl.) (2003) 517–525.

[41] D. Lannes, Secular growth estimates for hyperbolic systems, J. Differ. Equ. 190 (2) (2003) 466–503.

[42] D. Lannes, J.-C. Saut, Weakly transverse Boussinesq systems and the Kadomtsev–Petviashvili approximation, Nonlinearity 19 (2006) 2853–2875.

[43] H. Leblond, KP lumps in ferromagnets: a three-dimensional KdV-Burgers model, J. Phys. A 35 (47) (2002) 10149–10161.

[44] K. Nakamura, T. Sasada, Quantum kink in the continuous one-dimensional Heisenberg ferromagnet with easy plane: a picture of the antiferromagnetic magnon, J. Phys. C, Solid State Phys. 15 (1982) L1013–L1017.

[45] N. Papanicolaou, P. Spathis, Semitopological solitons in planar ferromagnets, Nonlinearity 12 (2) (1999) 285–302.

[46] P. Roberts, N. Berloff, Nonlinear Schrödinger equation as a model of superfluid helium, in: C.F. Barenghi, R.J. Donnelly, W.F. Vinen (Eds.), Quantized Vortex Dynamics and Superfluid Turbulence, in: Lect. Notes Phys., vol. 571, Springer-Verlag, 2001.

[47] J.-C. Saut, L. Xu, The Cauchy problem on large time for surface waves Boussinesq systems, J. Math. Pures Appl. (9) 97 (6) (2012) 635–662.

[48] S. Schochet, Asymptotics for symmetric hyperbolic systems with a large parameter, J. Differ. Equ. 75 (1) (1988) 1–27.

[49] J. Shatah, C. Zeng, Schrödinger maps and anti-ferromagnetic chains, Commun. Math. Phys. 262 (2006) 299–315.

[50] P.-L. Sulem, C. Sulem, C. Bardos, On the continuous limit for a system of classical spins, Commun. Math. Phys. 107 (3) (1986) 431–454.

[51] M. Taylor, Partial Differential Equations (III), Appl. Math. Sci., vol. 117, Springer-Verlag, New York, 1997.

[52] T. Tsuzuki, Nonlinear waves in the Pitaevskii–Gross equation, J. Low Temp. Phys. 4 (4) (1971) 441–457.

[53] S. Ukaï, Local solutions to the Kadomtsev–Petviashvili equation, J. Fac. Sci., Univ. Tokyo, Sect. 1A, Math. 36 (1989) 193–209.

[54] L. Xu, Intermediate long wave systems for internal waves, Nonlinearity 25 (3) (2012) 597–640.

[55] V. Zakharov, A. Kuznetsov, Multi-scale expansion in the theory of systems integrable by the inverse scattering transform, Physica D 18 (1–3) (1986) 455–463.

[56] P. Zhidkov, Korteweg–de Vries and Nonlinear Schrödinger Equations: Qualitative Theory, Lect. Notes Math., vol. 1756, Springer-Verlag, 2001.