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*Ann. I. H. Poincaré – AN* 31 (2014) 1101–1107



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# Almost reduction and perturbation of matrix cocycles

Jairo Bochi <sup>a,\*</sup>, Andrés Navas <sup>b,2</sup>

<sup>a</sup> PUC-Rio, Rua Marquês de S. Vicente, 225, Rio de Janeiro, Brazil

<sup>b</sup> Universidad de Santiago, Alameda 3363, Estación Central, Santiago, Chile

Received 31 January 2013; received in revised form 15 August 2013; accepted 23 August 2013

Available online 13 September 2013

## Abstract

In this note, we show that if all Lyapunov exponents of a matrix cocycle vanish, then it can be perturbed to become cohomologous to a cocycle taking values in the orthogonal group. This extends a result of Avila, Bochi and Damanik to general base dynamics and arbitrary dimension. We actually prove a fibered version of this result, and apply it to study the existence of dominated splittings into conformal subbundles for general matrix cocycles.

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## 1. From zero Lyapunov exponents to rotation cocycles

### 1.1. Basic definitions

Let  $F: \Omega \rightarrow \Omega$  be a homeomorphism of a compact metric space  $\Omega$ . Let  $V$  be a finite-dimensional real vector bundle over  $\Omega$ , whose fiber over  $\omega$  is denoted by  $V_\omega$ . Let  $\mathcal{A}$  be a vector-bundle automorphism that fibers over  $F$ ; this means that the restriction of  $\mathcal{A}$  to each fiber  $V_\omega$  is a linear automorphism  $A(\omega)$  onto  $V_{F\omega}$ . In the case of trivial vector bundles,  $\mathcal{A}$  is usually called a *linear cocycle*.

As a convention, automorphisms of  $V$  will be denoted by calligraphic letters, and the restrictions to the fibers will be denoted by the corresponding roman letters. Analogously, for any integer  $n$ , the restriction of the power  $\mathcal{A}^n$  to the fiber  $V_\omega$  is denoted by  $A^n(\omega)$ ; thus  $A^n(\omega) = A(F^{n-1}\omega) \circ \dots \circ A(\omega)$  for  $n > 0$ .

A *Riemannian metric* on  $V$  is a continuous choice of inner product  $\langle \cdot, \cdot \rangle_\omega$  on each fiber  $V_\omega$ . It induces a *Riemannian norm*  $\|v\|_\omega = \sqrt{\langle v, v \rangle_\omega}$ . Given a linear map  $L: V_\omega \rightarrow V_{\omega'}$ , its *norm*  $\|L\|$  and its *mininorm*  $m(L)$  are defined respectively as the supremum and the infimum of  $\|Lv\|_{\omega'}$  over all unit vectors  $v \in V_\omega$ .

Let  $\text{Aut}(V, F)$  denote the space of all automorphisms of  $V$  that fiber over  $F$ , endowed with the topology induced by the distance  $d(\mathcal{A}, \mathcal{B}) = \sup_\omega \|A(\omega) - B(\omega)\|$ , for some choice of a Riemannian norm on  $V$ .

\* Corresponding author.

E-mail addresses: [jairo@mat.puc-rio.br](mailto:jairo@mat.puc-rio.br) (J. Bochi), [andres.navas@usach.cl](mailto:andres.navas@usach.cl) (A. Navas).

URL: <http://www.mat.puc-rio.br/~jairo> (J. Bochi).

<sup>1</sup> Partially supported by CNPq (Brazil) and FAPERJ (Brazil).

<sup>2</sup> Partially supported by the Fondecyt Project 1120131 and the “Center of Dynamical Systems and Related Topics” (DySyRF; ACT-Project 1103, Conicyt).

### 1.2. Uniform subexponential growth and its consequences

Define

$$\lambda^+(\mathcal{A}) = \lim_{n \rightarrow +\infty} \frac{1}{n} \sup_{\omega \in \Omega} \log \|A^n(\omega)\| \quad \text{and} \quad \lambda^-(\mathcal{A}) = \lim_{n \rightarrow +\infty} \frac{1}{n} \inf_{\omega \in \Omega} \log m(A^n(\omega)),$$

which exist by subadditivity and supraadditivity, respectively.

If  $\mu$  is an ergodic probability measure for  $F: \Omega \rightarrow \Omega$ , then there are constants  $\lambda^+(\mathcal{A}, \mu), \lambda^-(\mathcal{A}, \mu)$ , called the *top* and *bottom Lyapunov exponents*, such that, for  $\mu$ -almost every  $\omega \in \Omega$ ,

$$\frac{1}{n} \log \|A^n(\omega)\| \rightarrow \lambda^+(\mathcal{A}, \mu) \quad \text{and} \quad \frac{1}{n} \log m(A^n(\omega)) \rightarrow \lambda^-(\mathcal{A}, \mu) \quad \text{as } n \rightarrow +\infty.$$

Moreover, the following “variational principle” holds<sup>3</sup>:

$$\lambda^+(\mathcal{A}) = \sup_{\mu} \lambda^+(\mathcal{A}, \mu) \quad \text{and} \quad \lambda^-(\mathcal{A}) = \inf_{\mu} \lambda^-(\mathcal{A}, \mu), \tag{1.1}$$

where  $\mu$  runs over all invariant ergodic probabilities for  $F$ .

Let us say that the automorphism  $\mathcal{A}$  has *uniform subexponential growth* if  $\lambda^+(\mathcal{A}) = \lambda^-(\mathcal{A}) = 0$ . By (1.1), this is equivalent to the vanishing of all Lyapunov exponents with respect to all ergodic probability measures.

Our first result is:

**Theorem 1.1.** *Assume that  $\mathcal{A} \in \text{Aut}(V, F)$  has uniform subexponential growth. Then:*

- (a) *For any  $\varepsilon > 0$ , there exists a Riemannian norm  $\|\cdot\|$  on  $V$  such that*

$$e^{-\varepsilon} \|v\|_{\omega} < \|A(\omega)v\|_{F\omega} < e^{\varepsilon} \|v\|_{\omega}, \quad \text{for all } \omega \in \Omega, v \in V_{\omega}. \tag{1.2}$$

- (b) *There exists an arbitrarily small perturbation of  $\mathcal{A}$  that preserves some Riemannian norm on  $V$ .*

As we will see, part (a) follows from a standard construction in Pesin theory, and part (b) follows from part (a). However, the latter implication is *not* straightforward, because if  $\varepsilon$  is small then the Riemannian norm constructed in part (a) may be very distorted with respect to a fixed reference Riemannian norm on  $V$ .

For a reformulation of the theorem in terms of conjugacy to isometric automorphisms, see Section 1.4.

Despite making stringent assumptions about the automorphism  $\mathcal{A}$ , Theorem 1.1 can be used to obtain very strong properties for a dense subset  $D$  of  $\text{Aut}(V, F)$ , under the assumption that  $F$  is uniquely ergodic (or, in some cases, minimal). More precisely, we show that for every automorphism  $\mathcal{A}$  in the subset  $D$  there exist a Riemannian metric norm  $\|\cdot\|$  on  $V$  and a splitting of  $V$  as a Whitney sum of  $\mathcal{A}$ -invariant subbundles where  $\mathcal{A}$  acts conformally with respect to the norm  $\|\cdot\|$ . Moreover, this splitting is either trivial or dominated. See Section 2.2 for details.

In the paper [6], we prove results about cocycles of isometries of spaces of nonpositive curvature that generalize Theorem 1.1. Actually, we first obtained Theorem 1.1 as a corollary of the geometrical results of [6]. Later, we realized that the constructions could be modified or adapted to produce an elementary proof of Theorem 1.1, which we present, together with its applications, in this note.

### 1.3. Proof of Theorem 1.1

We need a few preliminaries.

Recall that  $V$  is a finite-dimensional vector bundle over the compact space  $\Omega$ . We choose and fix a Riemannian metric  $\langle \cdot, \cdot \rangle$  on  $V$ . Let  $\mathcal{B}$  be an automorphism of  $V$  over a homeomorphism  $G: \Omega \rightarrow \Omega$ . The *transpose* of  $\mathcal{B}$  is the automorphism  $\mathcal{B}^*$  over  $G^{-1}$  defined by

$$\langle B(\omega)u, v \rangle_{G\omega} = \langle u, B^*(G\omega)v \rangle_{\omega}, \quad \text{for all } u \in V_{\omega}, v \in V_{G\omega}.$$

<sup>3</sup> This follows from [16, Thm. 1] or [17, Thm. 1.7]. Although these references assume  $\Omega$  to be compact metrizable, the proofs also work for compact Hausdorff  $\Omega$ . (See also the proof of Proposition 1 in [1].) A particular case was considered in [10].

If  $\mathcal{B}^* = \mathcal{B}$  (and thus  $G$  is the identity), then  $\mathcal{B}$  is called *symmetric*. An automorphism  $\mathcal{P}$  is called *positive* if it is symmetric and  $\langle P(\omega)v, v \rangle_\omega > 0$  for all nonzero  $v \in V_\omega$ . We write  $\mathcal{B} < \mathcal{C}$  if  $\mathcal{B}$  and  $\mathcal{C}$  are symmetric and  $\mathcal{C} - \mathcal{B}$  is positive.

The following proposition collects some useful properties:

### Proposition 1.2.

- (a) If  $\mathcal{A}$  is any automorphism and  $\mathcal{B}$  is symmetric, then  $\mathcal{A}^*\mathcal{B}\mathcal{A}$  is symmetric; moreover, if  $\mathcal{B} < \mathcal{C}$ , then  $\mathcal{A}^*\mathcal{B}\mathcal{A} < \mathcal{A}^*\mathcal{C}\mathcal{A}$ .
- (b) Each positive automorphism  $\mathcal{P}$  has a unique positive square root  $\mathcal{P}^{1/2}$ ; moreover,  $\mathcal{P}^{1/2}$  commutes with  $\mathcal{P}$ , and the map  $\mathcal{P} \mapsto \mathcal{P}^{1/2}$  is continuous.
- (c) The square root map is monotonic: if  $\mathcal{P}, \mathcal{Q}$  are positive and  $\mathcal{P} < \mathcal{Q}$ , then  $\mathcal{P}^{1/2} < \mathcal{Q}^{1/2}$ .

Properties (a) and (b) above are easy exercises. For a proof of property (c), see [4, p. 9].

**Proof of Theorem 1.1.** Let  $\mathcal{A}$  be an automorphism of  $V$  over the homeomorphism  $F$  having uniform subexponential growth. Fix a small  $\varepsilon > 0$ .

To prove part (a), we will use a standard construction in Pesin theory called *Lyapunov norms* (see e.g. [13, p. 667]). Define

$$\|v\|_\omega^2 := \sum_{n \in \mathbb{Z}} e^{-2\varepsilon|n|} \|A^n(\omega)v\|_{F^n\omega}^2. \quad (1.3)$$

Since the cocycle has uniform subexponential growth, the series converges uniformly on compact subsets of  $V$ , and hence defines a (continuous) Riemannian norm. Property (1.2) is straightforward to check. This proves part (a).

To prove part (b), let  $\langle \cdot, \cdot \rangle$  be the inner product that induces the norm (1.3). Then there are positive automorphisms  $\mathcal{R}, \mathcal{Q}$  such that for all  $u, v \in V_\omega$ ,

$$\langle u, v \rangle_\omega = \langle R(\omega)u, v \rangle_\omega, \quad (1.4)$$

$$\langle A(\omega)u, A(\omega)v \rangle_{F\omega} = \langle Q(\omega)u, v \rangle_\omega. \quad (1.5)$$

The almost-invariance property (1.2) can now be expressed as:

$$e^{-2\varepsilon}\mathcal{R} < \mathcal{Q} < e^{2\varepsilon}\mathcal{R}. \quad (1.6)$$

We want to find an automorphism  $\tilde{\mathcal{A}}$  over  $F$  that is close to  $\mathcal{A}$  and leaves the inner product  $\langle \cdot, \cdot \rangle$  invariant. As it is straightforward to check, invariance means that the automorphism  $\mathcal{P} = \mathcal{A}^{-1}\tilde{\mathcal{A}}$  (over the identity) satisfies:

$$\mathcal{P}^*\mathcal{Q}\mathcal{P} = \mathcal{R}. \quad (1.7)$$

Equivalently,

$$(\mathcal{Q}^{1/2}\mathcal{P}\mathcal{Q}^{1/2})^*(\mathcal{Q}^{1/2}\mathcal{P}\mathcal{Q}^{1/2}) = \mathcal{Q}^{1/2}\mathcal{R}\mathcal{Q}^{1/2}.$$

Let us try to find a *positive* solution  $\mathcal{P}$ . Then the relation above becomes  $(\mathcal{Q}^{1/2}\mathcal{P}\mathcal{Q}^{1/2})^2 = \mathcal{Q}^{1/2}\mathcal{R}\mathcal{Q}^{1/2}$ , and using the uniqueness of positive square roots (property (b) in Proposition 1.2), we obtain

$$\mathcal{P} = \mathcal{Q}^{-1/2}(\mathcal{Q}^{1/2}\mathcal{R}\mathcal{Q}^{1/2})^{1/2}\mathcal{Q}^{-1/2}. \quad (1.8)$$

One checks directly that this formula solves the invariance equation (1.7), and thus gives the unique positive solution.

To estimate  $\mathcal{P}$ , we follow the steps of [14]. By the first inequality in (1.6) and property (a) in Proposition 1.2, we have  $\mathcal{Q}^{1/2}\mathcal{R}\mathcal{Q}^{1/2} < e^{2\varepsilon}\mathcal{Q}^2$ . So, by property (c) in that proposition,  $(\mathcal{Q}^{1/2}\mathcal{R}\mathcal{Q}^{1/2})^{1/2} < e^\varepsilon\mathcal{Q}$ . Applying property (a) again, we obtain  $\mathcal{P} < e^\varepsilon\mathcal{I}$ , where  $\mathcal{I}$  is the identity automorphism. This means that  $\|P(\omega)\| < e^\varepsilon$  for every  $\omega$ . An analogous argument starting from the second inequality in (1.6) gives  $\text{m}(P(\omega)) > e^{-\varepsilon}$  for every  $\omega$ . This shows that  $\mathcal{P}$  is close to the identity, and therefore the automorphism  $\tilde{\mathcal{A}} := \mathcal{A}\mathcal{P}$  is close to  $\mathcal{A}$ . As we have seen,  $\tilde{\mathcal{A}}$  preserves the new Riemannian metric, thus completing the proof of the theorem.  $\square$

**Remark 1.3.** Eq. (1.7) obviously has infinitely many solutions  $\mathcal{P}$ , not all of them close to the identity. As we have seen, restricting to positive automorphisms we have a unique solution, which is close to the identity and varies continuously with the data.

In [6], we obtain a generalization of Theorem 1.1 to cocycles of isometries of symmetric spaces of non-positive curvature. If specialized to the present situation, the construction presented in [6] is the same as the one given here for part (b), thus “explaining” the efficiency of positive matrices.

**Remark 1.4.** Notice that the Riemannian norm and the perturbed automorphism constructed in the proof of Theorem 1.1 depend continuously on the parameter  $\varepsilon$  and also on the automorphism  $\mathcal{A}$  itself. These properties are relevant for the applications obtained in [3].

#### 1.4. Conjugacy

Let us put Theorem 1.1 under a different perspective.

Two automorphisms  $\mathcal{A}, \mathcal{B} \in \text{Aut}(V, F)$  are said to be *conjugate* if there exists  $\mathcal{U} \in \text{Aut}(V, \text{id})$  such that  $\mathcal{A} = \mathcal{U}\mathcal{B}\mathcal{U}^{-1}$ . (In the case of a trivial vector bundle, we say that the two linear cocycles are *cohomologous*.)

Fixed a Riemannian metric on  $V$ , we say that an automorphism  $\mathcal{A}$  is *isometric* if it preserves this metric. (In the case of a trivial vector bundle, the cocycle will take values in the orthogonal group, i.e., it will be a *rotation cocycle*.)

Then we have:

**Theorem 1.5.** Fix a Riemannian metric on the vector bundle  $V$ . Assume that  $\mathcal{A} \in \text{Aut}(V, F)$  has uniform subexponential growth. Then:

- (a) There exists an automorphism conjugate to  $\mathcal{A}$  that is close to an isometric automorphism. More precisely, every neighborhood of the set of isometric automorphisms contains a conjugate of  $\mathcal{A}$ .
- (b) There exists an automorphism close to  $\mathcal{A}$  that is conjugate to an isometric automorphism. More precisely, every neighborhood of  $\mathcal{A}$  contains a conjugate of an isometric automorphism.

For  $\text{SL}(2, \mathbb{R})$ -cocycles and under extra assumptions on the dynamics  $F$ , the result above was shown by Avila, Bochi and Damanik as a step in the proofs of their results about spectra of Schrödinger operators, see [2,3].<sup>4</sup>

In the case of cocycles (i.e., trivial vector bundles), it is natural to look for conditions under which we can improve the conclusion of Theorem 1.5(b) and find a perturbed cocycle cohomologous to a constant rotation, or even to the identity. The case of  $\text{SL}(2, \mathbb{R})$ -cocycles is studied in [3].

**Proof of Theorem 1.5.** Let  $\varepsilon > 0$  be small. We follow the notation of the proof of Theorem 1.1.

It follows from (1.4) that  $\|v\|_\omega = \|R(\omega)^{1/2}v\|_\omega$  for every  $v \in V_\omega$ . Let  $\mathcal{B} := \mathcal{R}^{1/2}\mathcal{A}\mathcal{R}^{-1/2}$ . Then, by (1.2),

$$e^{-\varepsilon}\|v\|_\omega < \|B(\omega)v\|_{F_\omega} < e^\varepsilon\|v\|_\omega.$$

This implies that  $\mathcal{B}$  is close to an isometric isomorphism, thus proving part (a).

To prove part (b), it suffices to notice that  $\mathcal{R}^{-1/2}\tilde{\mathcal{A}}\mathcal{R}^{1/2}$  is an isometric isomorphism.  $\square$

There is another property which is closely related to what we have seen so far. Let us say that  $\mathcal{A} \in \text{Aut}(V, F)$  is *product-bounded* if

$$0 < \inf_{\substack{\omega \in \Omega \\ n \in \mathbb{Z}}} \mathfrak{m}(A^n(\omega)) \leqslant \sup_{\substack{\omega \in \Omega \\ n \in \mathbb{Z}}} \|A^n(\omega)\| < \infty.$$

If an automorphism  $\mathcal{A}$  is conjugate to an isometric automorphism then  $\mathcal{A}$  is product-bounded, as it is easy to check. Although product-bounded cocycles are not always conjugate to isometric automorphisms,<sup>5</sup> this happens whenever  $F$  is minimal, according to a result shown by Coronel, Navas and Ponce in [9].<sup>6</sup>

<sup>4</sup> However, they haven't explicitly stated the result: see the proof of Theorem 1 in [2] and Proposition 6.3 in [3].

<sup>5</sup> See e.g. [13, Exercise 2.9.2], [15].

<sup>6</sup> In the non-minimal case, one can still ensure the existence of a bounded and measurable conjugacy.

## 2. Conformality properties

### 2.1. Extensions of the previous results for the case of coinciding Lyapunov exponents

The following is an immediate consequence of [Theorem 1.1](#):

**Corollary 2.1.** *Let  $\mathcal{A} \in \text{Aut}(V, F)$  be such that  $\lambda^+(\mathcal{A}) = \lambda^-(\mathcal{A}) =: \lambda$ . Then there exist an arbitrarily small perturbation  $\tilde{\mathcal{A}}$  of  $\mathcal{A}$  and a Riemannian norm  $\|\cdot\|$  on  $V$  such that*

$$\|\tilde{\mathcal{A}}(\omega)v\|_{F_\omega} = e^\lambda \|v\|_\omega, \quad \text{for all } \omega \in \Omega, v \in V_\omega.$$

In other words, if all Lyapunov exponents of an automorphism are equal to some  $\lambda$ , then we can perturb it to become conformal with respect to a new Riemannian metric; moreover it dilates the metric by the constant factor  $e^\lambda$ .

Actually, a weaker assumption is sufficient to obtain conformality:

**Corollary 2.2.** *Let  $\mathcal{A} \in \text{Aut}(V, F)$  be such that  $\lambda^+(\mathcal{A}, \mu) = \lambda^-(\mathcal{A}, \mu)$  for every ergodic probability measure  $\mu$  for  $F$ . Then there exist an arbitrarily small perturbation  $\tilde{\mathcal{A}}$  of  $\mathcal{A}$ , a Riemannian norm  $\|\cdot\|$  on  $V$ , and a continuous function  $\lambda: \Omega \rightarrow \mathbb{R}$  such that*

$$\|\tilde{\mathcal{A}}(\omega)v\|_{F_\omega} = e^{\lambda(\omega)} \|v\|_\omega, \quad \text{for all } \omega \in \Omega, v \in V_\omega.$$

See [\[12\]](#) for a non-perturbative result with a similar conclusion.

**Proof of Corollary 2.2.** First of all, notice that for any  $\mathcal{A} \in \text{Aut}(V, F)$ , we have

$$\mathfrak{m}(A(\omega))^d \leq |\det A(\omega)| \leq \|A(\omega)\|^d,$$

where  $d$  be the fiber dimension of  $V$ . So, by submultiplicativity of norms,

$$\lambda^-(\mathcal{A}, \mu) \leq \int \lambda d\mu \leq \lambda^+(\mathcal{A}, \mu) \quad \text{for every ergodic measure } \mu, \tag{2.1}$$

where

$$\lambda(\omega) := \frac{1}{d} \log |\det A_i(\omega)|. \tag{2.2}$$

Now assume that equalities hold in [\(2.1\)](#). Let  $\mathcal{B} = e^{-\lambda} \mathcal{A}$ . Then  $\lambda^\pm(\mathcal{B}, \mu) = \lambda^\pm(\mathcal{A}, \mu) - \int \lambda d\mu = 0$  for every ergodic measure  $\mu$ . By the “variational principle” [\(1.1\)](#), this implies  $\lambda^+(\mathcal{B}) = \lambda^-(\mathcal{B}) = 0$ . Therefore, by [Theorem 1.1](#), there is a Riemannian norm  $\|\cdot\|$  on  $V$  that is preserved by a perturbation  $\tilde{\mathcal{B}}$  of  $\mathcal{B}$ .

Let  $\tilde{\mathcal{A}} = e^\lambda \tilde{\mathcal{B}}$ . This is a perturbation of  $\mathcal{A}$  with the desired conformality property.  $\square$

### 2.2. Existence of conformal subbundles

Using [Corollary 2.1](#) and a theorem from [\[7\]](#), we will obtain the following result:

**Theorem 2.3.** *Assume that  $F: \Omega \rightarrow \Omega$  is a uniquely ergodic homeomorphism with an invariant probability measure of full support. Then for every automorphism  $\mathcal{A}$  in a dense subset of  $\text{Aut}(V, F)$ , there exist:*

- a Riemannian norm  $\|\cdot\|$  on  $V$ ;
- a continuous  $\mathcal{A}$ -invariant splitting  $V = V^1 \oplus \cdots \oplus V^k$  which is orthogonal with respect to the Riemannian norm;
- and constants  $\lambda_1 > \cdots > \lambda_k$ ;

such that

$$\|A(\omega)v_i\|_{F_\omega} = e^{\lambda_i} \|v_i\|_\omega, \quad \text{for all } \omega \in \Omega, i = 1, \dots, k, v_i \in V_\omega^i.$$

Weakening the assumption of unique ergodicity to minimality, we have the following result:

**Theorem 2.4.** *Assume that  $F: \Omega \rightarrow \Omega$  is a minimal homeomorphism of a compact space of finite dimension.<sup>7</sup> Then for every  $\mathcal{A}$  in a dense subset of  $\text{Aut}(V, F)$ , there exist:*

- a Riemannian norm  $\|\cdot\|$  on  $V$ ;
- a continuous  $\mathcal{A}$ -invariant splitting  $V = V^1 \oplus \cdots \oplus V^k$  which is orthogonal with respect to the Riemannian norm;
- and continuous functions  $\lambda_1 > \cdots > \lambda_k$  on  $\Omega$ ;

such that

$$\|A(\omega)v_i\|_{F_\omega} = e^{\lambda_i(\omega)}\|v_i\|_\omega, \quad \text{for all } \omega \in \Omega, i = 1, \dots, k, v_i \in V_\omega^i.$$

As we will see, this has a similar proof as [Theorem 2.3](#), basically replacing [Corollary 2.1](#) by [Corollary 2.2](#) and the result from [7] by the result from [5].

We expect that [Theorems 2.3, 2.4](#) will be useful to answer the following question: *When can a linear cocycle over a uniquely ergodic or minimal base dynamics be approximated by a cocycle with a dominated (non-trivial) splitting?* Results on the 2-dimensional case were obtained in [2,3].

### 2.3. Proofs

**Proof of Theorem 2.3.** Assume that  $F: \Omega \rightarrow \Omega$  has a unique invariant probability  $\mu$ , and its support is  $\Omega$ . Take any  $\mathcal{A} \in \text{Aut}(V, F)$ ; we will explain how to perturb it so that it has the desired properties. First, by [7], one can perturb  $\mathcal{A}$  so that along  $\mu$ -almost every orbit, the Oseledets splitting is trivial or dominated. Let

$$V_\omega^1 \oplus \cdots \oplus V_\omega^k = V_\omega, \quad \omega \in \Omega,$$

be the *finest dominated splitting* of the cocycle, that is, the unique everywhere defined global dominated splitting with a maximal number  $k$  of bundles (with  $k = 1$  if there is no dominated splitting).<sup>8</sup>

We claim that for almost every point, there are exactly  $k$  different Lyapunov exponents. Indeed, on the one hand, there are at least  $k$  different exponents because there is a dominated splitting with  $k$  bundles. On the other hand, if there is a positive measure set of points with more than  $k$  different Lyapunov exponents, then select an orbit along which the Oseledets splitting is dominated. This orbit is dense on  $\Omega$  (because the invariant measure has full support). Since dominated splittings extend to the closure (see [8]), one gets a global dominated splitting with more than  $k$  bundles; this is a contradiction.

For each  $i = 1, \dots, k$ , let  $\mathcal{A}_i$  be the restriction of  $\mathcal{A}$  to the bundle  $V^i$ ; this is a (continuous) vector bundle automorphism. By the claim above,

$$\lambda^+(\mathcal{A}_i) = \lambda^+(\mathcal{A}_i, \mu) = \lambda^-(\mathcal{A}_i, \mu) = \lambda^-(\mathcal{A}_i) =: \lambda_i.$$

Therefore, by [Corollary 2.1](#), for each  $i$  there is a perturbation  $\tilde{\mathcal{A}}_i$  of  $\mathcal{A}_i$  and a Riemannian norm  $\|\cdot\|_i$  on  $V^i$  such that

$$\|\tilde{\mathcal{A}}_i(\omega)v_i\|_{i, F_\omega} = e^{\lambda_i}\|v_i\|_{i, \omega}, \quad \text{for all } \omega \in \Omega, v_i \in V_\omega^i.$$

Let  $\|\cdot\|$  be the Riemannian norm that makes the subbundles orthogonal and that coincides with  $\|\cdot\|_i$  on  $V^i$ . Let  $\tilde{\mathcal{A}}$  be the automorphism of  $V$  whose restriction to the subbundles  $V^i$  are the automorphisms  $\tilde{\mathcal{A}}_i$ . This automorphism has the desired properties, thus completing the proof.  $\square$

For the proof of [Theorem 2.4](#), we need the following result:

**Theorem 2.5.** *Assume that  $F: \Omega \rightarrow \Omega$  is a minimal homeomorphism of a compact space of finite dimension. Then every  $\mathcal{A}$  in a residual subset of  $\text{Aut}(V, F)$  has the following property: the Oseledets splitting with respect to any invariant probability measure coincides almost everywhere with the finest dominated splitting of  $\mathcal{A}$ .*

<sup>7</sup> We say that  $\Omega$  has *finite dimension* if it is homeomorphic to a subset of an euclidean space  $\mathbb{R}^d$ .

<sup>8</sup> See [8] for details on finest dominated splittings.

This result is proved in full generality in [5]. (The case of  $\mathrm{SL}(2, \mathbb{R})$ -cocycles was previously considered in [1].) Notice that, as we have seen in the proof of [Theorem 2.3](#) above, under the additional assumption of unique ergodicity, [Theorem 2.5](#) follows from [7].

**Proof of Theorem 2.4.** Assume that  $F: \Omega \rightarrow \Omega$  is minimal. Take any  $\mathcal{A} \in \mathrm{Aut}(V, F)$ ; we will explain how to perturb it so that it has the desired properties. First, perturb  $\mathcal{A}$  so that it has the property from [Theorem 2.5](#). This means that if

$$V_\omega^1 \oplus \cdots \oplus V_\omega^k = V_\omega \quad (\omega \in \Omega)$$

is the finest dominated splitting of the cocycle and  $\mathcal{A}_i$  is the restriction of  $\mathcal{A}$  to the bundle  $V^i$ , then

$$\lambda^+(\mathcal{A}_i, \mu) = \lambda^-(\mathcal{A}_i, \mu) \quad \text{for every ergodic probability } \mu \text{ for } F. \quad (2.3)$$

By [11], we can choose a Riemannian metric on  $V$  that is *adapted* to the dominated splitting, which means that

$$\inf_{\omega \in \Omega} \frac{\mathrm{m}(A_i(\omega))}{\|A_{i+1}(\omega)\|} > 1, \quad \text{for every } i = 1, 2, \dots, k-1.$$

Let  $d_i$  be the fiber dimension of  $V^i$ , and let

$$\lambda_i(\omega) := \frac{1}{d_i} \log |\det A_i(\omega)|; \quad (2.4)$$

here determinants are computed with respect to the adapted metric, and in particular  $\lambda_1 > \lambda_2 > \cdots > \lambda_d$  pointwise.<sup>9</sup>

For each  $i$ , property (2.3) permits us to apply [Corollary 2.2](#) and find a perturbation  $\tilde{\mathcal{A}}_i$  of  $\mathcal{A}_i$  that is conformal with respect to some Riemannian norm  $\|\cdot\|_i$  on  $V^i$ . Recalling formula (2.2) from the proof of [Corollary 2.2](#), we see that  $\|A(\omega)v\|_{i,F\omega} = e^{\lambda_i(\omega)} \|v\|_{i,\omega}$  where the function  $\lambda_i$  is given by (2.4).

Let  $\|\cdot\|$  be the Riemannian norm that makes the subbundles orthogonal and that coincides with  $\|\cdot\|_i$  on  $V^i$ . Let  $\tilde{\mathcal{A}}$  be the automorphism of  $V$  whose restrictions to the subbundles  $V^i$  are the automorphisms  $\tilde{\mathcal{A}}_i$ . This automorphism has the desired properties, thus completing the proof.  $\square$

## Acknowledgement

We thank the referee for several suggestions and corrections.

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<sup>9</sup> One can avoid using adapted metrics in the proof of [Theorem 2.4](#) by using the following fact: if the functions  $\lambda_1, \dots, \lambda_k$  satisfy  $\int \lambda_1 d\mu > \cdots > \int \lambda_k d\mu$  for every ergodic probability  $\mu$ , then there are functions  $\hat{\lambda}_i$  cohomologous to the  $\lambda_i$ 's such that  $\hat{\lambda}_1 > \cdots > \hat{\lambda}_d$  pointwise.