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Nondegeneracy of blow-up points for the parabolic Keller–Segel system

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Abstract

This paper is concerned with the parabolic Keller–Segel system

$$\begin{cases} u_t = \nabla \cdot (\nabla u - u^m \nabla v) & \text{in } \Omega \times (0,T), \\ \Gamma v_t = \Delta v - \lambda v + u & \text{in } \Omega \times (0,T), \end{cases}$$

in a domain Ω of $\mathbb{R}^N$ with $N \geqslant 1$, where $m, \Gamma > 0$, $\lambda \geqslant 0$ are constants and $T > 0$. When $\Omega \neq \mathbb{R}^N$, we impose the Neumann boundary conditions on the boundary. Under suitable assumptions, we prove the local nondegeneracy of blow-up points. This seems new even for the classical Keller–Segel system ($m = 1$). Lower global blow-up estimates are also obtained. In the singular case $0 < m < 1$, as a prerequisite, local existence and regularity properties are established.

Résumé

Dans cet article, nous étudions le système parabolique de Keller–Segel

$$\begin{cases} u_t = \nabla \cdot (\nabla u - u^m \nabla v) & \text{dans } \Omega \times (0,T), \\ \Gamma v_t = \Delta v - \lambda v + u & \text{dans } \Omega \times (0,T), \end{cases}$$

avec Ω un domaine de $\mathbb{R}^N$, $N \geqslant 1$, où $m, \Gamma > 0$, $\lambda \geqslant 0$ sont des constantes et $T > 0$. Lorsque $\Omega \neq \mathbb{R}^N$, les conditions aux limites de Neumann sont prescrites sur le bord. Sous des hypothèses convenables, nous prouvons la non-dégénérescence locale des points d'explosion. Ce résultat semble nouveau même dans le cas du système de Keller–Segel classique ($m = 1$). Des estimations inférieures globales de la vitesse d'explosion sont également obtenues. Dans le cas singulier $0 < m < 1$, nous établissons les propriétés nécessaires d'existence locale et de régularité.
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1. Introduction

1.1. Problem and main results

This paper is concerned with the Keller–Segel type system

$$\begin{cases} u_t = \nabla \cdot \big(\nabla u - u^m \nabla v\big), & x \in \Omega,\ t > 0, \\ \Gamma v_t = \Delta v - \lambda v + u, & x \in \Omega,\ t > 0. \end{cases} \tag{1.1}$$

Throughout this paper, N is a positive integer, Ω is either the whole space $\Omega = \mathbb{R}^N$ or a bounded domain of $\mathbb{R}^N$ of class $C^{3+\eta}$ for some $\eta > 0$ and $m, \Gamma > 0$ and $\lambda \geqslant 0$ are constants. System (1.1) is complemented with the initial conditions

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \quad x \in \Omega, \tag{1.2}$$

and the Neumann boundary conditions

$$\frac{\partial u}{\partial \nu}(x,t) = \frac{\partial v}{\partial \nu}(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \tag{1.3}$$

where ν is the outward normal vector. Condition (1.3) is of course understood to be empty in case $\Omega = \mathbb{R}^N$ (so that the system (1.1)–(1.3) covers both the Cauchy–Neumann and the Cauchy problems).

Throughout this paper, the initial data are assumed to satisfy

$$u_0 \in L^\infty(\Omega), \qquad v_0 \in W^{1,\infty}(\Omega), \quad u_0, v_0 \geqslant 0. \tag{1.4}$$

By a solution of (1.1)–(1.3) on $[0,T)$, we understand a nonnegative mild solution such that $(u,v) \in L^\infty_{loc}([0,T); L^\infty(\Omega) \times W^{1,\infty}(\Omega))$; see Section 2 for details. Problem (1.1)–(1.3) admits at least a maximal in time solution. For a given maximal in time solution, we denote by $T = T_{\max}(u,v) \in (0,\infty]$ its existence time. For $m \geqslant 1$, it was already known before that this solution exists and is unique and classical. For $0 < m < 1$, the local existence and regularity issues are nontrivial and require significant effort (see Section 2). We note that in that case the solution need not be classical nor positive and, moreover, it is not known if it is unique. However, our main results below will apply to any nonglobal, maximal solution.

The main goal of this paper is to prove a local nondegeneracy property for blow-up points. We recall that, for a solution (u,v) of (1.1)–(1.3) such that $T = T_{\max}(u,v) < \infty$, a is a *blow-up point* if it belongs to the set

$$\mathcal{B} = \Big\{ a \in \overline{\Omega};\ \limsup_{t \to T,\ \Omega \ni x \to a} \big(u(x,t) + \big|\nabla v(x,t)\big|\big) = \infty \Big\}.$$

For $a \in \overline{\Omega}$ and $\rho > 0$, we define

$$\Omega_{a,\rho} = B_\rho(a) \cap \overline{\Omega}.$$

Our main result is the following.

Theorem 1.1. *Assume either $1 \leqslant m < 2$ and $\Gamma > 0$, or $0 < m < 1$ and $\Gamma = 1$. Let u_0, v_0 satisfy (1.4) and, in case $\Omega = \mathbb{R}^N$,*

$$\begin{cases} u_0, v_0 \in L^1\big(\mathbb{R}^N\big), \\ \nabla u_0 \in L^r\big(\mathbb{R}^N\big) \quad \text{for some } r \in [1,\infty) \text{ if } m < 1. \end{cases} \tag{1.5}$$

Let (u,v) be any solution of (1.1)–(1.3) such that $T = T_{\max}(u,v) < \infty$. Let $a \in \overline{\Omega}$, $t_0 \in (0,T)$ and $\rho > 0$. There exists a constant $\varepsilon = \varepsilon(N,m,\Gamma) > 0$ such that, if

$$u(x,t) \leqslant \varepsilon (T-t)^{-1/m} \quad \text{for all } (x,t) \in \Omega_{a,\rho} \times (t_0,T), \tag{1.6}$$

then a is not a blow-up point.

Theorem 1.1 can be restated as the following local, lower estimate:

$$\limsup_{t\to T,\ \Omega\ni x\to a} (T-t)^{1/m} u(x,t) \geqslant \varepsilon, \tag{1.7}$$

near any blow-up point a. It is an analogue of the classical result of Giga and Kohn [15] on the nondegeneracy of blow-up points for the semilinear heat equations $u_t - \Delta u = u^p$, $p > 1$. As a second motivation, we study the global-in-space lower blow-up estimate and obtain the following.

Theorem 1.2. *Let $m, \Gamma > 0$ and let (u, v) be any solution of (1.1)–(1.3) such that $T = T_{\max}(u, v) < \infty$. Then*

$$\|u(t)\|_\infty^m + \|u(t)\|_\infty^{2(m-1)} \|\nabla v(t)\|_\infty^2 \geqslant c(T-t)^{-1} \quad \textit{for all } t \in [0, T), \tag{1.8}$$

where $c = c(N, m, \Gamma) > 0$. (Here, when $m < 1$, we make the convention $\infty/\infty = \infty$.)

If $1 \leqslant m < 2$, then we have in particular

$$\|u(t)\|_\infty + \|\nabla v(t)\|_\infty^{\frac{2}{2-m}} \geqslant c(T-t)^{-1/m} \quad \textit{for all } t \in [0, T). \tag{1.9}$$

Unlike the local lower estimate (1.7) from Theorem 1.1, Theorem 1.2 provides information on the solution at each time $t \in (0, T)$. However, it does not estimate the size of u alone, but of the couple $(u, |\nabla v|)$.

Remark 1.1.

(a) Theorems 1.1 and 1.2 seem new even for the classical Keller–Segel system ($m = 1$). On the other hand, unlike in the case $m \geqslant 1$, there seems to have been almost no mathematical results on system (1.1) with $0 < m < 1$. Indeed, this range exhibits a number of additional difficulties, in the study of both local existence-regularity and of the nondegeneracy of blow-up. Some of these difficulties are connected with the necessity to work with suitable weak solutions.
The proofs of Theorem 1.1 for $1 \leqslant m \leqslant 2$ and for $m < 1$ are rather different. In the former range, it is based on multiplier arguments and on various heat kernel estimates. In the latter, due to our assumption of equal diffusivities ($\Gamma = 1$), it turns out that a very helpful auxiliary function is available (see formula (4.1)), which enables one to rely on scalar maximum principle arguments which are not directly applicable to system (1.1) itself. On the other hand, in both ranges, the proof of Theorem 1.1 uses an auxiliary result (Lemma 3.1), which provides a local upper blow-up estimate on $|\nabla v|$, assuming a local upper blow-up estimate on u. Its proof relies on heat kernel estimates and on the mass conservation property for system (1.1).
We stress that, although the assumption $\Gamma = 1$ for $m < 1$ is rather restrictive, the question seems completely open otherwise. The case $m \geqslant 2$ seems also open.

(b) When $\Gamma = 1$ and $0 < m < 1$, Theorem 1.1 remains true if the nonlinearity u^m is replaced with a more general function behaving like u^m for large u and satisfying some mild technical assumptions. This can be achieved by suitably modifying the second term in the auxiliary function H from (4.1) (see Lemma 4.1 below).

(c) As a consequence of properties of the auxiliary function H, we note that the global estimate (1.9) remains true when $0 < m < 1$ and $\Gamma = 1$.

(d) Like in [15], our nondegeneracy criterion involves the blow-up rate of the local L^∞ norm, and hence does not relate to the space dimension. In connection with Theorem 1.1, but from a different point of view and for $\Gamma = 0$, one could mention the so-called ε-regularity property (see [46] and the references therein), which is dimension-dependent and involves a suitable, local, critical norm. Namely, under suitable assumptions, it asserts that no singularity occurs at a point x_0 provided the quantity $\int_{B_r(x_0)} u^{Nm}(x,t)\,dx$ for some $r > 0$ remains small enough for t close to T.

In the following subsection, in order to motivate our results, we summarize some known facts about blow-up for system (1.1).

1.2. Background on blow-up for Keller–Segel type systems

In the works described hereafter, for simplicity, we shall not be always specific about whether Ω is bounded or $\Omega = \mathbb{R}^N$. Although the two cases share many common features, some of the results have been proven only in one of them – see the original references for details. System (1.1) with $m = 1$ was introduced by Keller and Segel in [26] to describe the motion of cells which are diffusing and moving towards the gradient of a substance called chemoattractant, the latter being produced by the cells themselves. The motivation of this model was to describe the – experimentally observed – phenomenon of chemotactic collapse or aggregation, which refers to the spatial concentration of the total population to a finite number of points. In mathematical terms this is expected to correspond to the finite time blow-up of the cell density u near one or several points, along with the formation of one or several Dirac measures (recall that the total cell mass $M =: \int_\Omega u_0\,dx$ is conserved in time). For the two-dimensional problem with $m = 1$, the existence of a mass threshold was conjectured in [40,7,8]. Namely, on the basis of heuristic arguments and numerical simulations, it has been predicted that chemotactic collapse should occur if and only if M is greater than 8π. This conjecture has since then been partially proven in a rigorous manner (see below).

In variants of the Keller–Segel model, the taxis term $-\nabla\cdot(u\nabla v)$ is replaced by a more general term $-\nabla\cdot(\phi(u)\nabla v)$, where the chemosensitivity function ϕ may be nonlinear. Such a feature may be used to model the so-called volume filling effect, see [20,21] for a detailed discussion. For simplicity, we shall mainly consider the class of pure power chemosensitivity functions $\phi(s) = s^m$, $m > 0$. Beside the original parabolic–parabolic model ($\Gamma > 0$), the corresponding parabolic–elliptic system ($\Gamma = 0$) was later proposed and studied in [23] as a simplified model in the limit where the diffusion of the chemical is much faster than that of cells. However, it is worth pointing out that proving the existence of blow-up solutions, not even to mention describing the blow-up singularity formation, has turned out to be much more difficult for $\Gamma > 0$ than for $\Gamma = 0$. In a related direction, it is also worth mentioning that chemotaxis systems involving nonlinear diffusion (replacing ∇u by $\nabla(\psi(u))$ with, e.g., $\psi(u) = u^\sigma$) have recently attracted a lot of interest (see, e.g., [20,29,21,12]).

For about the last fifteen years, the only existing result on finite time blow-up for $\Gamma > 0$ had been that of Herrero and Velázquez in [19], where an example of a special blowing-up, radial solution was constructed in dimension $N = 2$. Very recently, a breakthrough was made by M. Winkler [50], who obtained in dimensions $N \geqslant 3$ an explicit criterion on (radial) initial data which guarantees finite time blow-up. His technique was subsequently generalized in [9,10] to more general chemosensitivity functions ϕ behaving for large s like s^m with $m > 2/N$ (plus an additional restriction $m \geqslant 1$, which is probably technical). On the other hand, in dimension $N = 2$, global existence was proved to hold for $M < 4\pi$ in general bounded domains [1,14,38], and for $M < 8\pi$ for radial solutions in a ball [38] or for general solutions in the whole space [6]. And indeed, in the example from [19] the concentrated mass near the origin at $t = T$ is precisely equal to 8π and the total mass M is greater than 8π. However, it seems to remain open whether arbitrary values of $M > 8\pi$ can be realized through the construction in [19], so as to fully confirm the mass threshold conjecture for $\Gamma > 0$. On the contrary, no mass threshold phenomenon occurs for dimensions $N \geqslant 3$, since the result in [50] applies for arbitrary $M > 0$. As for the description of the blow-up singularity formation, the only known result is again that of [19], which gives for $N = 2$ a very precise asymptotic description of u as $t \to T$ (see after formula (1.10) below), but only for the above mentioned special solution.

Let us turn to the case $\Gamma = 0$, where more is known. We will mention only a few results. For $m = 1$ and $N = 2$, small mass global existence, as well as large mass blow-up in the radial case, was first established by Jäger and Luckhaus in [23]. The 8π mass threshold conjecture was later proved in [35] for radial solutions in a ball and in [5,11] for general solutions in the whole space. In the nonradial bounded domain case, the threshold phenomenon was also established, and it was shown that the critical mass is actually 4π instead of 8π, due to the possibility of boundary blow-up points (see [3,36,38]). As for the critical case $M = 8\pi$, an infinite time aggregation phenomenon may occur (see [2,4,25,44] and the references therein). When the chemosensitivity function $\phi(s)$ behaves for large s like s^m with $m > 2/N$, blow-up in finite time occurs independently of the magnitude of initial mass provided the data are concentrated enough, whereas all solutions exist globally if $m < 2/N$ (see [12,29,35,45]). Moreover, critical mass phenomena appear for $m = 2/N$ (see [33]).

For $\Gamma = 0$ and $m = 1$, the asymptotic blow-up behavior has been studied by several authors. When $N = 2$, it is known (see [47] and the references therein) that blow-up points are isolated and that, near each blow-up point x_0, $u(t,\cdot)$ converges to a multiple $K\delta_{x_0}$ of the Dirac mass as $t \to T$, with $K = 8\pi$ if $x_0 \in \Omega$ and $K = 4\pi$ if $x_0 \in \partial\Omega$. Moreover, for radial solutions, the origin is the only possible blow-up point. Regarding the temporal blow-up rate, the central issue is that of type I vs. type II blow-up, defined by whether $\limsup_{t\to T}(T-t)\|u(t)\|_\infty$ is finite or infinite. Recall that this notion is motivated by the self-similar scale invariance of the problem, namely the fact that for any solution of (1.1), the couple (u_α, v_α), defined by

$$u_\alpha(x,t) := \alpha u\big(\alpha^{1/2}x, \alpha t\big), \qquad v_\alpha(x,t) := v\big(\alpha^{1/2}x, \alpha t\big), \quad \alpha > 0,$$

is also a solution (taking $\Omega = \mathbb{R}^N$ and discarding the term λv which is irrelevant for blow-up). The lower blow-up rate estimate $\|u(t)\|_\infty \geqslant C(T-t)^{-1}$ was obtained for all solutions when $m=1$ and $N=2$ in [27]. For $N \geqslant 3$, there exist radial type I blow-up solutions which are backward self-similar [17,39,43], i.e. of the form

$$u(x,t) = (T-t)^{-1}U(y), \qquad v(x,t) = V(y), \quad \text{where } y = x(T-t)^{-1/2}.$$

On the other hand, if $N=2$, then any blow-up is type II (see [48, Theorem 8.19]) whereas, for $N \geqslant 11$, radial type II blow-up solutions are known to exist [31]. For $3 \leqslant N \leqslant 9$, a sufficient condition on the initial data ensuring type I blow-up was found in [16] (revealing a situation different from the cases $N=2$ and $N \geqslant 11$). Moreover, it was shown in [16] that any type I blow-up solution which blows up only at the origin behaves asymptotically like a backward self-similar solution around 0 near the blow-up time. We note that, in this connection, our Theorem 1.1 gives a backward self-similar lower bound near any blow-up point for $\Gamma > 0$. In [18], for $N=2$, a special type II blow-up solution was constructed, whose blow-up rate was found to be faster than self-similar only by a logarithmic correction. Its asymptotic behavior is given by

$$u(x,t) \sim M(t)\bar{u}\big(x\sqrt{M(t)}\big), \quad M(t) \sim C(T-t)^{-1}\exp\big[\big(2\big|\log(T-t)\big|\big)^{1/2}\big], \tag{1.10}$$

for $x\sqrt{M(t)}$ bounded, where $\bar{u}(r) = 8(1+r^2)^{-2}$. We note that the couple $(\bar{u}, \bar{v})$ with $\bar{v}(r) = 2\log(1+r^2)$ turns out to be a stationary solution of system (1.1) with $\lambda = 0$. It was later proved in [32] that, for $N=2$, any radial blow-up solution blows up with this rate and recently, in [42], the corresponding blow-up profile was shown to be stable. The asymptotic behavior of the blow-up solutions constructed in [19] for $\Gamma > 0$ is essentially similar to (1.10). Finally, let us mention that the question of the continuation of solutions after blow-up for $\Gamma = 0$, with persistence of moving Dirac masses, has also been studied (see [13,30,49] and the references therein).

2. Local existence

Notation. Throughout this paper, $G_\Gamma = G_\Gamma(x,y;t)$ and $(S_\Gamma(t))_{t\geqslant 0}$ respectively denote the kernel and the semigroup associated with the operator $\Gamma^{-1}\Delta$, with Neumann boundary conditions (unless $\Omega = \mathbb{R}^N$). Recall that for all $\phi \in L^\infty(\Omega)$, we have $(S_\Gamma(t)\phi)(x) = \int_\Omega G_\Gamma(x,y;t)\phi(y)\,dy$, $x \in \Omega$, $t > 0$. Also, we will write $G = G_1$, $S(t) = S_1(t)$ if no confusion arises and we recall that $G_\Gamma(x,y;t) = G(x,y;\Gamma^{-1}t)$, $S_\Gamma(t) = S(\Gamma^{-1}t)$.

The first result of this section asserts the local existence of a solution of problem (1.1)–(1.3). In the case $m \geqslant 1$ and Ω bounded, this is a special case of [22]. When $m \geqslant 1$ and $\Omega = \mathbb{R}^N$, the proof is completely similar to the case Ω bounded (see also [45]) and we shall omit it. On the other hand, the case $0 < m < 1$ seems new. In this case the nonlinearity is non-Lipschitz and the solution if not expected to be unique, nor u to be positive or smooth at the level $u = 0$. The proof of the regularity, of the nonnegativity and, even more, of the L^1 property of the solution is nontrivial. The latter requires the use of the auxiliary function defined in (4.1) and of Lemma 4.1 below and is therefore restricted to $\Gamma = 1$.

Since the nonnegativity of (u,v) is not a priori guaranteed (for $0 < m < 1$), we need to redefine the nonlinearity, and we choose to do so as u_+^m (here and in the rest of the paper, $u_+ = \max(u,0)$ denotes the positive part). Of course, any nonnegative solution will solve the original problem. By a *mild solution* of (1.1)–(1.3) on $[0,T)$ we thus understand a couple (u,v) of functions satisfying

$$u \in L^\infty_{loc}\big([0,T); L^\infty(\Omega)\big), \qquad v \in L^\infty_{loc}\big([0,T); W^{1;\infty}(\Omega)\big) \tag{2.1}$$

and

$$\begin{cases} u(t) = S(t)u_0 - \displaystyle\int_0^t S(t-s)\nabla \cdot \left[u_+^m \nabla v\right](s)\,ds, & 0 < t < T, \\ v(t) = S_\Gamma(t)v_0 + \Gamma^{-1} \displaystyle\int_0^t S_\Gamma(t-s)[u - \lambda v](s)\,ds, & 0 < t < T. \end{cases} \tag{2.2}$$

Here, for each $t > 0$, the operator $S(t)\nabla\cdot : (L^\infty(\Omega))^N \to L^\infty(\Omega)$ is defined by

$$S(t)\nabla \cdot h := -\int_\Omega \nabla_y G(x, y; t) \cdot h(y)\,dy \tag{2.3}$$

(see Remark 2.2 at the end of this section for the justification of the definition (2.3)). As for the integral in the first equation of (2.2), it is understood as an absolutely convergent integral in $L^\infty(\Omega)$. We note (see Remark 2.2) that if Ω is bounded, then, for any $h \in (L^\infty(\Omega))^N$, we have

$$\int_\Omega \big(S(t)\nabla \cdot h\big)(x)\,dx = 0. \tag{2.4}$$

If $\Omega = \mathbb{R}^N$ and $h \in (L^\infty \cap L^1(\mathbb{R}^N))^N$, then $S(t)\nabla \cdot h \in L^1(\mathbb{R}^N)$ and (2.4) remains true.

We begin with the local existence of a mild solution and its continuation property.

Theorem 2.1. *Let $m > 0$ and let u_0, v_0 satisfy (1.4). Then there exist $T = T_{\max}(u, v) \in (0, \infty]$ and functions u, v on $\overline{\Omega} \times [0, T)$ with the following properties*:

$$(u, v) \text{ is a solution of (2.1)–(2.2) on } [0, T); \tag{2.5}$$

$$\text{either } T = \infty, \text{ or } T < \infty \text{ and } \limsup_{t\to T} \|u(t)\|_\infty = \infty. \tag{2.6}$$

If $m \geqslant 1$, then the solution of (2.1)–(2.2) is unique, locally in time.

Our next result is concerned with positivity and regularity of mild solutions, for which we need to separate the cases $m \geqslant 1$ and $0 < m < 1$.

Proposition 2.1.

(i) *Assume $m \geqslant 1$ and let (u, v) be the unique, maximal solution of (2.1)–(2.2), given by Theorem 2.1. Then $u, v \in C(\overline{\Omega} \times [0, T)) \cap C^{2,1}(\overline{\Omega} \times (0, T))$ and (u, v) is a classical solution of (1.1) in $\overline{\Omega} \times (0, T)$. Furthermore, if $u_0 \not\equiv 0$, then $u, v > 0$ in $\overline{\Omega} \times (0, T)$.*

(ii) *Assume $0 < m < 1$ and let (u, v) be any maximal in time solution of (2.1)–(2.2). Then*

$$u \in C\big(\overline{\Omega} \times (0, T)\big), \qquad v \in C\big(\overline{\Omega} \times [0, T)\big), \qquad v, \nabla v \in C^{2,1}\big(\overline{\Omega} \times (0, T)\big), \tag{2.7}$$

$$v \text{ is a classical solution of } \Gamma v_t = \Delta v + u - \lambda v \text{ in } \overline{\Omega} \times (0, T), \tag{2.8}$$

$$u \geqslant 0 \quad \text{in } \overline{\Omega} \times (0, T), \tag{2.9}$$

$$\text{if } v_0 \not\equiv 0, \quad \text{then } v > 0 \text{ in } \overline{\Omega} \times (0, T) \tag{2.10}$$

and

$$u \text{ is a classical solution of } u_t = \nabla \cdot \big(\nabla u - u^m \nabla v\big) \text{ on the (relatively open) set } \big\{(x, t) \in \overline{\Omega} \times (0, T);\ u(x, t) > 0\big\}. \tag{2.11}$$

Furthermore, we have

$$u(\cdot, t) \in BC^1(\overline{\Omega}) \quad \text{for all } t \in (0, T), \ u \in L^\infty_{loc}\big((0, T); BC^1(\overline{\Omega})\big) \tag{2.12}$$

and, when Ω is bounded,

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0, T). \tag{2.13}$$

Finally, we give L^1 properties of mild solutions that will be useful in the sequel.

Proposition 2.2. *Assume either Ω bounded or $\Omega = \mathbb{R}^N$ and $u_0, v_0 \in L^1(\mathbb{R}^N)$. If $\Omega = \mathbb{R}^N$ and $m < 1$, assume in addition that $\Gamma = 1$ and that $\nabla v_0 \in L^q(\mathbb{R}^N)$ for some $q \in [1, \infty)$. Then, for any maximal in time solution (u, v) of (2.1)–(2.2), we have*

$$\|u(t)\|_1 = \|u_0\|_1 \quad \text{for all } t \in (0, T) \tag{2.14}$$

and

$$\|v(t)\|_1 \leqslant \|v_0\|_1 + \Gamma^{-1} t \|u_0\|_1 \quad \text{for all } t \in (0, T). \tag{2.15}$$

Before giving the proofs, we recall the following Gaussian bounds for the Neumann heat kernel G and its derivatives (see e.g. [28,34] and the references therein), which will be also used in the next section.

Proposition 2.3. *There exist constants $C, \widetilde{C} > 0$ such that*

$$\left|\partial_t^k D_x^\alpha D_y^\beta G(x, y; t)\right| \leqslant C t^{-k-\frac{N+|\alpha|+|\beta|}{2}} \exp\left(-\frac{\widetilde{C}|x-y|^2}{t}\right) \quad \text{for all } x, y \in \Omega \text{ and } t > 0,$$

where $k \in \{0, 1\}$, $\alpha, \beta \in \mathbb{N}^N$, D^α denotes the differentiation corresponding to α with respect to space variables, and $2k + |\alpha| + |\beta| \leqslant 3$ with $|\alpha| = \sum_{i=1}^N \alpha_i$.

Proof of Theorem 2.1. As mentioned above, we need only consider the case $0 < m < 1$. We may also assume $u_0 \not\equiv 0$ since otherwise $(u(t), v(t)) = (0, S_\Gamma(t)v_0)$ is a solution and there is nothing to prove.

Step 1. *Small time existence*. Set $f_\varepsilon(s) = (s_+^2 + \varepsilon^2)^{m/2} - \varepsilon^m$. By [22, Theorem 3.1], there exist a time $\tau_\varepsilon > 0$ and a nonnegative mild solution $(u_\varepsilon, v_\varepsilon)$ of the regularized problem

$$\begin{cases} u_\varepsilon(t) = S(t)u_0 + g_\varepsilon^1(t), & g_\varepsilon^1(\cdot, t) := -\displaystyle\int_0^t S(t-s)\nabla \cdot \left[f_\varepsilon(u_\varepsilon)\nabla v_\varepsilon\right](s)\,ds, \\ v_\varepsilon(t) = S_\Gamma(t)v_0 + g_\varepsilon^2(t), & g_\varepsilon^2(\cdot, t) := \Gamma^{-1}\displaystyle\int_0^t S_\Gamma(t-s)[u_\varepsilon - \lambda v_\varepsilon](s)\,ds, \end{cases} \tag{2.16}$$

for all $t \in (0, \tau_\varepsilon)$. Since $|f_\varepsilon(s)| \leqslant s_+^m$, we deduce from the proof of [22, Theorem 3.1] (see formulae (9)–(10) in [22]) that there exist $\tau_0, C > 0$ independent of ε such that $\tau_\varepsilon > \tau_0$ and

$$\|u_\varepsilon(t)\|_\infty + \|v_\varepsilon(t)\|_\infty + \|\nabla v_\varepsilon(t)\|_\infty \leqslant C \quad \text{for all } t \in (0, \tau_0]. \tag{2.17}$$

Now we claim that, for each $0 < \alpha < 1$,

$$\left(g_\varepsilon^1\right)_\varepsilon \text{ is bounded in } C^{\alpha,\alpha/2}(Q_{\tau_0}) \quad \text{and} \quad \left(g_\varepsilon^2\right)_\varepsilon \text{ is bounded in } C^{1+\alpha,\alpha/2}(Q_{\tau_0}), \tag{2.18}$$

where $Q_{\tau_0} = \overline{\Omega} \times [0, \tau_0]$. To prove the claim, first observe that, owing to standard heat kernel bounds (cf. Proposition 2.3), for all $t, h > 0$ and $x \in \overline{\Omega}$, we have the estimate

$$\delta(x, t, h) := h^{-\alpha/2} \int_\Omega \left|\nabla_y G(x, y; t+h) - \nabla_y G(x, y; t)\right| dy \leqslant C t^{-(\alpha+1)/2}. \tag{2.19}$$

Indeed, if $h \geqslant t$, then $\delta(x, t, h) \leqslant C h^{-\alpha/2} t^{-1/2}$, hence (2.19) and, on the other hand, if $h \leqslant t$, then

$$\delta(x,t,h) \leqslant h^{1-\alpha/2} \int_\Omega \sup_{t\leqslant\sigma\leqslant t+h} \left|\partial_t \nabla_y G(x,y;\sigma)\right| dy, \leqslant Ch^{1-\alpha/2} t^{-3/2},$$

hence again (2.19). Setting $\psi_\varepsilon = f_\varepsilon(u_\varepsilon)\nabla v_\varepsilon$, it follows from (2.17), (2.19) and Proposition 2.3 that, for each $0 \leqslant t < t+h \leqslant \tau_0$ and $x \in \overline{\Omega}$,

$$\begin{aligned}
&h^{-\alpha/2}\left|g_\varepsilon^1(x,t+h) - g_\varepsilon^1(x,t)\right| \\
&\quad\leqslant h^{-\alpha/2} \int_t^{t+h} \int_\Omega \left|\nabla_y G(x,y;t+h-s)\psi_\varepsilon(y,s)\right| dy\, ds \\
&\qquad + h^{-\alpha/2} \int_0^t \int_\Omega \left|\big(\nabla_y G(x,y;t+h-s) - \nabla_y G(x,y;t-s)\big)\psi_\varepsilon(y,s)\right| dy\, ds \\
&\quad\leqslant Ch^{-\alpha/2} \int_t^{t+h} (t+h-s)^{-1/2}\, ds + \int_0^t (t-s)^{-(\alpha+1)/2}\, ds \\
&\quad\leqslant Ch^{(1-\alpha)/2} + Ct^{(1-\alpha)/2} \leqslant C\tau_0^{(1-\alpha)/2}.
\end{aligned}$$

This gives the temporal part of the claimed Hölder estimate in (2.18) for g_ε^1. The spatial part follows similarly by using the estimate

$$\sup_{x_1,x_2\in\overline{\Omega},\, x_1\neq x_2} |x_1-x_2|^{-\alpha} \int_\Omega \left|\nabla_y G(x_1,y;t) - \nabla_y G(x_2;y,t)\right| dy \leqslant Ct^{-(\alpha+1)/2} \quad \text{for all } t>0,$$

which also follows from Proposition 2.3. The proof of the Hölder estimate in (2.18) for g_ε^2 is completely similar. This proves claim (2.18).

Going back to (2.16), by (2.18) and Ascoli's theorem, we deduce the existence of a subsequence $\varepsilon = \varepsilon_j$ and of nonnegative functions $u \in BC(\overline{\Omega}\times(0,\tau_0])$ and $v \in BC^{1,0}(\overline{\Omega}\times(0,\tau_0])$, such that $u_\varepsilon, v_\varepsilon, \nabla v_\varepsilon$ converge to $u, v, \nabla v$, locally uniformly on $\overline{\Omega}\times(0,\tau_0]$. We may then pass to the limit, using the Gaussian heat kernel bound in Proposition 2.3 and dominated convergence, and we end up with a solution of (2.1)–(2.2) with $T=\tau_0$.

Step 2. *Continuation property*. By Zorn's lemma, (u,v) can be extended maximally in time as a (non-necessarily unique) solution of (2.1)–(2.2), in such a way that (2.6) holds. Indeed, if (2.6) fails, it first easily follows from the second equation in (2.2) that $\sup_{0<t<T}\|v(t)\|_{W^{1,\infty}} < \infty$. Then by similar Hölder estimates as in Step 1, one can show that $(u(t), v(t), \nabla v(t))$ converges as $t\to T$, locally uniformly on $\overline{\Omega}$, and that the limit $(u(T),v(T)) \in L^\infty(\Omega)\times W^{1,\infty}(\Omega)$. Moreover, (2.2) is satisfied at $t=T$. Taking this limit as new initial data, one can extend the solution beyond T, contradicting the definition of T. □

Proof of Proposition 2.1. Again we need only consider the case $0<m<1$. Note that, at this stage, we do not know if $u, v \geqslant 0$ on the whole maximal interval of existence (but only for small time).

Step 1. *Regularity of maximal in time solutions*. Fix $0<t_0<\tau<T$. Since $u, u_+^m\nabla v \in L^\infty(Q_\tau)$, by the argument leading to (2.18), we obtain $u \in BC^{\alpha,\alpha/2}(\overline{\Omega}\times[t_0,\tau])$ and $v \in BC^{1+\alpha,\alpha/2}(\overline{\Omega}\times[t_0,\tau])$ for each $0<\alpha<1$. By standard parabolic regularity, we then deduce (2.7), (2.8), along with

$$\begin{aligned}
&u \text{ is a classical solution of } u_t = \nabla\cdot\big(\nabla u - u_+^m \nabla v\big) \text{ on the (relatively open) set} \\
&\quad \big\{(x,t)\in\overline{\Omega}\times(0,T);\ u(x,t)\neq 0\big\}.
\end{aligned} \tag{2.20}$$

(Note that this will imply (2.11) once we have shown $u \geqslant 0$ at Step 2 below.)

Let us now check properties (2.12)–(2.13). We shall need the following smoothing estimate:

$$\big\|S(t)\nabla\cdot h\big\|_{BC^k(\Omega)} \leqslant C(\varepsilon)\big(1+t^{-(1+\varepsilon+k-\theta)/2}\big)\|h\|_{BC^\theta(\Omega)}, \quad h\in\big(BC^\theta(\Omega)\big)^N,\ t>0, \tag{2.21}$$

for any $0 \leqslant \theta \leqslant 1$, $0 \leqslant k \leqslant 2$ with $k \geqslant \theta - 1$, and any $\varepsilon > 0$. We note that a similar estimate in Sobolev spaces is proved in [22, Lemma 2.1]. Here, in the case $\theta = 1$, estimate (2.21) (actually true for $\varepsilon = 0$) is a consequence of definition (2.3), formula (2.30) below and Proposition 2.3. In the case $\theta = 0$, estimate (2.21) (again true for $\varepsilon = 0$) is a consequence of (2.3) and Proposition 2.3. The general case (with any $\varepsilon > 0$) then follows by interpolation.

Fix $0 < t_0 < \tau < T$. Since $u_+^m \nabla v \in BC(\overline{\Omega} \times [t_0, \tau])$, it follows from (2.21) with $\theta = 0$ and the first equation in (2.2) (shifted in time) that $u \in L^\infty(t_0, \tau; BC^k(\Omega))$ for all $k \in (0, 1)$. Consequently, $u_+^m \in L^\infty(t_0, \tau; BC^{mk}(\Omega))$, hence

$$u_+^m \nabla v \in L^\infty\big(t_0, \tau; BC^\theta(\Omega)\big) \quad \text{for all } \theta \in (0, m).$$

Applying (2.21) again, this time with $\theta \in (0, m)$, we deduce that, for all $t \in (t_0, \tau)$, $u(t) \in BC^k(\Omega)$ for all $k \in (0, 1 + m)$, hence in particular (2.12).

Moreover, when Ω is bounded, for each given $t \in (0, T)$, the function $z_s(\cdot) := S(t - s)\nabla \cdot (u_+^m \nabla v)(s)$ belongs to $C^1(\overline{\Omega})$ and satisfies $\frac{\partial z_s}{\partial \nu}(x) = 0$ on $\partial\Omega$. Since the estimates in the previous paragraph guarantee that the integral $\int_0^t z(\cdot, s)\, ds$ is absolutely convergent in $C^1(\overline{\Omega})$, property (2.13) follows.

Step 2. *Nonnegativity of maximal in time solutions.* Although u is not smooth at the level $u = 0$, one can use the following maximum principle argument on alleged negative values of u.

Let us first show $u \geqslant 0$ in the case Ω bounded. Assume that the property $u \geqslant 0$ is not true. Set $w = e^{-t}u$. Then there exist $t_0 \in (0, T)$ and $x_0 \in \overline{\Omega}$ such that $w(x_0, t_0) = \min_{Q_0} w < 0$, where $Q_0 = \overline{\Omega} \times [0, t_0]$. By continuity, the exists $\varepsilon > 0$ such that $w < 0$, hence $u_+ = 0$, in $V := Q_0 \cap (B(x_0, \varepsilon) \times [t_0 - \varepsilon, t_0])$. In view of (2.20), we deduce that $w_t - \Delta w = -w > 0$ in V. This yields a direct contradiction at (x_0, t_0) if $x_0 \in \Omega$. If $x_0 \in \partial\Omega$, since $\frac{\partial w}{\partial \nu}(x_0, t_0) = 0$, we get a contradiction with the Hopf lemma.

In the case $\Omega = \mathbb{R}^N$, one can modify the above proof by a using a perturbation argument from [24]. Namely, we fix $\varepsilon > 0$ and set

$$\hat{w} = u + \varepsilon\big((2N + 1)t + |x|^2\big).$$

Fix $\tau \in (0, T)$ and assume that the property $\hat{w} \geqslant 0$ is not true on $Q_\tau := \mathbb{R}^N \times [0, \tau]$. Since u is bounded in Q_τ, there exists $R > 0$ such that $\hat{w} \geqslant 0$ in $(\mathbb{R}^N \setminus \overline{B}_R) \times [0, \tau]$. Consequently, there exist $t_0 \in (0, \tau]$ and $x_0 \in \overline{B}_R$ such that

$$\hat{w}(x_0, t_0) = \min_{\overline{B}_R \times [0,\tau]} \hat{w} = \min_{Q_\tau} \hat{w} < 0.$$

Observe that $(\partial_t - \Delta)[(2N + 1)t + |x|^2] = 1$. Due to (2.20), the function $\hat{w}$ thus satisfies

$$\hat{w}_t - \Delta\hat{w} = u_t - \Delta u + \varepsilon > -\nabla \cdot \big[\big(\hat{w} - \varepsilon\big((2N + 1)t + |x|^2\big)\big)_+^m \nabla v\big] \tag{2.22}$$

at any point $(x, t) \in \mathbb{R}^N \times (0, T)$ where $\hat{w}(x, t) - \varepsilon((2N + 1)t + |x|^2) \neq 0$. This is in particular true at (x_0, t_0) and, since $\hat{w}(x_0, t_0) < 0$, the RHS of (2.22) vanishes at (x_0, t_0) and we get $0 \geqslant (\hat{w}_t - \Delta\hat{w})(x_0, t_0) > 0$, a contradiction. We deduce that $\hat{w} \geqslant 0$ in $\mathbb{R}^N \times [0, \tau]$ for each $\tau \in (0, T)$ and $\varepsilon > 0$. Letting $\tau \to T$ and then $\varepsilon \to 0$, we conclude that $u \geqslant 0$ in $\mathbb{R}^N \times [0, T)$.

The positivity of v (cf. (2.10)) then follows from the strong maximum principle. □

Proof of Proposition 2.2. If Ω is bounded, then (2.14) and (2.15) directly follow from the mass preserving property of the Neumann heat semigroup, Fubini's theorem and (2.4), by integrating each equation of (2.2) in space.

Let us thus assume $\Omega = \mathbb{R}^N$. By our hypotheses, we have $\Gamma = 1$, $u_0, v_0 \in L^1 \cap L^\infty(\mathbb{R}^N)$ and $\nabla v_0 \in L^q \cap L^\infty(\mathbb{R}^N)$ for some $q \in [1, \infty)$. We may assume $q \geqslant 2$ without loss of generality. It suffices to show that

$$u(t), \nabla v(t) \in L^1_{loc}\big([0, T); L^1\big(\mathbb{R}^N\big)\big). \tag{2.23}$$

Indeed, once (2.23) is proved, (2.14) and (2.15) follow similarly as in the bounded domain case.

We shall prove (2.23) by a boostrap argument. Fix $\tau \in (0, T)$. We have

$$u, v, |\nabla v| \leqslant C \quad \text{in } \mathbb{R}^N \times (0, \tau). \tag{2.24}$$

Here and in what follows, C denotes a generic constant possibly depending on τ. In order to initialize our bootstrap argument, we shall first prove that

$$\|u(t)\|_{(2-m)q/2} \leqslant C \quad \text{for all } t \in (0,\tau). \tag{2.25}$$

In view of the proof of (2.25), we introduce the auxiliary function

$$\widetilde{H} := e^{-Kt}\left(\frac{m}{2}|\nabla v|^2 + \frac{u^{2-m}}{2-m}\right).$$

It follows from Lemma 4.1 below that for some constant $K = K(\tau) > 0$, the function $\widetilde{H}$ satisfies

$$\partial_t \widetilde{H} - \Delta \widetilde{H} \leqslant 0 \quad \text{in } \mathbb{R}^N \times (0,\tau)$$

in the weak sense (cf. Remark 4.1). Fix $\varepsilon > 0$. Then the function

$$H_\varepsilon := \widetilde{H} - \varepsilon\big(2Nt + |x|^2\big)$$

also satisfies $\partial_t H_\varepsilon - \Delta H_\varepsilon \leqslant 0$ in $\mathbb{R}^N \times (0,\tau)$ in the weak sense. Moreover, since $\widetilde{H} \leqslant C$ in $\mathbb{R}^N \times (0,\tau)$ due to (2.24), we have $H_\varepsilon \leqslant 0$ on $\partial B_R \times (0,\tau)$ for all $R \geqslant R_0(\varepsilon) \gg 1$. We may then apply the weak maximum principle (in duality form) in bounded domains (cf. [41, Proposition 52.13]) to deduce that, for all $t_0 \in (0,\tau)$ and all $R \geqslant R_0(\varepsilon)$,

$$H_\varepsilon(\cdot,t) \leqslant T_R(t-t_0)H_\varepsilon(t_0) \quad \text{in } B_R \times (t_0,\tau), \tag{2.26}$$

where $(T_R(t))_{t\geqslant 0}$ denotes the Dirichlet heat semigroup on B_R. On the other hand, it follows from (2.2), (1.4) and properties of the Cauchy heat semigroup that $u, |\nabla v| \in C([0,T]; L^1_{loc}(\mathbb{R}^N))$, hence $H_\varepsilon \in C([0,T]; L^1_{loc}(\mathbb{R}^N))$. We may thus let $t_0 \to 0$ in (2.26) to deduce that, for all $R \geqslant R_0(\varepsilon)$,

$$H_\varepsilon(\cdot,t) \leqslant T_R(t)H_\varepsilon(0) \quad \text{in } B_R \times (0,\tau),$$

hence

$$H_\varepsilon(\cdot,t) \leqslant T_R(t)\widetilde{H}(0) \leqslant S(t)\widetilde{H}(0) \quad \text{in } B_R \times (0,\tau).$$

Letting $R \to \infty$ and then $\varepsilon \to 0$, we deduce that $\widetilde{H}(\cdot,t) \leqslant S(t)\widetilde{H}(0)$ in $\mathbb{R}^N \times (0,\tau)$. Therefore, $\sup_{t\in(0,\tau)} \|\widetilde{H}(t)\|_{q/2} \leqslant \|\widetilde{H}(0)\|_{q/2} < \infty$, hence (2.25).

By (2.25) and the second equation in (2.2), we have

$$\begin{aligned}\|\nabla v(t)\|_{(2-m)q/2} &\leqslant Ct^{-1/2}\|v_0\|_{(2-m)q/2} + \int_0^t \|u(s)\|_{(2-m)q/2}\,ds\\ &\leqslant Ct^{-1/2} \quad \text{for all } t \in (0,\tau).\end{aligned} \tag{2.27}$$

In view of a bootstrap argument, we now assume that

$$\|u(t)\|_k + t^{1/2}\|\nabla v(t)\|_k \leqslant C \quad \text{for all } t \in (0,\tau), \tag{2.28}$$

for some $k \in [1,\infty)$. Interpolating with (2.24), we see that (2.28) is also true with k replaced by $\tilde{k} = \max(k, m+1)$. Letting $r = \max(1, k/(m+1))$, it follows from Hölder's inequality and (2.28) that

$$\begin{aligned}\|u^m\nabla v(t)\|_r &= \|u^m\nabla v(t)\|_{\tilde{k}/(m+1)}\\ &\leqslant \|u^m(t)\|_{\tilde{k}/m}\|\nabla v(t)\|_{\tilde{k}} \leqslant Ct^{-1/2} \quad \text{for all } t \in (0,\tau).\end{aligned} \tag{2.29}$$

By the first equation in (2.2), we have

$$\begin{aligned}\|u(t)\|_r &\leqslant \|u_0\|_r + \int_0^t \|\nabla e^{(t-s)\Delta}\cdot u^m\nabla v(s)\|_r\,ds\\ &\leqslant \|u_0\|_r + C\int_0^t (t-s)^{-1/2}s^{-1/2}\,ds \leqslant C \quad \text{for all } t \in (0,\tau).\end{aligned}$$

Arguing on the second equation as for (2.27), we see that (2.28) is true with k replaced by $r = \max(1, k/(m+1))$. Since $m+1 > 1$ and (2.28) is true with $k = (2-m)q/2$ by (2.25) and (2.27), after a finite number of steps we obtain (2.28) with $k = 1$. This in particular shows (2.23) and concludes the proof of (2.14) and (2.15). □

Remark 2.1.

(a) When $m < 1$, we point out that for the particular solution obtained by our approximation procedure, the additional assumption that $\nabla v_0 \in L^q(\mathbb{R}^N)$ for some $q \in [1, \infty)$ is not restrictive. Indeed, after a small time-shift, it is satisfied along with $u_0, v_0 \in L^1 \cap L^\infty(\mathbb{R}^N)$. However it is not clear whether this is true for other – nonconstructive – solutions.

(b) It follows from the proof of Proposition 2.1 that, when $0 < m < 1$, for each $0 < \beta < 1 + m$, we have $u(\cdot, t) \in C^\beta(\overline{\Omega})$ for all $t \in (0, T)$.

Remark 2.2. To see that the definition (2.3) is natural, observe that if Ω is bounded and $h \in (BC^1(\overline{\Omega}))^N$ then, by integration by parts, we have

$$\begin{aligned}\big(S(t)(\nabla\cdot h)\big)(x) &= \int_\Omega G(x,y;t)(\nabla_y\cdot h)(y)\,dy\\ &= -\int_\Omega \nabla_y G(x,y;t)\cdot h(y)\,dy + \int_{\partial\Omega} G(x,y;t)\big(h(y)\cdot\nu(y)\big)\,d\sigma,\end{aligned} \tag{2.30}$$

for all $t > 0$ and $x \in \Omega$. Now, if (u, v) is a positive classical solution of (1.1)–(1.3), the function $h := u^m\nabla v \in (BC^1(\overline{\Omega}))^N$ satisfies $h\cdot\nu = 0$ on $\partial\Omega$, hence

$$\big(S(t)(\nabla\cdot h)\big)(x) = -\int_\Omega \nabla_y G(x,y;t)\cdot h(y)\,dy. \tag{2.31}$$

When $\Omega = \mathbb{R}^N$ and $h \in (BC^1(\mathbb{R}^N))^N$, (2.31) remains true due to the decay of the heat kernel at space infinity.

Let us next justify the property (2.4) of the operator $S(t)\nabla\cdot$. First note that for all $h \in (C_0^\infty(\Omega))^N$, owing to (2.3), (2.31) and the mass preserving property of the Neumann (or Cauchy) heat semigroup, we have

$$\int_\Omega \big(S(t)\nabla\cdot h\big)(x)\,dx = \int_\Omega \big(S(t)(\nabla\cdot h)\big)(x)\,dx = \int_\Omega \nabla\cdot h(x)\,dx = 0. \tag{2.32}$$

On the other hand, by (2.3), Proposition 2.3 and Fubini's theorem, for any $h, \tilde h \in (L^\infty \cap L^1(\Omega))^N$, we have

$$\big\|S(t)\nabla\cdot h - S(t)\nabla\cdot\tilde h\big\|_1 \leqslant \int_\Omega\int_\Omega \big|\nabla_y G(x,y;t)(h-\tilde h)(y)\big|\,dy\,dx \leqslant Ct^{-\frac12}\|h-\tilde h\|_1. \tag{2.33}$$

Property (2.4) then follows from (2.32), (2.33) and the density of $C_0^\infty(\Omega)$ in $L^1(\Omega)$.

3. Local nondegeneracy for $1 \leqslant m < 2$

We begin with the following lemma, a property of the inhomogeneous, linear heat equation, which will be used again in Section 3. It gives an upper blow-up estimate of the gradient of the solution, assuming an upper blow-up estimate of the RHS.

Lemma 3.1. *Let v be a classical solution of*

$$\begin{cases} \Gamma v_t = \Delta v - \lambda v + f, & x \in \Omega,\ 0 < t < T,\\ \dfrac{\partial v}{\partial\nu}(x,t) = 0, & x \in \partial\Omega,\ 0 < t < T,\end{cases} \tag{3.1}$$

with $f \in L^\infty_{loc}([0,T); L^\infty(\Omega))$ *(the boundary conditions being as usual understood to be empty in case* $\Omega = \mathbb{R}^N$*). For any real number* $\mu > 1/2$*, there exists a constant* $C_0 = C_0(\Omega, \mu, \Gamma, \lambda) > 0$ *such that the following holds. Let* $a \in \overline{\Omega}$*,* $t_0 \in (0,T)$*,* $\rho, \varepsilon, c > 0$ *and assume that*

$$\left|f(x,t)\right| \leqslant \varepsilon (T-t)^{-\mu} \quad \text{for all } (x,t) \in \Omega_{a,\rho} \times (t_0,T) \tag{3.2}$$

and

$$\left\|v(t)\right\|_{L^1(\Omega_{a,\rho})} \leqslant c \quad \text{for all } t \in (t_0,T). \tag{3.3}$$

Then, for any $\tilde\rho \in (0,\rho)$*, there exists a real number* $K > 0$ *such that the function* v *satisfies*

$$\left|\nabla v(x,t)\right| \leqslant C_0 \varepsilon (T-t)^{-\mu+\frac{1}{2}} + K \quad \text{for all } (x,t) \in \Omega_{a,\tilde\rho} \times (t_0,T). \tag{3.4}$$

Proof. It suffices to show that (3.4) holds for *some* $\tilde\rho \in (0,\rho)$. Indeed, assume that this is true and fix any $\hat\rho \in (0,\rho)$. Then, for any $b \in \overline{\Omega_{a,\hat\rho}}$, we have $|f(x,t)| \leqslant \varepsilon(T-t)^{-\mu}$ for all $(x,t) \in \Omega_{b,\rho-\hat\rho} \times (t_0,T)$ and consequently there exist $\rho_b > 0$ and $K_b > 0$ such that

$$\left|\nabla v(x,t)\right| \leqslant C_0 \varepsilon (T-t)^{-\mu+\frac{1}{2}} + K_b \quad \text{for all } (x,t) \in \Omega_{b,\rho_b} \times (t_0,T).$$

Since the compact $\overline{\Omega_{a,\hat\rho}}$ can be covered by finitely many balls $B(b_i, \rho_{b_i})$, we then conclude that (3.4) is true with $\tilde\rho$ replaced by $\hat\rho$.

To show that (3.4) holds for some $\tilde\rho \in (0,\rho)$, we consider the cases $a \in \Omega$ and $a \in \partial\Omega$ separately.

Case 1: $a \in \Omega$. Set $\delta = \min(\rho, \operatorname{dist}(a, \partial\Omega))$. Take a function $\varphi \in C^2(\mathbb{R}^N)$, $0 \leqslant \varphi \leqslant 1$, such that $\varphi(x) = 1$ for all $x \in B_{\delta/2}(a)$ and $\varphi(x) = 0$ for all $x \in \mathbb{R}^N \setminus B_\delta(a)$. Put $\tilde v(x,t) = e^{\lambda t} v(x,t)\varphi(x)$ for $(x,t) \in \Omega \times (0,T)$. Multiplying the second equation of (1.1) by φ yields

$$\tilde v_t = \Delta \tilde v + e^{\lambda t}(f\varphi - 2\nabla v \cdot \nabla\varphi - v\Delta\varphi) \quad \text{in } \Omega \times (0,T). \tag{3.5}$$

Now pick $x_0 \in B_{\delta/4}(a)$ and $t \in (t_0,T)$. Then $\nabla\tilde v(x_0,t)$ is represented as

$$\nabla\tilde v(x_0,t) = J_0(x_0,t) + J_1(x_0,t) - 2J_2(x_0,t) - J_3(x_0,t), \tag{3.6}$$

where

$$J_0(x_0,t) = \nabla V(x_0,t) \quad \text{with } V(\cdot,t) = S(t-t_0)\tilde v(t_0), \tag{3.7}$$

and

$$J_1(x_0,t) = \int_{t_0}^{t} e^{\lambda s} \int_{\Omega} \nabla_x G(x_0,y;t-s) f(y,s)\varphi(y)\,dy\,ds, \tag{3.8}$$

$$J_2(x_0,t) = \int_{t_0}^{t} e^{\lambda s} \int_{\Omega} \nabla_x G(x_0,y;t-s)\big(\nabla v(y,s)\cdot\nabla\varphi(y)\big)\,dy\,ds, \tag{3.9}$$

$$J_3(x_0,t) = \int_{t_0}^{t} e^{\lambda s} \int_{\Omega} \nabla_x G(x_0,y;t-s) v(y,s)\,dy\,ds. \tag{3.10}$$

Here and below, we denote $G = G_\Gamma$ and $S(t) = S_\Gamma(t)$ for brevity. Also, for $i = 1,2,\ldots,$ we shall denote by C_i a constant depending only on $\Omega, m, \Gamma, \lambda$ and by K_i a constant independent of $x_0 \in B_{\delta/4}(a)$ and $t \in (t_0,T)$. By standard linear parabolic regularity properties, there exists K_1 such that

$$\left|J_0(x_0,t)\right| \leqslant K_1. \tag{3.11}$$

Using Proposition 2.3 and assumption (3.2), there exist $C_1, C_2, C_3 > 0$ such that

$$\begin{aligned}\left|J_1(x_0,t)\right| &\leqslant C_1 e^{\lambda t}\int_{t_0}^{t}(t-s)^{-\frac{N+1}{2}}\int_{\Omega}\left|f(y,s)\right|\varphi(y)\exp\left(-\frac{C_2|x_0-y|^2}{4(t-s)}\right)dy\,ds\\ &\leqslant C_3 e^{\lambda t}\varepsilon\int_{t_0}^{t}(T-s)^{-\mu}(t-s)^{-\frac12}\,ds.\end{aligned}$$

Recalling $\mu > 1/2$, for all $t \in (t_0, T)$, we have

$$\begin{aligned}\int_{t_0}^{t}(T-s)^{-\mu}(t-s)^{-\frac12}\,ds &= \int_{t_0<s<2t-T}(T-s)^{-\mu}(t-s)^{-\frac12}\,ds+\int_{2t-T<s<t}(T-s)^{-\mu}(t-s)^{-\frac12}\,ds\\ &\leqslant \int_{t_0<s<2t-T}(t-s)^{-\mu-\frac12}\,ds+(T-t)^{-\mu}\int_{2t-T<s<t}(t-s)^{-\frac12}\,ds\\ &\leqslant \left(\mu-\frac12\right)^{-1/2}(T-t)^{-\mu+\frac12}+2(T-t)^{-\mu+\frac12}.\end{aligned}$$

Therefore, there exists $C_4 > 0$ such that

$$\left|J_1(x_0,t)\right| \leqslant C_4 e^{\lambda t}\varepsilon(T-t)^{-\mu+\frac12}. \tag{3.12}$$

Integrating $J_2(x_0, t)$ by parts in y, making use of $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega$, we obtain

$$\begin{aligned}J_2(x_0,t) = -e^{\lambda t}\Bigg\{&\int_{t_0}^{t}\int_{\Omega}v(y,s)\Delta\varphi(y)\nabla_x G(x_0,y;t-s)\,dy\,ds\\ &+\int_{t_0}^{t}\int_{\Omega}v(y,s)\nabla_y\nabla_x G(x_0,y;t-s)\nabla\varphi(y)\,dy\,ds\Bigg\}.\end{aligned}$$

Using Proposition 2.3 again, we deduce that

$$\left|J_2(x_0,t)\right| \leqslant C_1 e^{\lambda t}\left\{J_{2,1}(x_0,t)+J_{2,2}(x_0,t)\right\}, \tag{3.13}$$

where

$$J_{2,1}(x_0,t) = \int_{t_0}^{t}(t-s)^{-\frac{N+1}{2}}\int_{\Omega}\left|v(y,s)\right|\cdot\left|\Delta\varphi(y)\right|\exp\left(-\frac{C_2|x_0-y|^2}{4(t-s)}\right)dy\,ds$$

and

$$J_{2,2}(x_0,t) = \int_{t_0}^{t}(t-s)^{-\frac{N+2}{2}}\int_{\Omega}\left|v(y,s)\right|\cdot\left|\nabla\varphi(y)\right|\exp\left(-\frac{C_2|x_0-y|^2}{4(t-s)}\right)dy\,ds.$$

Since $\nabla\varphi(y) = \Delta\varphi(y) = 0$ for all $y \in B_{\delta/4}(x_0) \subset B_{\delta/2}(a)$, there exists $K_2 > 0$ such that

$$\tau^{-\frac{N+1}{2}}\left|\Delta\varphi(y)\right|\exp\left(-\frac{C_2|x_0-y|^2}{4\tau}\right) \leqslant K_2 \quad \text{for all } y\in\Omega \text{ and } \tau>0 \tag{3.14}$$

and

$$\tau^{-\frac{N+2}{2}}\left|\nabla\varphi(y)\right|\exp\left(-\frac{C_2|x_0-y|^2}{4\tau}\right) \leqslant K_2 \quad \text{for all } y\in\Omega \text{ and } \tau>0. \tag{3.15}$$

It follows from (3.3), (3.13)–(3.15) that

$$\left|J_2(x_0,t)\right| \leqslant K_3. \tag{3.16}$$

We can similarly show that

$$\left|J_3(x_0,t)\right| \leqslant K_4. \tag{3.17}$$

Consequently, from (3.6)–(3.12), (3.16) and (3.17), we obtain $K_5 > 0$ such that

$$\left|\nabla\tilde{v}(x_0,t)\right| \leqslant C_4 e^{\lambda t}\varepsilon(T-t)^{\frac{1}{m}-\frac{1}{2}} + K_5 \quad \text{for all } t \in (t_0,T).$$

Since $\nabla\tilde{v}(x,t) = e^{\lambda t}\{\nabla v(x,t)\varphi(x) + v(x,t)\nabla\varphi(x)\}$, we get

$$\left|\nabla v(x_0,t)\right| \leqslant C_4\varepsilon(T-t)^{\frac{1}{m}-\frac{1}{2}} + K_5 \quad \text{for all } t \in (t_0,T),$$

hence (3.4) with $\tilde{\rho} = \delta/4$.

Case 2: $a \in \partial\Omega$. Set $\delta = \min(\rho,\rho_0)$, where ρ_0 is given by Lemma A.1. By that lemma, there exists a function $\tilde{\varphi} \in C^2(\mathbb{R}^N)$ such that $\tilde{\varphi}(x) = 1$ for all $x \in B_{\delta/2}(a)$, $\tilde{\varphi}(x) = 0$ for all $x \in \mathbb{R}^N \setminus B_\delta(a)$ and $\frac{\partial\tilde{\varphi}}{\partial\nu}(x) = 0$ for all $x \in \partial\Omega$. Now, the above argument with φ replaced by $\tilde{\varphi}$ gives a proof in this case with $\tilde{\rho} = \delta/4$. □

Proof of Theorem 1.1 with $1 \leqslant m < 2$. Let (u,v) be a solution of (1.1)–(1.3) such that $T = T_{\max}(u,v) < \infty$. Let $a \in \overline{\Omega}$, $t_0 \in (0,T)$, $\rho > 0$ and assume that

$$u(x,t) \leqslant \varepsilon(T-t)^{-1/m} \quad \text{for all } (x,t) \in \Omega_{a,\rho} \times (t_0,T). \tag{3.18}$$

We first consider the case $a \in \Omega$. We put $\delta = \min(\rho, \operatorname{dist}(a,\partial\Omega))$. We divide the proof into three steps.

Step 1. We claim that for each $p > 1$ there exist $C_1, K_1 > 0$ such that

$$\int\limits_{B_{\delta/2}(a)} u^{p+1}\,dx \leqslant K_1(T-t)^{-C_1\varepsilon^{2m}} \quad \text{for all } t \in [t_0,T). \tag{3.19}$$

Here and hereafter, for $i = 1,2,\ldots$, we denote by C_i a constant depending only on $\Omega, m, \Gamma, \lambda, p$ and by K_i a constant independent of $t \in (t_0,T)$ (and of x_0 in Steps 2 and 3).

Choose $1 - 1/p < k < 1$. Take a function $\psi \in C^2(\mathbb{R}^N)$, $0 \leqslant \psi \leqslant 1$, such that $\psi(x) = 1$ for all $x \in B_{\delta/2}(x_0)$, $\psi(x) = 0$ for all $x \in \mathbb{R}^N \setminus B_\delta(x_0)$, and $|\nabla\psi(x)| \leqslant A\psi(x)^{(k+1)/2}$ for all $x \in \mathbb{R}^N$, with some constant $A > 0$. Multiplying the first equation of (1.1) by $u^p\psi^{p+1}$ and integrating by parts yields

$$\frac{1}{p+1}\frac{d}{dt}\int\limits_\Omega \{u(t)\psi\}^{p+1}\,dx = -I_1(t) + I_2(t), \tag{3.20}$$

where

$$I_1(t) = p\int\limits_\Omega u^{p-1}|\nabla u|^2\psi^{p+1}\,dx + (p+1)\int\limits_\Omega u^p\psi^p\nabla u\cdot\nabla\psi\,dx$$

and

$$I_2(t) = p\int\limits_\Omega u^{p+m-1}\psi^{p+1}\nabla u\cdot\nabla v\,dx + (p+1)\int\limits_\Omega u^{p+m}\psi^p\nabla v\cdot\nabla\psi\,dx.$$

Since $\nabla(u^{\frac{p+1}{2}}) = \frac{p+1}{2}u^{\frac{p-1}{2}}\nabla u$, we have

$$I_1(t) = \left\{\frac{4p}{(p+1)^2} - \frac{2}{p+1}\right\}\int\limits_\Omega \left|\nabla u^{\frac{p+1}{2}}\right|^2\psi^{p+1}\,dx + \frac{2}{p+1}\int\limits_\Omega \left|\nabla(u\psi)^{\frac{p+1}{2}}\right|^2 dx$$
$$- \frac{2}{p+1}\int\limits_\Omega u^{p+1}\left|\nabla\psi^{\frac{p+1}{2}}\right|^2 dx.$$

By the choice of p and ψ, we get

$$\begin{aligned} I_1(t) &\geqslant \frac{2}{p+1}\int_\Omega \left|\nabla(u\psi)^{\frac{p+1}{2}}\right|^2 dx - \frac{p+1}{2}\int_\Omega u^{p+1}\psi^{p-1}|\nabla\psi|^2\,dx \\ &\geqslant \frac{2}{p+1}\int_\Omega \left|\nabla(u\psi)^{\frac{p+1}{2}}\right|^2 dx - \frac{p+1}{2}A^2\int_\Omega u^{p+1}\psi^{p+k}\,dx \\ &= \frac{2}{p+1}\int_\Omega \left|\nabla(u\psi)^{\frac{p+1}{2}})\right|^2 dx - \frac{p+1}{2}A^2\int_\Omega (u\psi)^{p+k}u^{1-k}\,dx \\ &\geqslant \frac{2}{p+1}\int_\Omega \left|\nabla(u\psi)^{\frac{p+1}{2}}\right|^2 dx - \frac{p+1}{2}A^2M^{1-k}\left(\int_\Omega (u\psi)^{1+\frac{p}{k}}\,dx\right)^k, \end{aligned} \tag{3.21}$$

where $M = \|u_0\|_1$.

Since

$$\begin{aligned} I_2(t) &= \int_\Omega u^{m-1}\nabla v\cdot\left[pu^p\psi^{p+1}\nabla u + (p+1)u^{p+1}\psi^p\nabla\psi\right]dx \\ &= \int_\Omega u^{m-1}\nabla v\cdot\left[\frac{p}{p+1}\nabla(u\psi)^{p+1} + u^{p+1}\psi^p\nabla\psi\right]dx, \end{aligned}$$

we have

$$\begin{aligned} I_2(t) &\leqslant \|\nabla v\|_{L^\infty(B_\delta(x_0))}\|u\|^{m-1}_{L^\infty(B_\delta(x_0))}\left\{\frac{p}{p+1}\int_\Omega \left|\nabla(u\psi)^{p+1}\right|dx + \int_\Omega u^{p+1}\psi^p|\nabla\psi|\,dx\right\} \\ &= \|\nabla v\|_{L^\infty(B_\delta(x_0))}\|u\|^{m-1}_{L^\infty(B_\delta(x_0))}\left\{\frac{2p}{p+1}\int_\Omega (u\psi)^{\frac{p+1}{2}}\left|\nabla(u\psi)^{\frac{p+1}{2}}\right|dx + \int_\Omega u^{p+1}\psi^p|\nabla\psi|\,dx\right\}. \end{aligned}$$

By the choice of ψ, we get

$$\begin{aligned} \int_\Omega u^{p+1}\psi^p|\nabla\psi|\,dx &\leqslant A\int_\Omega u^{p+1}\psi^{p+\frac{k+1}{2}}\,dx \leqslant A\int_\Omega (u\psi)^{p+\frac{k+1}{2}}u^{\frac{1-k}{2}} \\ &\leqslant AM^{\frac{1-k}{2}}\left(\int_\Omega (u\psi)^{1+\frac{2p}{k+1}}\,dx\right)^{\frac{k+1}{2}}. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} I_2(t) &\leqslant \frac{1}{p+1}\int_\Omega \left|\nabla(u\psi)^{\frac{p+1}{2}}\right|^2 dx + \frac{p^2}{p+1}\|\nabla v\|^2_{L^\infty(B_\delta(x_0))}\|u\|^{2(m-1)}_{L^\infty(B_\delta(x_0))}\int_\Omega (u\psi)^{p+1}\,dx \\ &\quad + AM^{\frac{1-k}{2}}\|\nabla v\|_{L^\infty(B_\delta(x_0))}\|u\|^{m-1}_{L^\infty(B_\delta(x_0))}\left(\int_\Omega (u\psi)^{1+\frac{2p}{k+1}}\,dx\right)^{\frac{k+1}{2}}. \end{aligned} \tag{3.22}$$

It follows from (3.20), (3.21), (3.22) that

$$\begin{aligned} \frac{d}{dt}\int_\Omega (u\psi)^{p+1}\,dx &\leqslant p^2\|\nabla v\|^2_{L^\infty(B_\delta(x_0))}\|u\|^{2(m-1)}_{L^\infty(B_\delta(x_0))}\int_\Omega (u\psi)^{p+1}\,dx + \frac{(p+1)^2A^2}{2}M^{1-k}\left(\int_\Omega (u\psi)^{1+\frac{p}{k}}\,dx\right)^k \\ &\quad + (p+1)AM^{\frac{1-k}{2}}\|\nabla v\|_{L^\infty(B_\delta(x_0))}\|u\|^{m-1}_{L^\infty(B_\delta(x_0))}\left(\int_\Omega (u\psi)^{1+\frac{2p}{k+1}}\,dx\right)^{\frac{k+1}{2}}. \end{aligned}$$

By (2.15), (3.18) and Lemma 3.1, there exist $C_2, K_2 > 0$ such that

$$\big|u(t)\big|_{L^\infty(B_\delta(x_0))} \leqslant \varepsilon(T-t)^{-\frac{1}{m}} \quad \text{and} \quad \big|\nabla v(t)\big|^2_{L^\infty(B_\delta(x_0))} \leqslant C_2\varepsilon^2(T-t)^{1-\frac{2}{m}} + K_2.$$

In particular (assuming $\varepsilon \leqslant 1$ without loss of generality), we have

$$(u\psi)^{1+\frac{p}{k}} \leqslant (u\psi)^{p+1}(T-t)^{-\frac{p(1-k)}{km}}, \qquad (u\psi)^{1+\frac{k+1}{2}} \leqslant (u\psi)^{p+1}(T-t)^{-\frac{p(1-k)}{(k+1)m}}.$$

It follows that

$$\begin{aligned}\frac{d}{dt}\int_\Omega (u\psi)^{p+1}\,dx &\leqslant p^2\|\nabla v\|^2_{L^\infty(B_\delta(x_0))}\|u\|^{2(m-1)}_{L^\infty(B_\delta(x_0))}\int_\Omega (u\psi)^{p+1}\,dx \\ &\quad + \frac{(p+1)^2A^2}{2}M^{1-k}(T-t)^{-\frac{p(1-k)}{m}}\bigg(\int_\Omega (u\psi)^{p+1}\,dx\bigg)^k \\ &\quad + (p+1)AM^{\frac{1-k}{2}}\|\nabla v\|_{L^\infty(B_\delta(x_0))}\|u\|^{m-1}_{L^\infty(B_\delta(x_0))}(T-t)^{-\frac{p(1-k)}{2m}}\bigg(\int_\Omega (u\psi)^{p+1}\,dx\bigg)^{\frac{k+1}{2}}.\end{aligned}$$

Setting $F(t) = 1 + \int_\Omega (u\psi)^{p+1}\,dx$ and using $p(1-k)/m < 2(m-1)/m < 1$, we obtain

$$\begin{aligned}F'(t) &\leqslant \Big[(p+1)^2A^2M^{1-k}(T-t)^{-\frac{p(1-k)}{m}} + \big(p^2+1\big)\|\nabla v\|^2_{L^\infty(B_\delta(x_0))}\|u\|^{2(m-1)}_{L^\infty(B_\delta(x_0))}\Big]F(t) \\ &\leqslant \Big[K_3\big((T-t)^{-\frac{p(1-k)}{m}} + (T-t)^{-\frac{2(m-1)}{m}}\big) + C_3\varepsilon^{2m}(T-t)^{-1}\Big]F(t) \\ &\leqslant \Big[K_4(T-t)^{\frac{2}{m}-2} + C_3\varepsilon^{2m}(T-t)^{-1}\Big]F(t).\end{aligned}$$

After integration, recalling $m < 2$, we obtain

$$F(t) \leqslant F(t_0)\exp\big[K_5T^{\frac{2}{m}-1}\big](T-t)^{-C_3\varepsilon^{2m}} = K_1(T-t)^{-C_1\varepsilon^{2m}},$$

hence the claim.

Step 2. We claim that

$$\big|\nabla v(x_0,t)\big| \leqslant K_5 \quad \text{for all } x_0 \in B_{\delta/8}(a) \text{ and } t \in [t_0,T). \tag{3.23}$$

Take a function $\varphi \in C^2(\mathbb{R}^N)$, $0 \leqslant \varphi(x) \leqslant 1$, such that $\varphi(x) = 1$ for $x \in B_{\delta/4}(a)$ and $\varphi(x) = 0$ for $x \in \mathbb{R}^N \setminus B_{\delta/2}(a)$. Put $\tilde v(x,t) = e^{\lambda t}v(x,t)\varphi(x)$ for $x \in \Omega \times (0,T)$. Pick $x_0 \in B_{\delta/8}(a)$ and $t \in (t_0,T)$. Like in the proof of Lemma 3.1, $\nabla\tilde v(x_0,t)$ is represented according to

$$\nabla\tilde v(x_0,t) = J_0(x_0,t) + J_1(x_0,t) - 2J_2(x_0,t) - J_3(x_0,t), \tag{3.24}$$

where the terms J_i are defined by (3.7)–(3.10) with $f = u$. Similarly to the proof of Lemma 3.1, we get the boundedness of $J_0(x_0,t)$, $J_2(x_0,t)$ and $J_3(x_0,t)$ for $x_0 \in B_{\delta/8}(a)$ and $t \in [t_0,T)$.

To control the term $J_1(x_0,t)$, we proceed as follows. Take $p, q \geqslant 1$ with $1/p + 1/q = 1$ and $q < 1 + 1/(N-1)$. Let $\varepsilon > 0$ such that

$$\frac{1}{2} - \frac{N}{2}\bigg(1 - \frac{1}{q}\bigg) - \frac{C_1}{p}\varepsilon^{2m} > 0, \tag{3.25}$$

where C_1 is the constant in (3.19). By Proposition 2.3 and (3.19), there exist $C_i > 0$ for $i = 4,5,6$ such that

$$\begin{aligned}\big|J_1(x_0,t)\big| &\leqslant C_4\int_{t_0}^{t}(t-s)^{-\frac{N+1}{2}}\int_\Omega u(y,s)\varphi(y)\exp\bigg(-\frac{C_5|x-y|^2}{4(t-s)}\bigg)\,dy\,ds \\ &\leqslant C_4\int_{t_0}^{t}(t-s)^{-\frac{1}{2}(N+1-\frac{N}{q})}\end{aligned}$$

$$\times \left\{ \frac{1}{(t-s)^{N/2}} \int_\Omega \exp\left(-\frac{C_5 q |x-y|^2}{4(t-s)} \right) dy \right\}^{1/q} \left(\int_\Omega \left\{ u(y,s)\varphi(y) \right\}^p dy \right)^{1/p}$$

$$\leqslant C_6 \int_{t_0}^{t} (t-s)^{-\frac{1}{2}(N+1-\frac{N}{q})} \left\{ K_1 (T-t)^{-C_1 \varepsilon^{2m}} \right\}^{1/p} ds.$$

By the choice of p, q, ε, $J_1(x_0,t)$ is bounded for $x_0 \in B_{\delta/8}(a)$ and $t \in [t_0,T)$. This proves the claim.

Step 3. We claim that $u(x_0,t) \leqslant K_6$ for $x_0 \in B_{\delta/32}(a)$ and $t \in [t_0,T)$.

Take a function $\phi \in C^2(\mathbb{R}^N)$, $0 \leqslant \phi(x) \leqslant 1$, such that $\phi(x) = 1$ for $x \in B_{\delta/16}(x_0)$ and $\phi(x) = 0$ for $x \in \mathbb{R}^N \setminus B_{\delta/8}(x_0)$. Put $\tilde{u}(x,t) = u(x,t)\phi(x)$. Then $\tilde{u}$ satisfies

$$\tilde{u}_t = \Delta \tilde{u} - \phi \nabla\left(u^m \nabla v\right) - 2\nabla u \cdot \nabla \phi - u \Delta \phi.$$

Pick $x_0 \in B_{\delta/32}(a)$ and $t \in (t_0,T)$. We represent $\tilde{u}(x_0,t)$ as

$$\begin{aligned} \tilde{u}(x_0,t) &= \left[S(t-t_0)\left(\tilde{u}(t_0)\right)\right](x_0) \\ &\quad - \int_{t_0}^{t} \int_\Omega G(x_0,y;t-s) \left\{ \phi(y) \nabla\left[u^m \nabla v\right](y,s) + 2\nabla u(y,s) \cdot \nabla \phi(y) + u(y,s)\Delta\phi(y) \right\} dy\, ds \\ &= K_0(x_0,t) + K_1(x_0,t) + K_2(x_0,t) - 2K_3(x_0,t) - K_4(x_0,t), \end{aligned} \tag{3.26}$$

where

$$K_0(x_0,t) = \left[S(t-t_0)\left(\tilde{u}(t_0)\right)\right](x_0),$$

$$K_1(x_0,t) = \int_{t_0}^{t} \int_\Omega \left(\nabla_y G(x_0,y;t-s) \cdot \nabla v(y,s) \right) u^m(y,s) \phi(y)\, dy\, ds,$$

$$K_2(x_0,t) = \int_{t_0}^{t} \int_\Omega G(x_0,y;t-s) u^m(y,s) \left(\nabla v(y,s) \cdot \nabla \phi(y) \right) dy\, ds,$$

$$K_3(x_0,t) = \int_{t_0}^{t} \int_\Omega G(x_0,y;t-s) \left(\nabla u(y,s) \cdot \nabla \phi(y) \right) dy\, ds$$

and

$$K_4(x_0,t) = \int_{t_0}^{t} \int_\Omega G(x_0,y;t-s) u(y,s) \Delta \phi(y)\, dy\, ds.$$

By (3.23) in Step 2, we obtain the boundedness of $K_1(x_0,t)$ for $x_0 \in B_{\delta/32}(a)$ and $t \in (t_0,T)$ in the same way as for $J_1(x_0,t)$ in Step 2. Similar arguments to those in the proof of Lemma 3.1 imply that $K_0(x_0,t)$, $K_2(x_0,t)$, $K_3(x_0,t)$, $K_4(x_0,t)$ are bounded for $x_0 \in B_{\delta/32}(a)$ and $t \in (t_0,T)$. Consequently, $\tilde{u}(x_0,t)$ is bounded for $x_0 \in B_{\delta/32}(a)$ and $t \in (t_0,T)$. This proves the claim, hence the theorem in the case $a \in \Omega$.

Finally, in the case of $a \in \partial\Omega$, set $\delta = \min(\rho,\rho_0)$, where ρ_0 is given by Lemma A.1. Thanks to that lemma, we may find functions $\tilde{\psi}$, $\tilde{\varphi}$, $\tilde{\phi}$ in Steps 1, 2, 3 satisfying $\frac{\partial \tilde{\varphi}}{\partial \nu} = \frac{\partial \tilde{\phi}}{\partial \nu} = 0$ on $\partial\Omega$ in addition to the properties of ψ, φ, ϕ, respectively. The above arguments give a proof in this case if ψ, φ, ϕ are replaced by $\tilde{\psi}$, $\tilde{\varphi}$, $\tilde{\phi}$.

The Cauchy problem is similarly treated without Proposition 2.3. □

Remark 3.1. A similar calculation for $\int_\Omega u^p u_t \psi^{p+1}\, dx$ in Step 1 of the proof was done in the proof of Proposition 4.2 of [37]. However their method does not work well to prove Theorem 1.1 since they treated (1.1) with $m = 1$ when $\limsup_{t\to T} \int_{B_R(x_0)} u(t) \log u(t)\, dx < \infty$ for some $R > 0$, which yields $\limsup_{t\to T} \|\nabla v(t)\|_{L^\infty(B_R(x_0))} < +\infty$.

4. Local nondegeneracy for $0 < m < 1$

The key to our proof of local nondegeneracy when $0 < m < 1$ is the following auxiliary function:

$$H = \frac{m}{2}|\nabla v|^2 + \frac{u^{2-m}}{2-m} \tag{4.1}$$

and the following lemma, which states that H satisfies a suitable, scalar parabolic inequality.

Lemma 4.1. *Assume $0 < m \leqslant 1$, $\Gamma = 1$ and let $p = 2/(2-m) > 1$. Then the function $H \in L^\infty_{loc}((0,T); BC^1(\overline{\Omega}))$ satisfies*

$$\partial_t H - \Delta H \leqslant C_1 H^p \quad \text{in } \Omega \times (0,T) \tag{4.2}$$

in the weak sense (cf. Remark 4.1), where $C_1 = (2-m)^{2/(2-m)} \frac{N}{4m} > 0$.

Remark 4.1. If Ω is bounded, the weak formulation of (4.2) is understood as

$$\int_{t_0}^{\tau}\int_{\Omega} \big(-H\partial_t\varphi + \nabla H\cdot\nabla\varphi - C_1 H^p\varphi\big)\,dx\,dt \leqslant -\Big[\int_{\Omega} H\varphi(x,t)\,dx\Big]_{t_0}^{\tau} + \int_{t_0}^{\tau}\int_{\partial\Omega} \varphi\partial_\nu H\,d\sigma\,dt,$$

for all $0 < t_0 < \tau < T$ and all $0 \leqslant \varphi \in C^{1,1}(\overline{\Omega} \times [t_0,\tau])$. When $\Omega = \mathbb{R}^N$, we say that (4.2) is true if is satisfied in $\omega \times (0,T)$ for each smooth compact subdomain ω of $\mathbb{R}^N$.

Proof of Lemma 4.1. We only consider the case Ω bounded, the case $\Omega = \mathbb{R}^N$ being completely similar. Let $(G_k)_{k\in\mathbb{N}^*}$ be a sequence of approximations of the positive sign function with the following properties:

$$\begin{aligned} &G_k \in C^2(\mathbb{R}), \quad G_k(s) = 0 \quad \text{for all } s \leqslant 1/k,\ 0 \leqslant G_k' \leqslant 1,\ G_k'' \geqslant 0,\\ &\lim_{k\to\infty} G_k(s) = s \quad \text{and} \quad \lim_{k\to\infty} G_k'(s) = \chi_{(0,\infty)}(s) \quad \text{for each } s \geqslant 0. \end{aligned} \tag{4.3}$$

Let

$$J = \frac{u^{2-m}}{2-m}$$

and note that, for each k, $J_k := G_k \circ J$ is smooth (since u is smooth on the set $\{u > 0\}$). We compute

$$\begin{aligned} (\partial_t - \Delta)J_k &= \big(G_k' \circ J\big)(\partial_t J - \Delta J) - \big(G_k'' \circ J\big)|\nabla J|^2\\ &= \big(G_k' \circ J\big)\big[u^{1-m}(u_t - \Delta u) - (1-m)|\nabla u|^2 u^{-m}\big] - \big(G_k'' \circ J\big)|\nabla J|^2\\ &\leqslant -\big(G_k' \circ J\big)u^{1-m}\nabla\cdot\big(u^m\nabla v\big), \end{aligned}$$

hence

$$\begin{aligned} &\int_{t_0}^{\tau}\int_{\Omega} \big(-J_k\partial_t\varphi + \big(G_k' \circ J\big)[\nabla J\cdot\nabla\varphi + u\Delta v + m\nabla u\cdot\nabla v]\varphi\big)\,dx\,dt\\ &\quad \leqslant -\Big[\int_{\Omega} J_k\varphi(x,t)\,dx\Big]_{t_0}^{\tau} + \int_{t_0}^{\tau}\int_{\partial\Omega} \varphi\big(G_k' \circ J\big)\partial_\nu J\,d\sigma\,dt, \end{aligned} \tag{4.4}$$

for any $0 < t_0 < \tau < T$ and any test-function $0 \leqslant \varphi \in C^{1,1}(\overline{\Omega} \times [t_0,\tau])$. Also, we observe that, for a.e. $t \in (t_0,T)$, $\nabla u(x,t) = \nabla J(x,t) = 0$ at each point $x \in \Omega$ such that $u(x,t) = 0$ (due to $u \geqslant 0$) and $\partial_\nu J(x,t) = 0$ at each point $x \in \partial\Omega$ such that $u(x,t) = 0$. Therefore, using (4.3), we deduce that, as $k \to \infty$:

$$\lim_{k\to\infty} \big(G_k' \circ J\big)[\nabla J\cdot\nabla\varphi + u\Delta v + m\nabla u\cdot\nabla v] = \nabla J\cdot\nabla\varphi + u\Delta v + m\nabla u\cdot\nabla v$$

in Ω for a.e. $t \in (t_0, \tau)$ and

$$\lim_{k\to\infty} \big(G_k' \circ J\big)\partial_\nu J = \partial_\nu J$$

on $\partial\Omega$ for a.e. $t \in (t_0, \tau)$. Since also $\lim_{k\to\infty} J_k = J$ uniformly in $\overline{\Omega} \times [t_0, \tau]$, we may pass to the limit in (4.4) via dominated convergence and we obtain, in the weak sense (cf. Remark 4.1),

$$(\partial_t - \Delta)J \leqslant -[u\Delta v + m\nabla u \cdot \nabla v] \quad \text{in } \Omega \times (0, T). \tag{4.5}$$

On the other hand, denoting $|D^2v|^2 = \sum_{ij}(\partial_{ij}^2 v)^2$, we have

$$(\Gamma\partial_t - \Delta)\frac{|\nabla v|^2}{2} = \nabla v \cdot \nabla(v_t - \Delta v) - \big|D^2v\big|^2 = \nabla v \cdot \nabla u - \big|D^2v\big|^2. \tag{4.6}$$

(This is satisfied in the classical sense in $\Omega \times (0, T)$, recalling (2.7).) Combining (4.5), (4.6) and using $\Gamma = 1$, we obtain

$$\partial_t H - \Delta H \leqslant -u\Delta v - m\big|D^2v\big|^2.$$

Using the inequality $|\Delta v|^2 \leqslant N|D^2v|^2$, it follows that

$$\partial_t H - \Delta H \leqslant \frac{N}{4m}u^2$$

in the weak sense, hence (4.2). □

Next, to obtain suitable boundary conditions on the function H, we shall rely on the following simple, differential geometric property. It is probably known but we give a proof in Appendix A for completeness.

Lemma 4.2. *Assume that Ω is bounded and let $w \in C^2(\overline{\Omega})$ satisfy $\frac{\partial w}{\partial \nu} = 0$ on $\partial\Omega$. Then we have*

$$\frac{\partial|\nabla w|^2}{\partial\nu} \leqslant 2\kappa|\nabla w|^2 \quad \text{on } \partial\Omega, \tag{4.7}$$

where $\kappa = \kappa(\Omega) > 0$ is an upper bound for the curvatures of $\partial\Omega$.

With Lemmas 3.1, 4.1, 4.2 at hand, we can then reduce the proof of Theorem 1.1 for $m < 1$ to the following nondegeneracy result for parabolic scalar equations. It was proved in [15] for classical subsolutions, in the case of interior points or of boundary points under Dirichlet boundary conditions (see also [41, Section 25] for a simpler, alternative proof in the interior case). In Appendix A, we give a proof for weak subsolutions in the boundary case under our current Neumann boundary conditions, by adapting the arguments from [15] (the argument works for weak subsolutions in the interior case as well).

Proposition 4.1. *Let $p > 1$, $M_1, M_2, T > 0$, $t_0 \in (0, T)$, $\rho > 0$, $a \in \overline{\Omega}$. Let $0 \leqslant w \in L^\infty_{loc}((0, T); BC^1(\overline{\Omega}))$ satisfy*

$$w_t - \Delta w \leqslant M_1 w^p \quad \text{in } \Omega \times (t_0, T) \tag{4.8}$$

in the weak sense. If $a \in \partial\Omega$, assume in addition that

$$\frac{\partial w}{\partial \nu} \leqslant M_2 w \quad \text{on } \partial\Omega \cap B(a, \rho) \text{ for a.e. } t \in (t_0, T). \tag{4.9}$$

There exists $\varepsilon_0 > 0$ depending only on p, M_1, M_2 such that if

$$w(x, t) \leqslant \varepsilon_0(T - t)^{-1/(p-1)}, \quad (x, t) \in \Omega_\rho \times (t_0, T), \tag{4.10}$$

then w is uniformly bounded in a neighborhood of (a, T).

Proof of Theorem 1.1 for $0 < m < 1$. (The proof is valid also for $m = 1$ but, unlike that in Section 3 for $1 \leqslant m < 2$, it requires $\Gamma = 1$.) Assume that (1.6) holds. By Lemma 3.1, it follows that the function H, defined in (4.1), satisfies

$$H(x,t) \leqslant C\varepsilon(T-t)^{-1/m} \quad \text{for all } (x,t) \in \big(B(a,\rho)\cap\Omega\big)\times[t_0,T), \tag{4.11}$$

with $C > 0$ independent of ε and with some $\rho > 0$.

On the other hand, by Lemma 4.2 and (2.13), we have

$$\frac{\partial H}{\partial \nu} = \frac{m}{2}\frac{\partial|\nabla v|^2}{\partial \nu} + u^{1-m}\frac{\partial u}{\partial \nu} \leqslant m\kappa|\nabla v|^2 \leqslant 2\kappa H \quad \text{on } \partial\Omega\times(0,T).$$

The conclusion then follows from Lemma 4.1 and Proposition 4.1.

5. Global lower estimate for all time: proof of Theorem 1.2

Let us first consider the case $u_0 \not\equiv 0$ and set

$$t_0 = \min\big\{t > 0;\ \big\|u(t_0)\big\|_\infty = 2\|u_0\|_\infty\big\}.$$

Note that, due to (2.6), we have $t_0 < \infty$. For $t \in (0,t_0)$, we first use the second equation in (2.2) to estimate

$$\big\|\nabla v(t)\big\|_\infty \leqslant C\|\nabla v_0\|_\infty + \int_0^t C(t-s)^{-1/2}\big\|u(s)\big\|_\infty\,ds \leqslant C\|\nabla v_0\|_\infty + Ct^{1/2}\|u_0\|_\infty.$$

Next, plugging this into the first equation in (2.2), we obtain

$$2\|u_0\|_\infty = \big\|u(t_0)\big\|_\infty \leqslant \|u_0\|_\infty + \int_0^{t_0} C(t_0-s)^{-1/2}\big(2\|u_0\|_\infty\big)^m\big[\|\nabla v_0\|_\infty + s^{1/2}\|u_0\|_\infty\big]\,ds,$$

hence

$$\begin{aligned}
\|u_0\|_\infty &\leqslant \int_0^{t_0} C(t_0-s)^{-1/2}\big(2\|u_0\|_\infty\big)^m\big[\|\nabla v_0\|_\infty + s^{1/2}\|u_0\|_\infty\big]\,ds \\
&\leqslant C\|u_0\|_\infty^m\|\nabla v_0\|_\infty\int_0^{t_0}(t_0-s)^{-1/2}\,ds + C\|u_0\|_\infty^{m+1}\int_0^{t_0}(t_0-s)^{-1/2}s^{1/2}\,ds \\
&\leqslant Ct_0^{1/2}\|u_0\|_\infty^m\|\nabla v_0\|_\infty + Ct_0\|u_0\|_\infty^{m+1} \\
&\leqslant CT^{1/2}\|u_0\|_\infty^m\|\nabla v_0\|_\infty + CT\|u_0\|_\infty^{m+1}.
\end{aligned}$$

Therefore, we have either

$$\|u_0\|_\infty^m \geqslant cT^{-1} \quad \text{or} \quad \|u_0\|_\infty^{m-1}\|\nabla v_0\|_\infty \geqslant cT^{-1/2},$$

hence

$$\|u_0\|_\infty^m + \|u_0\|_\infty^{2(m-1)}\|\nabla v_0\|_\infty^2 \geqslant cT^{-1}. \tag{5.1}$$

We note that, if $u_0 \equiv 0$ and $m < 1$, then (5.1) is still true in view of our convention $\infty/\infty = \infty$. On the other hand, we may exclude the case $u_0 \equiv 0$ and $m \geqslant 1$, since then $(u,v) = (0, S(t)v_0)$ by local uniqueness, hence $T = \infty$.

Next, shifting the time origin, with $u(t)$ considered as initial data at time t, we deduce from (5.1) that

$$\big\|u(t)\big\|_\infty^m + \big\|u(t)\big\|_\infty^{2(m-1)}\big\|\nabla v(t)\big\|_\infty^2 \geqslant c(T-t)^{-1}.$$

Finally, assume $1 \leqslant m < 2$. For a given $t \in [0,T)$, if $\|u(t)\|_\infty^m \leqslant (c/2)(T-t)^{-1}$, then we infer that

$$(c/2)(T-t)^{-1} \leqslant \big\|u(t)\big\|_\infty^{2(m-1)}\big\|\nabla v(t)\big\|_\infty^2 \leqslant \big[(c/2)(T-t)^{-1}\big]^{2(m-1)/m}\big\|\nabla v(t)\big\|_\infty^2,$$

hence

$$\left\|\nabla v(t)\right\|_\infty \geqslant \tilde{c}(T-t)^{(m-2)/2m}.$$

It follows that

$$\left\|u(t)\right\|_\infty + \left\|\nabla v(t)\right\|_\infty^{2/(2-m)} \geqslant c(T-t)^{-1/m}. \qquad \square$$

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Appendix A. Proof of Lemma 4.2 and Proposition 4.1

Proof of Lemma 4.2. We may assume $N \geqslant 2$ since otherwise the result is obvious (with the convention $\kappa = 0$). Pick $a \in \partial\Omega$ and assume $a = 0$ without loss of generality. Set $\tilde{x} = (x_1, \dots, x_{N-1})$, so that $x = (\tilde{x}, x_N)$, and denote by $(e_1, \dots, e_N)$ the canonical basis of vectors in $\mathbb{R}^N$. After a rotation, we may assume that, locally near 0, Ω and $\partial\Omega$ are respectively given by $\{x_N \leqslant f(\tilde{x})\}$ and $\{x_N = f(\tilde{x})\}$, where f is a C^2-function such that $f(0) = 0$ and $\nabla f(0) = 0$. At a point $(\tilde{x}, f(\tilde{x}))$ of $\partial\Omega$ close to 0, the tangent vectors to $\partial\Omega$ are given by

$$\tau_i(\tilde{x}) = \left(1 + \left|\nabla f(\tilde{x})\right|^2\right)^{-1/2}\left(e_i + \frac{\partial f}{\partial x_i}(\tilde{x})e_N\right), \quad i = 1, \dots, N-1.$$

Since $\nabla w_{|\partial\Omega}$ is tangential to $\partial\Omega$ due to $\frac{\partial w}{\partial \nu}$, setting $\alpha_i(\tilde{x}) = \nabla w(\tilde{x}, f(\tilde{x})) \cdot \tau_i(\tilde{x})$, we have

$$\begin{aligned}\nabla w\big(\tilde{x}, f(\tilde{x})\big) &= \sum_{i=1}^{N-1} \alpha_i(\tilde{x})\tau_i(\tilde{x}) \\ &= \left(1 + \left|\nabla f(\tilde{x})\right|^2\right)^{-1/2}\left[\left(\sum_{i=1}^{n-1} \alpha_i(\tilde{x})e_i\right) + \left(\sum_{i=1}^{n-1} \alpha_i(\tilde{x})\frac{\partial f}{\partial x_i}(\tilde{x})\right)e_N\right],\end{aligned}$$

hence

$$\frac{\partial w}{\partial x_N}\big(\tilde{x}, f(\tilde{x})\big) = \left(1 + \left|\nabla f(\tilde{x})\right|^2\right)^{-1/2}\sum_{i=1}^{n-1} \alpha_i(\tilde{x})\frac{\partial f}{\partial x_i}(\tilde{x}). \tag{A.1}$$

On the other hand, using $\frac{\partial w}{\partial x_N}(0,0) = \frac{\partial w}{\partial \nu}(0) = 0$, we see that

$$\frac{1}{2}\frac{\partial|\nabla w|^2}{\partial \nu}(0) = \frac{1}{2}\sum_{j=1}^{N} \frac{\partial}{\partial x_N}\left(\frac{\partial w}{\partial x_j}\right)^2(0,0) = \sum_{j=1}^{N-1} \frac{\partial^2 w}{\partial x_N \partial x_j}\frac{\partial w}{\partial x_j}(0,0). \tag{A.2}$$

Differentiating (A.1) and taking into account that $\nabla f(0) = 0$ and $\alpha_i(0) = \frac{\partial w}{\partial x_i}(0,0)$, we obtain, for $j = 1, \dots, N-1$,

$$\frac{\partial^2 w}{\partial x_N \partial x_j}(0,0) = \frac{\partial}{\partial x_j}\left(\sum_{i=1}^{N-1} \alpha_i \frac{\partial f}{\partial x_i}\right)(0) = \left(\sum_{i=1}^{N-1} \alpha_i \frac{\partial^2 f}{\partial x_j \partial x_i}\right)(0) = \left(\sum_{i=1}^{N-1} \frac{\partial w}{\partial x_i}(0,0)\frac{\partial^2 f}{\partial x_j \partial x_i}(0)\right).$$

Plugging this into (A.2) and denoting ${}^t\tilde{\nabla} = (\partial/\partial x_1, \dots, \partial/\partial x_{N-1})$, we obtain

$$\frac{1}{2}\frac{\partial|\nabla w|^2}{\partial \nu}(0) = \left[{}^t\tilde{\nabla} w\big(D^2 f\big)\tilde{\nabla} w\right](0) \leqslant \kappa\left|\nabla w(0)\right|^2,$$

hence (4.7). $\square$

We now turn to the proof of Proposition 4.1. We shall only consider the case $a \in \partial\Omega$, the interior case being easier. We need a suitable Neumann cut-off function, given by the following lemma.

Lemma A.1. *Let $M \leqslant 0$, $0 < \gamma < 1$ and $a \in \partial\Omega$. There exists $\rho_0 > 0$ such that for each $\rho \in (0, \rho_0]$, there exists a function $0 \leqslant \phi \in C^2(\mathbb{R}^N)$ such that*

$$\phi(x) = 0 \quad \text{for all } x \in B_{\rho/2}(a), \tag{A.3}$$

$$\left.\begin{array}{ll} \text{if } M = 0: & \phi(x) = 1 \\ \text{if } M > 0: & \phi(x) \geqslant 1 \end{array}\right\} \quad \text{for all } x \in \mathbb{R}^N \setminus B_\rho(a), \tag{A.4}$$

$$\frac{\partial\phi}{\partial\nu} = M\phi \quad \text{on } \partial\Omega, \tag{A.5}$$

and

$$|\nabla\phi| \leqslant A\phi^\gamma \quad \text{in } \mathbb{R}^N \text{ for some } A > 0. \tag{A.6}$$

Proof. We assume $N \geqslant 2$, the case $N = 1$ being much easier. To construct ϕ, we shall use local flow-coordinates near $\partial\Omega$.

For $x \in \overline{\Omega}$, denote by $\delta(x) = d(x, \partial\Omega)$ the distance to the boundary and, for $\eta > 0$, set $\Omega'_\eta = \{x \in \overline{\Omega};\ \delta(x) \leqslant \eta\}$. Due to the regularity of Ω, there exists $\eta > 0$ such that for all $x \in \Omega'_\eta$, the projection $p(x)$ of x onto $\partial\Omega$ is unique. Moreover, for all $x \in \Omega'_\eta$, we have $x = p(x) - \delta(x)\nu(p(x))$, where ν is the outer normal vector field on $\partial\Omega$. We denote the canonical basis of $\mathbb{R}^N$ by $(e_1, \ldots, e_N)$ and the current point of $\mathbb{R}^N$ by $y = (\tilde{y}, y_N)$, where $\tilde{y} = (y_1, \ldots, y_{N-1})$.

Assuming without loss of generality that $\nu(a) = e_N$ and reducing η if necessary, there exists a local parametrization of $\partial\Omega$ near a, denoted by $X_0 = X_0(\tilde{y})$, defined on $\widetilde{B}_\eta := \{\tilde{y} \in \mathbb{R}^{N-1};\ |\tilde{y}| < \eta\}$, such that $X_0(0) = a$,

$$\frac{\partial X_0}{\partial y_i}(0) = e_i, \quad i = 1, \ldots, N-1, \tag{A.7}$$

and the map

$$X : B_\eta \ni y \mapsto X(y) = X_0(\tilde{y}) + y_N\nu\big(X_0(\tilde{y})\big)$$

(the flow-coordinates) is a C^2-diffeomorphism from B_η onto a neighborhood U of a. Moreover

$$U \cap \Omega = X\big(B_\eta \cap \{y_N < 0\}\big) \quad \text{and} \quad U \cap \partial\Omega = X\big(B_\eta \cap \{y_N = 0\}\big). \tag{A.8}$$

Furthermore, owing to (A.7), by taking $\rho_0 \in (0, \eta)$ sufficiently small, we have

$$B_{\rho/2}(a) \subset X(B_{2\rho/3}) \subset X(B_{3\rho/4}) \subset B_\rho(a) \subset U \quad \text{for all } \rho \in (0, \rho_0]. \tag{A.9}$$

Now fix $\rho \in (0, \rho_0]$. It is easy to check that there exists a C^2-function $f : \mathbb{R} \to [0, 1]$ such that

$$f(s) = \begin{cases} 1 & \text{for } s \leqslant 2\rho/3, \\ 0 & \text{for } s \geqslant 3\rho/4, \end{cases} \tag{A.10}$$

along with

$$\big|f'(s)\big| \leqslant Bf^\gamma(s) \quad \text{for some } B > 0. \tag{A.11}$$

We then define the function $\phi = \phi(x)$ by

$$\phi\big(X(y)\big) = f\big(|y|\big)e^{M(y_N - \rho)}, \quad y \in B_{3\rho/4},$$

and we extend ϕ to be 0 on $\mathbb{R}^N \setminus X(B_{3\rho/4})$. As a consequence of (A.8)–(A.10), we obtain properties (A.3)–(A.4), whereas (A.11) implies (A.6). We compute

$$\left(f'\big(|y|\big)\frac{y_N}{|y|} + Mf\big(|y|\big)\right)e^{My_N} = (\nabla_x\phi)\big(X(y)\big) \cdot \frac{\partial X}{\partial y_N} = (\nabla_x\phi)\big(X(y)\big) \cdot \nu\big(X_0(\tilde{y})\big), \quad y \in B_\eta.$$

Evaluating this expression at $y = (\tilde{y}, 0)$ and using $f'(0) = 0$, we obtain

$$\partial_\nu\phi\big(X_0(\tilde{y})\big) = (\nabla_x\phi)\big(X_0(\tilde{y})\big) \cdot \nu\big(X_0(\tilde{y})\big) = Mf(\tilde{y})f(0) = M\phi\big(X_0(\tilde{y})\big), \quad \tilde{y} \in \widetilde{B}_\eta.$$

Therefore, $\partial_\nu\phi = M\phi$ on $U \cap \partial\Omega$, hence (A.5). □

Proof of Proposition 4.1. We treat only the case $a \in \partial\Omega$, the case $a \in \Omega$ being similar (and slightly easier). We shall adapt the arguments in [15, Theorem 2.1]. Consider the localization of w given by $z = w\phi$, where ϕ is provided by Lemma A.1 with $M = -M_2$ (the choice of γ is unimportant since property (A.6) is not used here). The function z satisfies

$$z_t - \Delta z \leqslant M_1 w^p \phi + g \quad \text{in } \Omega \times (t_0, T),$$

in the weak sense, where

$$g = -2\nabla \cdot (w\nabla\phi) + w\Delta\phi. \tag{A.12}$$

This computation, which is direct in the case of classical subsolutions, can be easily carried out in the case of weak subsolutions by applying the integral formulation in Remark 4.1 with the test-function $\varphi(x,t)$ replaced by $\varphi(x,t)\phi(x)$, and using the divergence theorem. Moreover, owing to (A.5) with $M = -M_2$, we have

$$\partial_\nu z = w\partial_\nu\phi + \phi\partial_\nu w \leqslant -wM_2\phi + \phi M_2 w = 0 \quad \text{on } \partial\Omega \text{ for a.e. } t \in (t_0, T).$$

Let now Z be the solution of the problem

$$\begin{cases} Z_t = \Delta Z + M_1 w^p\phi + g, & x \in \Omega,\ t_0 < t < T, \\ \dfrac{\partial Z}{\partial \nu}(x,t) = 0, & x \in \partial\Omega,\ t_0 < t < T, \\ Z(x,t_0) = z(x,t_0), & x \in \Omega. \end{cases} \tag{A.13}$$

Since this is a linear problem and $M_1 w^p \phi + g \in L^\infty_{loc}(\overline{\Omega} \times [t_0, T))$, it is clear that Z exists and is moreover a strong solution in $\Omega \times (t_0, T)$. We have $z \leqslant Z$ in $\Omega \times (t_0, T)$ by the weak maximum principle (see e.g. [41, Proposition 52.13], whose proof can be easily adapted to the case of Neumann boundary conditions).

By the variation-of-constants formula, we have

$$Z(t) = S(t-t_0)Z(t_0) + M_1 \int_{t_0}^{t} S(t-s)\phi w^p(s)\,ds + \int_{t_0}^{t} S(t-s)g(s)\,ds, \quad t_0 < t < T.$$

Denote $\tilde{w}(t) = w(t)_{|B(a,\rho)}$ and recall (A.12). Using formula (2.3), Proposition 2.3 and the estimate $\|S(t)h\|_\infty \leqslant \|h\|_\infty$, we obtain, for all $t \in (t_0, T)$,

$$\big\|Z(t)\big\|_\infty \leqslant \big\|Z(t_0)\big\|_\infty + M_1 \int_{t_0}^{t} \big\|w^{p-1}Z\big\|_\infty(s)\,ds + C\int_{t_0}^{t} \big(1 + (t-s)^{-1/2}\big)\big\|\tilde{w}(s)\big\|\,ds. \tag{A.14}$$

Using assumption (4.10), it follows that,

$$\big\|Z(t)\big\|_\infty \leqslant \big\|Z(t_0)\big\|_\infty + M_1\varepsilon^{p-1} \int_{t_0}^{t} (T-s)^{-1}\|Z\|_\infty(s)\,ds + C\varepsilon\int_{t_0}^{t} \big(1 + (t-s)^{-1/2}\big)(T-s)^{-1/(p-1)}\,ds.$$

Here and in what follows, C is a generic positive constant possibly depending on the solution u. If $p > 3$, we deduce that

$$\big\|Z(t)\big\|_\infty \leqslant C + M_1\varepsilon^{p-1} \int_{t_0}^{t} (T-s)^{-1}\|Z\|_\infty(s)\,ds, \quad t_0 < t < T.$$

Taking ε so small that $M_1\varepsilon^{p-1} < 1/(p+1)$ it follows from Gronwall's lemma that, for all $t \in (t_0, T)$, $\|z(t)\|_\infty \leqslant \|Z(t)\|_\infty \leqslant C(T-t)^{-1/(p+1)}$, hence $w(x,t) \leqslant C(T-t)^{-1/(p+1)}$ in $\overline{\Omega} \cap B(a, \rho/2)$, due to (A.5). Now let $\check{\phi}$ be the cut-off function given by Lemma A.1 with $\rho/2$ instead of ρ, and define $\check{Z}$, $\check{z}$ correspondingly. We deduce from (A.14) applied to $\check{Z}$ that

$$\big\|\check{Z}(t)\big\|_\infty \leqslant \big\|\check{Z}(t_0)\big\|_\infty + C\int_{t_0}^{t} (T-s)^{-p/(p+1)}\,ds + C\varepsilon\int_{t_0}^{t} \big(1 + (t-s)^{-1/2}\big)(T-s)^{-1/(p+1)}\,ds \leqslant C.$$

It follows that w is bounded near (a, T), hence a is not a blow-up point.

The case $1 < p \leqslant 3$ follows by adding a bootstrap argument to the above procedure – see the proof of [15, Theorem 2.1] for details. □

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