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On the "viscous incompressible fluid + rigid body" system with Navier conditions

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Abstract

In this paper we consider the motion of a rigid body in a viscous incompressible fluid when some Navier slip conditions are prescribed on the body's boundary. The whole system "viscous incompressible fluid + rigid body" is assumed to occupy the full space $\mathbb{R}^3$. We start by proving the existence of global weak solutions to the Cauchy problem. Then, we exhibit several properties of these solutions. First, we show that the added-mass effect can be computed which yields better-than-expected regularity (in time) of the solid velocity-field. More precisely we prove that the solid translation and rotation velocities are in the Sobolev space H^1. Second, we show that the case with the body fixed can be thought as the limit of infinite inertia of this system, that is when the solid density is multiplied by a factor converging to $+\infty$. Finally we prove the convergence in the energy space of weak solutions "à la Leray" to smooth solutions of the system "inviscid incompressible fluid + rigid body" as the viscosity goes to zero, till the lifetime T of the smooth solution of the inviscid system. Moreover we show that the rate of convergence is optimal with respect to the viscosity and that the solid translation and rotation velocities converge in $H^1(0,T)$.

1. Introduction

Recently several efforts have been made to establish a Cauchy theory for various models involving a fluid and an immersed structure. In particular in the case where the structure is a rigid body and where the fluid is incompressible, there now exists a quite satisfactory range of results, at least in view of what is known in the case of a fluid alone; we may cite, among others, [35,43,30,29,44,46,54,25,28,26,29,27] for the case of inviscid fluid and [31,34,48,14,15, 8,47,17–19] for the case of viscous fluid.

In this paper we deal with the issue of the inviscid limit for the system "viscous incompressible fluid + rigid body", which involves an immersed rigid body moving into a viscous incompressible fluid driven by the Navier–Stokes equations.

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Formally dropping the viscosity in the equations of the system yields the system "inviscid incompressible fluid + rigid body", which involves an immersed rigid body moving into an inviscid incompressible fluid driven by the Euler equations.

This paper aims at giving a justification of this formal procedure.

It is expected that the issue of justifying the inviscid limit is at least as difficult in the case of a moving body as in the case of a fixed body. Indeed the case of a fixed body can be seen as the limit case where the body's inertia becomes infinite, cf. Sections 3.5 and 4.4.

This is quite a bad news since the inviscid limit is already quite intricate in the case of a fixed boundary, because of the boundary layers phenomenon. Moreover it is not clear a priori if it is possible to pass to the limit in the body's dynamics, with such singular variations in the neighborhood.

These boundary layers issues are particularly involved in the case where one prescribes the no-slip condition on a fixed fluid–solid interface. In particular, it is not known at the time of writing if there is convergence in the energy space. A longstanding approach in this domain is Prandtl's theory, but this theory fails to model flows with too small viscosity in general, see for instance [9,2,16,32,21,33]. However a necessary and sufficient condition for the convergence to the Euler equations in the energy space has been given by Kato in [38]; it states that the energy dissipation rate of the viscous flows in a boundary strip of width proportional to the viscosity vanishes. Even if little is known about whether or not this condition is verified for a general given flow, this result gives a nice insight of the scale for which the description of the flow is necessary to understand the inviscid limit.

In the paper [50] the second author proved an extension of Kato's result in the case of the system "viscous incompressible fluid + rigid body" with the no-slip condition on the fluid–solid interface: the convergence to the system "inviscid incompressible fluid + rigid body" holds true in the energy space if and only if the energy dissipation rate of the viscous flows in a boundary's neighborhood of width proportional to the viscosity vanishes. As in the case of a fixed boundary it is not clear how to check this condition but this result seems to indicate that the issue of the inviscid limit in the case of a moving body is maybe not much harder than the one in the case of a fixed body.

In this paper we will prescribe some Navier conditions on the fluid–solid interface, which encode that the fluid slips with some friction on this boundary. In the case of a fixed boundary it is by now well understood that the issue of the inviscid limit is simplified compared to the no-slip conditions, at least when the friction coefficient is not too big; in particular the convergence holds true in the energy space (cf. for instance [1,3,7,36,37,39,45]). We prove here a similar result in the case of a moving body.

In fact we even prove a better convergence of the body's dynamics (with respect to the fluid's dynamics), i.e. a convergence in the Sobolev space H^1. This surprising result uses a well-known phenomenon in the theory of the systems involving an incompressible flow and a structure, namely the added-mass phenomenon, for which we refer for instance to [6,20], and which can be computed in the present case of the Navier conditions.

Let us say here for sake of clarity that we will consider the case of a physical space of three dimensions and we will assume that the system occupies the whole of $\mathbb{R}^3$ to avoid the extra difficulties which would be implied by an exterior boundary.

After finishing this paper we became aware of the work [22] by Gérard-Varet and Hillairet, which established the existence of weak solutions to the "viscous incompressible fluid + rigid body" system with Navier slip conditions in the case where the whole system occupies a bounded domain of $\mathbb{R}^3$, rather than the full space $\mathbb{R}^3$, up to collision. It would be therefore interesting to look for some extensions of the properties exhibited here in such a case.

The paper is organized as follows.

- In Section 2 we introduce the system "viscous incompressible fluid + rigid body" with Navier conditions.
- In Section 3 we establish the existence of an appropriate notion of weak solutions "à la Leray" of this system, after a change of variables. We will also establish a regularity property of the body's dynamics and we will discuss the infinite inertia limit for this system.
- In Section 4 we recall a result of [50] which establishes the existence of smooth local-in-time solutions of the inviscid system and we also discuss the infinite inertia limit.
- In Section 5 we state the main result of this paper about the convergence of the system "viscous incompressible fluid + rigid body" to the system "inviscid incompressible fluid + rigid body" as the viscosity goes to zero.
- Finally we give the proof of this result in Section 6.

2. The system "viscous incompressible fluid + rigid body" with Navier conditions

We consider a rigid body initially occupying a closed, bounded, connected and simply connected subset $\mathcal{S}_0 \subset \mathbb{R}^3$ with smooth boundary. It rigidly moves so that at time t it occupies an isometric domain denoted by $\mathcal{S}(t)$. More precisely if we denote by $h(t)$ the position of the center of mass of the body at time t, then there exists a rotation matrix $Q(t) \in SO(3)$, such that the position $\eta(t,x) \in \mathcal{S}(t)$ at the time t of the point fixed to the body with an initial position x is

$$\eta(t,x) := h(t) + Q(t)\big(x - h(0)\big).$$

Of course this yields that $Q(0) = \mathrm{Id}_3$.

Moreover since $Q^T Q'(t)$ is skew symmetric there exists only one $r(t)$ in $\mathbb{R}^3$ such that for any $x \in \mathbb{R}^3$,

$$Q^T Q'(t)x = r(t) \wedge x.$$

Accordingly, the solid velocity is given by

$$U_{\mathcal{S}}(t,x) := h'(t) + R(t) \wedge \big(x - h(t)\big) \quad \text{with } R(t) := Q(t)r(t).$$

Given a positive function $\rho_{\mathcal{S}_0}$, say in $L^\infty(\mathcal{S}_0;\mathbb{R})$, describing the density in the solid initially: the solid mass $m > 0$, the initial position h_0 of the center of mass, and the initial value of the inertial matrix $\mathcal{J}_0$ can be computed by it first moments:

$$m := \int_{\mathcal{S}_0} \rho_{\mathcal{S}_0}(x)\,dx > 0, \tag{1}$$

$$mh_0 := \int_{\mathcal{S}_0} x\rho_{\mathcal{S}_0}(x)\,dx, \tag{2}$$

$$\mathcal{J}_0 := \int_{\mathcal{S}_0} \rho_{\mathcal{S}_0}(x)\big(|x-h_0|^2\,\mathrm{Id}_3 - (x-h_0)\otimes(x-h_0)\big)\,dx. \tag{3}$$

At time $t > 0$, the density in the solid is given, for $x \in \mathcal{S}(t)$, by

$$\rho_{\mathcal{S}}(t,x) := \rho_{\mathcal{S}_0}\big(\eta(t,x)^{-1}(x)\big),$$

where $\eta(t,x)^{-1}$ denotes the inverse at time t of the diffeomorphism $x \mapsto \eta(t,x)$; so that, of course, the mass is preserved:

$$m = \int_{\mathcal{S}(t)} \rho_{\mathcal{S}}(t,x)\,dx.$$

Moreover, the position of the center of mass $h(t)$ and the inertial matrix $\mathcal{J}(t)$ are given by

$$mh(t) := \int_{\mathcal{S}(t)} x\rho_{\mathcal{S}}(t,x)\,dx,$$

$$\mathcal{J}(t) := \int_{\mathcal{S}(t)} \rho_{\mathcal{S}}(t,x)\big(\big|x-h(t)\big|^2\,\mathrm{Id}_3 - \big(x-h(t)\big)\otimes\big(x-h(t)\big)\big)\,dx,$$

so that $\mathcal{J}(t)$ is symmetric positive definite and satisfies Sylvester's law:

$$\mathcal{J}(t) = Q(t)\mathcal{J}_0 Q^T(t).$$

Let us assume that in the rest of the space, that is, in the open set $\mathcal{F}(t) := \mathbb{R}^3 \setminus \mathcal{S}(t)$, there evolves a viscous incompressible fluid. We denote correspondingly $\mathcal{F}_0 := \mathbb{R}^3 \setminus \mathcal{S}_0$ the initial fluid domain.

The complete system driving the dynamics reads

$$\frac{\partial U}{\partial t} + (U \cdot \nabla)U + \nabla P = \nu \Delta U \quad \text{for } x \in \mathcal{F}(t), \tag{4}$$

$$\operatorname{div} U = 0 \quad \text{for } x \in \mathcal{F}(t), \tag{5}$$

$$U \cdot n = U_{\mathcal{S}} \cdot n \quad \text{for } x \in \partial\mathcal{S}(t), \tag{6}$$

$$\big(D(U)n\big) \wedge n = -\alpha(U - U_{\mathcal{S}}) \wedge n \quad \text{for } x \in \partial\mathcal{S}(t), \tag{7}$$

$$mh''(t) = -\int_{\partial\mathcal{S}(t)} \Sigma n \, ds, \tag{8}$$

$$(\mathcal{J}R)'(t) = -\int_{\partial\mathcal{S}(t)} \big(x - h(t)\big) \wedge \Sigma n \, ds, \tag{9}$$

$$U|_{t=0} = U_0, \tag{10}$$

$$h(0) = 0, \qquad h'(0) = \ell_0, \qquad R(0) = r_0. \tag{11}$$

Here U and P denote the fluid velocity and pressure, which are defined on $\mathcal{F}(t)$ for each t, and $\nu > 0$ is the fluid viscosity. The fluid is supposed to be homogeneous of density 1, to simplify the notations and without any loss of generality. The Cauchy stress tensor is defined by

$$\Sigma := -P \operatorname{Id}_3 + 2\nu D(U),$$

where $D(U)$ is the deformation tensor

$$D(U) := \left[\frac{1}{2}(\partial_j U_i + \partial_i U_j)\right]_{1 \leqslant i,j \leqslant 3}.$$

Above n denotes the unit outward normal on the boundary of the fluid domain, ds denotes the integration element on this boundary and $\alpha \geqslant 0$ is a material constant (the friction coefficient). Let us precise that we choose here to consider the case where α is constant but it will be possible to consider the more general case where α depends smoothly on t, x as well with only a few modifications.

Observe that, without loss of generality, we have assumed that $h(0) = 0$ which means that the body is centered at the origin at the initial time $t = 0$.

In the sequel the integrals over open subsets of $\mathbb{R}^3$ will always be taken with respect to the Lebesgue measure dx and the integrals over hypersurfaces of $\mathbb{R}^3$ will always be taken with respect to the surface measure.

Eqs. (4)–(5) are the incompressible Navier–Stokes equations.

Eqs. (8) and (9) are the Newton's balance laws for linear and angular momenta: the fluid acts on the body through pressure forces.

Eqs. (6)–(7) are referred to as the Navier conditions and encode that the body's boundary is impermeable and that the fluid slips with some friction on this boundary.

This condition was introduced phenomenologically by Navier in 1823, cf. [42]. Let us mention some recent results about the derivation of such a condition, on the one hand from kinetic models (derivation from the Boltzmann equation with accommodation boundary conditions) see [10,40,1], and on the other hand from homogenization of rough boundaries [13,23].

3. Weak solutions "à la Leray"

In this section we start by use the change of variables introduced by Serre in [48] in order to fix the fluid domain. It is fair to point out that this change of variable is here particularly simple as there is no exterior boundary. Then we establish the existence of an appropriate notion of weak solutions "à la Leray" of this system. We will observe that the solid velocity benefits from extra regularity with respect to what is expected from the energy estimate. We will also discuss the infinite inertia limit.

3.1. A change of variables

In order to write the equations of the fluid in a fixed domain, we are going to use the following changes of variables:

$$\ell(t) := Q(t)^T h'(t), \qquad u(t,x) := Q(t)^T U\big(t, Q(t)x + h(t)\big), \qquad p(t,x) := P\big(t, Q(t)x + h(t)\big),$$

and we introduce

$$\sigma := -p\,\mathrm{Id}_3 + 2\nu D(u), \quad \text{where } D(u) := \left[\frac{1}{2}(\partial_j u_i + \partial_i u_j)\right]_{1\leqslant i,j\leqslant 3}.$$

Therefore the system (4)–(11) now reads

$$\frac{\partial u}{\partial t} + (u - u_{\mathcal{S}})\cdot\nabla u + r\wedge u + \nabla p = \nu\Delta u \quad \text{for } x\in\mathcal{F}_0, \tag{12}$$

$$\operatorname{div} u = 0 \quad \text{for } x\in\mathcal{F}_0, \tag{13}$$

$$u\cdot n = u_{\mathcal{S}}\cdot n \quad \text{for } x\in\partial\mathcal{S}_0, \tag{14}$$

$$\big(D(u)n\big)\wedge n = -\alpha(u-u_{\mathcal{S}})\wedge n \quad \text{for } x\in\partial\mathcal{S}_0, \tag{15}$$

$$m\ell' = -\int_{\partial\mathcal{S}_0}\sigma n\,ds + m\ell\wedge r, \tag{16}$$

$$\mathcal{J}_0 r' = -\int_{\partial\mathcal{S}_0} x\wedge\sigma n\,ds + (\mathcal{J}_0 r)\wedge r, \tag{17}$$

$$u|_{t=0} = u_0, \tag{18}$$

$$h(0) = 0, \qquad h'(0) = \ell_0, \qquad r(0) = r_0, \tag{19}$$

with

$$u_{\mathcal{S}}(t,x) := \ell(t) + r(t)\wedge x. \tag{20}$$

3.2. A weak formulation

With the purpose of writing a weak formulation of the system (12)–(19) we introduce the following space

$$\mathcal{H} := \big\{\phi\in L^2(\mathbb{R}^3)\ \big|\ \operatorname{div}\phi = 0 \text{ in } \mathbb{R}^3 \text{ and } D(\phi) = 0 \text{ in } \mathcal{S}_0\big\}.$$

According to Lemma 1.1 in [51, p. 18], for all $\phi\in\mathcal{H}$, there exist $\ell_\phi\in\mathbb{R}^3$ and $r_\phi\in\mathbb{R}^3$ such that for any $x\in\mathcal{S}_0$, $\phi(x) = \ell_\phi + r_\phi\wedge x$. Therefore we extend the initial data u_0 by setting $u_0 := \ell_0 + r_0\wedge x$ for $x\in\mathcal{S}_0$.

Conversely, when $\phi\in\mathcal{H}$, we denote $\phi_{\mathcal{S}}$ its restriction to $\mathcal{S}_0$.

Let us give here a result which will be useful in the sequel.

Lemma 1. *For any $u, v\in\mathcal{H}$ with $u|_{\mathcal{F}_0}\in H^2$ and $v|_{\mathcal{F}_0}\in H^1$,*

$$\int_{\mathcal{F}_0}\Delta u\cdot v = -2\int_{\mathcal{F}_0} D(u):D(v) + 2\ell_v\cdot\int_{\partial\mathcal{S}_0} D(u)n\,ds + 2r_v\cdot\int_{\partial\mathcal{S}_0} x\wedge D(u)n\,ds$$
$$+ 2\int_{\partial\mathcal{S}_0}\big(\big(D(u)n\big)\wedge n\big)\cdot\big((v - v_{\mathcal{S}})\wedge n\big).$$

Proof. We have

$$\int_{\mathcal{F}_0}\Delta u\cdot v = 2\int_{\partial\mathcal{S}_0}\big(D(u)v\big)\cdot n - 2\int_{\mathcal{F}_0} D(u):D(v) = 2\int_{\partial\mathcal{S}_0}\big(D(u)n\big)\cdot v - 2\int_{\mathcal{F}_0} D(u):D(v),$$

since $D(u)$ is symmetric. Moreover

$$\int_{\partial\mathcal{S}_0} \big(D(u)n\big)\cdot v = \int_{\partial\mathcal{S}_0} \big(\big(D(u)n\big)\cdot n\big)(v\cdot n) + \int_{\partial\mathcal{S}_0} \big(\big(D(u)n\big)\wedge n\big)\cdot(v\wedge n).$$

But

$$\begin{aligned}
\int_{\partial\mathcal{S}_0} \big(\big(D(u)n\big)\cdot n\big)(v\cdot n) &= \int_{\partial\mathcal{S}_0} \big(\big(D(u)n\big)\cdot n\big)(v_{\mathcal{S}}\cdot n) \\
&= \int_{\partial\mathcal{S}_0} \big(\big(\big(D(u)n\big)\cdot n\big)n\big)\cdot v_{\mathcal{S}} \\
&= \int_{\partial\mathcal{S}_0} \big(D(u)n - \big(D(u)n\big)_{\tan}\big)\cdot v_{\mathcal{S}} \\
&= \ell_v\cdot\int_{\partial\mathcal{S}_0} D(u)n\,ds + r_v\cdot\int_{\partial\mathcal{S}_0} x\wedge D(u)n\,ds - \int_{\partial\mathcal{S}_0} \big(D(u)n\big)_{\tan}\cdot v_{\mathcal{S}} \\
&= \ell_v\cdot\int_{\partial\mathcal{S}_0} D(u)n\,ds + r_v\cdot\int_{\partial\mathcal{S}_0} x\wedge D(u)n\,ds - \int_{\partial\mathcal{S}_0} \big(D(u)n\wedge n\big)\cdot(v_{\mathcal{S}}\wedge n).
\end{aligned}$$

Above we use the index "tan" to denote the tangential part of a vector field defined on $\partial\mathcal{S}_0$. Gathering the previous identities yields the result. □

Now we endow the space $L^2(\mathbb{R}^3)$ with the following inner product, which is equivalent to the usual one,

$$(\phi,\psi)_{\mathcal{H}} := \int_{\mathcal{F}_0} \phi\cdot\psi\,dx + \int_{\mathcal{S}_0} \rho_{S_0}\phi\cdot\psi\,dx.$$

When ϕ, ψ are in $\mathcal{H}$ we obtain:

$$(\phi,\psi)_{\mathcal{H}} = \int_{\mathcal{F}_0} \phi\cdot\psi\,dx + m\ell_\phi\cdot\ell_\psi + \mathcal{J}_0 r_\phi\cdot r_\psi, \tag{21}$$

by using (1)–(2)–(3). The spaces $L^2(\mathbb{R}^3)$ and $\mathcal{H}$ are clearly Hilbert spaces for the scalar product $(\cdot,\cdot)_{\mathcal{H}}$.

Proposition 1. *A smooth solution u of* (12)–(19) *satisfies the following: for any $v\in C^\infty([0,T];\mathcal{H})$ such that its restriction $v|_{\overline{\mathcal{F}_0}}$ is in $C^\infty([0,T];C_c^\infty(\overline{\mathcal{F}_0}))$, for all $t\in[0,T]$,*

$$(u,v)_{\mathcal{H}}(t) - (u_0, v|_{t=0})_{\mathcal{H}} = \int_0^t \big[(u,\partial_t v)_{\mathcal{H}} + 2\nu a(u,v) + b(u,u,v)\big], \tag{22}$$

where

$$a(u,v) := -\alpha\int_{\partial\mathcal{S}_0}(u-u_{\mathcal{S}})\cdot(v-v_{\mathcal{S}}) - \int_{\mathcal{F}_0} D(u):D(v),$$

$$b(u,v,w) := m\det(r_u,\ell_v,\ell_w) + \det(\mathcal{J}_0 r_u, r_v, r_w) + \int_{\mathcal{F}_0}\big(\big[(u-u_{\mathcal{S}})\cdot\nabla w\big]\cdot v - \det(r_u,v,w)\big).$$

Moreover this solution u satisfies the following energy equality: for almost any $t\in[0,T]$,

$$\frac{1}{2}\big\|u(t,\cdot)\big\|_{\mathcal{H}}^2 + 2\nu\int_{(0,t)\times\mathcal{F}_0}\big|D(u)\big|^2 + 2\alpha\nu\int_0^t\int_{\partial\mathcal{S}_0}|u-u_{\mathcal{S}}|^2 = \frac{1}{2}\|u_0\|_{\mathcal{H}}^2. \tag{23}$$

In the sequel we will consider several asymptotics with respect to the parameters α, ν, m and $\mathcal{J}_0$. Let us therefore stress here that the forms a and b above depend respectively on α, $\mathcal{S}_0$ and on m, $\mathcal{J}_0$, $\mathcal{S}_0$.

Proof of Proposition 1. Let u be a smooth solution of (12)–(19) on $[0,T]$ and $v \in C^\infty([0,T];\mathcal{H})$ such that $v|_{\overline{\mathcal{F}_0}} \in C^\infty([0,T]; C_c^\infty(\overline{\mathcal{F}_0}))$. We first observe that the result of Proposition 1 will follow, by an integration by parts in time, from the following claim: for any $t \in [0,T]$,

$$(\partial_t u, v)_{\mathcal{H}} = 2\nu a(u,v) + b(u,u,v). \tag{24}$$

To prove the claim, we multiply Eq. (12) by v and integrate over $\mathcal{F}_0$:

$$\int_{\mathcal{F}_0} \frac{\partial u}{\partial t}\cdot v + \int_{\mathcal{F}_0} \big[\big((u-u_{\mathcal{S}})\cdot\nabla\big)u\big]\cdot v + \int_{\mathcal{F}_0} \big(r(t)\wedge u\big)\cdot v + \int_{\mathcal{F}_0} \nabla p\cdot v = \int_{\mathcal{F}_0} \nu\Delta u\cdot v.$$

We then use some integrations by parts, taking into account (13) and (14), to get

$$\int_{\mathcal{F}_0} \big[\big((u-u_{\mathcal{S}})\cdot\nabla\big)u\big]\cdot v = -\int_{\mathcal{F}_0} u\cdot\big((u-u_{\mathcal{S}})\cdot\nabla v\big),$$

$$\int_{\mathcal{F}_0} \big(r(t)\wedge u\big)\cdot v = \int_{\mathcal{F}_0} \det(r,u,v),$$

$$\int_{\mathcal{F}_0} \nabla p\cdot v = \int_{\partial\mathcal{S}_0} pn\cdot v.$$

Next, we observe that

$$\int_{\partial\mathcal{S}_0} pn\cdot v = \ell_v\cdot\int_{\partial\mathcal{S}_0} pn\,ds + r_v\cdot\int_{\partial\mathcal{S}_0} x\wedge pn\,ds.$$

Therefore, using Lemma 1, the Navier conditions and (16)–(17), we obtain

$$\begin{aligned}\int_{\partial\mathcal{S}_0} pn\cdot v - \int_{\mathcal{F}_0} \nu\Delta u\cdot v &= -\ell_v\cdot\int_{\partial\mathcal{S}_0} \sigma n\,ds - r_v\cdot\int_{\partial\mathcal{S}_0} x\wedge\sigma n\,ds \\ &\quad + 2\alpha\nu\int_{\partial\mathcal{S}_0} (u-u_{\mathcal{S}})\cdot(v-v_{\mathcal{S}}) + 2\nu\int_{\mathcal{F}_0} D(u):D(v) \\ &= m\ell_v\cdot\ell' + \mathcal{J}_0 r_v\cdot r' - \det(m\ell, r, \ell_v) - \det(\mathcal{J}_0 r, r, r_v) \\ &\quad + 2\alpha\nu\int_{\partial\mathcal{S}_0} (u-u_{\mathcal{S}})\cdot(v-v_{\mathcal{S}}) + 2\nu\int_{\mathcal{F}_0} D(u):D(v).\end{aligned}$$

Gathering all these equalities yields (24). Finally the energy equality (23) follows from (22) by specifying the test function as $v = u$. □

Let us recall that according to Korn's inequality, see for instance [41, Theorem 10.2], the energy equality (23) yields that $u \in L^2(0,T;\underline{\mathcal{V}})$, where $\underline{\mathcal{V}}$ is given by

$$\underline{\mathcal{V}} := \left\{\phi\in\mathcal{H}\ \Big|\ \int_{\mathcal{F}_0} \big|\nabla\phi(y)\big|^2\,dy < +\infty\right\} \quad \text{with norm } \|\phi\|_{\underline{\mathcal{V}}} := \|\phi\|_{\mathcal{H}} + \|\nabla\phi\|_{L^2(\mathcal{F}_0,dy)}.$$

We introduce now the concept of weak solutions "à la Leray".

Definition 1. We say that

$$u \in C_w\big([0,T];\mathcal{H}\big)\cap L^2(0,T;\underline{\mathcal{V}})$$

is a weak solution of the system (12)–(19) if for all $v \in C^\infty([0,T];\mathcal{H})$ such that $v|_{\overline{\mathcal{F}_0}} \in C^\infty([0,T];C_c^\infty(\overline{\mathcal{F}_0}))$ and for all $t \in [0,T]$, (22) holds true.

Let us remark that a standard density argument allows us to take less smooth test vector fields v in the above weak formulation. More precisely, to enlarge the space of the test functions, we introduce the space

$$\mathcal{V} := \left\{ \phi \in \mathcal{H} \;\middle|\; \int_{\mathcal{F}_0} \left|\nabla\phi(y)\right|^2 \left(1+|y|^2\right) dy < +\infty \right\},$$

endowed with the norm

$$\|\phi\|_{\mathcal{V}} := \|\phi\|_{\mathcal{H}} + \|\nabla\phi\|_{L^2(\mathcal{F}_0,(1+|y|^2)^{\frac{1}{2}}\,dy)}.$$

It is worth to notice from now on that b is a trilinear continuous form on $\underline{\mathcal{V}} \times \underline{\mathcal{V}} \times \mathcal{V}$: there exists a constant $C > 0$ such that for any $(u,v,w) \in \underline{\mathcal{V}} \times \underline{\mathcal{V}} \times \mathcal{V}$,

$$\left|b(u,v,w)\right| \leqslant C\|u\|_{\underline{\mathcal{V}}}\|v\|_{\underline{\mathcal{V}}}\|w\|_{\mathcal{V}}. \tag{25}$$

This follows easily from Hölder's inequality and the following interpolation inequality

$$\|v\|_{L^4(\mathcal{F}_0)} \leqslant \sqrt{2}\|v\|_{L^2(\mathcal{F}_0)}^{\frac{1}{4}}\|\nabla v\|_{L^2(\mathcal{F}_0)}^{\frac{3}{4}}. \tag{26}$$

Observe in particular that the weight in the definition of $\mathcal{V}$ allows to handle the rotation part of $u_{\mathcal{S}}$.

Moreover the trilinear form b satisfies the following crucial property

$$(u,v) \in \underline{\mathcal{V}} \times \mathcal{V} \quad \text{implies} \quad b(u,v,v) = 0. \tag{27}$$

On the other hand, for any u, v in $\underline{\mathcal{V}}$,

$$\left|a(u,v)\right| \leqslant C\|u\|_{\underline{\mathcal{V}}}\|v\|_{\underline{\mathcal{V}}}. \tag{28}$$

In fact, to deal with the boundary integral, we introduce a smooth cut-off function χ defined on $\overline{\mathcal{F}_0}$ such that $\chi = 1$ in Γ_c and $\chi = 0$ in $\mathcal{F}_0 \setminus \Gamma_{2c}$, where

$$\Gamma_c := \left\{ x \in \mathcal{F}_0 \;\middle|\; d(x) < c \right\} \quad \text{with } d(x) := \operatorname{dist}(x,\partial\mathcal{S}_0).$$

Let us denote

$$\psi_{\mathcal{S}}^u(t,x) := \frac{1}{2}\left(\ell_u(t) \wedge x - r_u(t)|x|^2\right) \quad \text{and} \quad \tilde{u}_{\mathcal{S}} := \operatorname{curl}\left(\chi\psi_{\mathcal{S}}^u\right),$$

and let us define similarly $\tilde{v}_{\mathcal{S}}$.

Thus, $\tilde{u}_{\mathcal{S}}$ and $\tilde{v}_{\mathcal{S}}$ are equal respectively, to $u_{\mathcal{S}}$ and $v_{\mathcal{S}}$ near $\partial\mathcal{S}_0$, vanish away $\mathcal{S}_0$, and are divergence free. Moreover, $\|\tilde{u}_{\mathcal{S}}\|_{H^1(\mathcal{F}_0)} \leqslant C(\|\ell_u\| + \|r_u\|)$, and a similar estimate holds for $\tilde{v}_{\mathcal{S}}$. Then, we apply the Hölder inequality and the trace theorem, to arrive at

$$\begin{aligned}\left|\int_{\partial\mathcal{S}_0}(u-u_{\mathcal{S}})\cdot(v-v_{\mathcal{S}})\right| &= \left|\int_{\partial\mathcal{S}_0}(u-\tilde{u}_{\mathcal{S}})\cdot(v-\tilde{v}_{\mathcal{S}})\right| \\ &\leqslant C\|u-\tilde{u}_{\mathcal{S}}\|_{H^1(\mathcal{F}_0)}\|v-\tilde{v}_{\mathcal{S}}\|_{H^1(\mathcal{F}_0)} \\ &\leqslant C\left(\|u\|_{H^1(\mathcal{F}_0)} + \|\ell_u\| + \|r_u\|\right)\left(\|v\|_{H^1(\mathcal{F}_0)} + \|\ell_v\| + \|r_v\|\right),\end{aligned}$$

so that, (28) follows.

These previous arguments allow us to take less smooth test vector fields v in the weak formulation (22), for instance, belonging to $H^1(0,T;\mathcal{H}) \cap L^4(0,T;\mathcal{V})$.

Finally let us mention that to a weak solution we may associate a pressure such that the equations are satisfied in the distribution sense, and prove that a regular weak solution is a solution in the classical sense; following for instance [48, Section III] and [52] with a few straightforward adaptations.

3.3. An extension of Leray's theorem

The following result establishes the existence of global weak solutions of the system (12)–(19).

Theorem 1. *Let be given $u_0 \in \mathcal{H}$ and $T > 0$. Then there exists a weak solution u of (12)–(19) in $C_w([0,T];\mathcal{H}) \cap L^2(0,T;\underline{\mathcal{V}})$. Moreover this solution satisfies the following energy inequality: for almost any $t \in [0,T]$,*

$$\frac{1}{2}\|u(t,\cdot)\|^2_{\mathcal{H}} + 2\nu \int\limits_{(0,t)\times\mathcal{F}_0} |D(u)|^2 + 2\alpha\nu \int\limits_0^t \int\limits_{\partial\mathcal{S}_0} |u-u_{\mathcal{S}}|^2 \leqslant \frac{1}{2}\|u_0\|^2_{\mathcal{H}}. \tag{29}$$

Theorem 1 is the counterpart of Theorem 4.5 of [48] for the Navier conditions instead of the no-slip conditions.

Proof of Theorem 1. We will proceed in several steps. In particular because the space of test functions $\mathcal{V}$ involves a weight which makes it smaller than the space $\underline{\mathcal{V}}$ involved by the energy estimates, we will first introduce a truncation of the solid velocity far from the solid. This strategy was already used in [44] in a slightly different context.

Truncation. Let $R_0 > 0$ be such that $\mathcal{S}_0 \subset B(0, \frac{R_0}{2})$. For $R > R_0$, let $\chi_R : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field such that $r \wedge \chi_R$ is divergence free, $\chi_R(x) = x$ for $x \in B(0,R)$ and satisfying $\|\chi_R\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)} \leqslant R$. Indeed one may define for example χ_R by the formula $\chi_R(x) = \frac{R}{|x|}x$ for $x \in \mathbb{R}^3 \setminus B(0,R)$.

Observe in particular that for any $r \in \mathbb{R}^3$, for any $w \in \mathcal{V}$,

$$(r \wedge \chi_R)\cdot\nabla w \to (r \wedge x)\cdot\nabla w \quad \text{in } L^2(\mathbb{R}^3), \text{ when } R \to +\infty, \tag{30}$$

by Lebesgue's dominated convergence theorem.

Then we truncate the solid velocity $u_{\mathcal{S}}$ defined in (20) by $u_{\mathcal{S},R}(t,x) := \ell(t) + r(t) \wedge \chi_R(x)$, and we introduce the form

$$b_R(u,v,w) := m \det(r_u, \ell_v, \ell_w) + \det(\mathcal{J}_0 r_u, r_v, r_w) + \int\limits_{\mathcal{F}_0} \big(\big[(u-u_{\mathcal{S},R})\cdot\nabla w\big]\cdot v - \det(r_u, v, w)\big).$$

The interest of such a truncation b_R of b is that it is now well-defined and trilinear on $\underline{\mathcal{V}} \times \underline{\mathcal{V}} \times \underline{\mathcal{V}}$ (note that the third argument is here taken not only in $\mathcal{V}$ but in the larger space $\underline{\mathcal{V}}$) and continuous in the sense that there exists a constant $C > 0$ such that for any $(u,v,w) \in \underline{\mathcal{V}} \times \underline{\mathcal{V}} \times \underline{\mathcal{V}}$,

$$|b_R(u,v,w)| \leqslant C\|u\|_{\underline{\mathcal{V}}}\|v\|_{\underline{\mathcal{V}}}\|w\|_{\underline{\mathcal{V}}}. \tag{31}$$

Of course the constant C in (31) depends on R. However, when restricting b_R to $\underline{\mathcal{V}} \times \underline{\mathcal{V}} \times \mathcal{V}$, there exists $C > 0$ such that for any $R > R_0$, for any $(u,v,w) \in \underline{\mathcal{V}} \times \underline{\mathcal{V}} \times \mathcal{V}$,

$$|b_R(u,v,w)| \leqslant C\|u\|_{\underline{\mathcal{V}}}\|v\|_{\underline{\mathcal{V}}}\|w\|_{\mathcal{V}}. \tag{32}$$

We also have that there exists $C > 0$ such that for any $R > R_0$, for any $(u,v) \in \underline{\mathcal{V}} \times \mathcal{V}$,

$$|b_R(u,u,v)| \leqslant C\big(\|u\|^2_{L^4(\mathcal{F}_0)} + \|u\|^2_{\mathcal{H}}\big)\|v\|_{\mathcal{V}}. \tag{33}$$

Actually, estimates (32) and (33) are proved by proceeding in the same way as for the proof of (25).

Moreover the cancellation property (27) is still correct:

$$(u,v) \in \underline{\mathcal{V}} \times \underline{\mathcal{V}} \quad \text{implies} \quad b_R(u,v,v) = 0. \tag{34}$$

Finally we deduce from (30) that for any $(u,v,w) \in \underline{\mathcal{V}} \times \underline{\mathcal{V}} \times \mathcal{V}$,

$$b_R(u,v,w) \to b(u,v,w) \quad \text{when } R \to +\infty. \tag{35}$$

Existence for the truncated system. Then, given $u_0 \in \mathcal{H}$ and $T > 0$, there exists u_R in $C_w([0,T];\mathcal{H}) \cap L^2(0,T;\underline{\mathcal{V}})$ satisfying, for any $v \in C^\infty([0,T];\mathcal{H})$ such that $v|_{\overline{\mathcal{F}_0}} \in C^\infty([0,T];C^\infty_c(\overline{\mathcal{F}_0}))$, and for all $t \in [0,T]$,

$$(u_R, v)_{\mathcal{H}}(t) - (u_0, v|_{t=0})_{\mathcal{H}} = \int_0^t \big[(u_R, \partial_t v)_{\mathcal{H}} + 2\nu a(u_R, v) + b_R(u_R, u_R, v)\big]. \tag{36}$$

Moreover u_R verifies for almost any $t \in [0, T]$ the energy inequality (29).

This can be proved with very standard methods, by example considering some Faedo–Galerkin approximations and passing to the limit. Let us therefore sketch a proof of it referring for example to [52] where a comprehensive study of the Leray theorem for the classical case of a fixed boundary is treated.

Let $(w_j)_{j\geqslant 1}$ be a Hilbert basis of $\underline{\mathcal{V}}$. For simplicity, since the set

$$\mathcal{Y} := \big\{\phi \in C_c^\infty(\mathbb{R}^3) \,\big|\, \operatorname{div}\phi = 0 \text{ in } \mathbb{R}^3 \text{ and } D(\phi) = 0 \text{ in } \mathcal{S}_0\big\}$$

is dense in $\underline{\mathcal{V}}$, we take $w_j \in \mathcal{Y}$, for all $j \geqslant 1$.

We define an approximate solution $u_N := u_{N,R}$ (in the sequel we will omit the dependence on R for the sake of clarity) of the form $u_N = \sum_{i=1}^N g_{iN}(t) w_i$ satisfying, for any $j = 1, \dots, N$,

$$(\partial_t u_N, w_j)_{\mathcal{H}} = 2\nu a(u_N, w_j) + b_R(u_N, u_N, w_j), \tag{37}$$

$$u_N|_{t=0} = u_{N0}, \tag{38}$$

where u_{N0} is the orthogonal projection in $\mathcal{H}$ of u_0 onto the space spanned by $w_1, \dots, w_N$. Let us explain why such $(u_N)_N$ do exist. First we introduce the matrices:

$$\mathcal{M}_N := \big[(w_i, w_j)_{\mathcal{H}}\big]_{1\leqslant i,j\leqslant N}, \qquad \mathcal{G}_N := [g_{1N} \quad \dots \quad g_{NN}], \qquad \mathcal{A}_N := \big[a(w_i, w_j)\big]_{1\leqslant i,j\leqslant N},$$

and for any $u, v \in \mathbb{R}^N$, $\mathcal{B}_N(u, v) := [\mathcal{B}_{Nj}(u, v)]_{1\leqslant j\leqslant N}$, where $\mathcal{B}_{Nj}(u, v) := \sum_{1\leqslant i,k\leqslant N} u_i v_k b_R(w_i, w_k, w_j)$. Then Eq. (37) can be recast as the following nonlinear differential system for the functions $(g_{iN})_{1\leqslant i\leqslant N}$:

$$\mathcal{G}_N'(t) = \mathcal{M}_N^{-1}\big(2\nu \mathcal{A}_N \mathcal{G}_N + \mathcal{B}_N(\mathcal{G}_N, \mathcal{G}_N)\big)$$

and the initial condition (38) is equivalent to an initial condition of the form $\mathcal{G}_N(0) = \mathcal{G}_{N,0}$. According to the Cauchy–Lipschitz theorem this system has a maximal solution defined on some time interval $[0, T_N]$ with $T_N > 0$. Moreover if $T_N < T$ then $\|u_N\|_{\mathcal{H}}$ must tend to $+\infty$ as $t \to T_N$.

The following energy estimate shows that this does not happen and therefore $T_N = T$. For any $j = 1, \dots, N$, we multiply (37) by $g_{jN}(t)$ and we sum the resulting identities to obtain, thanks to (34), $\frac{1}{2}\partial_t \|u_N\|_{\mathcal{H}}^2 = 2\nu a(u_N, u_N)$, so that, by integration in time, we have

$$\frac{1}{2}\big\|u_N(t,\cdot)\big\|_{\mathcal{H}}^2 + 2\nu \int_{(0,t)\times\mathbb{R}^3} \big|D(u_N)\big|^2 + 2\alpha\nu \int_0^t \int_{\partial\mathcal{S}_0} |u_N - u_{N,\mathcal{S}}|^2 \leqslant \frac{1}{2}\|u_{N0}\|_{\mathcal{H}}^2 \leqslant \frac{1}{2}\|u_0\|_{\mathcal{H}}^2.$$

In particular, by the Korn inequality, the sequence $(u_N)_N$ is bounded in $L^\infty(0, T; \mathcal{H}) \cap L^2(0, T; \underline{\mathcal{V}})$. Therefore, there exists a subsequence of $(u_N)_N$, relabelled the same, converging weakly-* in $L^\infty(0, T; \mathcal{H})$ and weakly in $L^2(0, T; \underline{\mathcal{V}})$ to $u \in L^\infty(0, T; \mathcal{H}) \cap L^2(0, T; \underline{\mathcal{V}})$, as $N \to +\infty$, which satisfies, for almost any $t \in [0, T]$, the energy inequality (29).

In order to pass to the limit in the nonlinear term in (37), we need a strong convergence. We will closely follow the classical arguments in Chapter 3 of [52]. To this end, we are going to bound a fractional derivative in time of the functions u_N by applying the Fourier transform. We therefore first extend the functions u_N to the whole time line as follows. For any $N > 1$ let us now denote by $\tilde{u}_N$ the function defined from $\mathbb{R}$ to $\mathcal{H}$ which is equal to u_N on $[0, T]$ and by 0 outside. We denote by $\hat{u}_N$ the Fourier transform of $\tilde{u}_N$, defined by $\hat{u}_N(\tau) := \int_{\mathbb{R}} e^{-2i\pi t\tau} \tilde{u}_N(t)\, dt$. Similarly we extend the functions g_{iN} by 0 outside $[0, T]$ and we denote by $\hat{g}_{iN}$ their respective Fourier transform.

According to Theorem 2.2 in [52] it is sufficient to prove that there exists $\gamma > 0$ such that $(|\tau|^\gamma \hat{u}_N(\tau))_N$ is bounded in $L^2(\mathbb{R}; \mathcal{H})$ to deduce that the sequence $(u_N)_N$ is relatively compact in $L^2(0, T; L^2_{\mathrm{loc}}(\mathbb{R}^3))$.

Let us denote, for $t \in \mathbb{R}$, by $\tilde{f}_N(t)$ the linear form on $\underline{\mathcal{V}}$ defined by

$$\big\langle \tilde{f}_N(t), w\big\rangle := 2\nu a\big(u_N(t), w\big) + b_R\big(u_N(t), u_N(t), w\big) \quad \text{for } t \in [0, T] \quad \text{and} \quad \tilde{f}_N(t) := 0 \quad \text{otherwise},$$

so that Eq. (37) becomes

$$(\partial_t \tilde{u}_N, w_j)_{\mathcal{H}} = -(u_{N0}, w_j)_{\mathcal{H}} \delta_0(t) + (u_N|_{t=T}, w_j)_{\mathcal{H}} \delta_T(t) + \langle \tilde{f}_N, w_j\rangle,$$

and then, taking the Fourier transform in time, we get for any $\tau \in \mathbb{R}$,

$$2\pi i\tau(\hat{u}_N, w_j)_{\mathcal{H}} = -(u_{N0}, w_j)_{\mathcal{H}} + (u_N|_{t=T}, w_j)_{\mathcal{H}} e^{-2i\pi T\tau} + \langle \hat{f}_N, w_j\rangle.$$

This yields, multiplying by $\hat{g}_{jN}$ and summing over $1 \leqslant j \leqslant N$, that, for any $\tau \in \mathbb{R}$,

$$2\pi i\tau \big\|\hat{u}_N(\tau)\big\|^2_{\mathcal{H}} = -(u_{N0}, \hat{u}_N)_{\mathcal{H}} + (u_N|_{t=T}, \hat{u}_N)_{\mathcal{H}} e^{-2i\pi T\tau} + \langle \hat{f}_N, \hat{u}_N\rangle.$$

Thanks to (28) and (31), there exists $C > 0$ such that for any $t \in [0, T]$, $\|f_N(t)\|_{\underline{\mathcal{V}}'} \leqslant C(\|u_N(t)\|_{\underline{\mathcal{V}}} + \|u_N(t)\|^2_{\underline{\mathcal{V}}})$. Moreover, for any $\tau \in \mathbb{R}$, $\|\hat{f}_N(\tau)\|_{\underline{\mathcal{V}}'} \leqslant \int_0^T \|f_N(t)\|_{\underline{\mathcal{V}}'}\,dt$. Thus, $(\sup_{\tau\in\mathbb{R}} \|\hat{f}_N(\tau)\|_{\underline{\mathcal{V}}'})$ is bounded, and the initial and final values u_{N0} and $u_N|_{t=T}$ are bounded as well. Therefore there exists $C > 0$ such that $\tau\|\hat{u}_N(\tau)\|^2_{\mathcal{H}} \leqslant C\|\hat{u}_N(\tau)\|_{\underline{\mathcal{V}}}$. Now, we observe that there exists $C > 0$ such that for any $\tau \in \mathbb{R}$, $|\tau|^{\frac{1}{4}} \leqslant C(1 + |\tau|)(1 + |\tau|)^{-\frac{3}{4}}$, to deduce that

$$\begin{aligned}
\int_{\mathbb{R}} |\tau|^{\frac{1}{4}} \big\|\hat{u}_N(\tau)\big\|^2_{\mathcal{H}}\,d\tau &\leqslant C\int_{\mathbb{R}} \frac{1+|\tau|}{1+|\tau|^{\frac{3}{4}}} \big\|\hat{u}_N(\tau)\big\|^2_{\mathcal{H}}\,d\tau \\
&\leqslant C\int_{\mathbb{R}} \frac{1}{1+|\tau|^{\frac{3}{4}}} \big\|\hat{u}_N(\tau)\big\|_{\underline{\mathcal{V}}}\,d\tau + C\int_{\mathbb{R}} \big\|\hat{u}_N(\tau)\big\|^2_{\underline{\mathcal{V}}}\,d\tau \\
&\leqslant C\int_{\mathbb{R}} \big\|\hat{u}_N(\tau)\big\|^2_{\underline{\mathcal{V}}}\,d\tau,
\end{aligned}$$

by the Cauchy–Schwarz inequality. Then it follows from the Parseval identity that $(|\tau|^{\frac{1}{8}}\hat{u}_N(\tau))_N$ is bounded in $L^2(\mathbb{R}; \mathcal{H})$.

Then, we can classically pass to the limit in (37) as $N \to \infty$, and obtain that (36) is satisfied.

Endgame. Since the bounds given by the energy estimate (29) are uniform with respect to $R > R_0$, there exists a subsequence $(u_{R_k})_k$ converging to $u \in C_w([0, T]; \mathcal{H}) \cap L^2(0, T; \underline{\mathcal{V}})$ for the weak (or weak-*) topologies, satisfying (29) for almost any $t \in [0, T]$. This allows to pass to the limit in all the terms involved in (36) except for the trilinear term.

On the other hand it is sufficient to prove for any $v \in C^\infty([0, T]; \mathcal{H})$ such that $v|_{(0,T)\times\overline{\mathcal{F}_0}} \in C^\infty_c((0, T) \times \overline{\mathcal{F}_0})$

$$0 = \int_0^t \big[(u, \partial_t v)_{\mathcal{H}} + 2\nu a(u, v) + b(u, u, v)\big],$$

to deduce, by standard arguments, that u is a weak solution of (12)–(19).

It therefore only remains to prove that there exists a subsequence, still denoted $(u_{R_k})_k$, such that for any $v \in C^\infty([0, T]; \mathcal{H})$ with $v|_{(0,T)\times\overline{\mathcal{F}_0}} \in C^\infty_c((0, T) \times \overline{\mathcal{F}_0})$, as $k \to \infty$,

$$\int_0^t b_{R_k}(u_{R_k}, u_{R_k}, v) \to \int_0^t b(u, u, v). \tag{39}$$

First let us observe that to prove (39) it will be enough to show that the sequence $(u_{R_k})_k$ is relatively compact in $L^2(0, T; L^2_{\text{loc}}(\mathbb{R}^3))$. Indeed this yields that there exists a subsequence, still denoted $(u_{R_k})_k$, converging to u in $L^2(0, T; L^2_{\text{loc}}(\mathbb{R}^3))$, and then we use the decomposition

$$b_{R_k}(u_{R_k}, u_{R_k}, v) - b(u, u, v) = b_{R_k}(u, u, v) - b(u, u, v) + b_{R_k}(u_{R_k} - u, u_{R_k}, v) - b_{R_k}(u, u - u_{R_k}, v).$$

Observe that we can bound

$$\big|b_{R_k}(u, \bar{u}, v)\big| \leqslant C\|u|_K\|_{\mathcal{H}_K}\|\bar{u}|_K\|_{\mathcal{H}_K}\|v\|_{\text{Lip}(K)},$$

where C is independent of R_k, the set K is such that $\operatorname{supp} v \subset K$ and $\|\cdot\|_{\mathcal{H}_K}$ is defined by

$$\|\phi\|^2_{\mathcal{H}_K} := \int_{\mathcal{F}_0\cap K} |\phi|^2\,dx + \int_{\mathcal{S}_0\cap K} \rho_{\mathcal{S}_0}|\phi|^2\,dx.$$

Hence, (39) follows from the local strong convergence and (35).

Now, with the purpose of proving that $(u_{R_k})_k$ is relatively compact in $L^2(0,T;L^2_{\mathrm{loc}}(\mathbb{R}^3))$, we are going to establish an a priori estimate of the time derivative of the functions u_{R_k}. Of course we already have such an estimate thanks to the Fourier transform in time applied above, but this estimate is not uniform in R, since we relied on the inequality (31) which is not uniform in R. Instead we are going to prove that $(\partial_t u_{R_k})_k$ is bounded in $L^{\frac{4}{3}}(0,T;\mathcal{V}')$, relying on the estimate (33), which is uniform in large R, rather than on (31). Then, by using a standard cut-off function, we can apply the Aubin–Lions lemma, see for instance [49, Corollary 4], to conclude the desired compactness.

The bound of $(\partial_t u_{R_k})_k$ is obtained as follows. We first combine the interpolation inequality (26) with the energy bounds, to see that $(u_{R_k})_k$ is bounded in $L^{\frac{8}{3}}(0,T;L^4(\mathcal{F}_0))$. Next we use (33) and Hölder's inequality to get that there exists $C>0$ such that for any $k\in\mathbb{N}$, for any $v\in L^4(0,T;\mathcal{V})$,

$$\left|\int_0^t b_{R_k}(u_{R_k},u_{R_k},v)\right| \leqslant C\|v\|_{L^4(0,T;\mathcal{V})}.$$

Then we easily infer from (36) the desired estimate of $(\partial_t u_{R_k})_k$, and therefore the proof of Theorem 1 is complete. □

3.4. A regularity property

In the present case of the Navier conditions, the dynamics of the body benefits from a remarkable regularity property stated in the proposition below. We will make use a slight variant of (25), which involves the space

$$\widehat{\mathcal{V}} := \{\phi\in\mathcal{V} \mid \phi|_{\mathcal{F}_0}\in \mathrm{Lip}(\overline{\mathcal{F}_0})\}, \quad \text{endowed with the norm } \|\phi\|_{\widehat{\mathcal{V}}} := \|\phi\|_{\mathcal{V}} + \|\phi\|_{\mathrm{Lip}(\overline{\mathcal{F}_0})}.$$

Then one may extend b to $\mathcal{H}\times\mathcal{H}\times\widehat{\mathcal{V}}$ such that there exists a constant $C>0$ such that for any $(u,v,w)\in\mathcal{H}\times\mathcal{H}\times\widehat{\mathcal{V}}$,

$$\big|b(u,v,w)\big| \leqslant C\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}}\|w\|_{\widehat{\mathcal{V}}}. \tag{40}$$

Let us emphasize for the comfort of the reader that $\widehat{\mathcal{V}}\subset\mathcal{V}\subset\underline{\mathcal{V}}$.

Let us also denote by λ_i, $i=1,2,3$, the eigenvalues of the inertial matrix $\mathcal{J}_0$, which is symmetric definite positive, so that, $\lambda_i>0$ for all $i=1,2,3$. Moreover, we consider the spectral norms $\|\mathcal{J}_0\| := \max(\lambda_i)$ and $\|\mathcal{J}_0^{-1}\|^{-1} := \min(\lambda_i)$.

Proposition 2. *Let be given $u_0\in\mathcal{H}$ and $T>0$. Consider a weak solution u of* (12)–(19) *given by Theorem* 1*. Then ℓ and r are in $H^1(0,T;\mathbb{R}^3)$ and satisfy the following*: *there exist*

1. *a 6×6 definite positive symmetric matrix $\mathcal{M}$ depending only on $\mathcal{S}_0$, m and $\mathcal{J}_0$ such that there exist $\underline{m}>0$ and $\beta>0$ depending only on $\mathcal{S}_0$ such that, for any F and T in $\mathbb{R}^3$,*

$$\left\|\mathcal{M}^{-1}\begin{bmatrix}F\\T\end{bmatrix}\right\| \leqslant 2\big(m^{-1}\|F\| + \big\|\mathcal{J}_0^{-1}\big\|\|T\|\big) \quad \textit{for } m\geqslant\underline{m}, \quad \textit{and} \quad \lambda_i\geqslant\beta, \quad i=1,2,3, \tag{41}$$

2. *some functions $(v_i)_{i\in\{1,\dots,6\}}$ in $\widehat{\mathcal{V}}$ depending only on $\mathcal{S}_0$,*

such that there holds in $L^2(0,T)$:

$$\mathcal{M}\begin{bmatrix}\ell\\r\end{bmatrix}' = \big[2\nu a(u,v_i) + b(u,u,v_i)\big]_{i\in\{1,\dots,6\}}. \tag{42}$$

Proof. The matrix $\mathcal{M}$ is usually referred to as virtual inertia tensor, it incorporates the added mass of the solid $\mathcal{M}_2$, and is defined by

$$\mathcal{M}_1 := \begin{bmatrix} m\,\mathrm{Id}_3 & 0\\ 0 & \mathcal{J}_0\end{bmatrix}, \qquad \mathcal{M}_2 := \left[\int_{\mathcal{F}_0}\nabla\Phi_i\cdot\nabla\Phi_j\,dx\right]_{i,j\in\{1,\dots,6\}} \quad \text{and} \quad \mathcal{M} := \mathcal{M}_1+\mathcal{M}_2,$$

where the functions Φ_i, usually referred to as the Kirchhoff potentials, as the solutions of the following problems:

$$-\Delta\Phi_i = 0 \quad \text{for } x \in \mathcal{F}_0,$$
$$\Phi_i \to 0 \quad \text{as } |x| \to \infty,$$
$$\frac{\partial \Phi_i}{\partial n} = K_i \quad \text{for } x \in \partial\mathcal{S}_0,$$

where

$$K_i := \begin{cases} n_i & \text{if } i = 1, 2, 3, \\ [x \wedge n]_{i-3} & \text{if } i = 4, 5, 6. \end{cases}$$

We observe that the matrix $\mathcal{M}_2$ depends only on $\mathcal{S}_0$ and is nonnegative symmetric so that $\mathcal{M}$ depends only on $\mathcal{S}_0$, m and $\mathcal{J}_0$, and is definite positive symmetric.

Now for any F and T in $\mathbb{R}^3$, let

$$\begin{bmatrix} x \\ y \end{bmatrix} := \mathcal{M}^{-1} \begin{bmatrix} F \\ T \end{bmatrix} = \begin{bmatrix} m^{-1}\,\mathrm{Id}_3 & 0 \\ 0 & \mathcal{J}_0^{-1} \end{bmatrix} \left(\begin{bmatrix} F \\ T \end{bmatrix} - \mathcal{M}_2 \begin{bmatrix} x \\ y \end{bmatrix} \right).$$

Since $\mathcal{M}_2$ depends only on $\mathcal{S}_0$, there exists $C > 0$ depending only on $\mathcal{S}_0$ such that $\|\mathcal{M}_2 \big[\begin{smallmatrix} x \\ y \end{smallmatrix}\big]\| \leqslant C \|\big[\begin{smallmatrix} x \\ y \end{smallmatrix}\big]\|$. Then, we can estimate

$$\left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\| \leqslant m^{-1} \|F\| + \left\| \mathcal{J}_0^{-1} \right\| \|T\| + \max\left\{ m^{-1}, \left\| \mathcal{J}_0^{-1} \right\| \right\} C \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|.$$

It is therefore sufficient to take $\underline{m} = \beta = 2C$ to obtain (41).

For $i = 1, \dots, 6$, we introduce the function v_i defined by

$$v_i := \nabla\Phi_i \quad \text{in } \mathcal{F}_0 \quad \text{and} \quad v_i := \begin{cases} e_i & \text{if } i = 1, 2, 3, \\ e_{i-3} \wedge x & \text{if } i = 4, 5, 6, \end{cases} \quad \text{in } \mathcal{S}_0.$$

These functions only depend on $\mathcal{S}_0$. Moreover they are in $\widehat{\mathcal{V}}$. Observe in particular that ∇v_i decays like $1/|\cdot|^4$ at infinity so that $\int_{\mathcal{F}_0} |\nabla v_i(y)|^2 (1 + |y|^2)\, dy < +\infty$, see for instance [6, 4.3.1]. We can therefore take them as test functions in (22). Indeed we apply (22) to $v = v_i$ and we derive in time to obtain, for all $t \in [0, T]$,

$$\partial_t (u, v_i)_{\mathcal{H}} = 2\nu a(u, v_i) + b(u, u, v_i). \tag{43}$$

Let us prove that

$$\big[(u, v_i)_{\mathcal{H}}\big]_{i,j \in \{1,\dots,6\}} = \mathcal{M} \begin{bmatrix} \ell \\ r \end{bmatrix}. \tag{44}$$

To this end, we first use (21) to arrive at

$$\big[(u, v_i)_{\mathcal{H}}\big]_{i,j \in \{1,\dots,6\}} = \left[\int_{\mathcal{F}_0} u \cdot \nabla\Phi_i \right]_{i,j \in \{1,\dots,6\}} + \mathcal{M}_1 \begin{bmatrix} \ell \\ r \end{bmatrix}. \tag{45}$$

Then, using an integration by parts, we observe that

$$\int_{\mathcal{F}_0} u \cdot \nabla\Phi_i = \int_{\partial\mathcal{S}_0} (u \cdot n)\Phi_i = \int_{\partial\mathcal{S}_0} (u_{\mathcal{S}} \cdot n)\Phi_i,$$

so that, expanding $u_{\mathcal{S}}$ and using another integration by parts, give us

$$\left[\int_{\mathcal{F}_0} u \cdot \nabla\Phi_i \right]_{i,j \in \{1,\dots,6\}} = \mathcal{M}_2 \begin{bmatrix} \ell \\ r \end{bmatrix}. \tag{46}$$

Gathering (45) and (46) yields (44). Then combining (43) and (44) furnishes (42).

Therefore it only remains to prove that ℓ and r are in $H^1(0, T; \mathbb{R}^3)$. Since the matrix $\mathcal{M}$ is time-independent it is sufficient to prove that the right-hand side of (42) is in $L^2(0, T; \mathbb{R}^3)$. Indeed this follows from (28), (40), from that $(v_i)_{i \in \{1,\dots,6\}}$ are in $\widehat{\mathcal{V}}$ and that u is in $C_w([0, T]; \mathcal{H}) \cap L^2(0, T; \underline{\mathcal{V}})$. $\square$

Let us emphasize that this property seems to be a particularity of the case of the Navier conditions. In particular in the case of the no-slip conditions, the corresponding weak formulation involves test functions continuous across the body's boundary, a feature which is not satisfied by the functions v_i. To our knowledge the counterpart of Proposition 2 in the case of the no-slip conditions is not known.

The estimate (41) will be useful for the next section. It is also interesting for the sequel to observe that

$$\big[b(u,u,v_i)\big]_{i\in\{1,\dots,6\}} = \begin{bmatrix} mr\wedge\ell \\ (\mathcal{J}_0 r)\wedge r \end{bmatrix} + \bigg[\int_{\mathcal{F}_0} \big(\big[\big((u-u_{\mathcal{S}})\cdot\nabla\big)\nabla\Phi_i\big]\cdot u - \det(r_u,u,\nabla\Phi_i)\big)\bigg]_{i\in\{1,\dots,6\}}. \tag{47}$$

3.5. The infinite inertia limit

Let us also mention that Theorem 1 extends to the case of a moving body some earlier results, in particular see [7, 37], about the existence of Leray solutions in the case where Navier conditions are considered but on a fixed boundary. In this case, the system reads

$$\frac{\partial u}{\partial t} + u\cdot\nabla u + \nabla p = \nu\Delta u \quad \text{for } x\in\mathcal{F}_0, \tag{48}$$

$$\operatorname{div} u = 0 \quad \text{for } x\in\mathcal{F}_0, \tag{49}$$

$$u\cdot n = 0 \quad \text{for } x\in\partial\mathcal{S}_0, \tag{50}$$

$$\big(D(u)n\big)\wedge n = -\alpha u\wedge n \quad \text{for } x\in\partial\mathcal{S}_0, \tag{51}$$

$$u|_{t=0} = u_0, \tag{52}$$

and a weak Leray solution of (48)–(52) is by definition a function

$$u\in C_w\big([0,T];L^2_\sigma(\mathcal{F}_0)\big)\cap L^2\big(0,T;H^1(\mathcal{F}_0)\big),$$

such that

1. for all $v\in H^1(0,T;L^2_\sigma(\mathcal{F}_0))\cap L^4(0,T;H^1(\mathcal{F}_0))$, and for all $t\in[0,T]$,

$$\int_{\mathcal{F}_0} u(t,\cdot)\cdot v(t,\cdot)\,dx - \int_{\mathcal{F}_0} u_0\cdot v|_{t=0}\,dx = \int_0^t\bigg[\int_{\mathcal{F}_0} u\cdot\partial_t v\,dx + 2\nu a^*(u,v) + b^*(u,u,v)\bigg], \tag{53}$$

   where

$$a^*(u,v) := -\alpha\int_{\partial\mathcal{S}_0} u\cdot v - \int_{\mathcal{F}_0} D(u):D(v), \quad \text{and} \quad b^*(u,v,w) := \int_{\mathcal{F}_0}[u\cdot\nabla w]\cdot v,$$

2. for any $t\in[0,T]$,

$$\frac{1}{2}\big\|u(t,\cdot)\big\|^2_{L^2(\mathcal{F}_0)} + 2\nu\int_{(0,t)\times\mathcal{F}_0}\big|D(u)\big|^2 + 2\alpha\nu\int_0^t\int_{\partial\mathcal{S}_0}|u|^2 \leqslant \frac{1}{2}\|u_0\|^2_{L^2(\mathcal{F}_0)}.$$

Here, $L^2_\sigma(\mathcal{F}_0)$ denotes the space of the divergence free vector fields in $L^2(\mathcal{F}_0)$ which are tangent to the solid's boundary $\partial\mathcal{S}_0$.

The following result shows that the case with the body fixed can be thought as *the limit of infinite inertia*, that is, when m and the eigenvalues $(\lambda_i)_{i=1,2,3}$ of $\mathcal{J}_0$ converge to $+\infty$ with $\lambda_i = O(\lambda_j)$ for any i, j. Let us observe that the eigenvalues of $\mathcal{J}_0$ are required to diverge at the same order. This last condition is quite natural if one thinks that the solid density $\rho_{\mathcal{S}_0}$ is multiplied by a factor converging to $+\infty$ in (1) and (3). This condition can alternatively be written as $\|\mathcal{J}_0\| = O(\|\mathcal{J}_0^{-1}\|^{-1})$ (using the spectral norms introduced in Section 3.4).

Let us point out that $\|\mathcal{M}^{-1}\|_{\mathbb{R}^{6\times 6}} \to 0$ as m and $(\lambda_i)_{i=1,2,3}$ converge to $+\infty$, as a consequence of the estimate (41), and therefore in particular $\|\mathcal{M}^{-1}\|_{\mathbb{R}^{6\times 6}} \to 0$ in the infinite inertia limit. Another observation that will be useful is that there holds for any $r \in \mathbb{R}^3$,

$$\big\|(\mathcal{J}_0 r) \wedge r\big\| \leqslant C(\mathcal{J}_0 r)\cdot r, \tag{54}$$

for a constant $C > 0$, uniform in the infinite inertia limit.

Indeed, introducing some normalized eigenvectors $(r_i)_{i=1,2,3}$, associated to $(\lambda_i)_{i=1,2,3}$, respectively, we can write for some real coefficients $(\alpha_i)_{i=1,2,3}$, that $r = \sum_{i=1}^3 \alpha_i r_i$ and therefore $(\mathcal{J}_0 r)\wedge r = \sum_{i,j} \alpha_i\alpha_j\lambda_i (r_i \wedge r_j)$. So that, for some constants uniform in the infinite inertia limit, one has:

$$\begin{aligned}\big\|(\mathcal{J}_0 r)\wedge r\big\| &\leqslant C\sum_{i,j}|\alpha_i||\alpha_j|\lambda_i \leqslant C'\sum_{i,j}|\alpha_i||\alpha_j|\sqrt{\lambda_i}\sqrt{\lambda_j} \leqslant C''\Big(\sum_i|\alpha_i|\sqrt{\lambda_i}\Big)^2 \leqslant C'''\sum_i \alpha_i^2\lambda_i \\ &= C'''(\mathcal{J}_0 r)\cdot r.\end{aligned}$$

Theorem 2. *Let be given $u_0 \in \mathcal{H}$ with $\ell_0 = r_0 = 0$ and $T > 0$. For any m and $\mathcal{J}_0$ we consider a weak solution u of (12)–(19) in $C_w([0,T];\mathcal{H}) \cap L^2(0,T;\underline{\mathcal{V}})$ given by Theorem 1. Then in the infinite inertia limit, $u|_{\mathcal{F}_0}$ converges, up to a subsequence, in $L^2(0,T;L^2_{\mathrm{loc}}(\mathcal{F}_0))$ to a weak solution of (48)–(52) and ℓ and r converge to 0 in $H^1(0,T;\mathbb{R}^3)$.*

Proof. We infer from (29) that u is bounded in $L^\infty(0,T;\mathcal{H}) \cap L^2(0,T;\underline{\mathcal{V}})$ uniformly with respect to the inertia. Not only that but we also obtain that ℓ and r converge to 0 in $L^\infty(0,T;\mathbb{R}^3)$ in the infinite inertia limit, because $(\mathcal{J}_0 r)\cdot r \geqslant \min(\lambda_i)\|r\|^2$.

The following lemma is quite simple to establish but will be useful in the sequel.

Lemma 2. *Let u be as in Theorem 2. Then for any $v \in H^1(0,T;L^2_\sigma(\mathcal{F}_0)) \cap L^4(0,T;\widetilde{\mathcal{V}})$, for any $t \in [0,T]$,*

$$\int_{\mathcal{F}_0} u(t,\cdot)\cdot v(t,\cdot)\,dx - \int_{\mathcal{F}_0} u_0\cdot v|_{t=0}\,dx = \int_0^t\Big[\int_{\mathcal{F}_0} u\cdot\partial_t v\,dx + 2\nu a^*(u,v) + b^*(u,u,v) + F(u,v)\Big], \tag{55}$$

where

$$F(u,v) := 2\alpha\nu\int_{\partial\mathcal{S}_0} u_{\mathcal{S}}\cdot v - \int_{\mathcal{F}_0}\big([u_{\mathcal{S}}\cdot\nabla v]\cdot u - \det(r_u,u,v)\big).$$

Above $\widetilde{\mathcal{V}}$ denotes the space

$$\widetilde{\mathcal{V}} := \Big\{\phi \in L^2_\sigma(\mathcal{F}_0)\ \Big|\ \int_{\mathcal{F}_0}\big|\nabla\phi(y)\big|^2\big(1+|y|^2\big)\,dy < +\infty\Big\},$$

endowed with the norm

$$\|\phi\|_{\widetilde{\mathcal{V}}} := \|\phi\|_{L^2(\mathcal{F}_0)} + \|\nabla\phi\|_{L^2(\mathcal{F}_0,(1+|y|^2)^{\frac{1}{2}}\,dy)}.$$

Proof of Lemma 2. It is sufficient to extend v by 0 in $\mathcal{S}_0$ to obtain a function in $H^1(0,T;\mathcal{H}) \cap L^4(0,T;\mathcal{V})$ which is used as a test function in (22). This provides (55). □

Then, proceeding as in Section 3.3, we obtain a bound of $\partial_t u|_{\mathcal{F}_0}$ in $L^{\frac{4}{3}}(0,T;\widetilde{\mathcal{V}}')$ which is uniform in the infinite inertia limit. We therefore deduce that the sequence of weak solutions u is relatively compact in $L^2(0,T;L^2_{\mathrm{loc}}(\mathcal{F}_0))$. Thus the restrictions of u to $\mathcal{F}_0$ converge, up to a subsequence, to a limit u^* weakly-* in $L^\infty(0,T;L^2(\mathcal{F}_0))$, weakly in $L^2(0,T;H^1(\mathcal{F}_0))$ and strongly in $L^2(0,T;L^2_{\mathrm{loc}}(\mathcal{F}_0))$.

Let us now prove that u^* is a weak solution of (48)–(52). We deduce from the above convergence that $F(u,v)$ converges to 0 in $L^1(0,T)$, and passing to the limit in the other terms of (55) we can conclude that u^* satisfies (53) for any $v \in H^1(0,T;L^2_\sigma(\mathcal{F}_0)) \cap L^4(0,T;H^1(\mathcal{F}_0))$ such that $\int_{\mathcal{F}_0} |\nabla v(y)|^2(1+|y|^2)\,dy < +\infty$. Then one easily removes this last condition by using Lebesgue's dominated convergence theorem. Thus u^* is a weak solution of (48)–(52).

In order to finish the proof of Theorem 2 it only remains to prove that ℓ' and r' converge to 0 in $L^2(0,T;\mathbb{R}^3)$. This relies on the regularity property established in the previous section. Indeed we define, for $i \in \{1,\dots,6\}$,

$$\mathcal{T}_{1,i} := \int\limits_{\mathcal{F}_0} \big(\big[\big((u-u_{\mathcal{S}})\cdot\nabla\big)\nabla\Phi_i\big]\cdot u - \det(r_u, u, \nabla\Phi_i)\big),$$

and $\mathcal{T}_2 := \left[\begin{smallmatrix} mr\wedge\ell \\ (\mathcal{J}_0 r)\wedge r \end{smallmatrix}\right]$, so that from (42) and (47) we infer that

$$\begin{bmatrix} \ell \\ r \end{bmatrix}' = 2\nu\mathcal{M}^{-1}\big[a(u,v_i)\big]_{i\in\{1,\dots,6\}} + \mathcal{M}^{-1}[\mathcal{T}_{1,i}]_{i\in\{1,\dots,6\}} + \mathcal{M}^{-1}\mathcal{T}_2. \tag{56}$$

Since u is bounded in $L^\infty(0,T;\mathcal{H}) \cap L^2(0,T;\underline{\mathcal{V}})$, the functions $(v_i)_{i\in\{1,\dots,6\}}$ are in $\widehat{\mathcal{V}}$ and depend only on $\mathcal{S}_0$, and $\|\mathcal{M}^{-1}\|_{\mathbb{R}^{6\times6}} \to 0$ in the infinite inertia limit, we infer easily from (28) that the first term of the right-hand side of (56) vanishes in $L^2(0,T;\mathbb{R}^6)$ in the infinite inertia limit.

On the other hand we can bound the second term as follows: for any $i \in \{1,\dots,6\}$, for any t, $\|\mathcal{T}_{1,i}\| \leqslant C(\|u\|^2_{L^2(\mathcal{F}_0)} + \|\ell\|^2 + \|r\|^2)$. Hence, thanks to the energy bound, we get that $\mathcal{T}_{1,i}$ is bounded in the infinite inertia limit. Therefore the second term of the right-hand side of (56) also vanishes in $L^2(0,T;\mathbb{R}^6)$ in the infinite inertia limit.

Finally, in order to deal with the last term, we use the estimate (41) to bound, for any t,

$$\begin{aligned} \|\mathcal{M}^{-1}\mathcal{T}_2\| &\leqslant 2\big(\|r\wedge\ell\| + \big(\min(\lambda_i)\big)^{-1}\big\|(\mathcal{J}_0 r)\wedge r\big\|\big) \\ &\leqslant C\big(\|\ell\|^2 + \|r\|^2 + \big(\min(\lambda_i)\big)^{-1}(\mathcal{J}_0 r)\cdot r\big), \end{aligned}$$

thanks to (54). We thus deduce from the energy bound that the last term of the right-hand side of (56) also vanishes in $L^2(0,T;\mathbb{R}^6)$ in the infinite inertia limit.

The proof of Theorem 2 is then complete. □

4. Smooth local-in-time solutions of the inviscid system

In this section we consider the system "inviscid incompressible fluid + rigid body".

4.1. The system "inviscid incompressible fluid + rigid body"

When the viscosity coefficient ν is set equal to 0, formally, the system (4)–(11) degenerates into the following equations:

$$\frac{\partial U^E}{\partial t} + \big(U^E\cdot\nabla\big)U^E + \nabla P^E = 0 \quad \text{for } x \in \mathcal{F}^E(t) = \mathbb{R}^3 \setminus \mathcal{S}^E(t), \tag{57}$$

$$\operatorname{div} U^E = 0 \quad \text{for } x \in \mathcal{F}^E(t), \tag{58}$$

$$U^E\cdot n = U^E_{\mathcal{S}}\cdot n \quad \text{for } x \in \partial\mathcal{S}^E(t), \tag{59}$$

$$m\big(h^E\big)'' = \int\limits_{\partial\mathcal{S}^E(t)} P^E n\,ds, \tag{60}$$

$$\big(\mathcal{J}^E R^E\big)' = \int\limits_{\partial\mathcal{S}^E(t)} P^E\big(x-h^E(t)\big)\wedge n\,ds, \tag{61}$$

$$U^E\big|_{t=0} = U^E_0, \tag{62}$$

$$h^E(0) = 0, \qquad \big(h^E\big)'(0) = \ell^E_0, \qquad R^E(0) = r^E_0, \tag{63}$$

where the solid velocity is given by

$$U_{\mathcal{S}}^E(t,x) := \left(h^E\right)'(t) + R^E(t) \wedge \left(x - h^E(t)\right),$$

and

$$\mathcal{S}^E(t) := \eta^E(t,\cdot)(\mathcal{S}_0), \quad \text{with } \eta^E(t,x) := h^E(t) + Q^E(t)x,$$

where the matrix Q^E solves the differential equation

$$\left(Q^E\right)'x = R^E \wedge \left(Q^E x\right) \quad \text{with } Q^E(0)x = x, \text{ for any } x \in \mathbb{R}^3.$$

Finally $\mathcal{J}^E$ is given by

$$\mathcal{J}^E = Q^E \mathcal{J}_0 \left(Q^E\right)^T.$$

Observe that we prescribe $h^E(0) = 0$ so that the initial position $\mathcal{S}^E(0)$ occupied by the solid also starts from $\mathcal{S}_0$ at $t = 0$. The mass m and the initial inertial matrix $\mathcal{J}_0$ are also the same as in the previous case of the Navier–Stokes equations.

Let us emphasize that in the boundary condition (59) there is only an impermeability condition, the slip-with-friction condition is no more prescribed. This loss of boundary condition generates a boundary layer which makes difficult the issue of the inviscid limit of the system since the fluid flow is drastically modified in a neighborhood of the body's boundary of thickness proportional to $\sqrt{\nu}$. The main goal of the paper is precisely to show that despite these layers the solution of (4)–(11) converges in a rather good manner to the solution of (57)–(63) as $\nu \to 0$. This will be achieved in the next section. Here we will first gather a few results about the inviscid system (57)–(63).

4.2. *A change of variables*

To write the system in a fixed domain, we perform the following change of coordinates:

$$\ell^E(t) := Q^E(t)^T \left(h^E\right)'(t), \qquad R^E(t) := Q^E(t) r^E(t),$$
$$u^E(t,x) := Q^E(t)^T U^E\left(t, Q^E(t)x + h^E(t)\right) \quad \text{and} \quad p^E(t,x) := P^E\left(t, Q^E(t)x + h^E(t)\right),$$

where $Q^E(t)$ is the rotation matrix associated to the motion of $\mathcal{S}^E(t)$ defined in the previous section.

Observe that this change of variable is analogous to the one that we have used for the Navier–Stokes equations in Section 3.1.

The system (57)–(63) now reads

$$\frac{\partial u^E}{\partial t} + \left(u^E - u_{\mathcal{S}}^E\right) \cdot \nabla u^E + r^E \wedge u^E + \nabla p^E = 0 \quad \text{for } x \in \mathcal{F}_0, \tag{64}$$

$$\operatorname{div} u^E = 0 \quad \text{for } x \in \mathcal{F}_0, \tag{65}$$

$$u^E \cdot n = u_{\mathcal{S}}^E \cdot n \quad \text{for } x \in \partial\mathcal{S}_0, \tag{66}$$

$$m\left(\ell^E\right)' = \int_{\partial\mathcal{S}_0} p^E n\, ds + \left(m\ell^E\right) \wedge r^E, \tag{67}$$

$$\mathcal{J}_0\left(r^E\right)' = \int_{\partial\mathcal{S}_0} p^E x \wedge n\, ds + \left(\mathcal{J}_0 r^E\right) \wedge r^E, \tag{68}$$

$$u^E|_{t=0} = u_0^E, \tag{69}$$

$$h^E(0) = 0, \qquad \left(h^E\right)'(0) = \ell_0^E, \qquad r^E(0) = r_0^E, \tag{70}$$

with

$$u_{\mathcal{S}}^E(t,x) := \ell^E(t) + r^E(t) \wedge x. \tag{71}$$

4.3. Smooth local-in-time solutions

Let us recall the following result from [50] about the existence and uniqueness of classical solutions to Eqs. (64)–(70), where, as previously, we extend the initial data u_0^E by setting $u_0^E := \ell_0^E + r_0^E \wedge x$ for $x \in \mathcal{S}_0$.

Theorem 3. *Let be given $\lambda \in (0,1)$ and $u_0^E \in \mathcal{H}$ such that $u_0^E|_{\mathcal{F}_0} \in H^1 \cap C^{1,\lambda}$ and $\operatorname{curl} u_0^E|_{\mathcal{F}_0}$ is compactly supported. Then there exist $T > 0$ and a unique solution u^E of (64)–(70) in $C^1([0,T];\mathcal{H})$ such that $(\nabla u^E)|_{[0,T]\times\mathcal{F}_0} \in C([0,T]; L^2(\mathcal{F}_0, (1+|x|^2)^{\frac{1}{2}}\,dx)) \cap C_{w*}([0,T]; C^{0,\lambda}(\mathcal{F}_0))$. Moreover for any $t \in [0,T]$,*

$$\|u^E(t,\cdot)\|_{\mathcal{H}}^2 = \|u_0^E\|_{\mathcal{H}}^2. \tag{72}$$

Here, $C^{k,\lambda}(\mathcal{F}_0)$, $k \in \mathbb{N}$, $\lambda \in (0,1)$, denotes the usual Hölder space.

A few comments are in order.

It is worth to point out here that the solution u^E given by Theorem 3 satisfies the following property: for any $t \in [0,T]$, for any $v \in \underline{\mathcal{V}}$, there holds

$$(\partial_t u^E, v)_{\mathcal{H}} = -b(u^E, v, u^E). \tag{73}$$

To see that, multiply (64) by v and integrate by parts in space using (65)–(66).

It is useful for the sequel to recall that the proof given in [50] of Theorem 3 relies on the following vorticity reformulation of (64)–(70):

$$\partial_t \omega^E + (u^E - u_{\mathcal{S}}^E)\cdot\nabla\omega^E = (\omega^E\cdot\nabla)(u^E - u_{\mathcal{S}}^E) \quad \text{in } [0,T]\times\mathcal{F}_0, \tag{74}$$

$$\begin{cases} \operatorname{curl} u^E = \omega^E & \text{in } [0,T]\times\mathcal{F}_0, \\ \operatorname{div} u^E = 0 & \text{in } [0,T]\times\mathcal{F}_0, \\ u^E\cdot n = u_{\mathcal{S}}^E\cdot n & \text{on } [0,T]\times\partial\mathcal{S}_0, \\ u^E \to 0 & \text{for } |x|\to\infty, \end{cases} \tag{75}$$

$$u_{\mathcal{S}}^E(t,x) := \ell^E(t) + r^E(t)\wedge x \quad \text{in } [0,T]\times\mathbb{R}^3, \tag{76}$$

$$\mathcal{M}\begin{bmatrix} \ell^E \\ r^E \end{bmatrix}' = \big[b(u^E, u^E, v_i)\big]_{i\in\{1,\dots,6\}}, \tag{77}$$

where $\mathcal{M}$ and v_i were introduced in Section 3.4.

The vorticity equation (74) can easily be inferred from (64), (65) and (71) whereas (77) is obtained from (27), (73) and some integration by parts. Observe that (77) can be seen as the inviscid counterpart of (42), and recall that the $b(u^E, u^E, v_i)$ can be computed thanks to the formula (47).

4.4. The infinite inertia limit

The following result is the counterpart of Theorem 2 for the inviscid system (64)–(70). It shows that in the limit of infinite inertia, that is, when m and the eigenvalues $(\lambda_i)_{i=1,2,3}$ of $\mathcal{J}_0$ converge to $+\infty$ with $\lambda_i = O(\lambda_j)$ for any i, j, the system (64)–(70) degenerates into the following classical Euler equations in $\mathcal{F}_0$:

$$\frac{\partial u^E}{\partial t} + u^E\cdot\nabla u^E + \nabla p^E = 0 \quad \text{for } x\in\mathcal{F}_0, \tag{78}$$

$$\operatorname{div} u^E = 0 \quad \text{for } x\in\mathcal{F}_0, \tag{79}$$

$$u^E\cdot n = 0 \quad \text{for } x\in\partial\mathcal{S}_0, \tag{80}$$

$$u^E|_{t=0} = u_0^E. \tag{81}$$

Theorem 4. *Let be given $\lambda \in (0,1)$ and $u_0^E \in \mathcal{H}$ such that $\ell_0^E = r_0^E = 0$, $u_0^E|_{\mathcal{F}_0} \in H^1 \cap C^{1,\lambda}$ and $\operatorname{curl} u_0^E|_{\mathcal{F}_0}$ is compactly supported. Let be given $\underline{m} > 0$ and $\beta > 0$.*

Then there exists $T>0$ such that for any $m \geqslant \underline{m}$ and for any symmetric positive 3×3 matrix $\mathcal{J}_0$ with eigenvalues $(\lambda_i)_{i=1,2,3}$ satisfying $\lambda_i \geqslant \beta$, the corresponding solution u^E of (64)–(70) given by Theorem 3 is defined up to the time T.

Moreover in the infinite inertia limit, $u^E|_{\mathcal{F}_0}$ converges in $L^\infty(0,T;C^{1,\tilde\lambda}(\mathcal{F}_0))$, for any $\tilde\lambda \in (0,\lambda)$, to the unique smooth solution of (78)–(81) and ℓ^E and r^E converge to 0 in $C^1(0,T;\mathbb{R}^3)$.

Proof. The key issue here is to obtain some estimates uniform in the infinite inertia limit. Let us stress in particular that, a priori, Theorem 3 only provides, for some inertia m and $\mathcal{J}_0$, the existence of a smooth solution on a time interval $[0,T_{m,\mathcal{J}_0}]$ which may shrink when m and the eigenvalues $(\lambda_i)_{i=1,2,3}$ of $\mathcal{J}_0$ go to infinity. However we are going to see below that this is not the case. Actually revisiting the proof of Theorem 3 given in [50] we will provide a time T common to any large enough inertia.

The first basic observation in this direction is that according to the energy identity (72), and because of the choice of the initial data (which are somehow "well-prepared") the energy $\|u^E(t,\cdot)\|^2_{\mathcal{H}}$ does not depend on time nor on the inertia:

$$\int_{\mathcal{F}_0} \big|u^E(t,\cdot)\big|^2\,dx + m\ell^E(t)\cdot \ell^E(t) + \mathcal{J}_0 r^E(t)\cdot r^E(t) = \int_{\mathcal{F}_0} \big|u_0^E\big|^2\,dx. \tag{82}$$

Let us now show how to obtain a uniform estimate of the velocity in $L^\infty(0,T;C^{1,\lambda}(\mathcal{F}_0))$ for some $T>0$, thanks to the vorticity formulation (74)–(77).

By standard transport estimates (cf. [24, Corollary 2.4]) one infers from (74) the estimate

$$\begin{aligned}\big\|\omega^E(t)\big\|_{C^{0,\lambda}(\mathcal{F}_0)} &\leqslant \big\|\omega_0^E\big\|_{C^{0,\lambda}(\mathcal{F}_0)} \exp\left(C\int_0^t \big(\big\|u^E\big\|_{C^{1,\lambda}(\mathcal{F}_0)}(s) + \big\|r^E\big\|(s)\big)\,ds\right)\\ &\leqslant \big\|u_0^E\big\|_{C^{1,\lambda}(\mathcal{F}_0)} \exp\big(C\big\|r^E\big\|_{L^\infty(0,T)}t\big) \exp\left(C\int_0^t \big\|u^E\big\|_{C^{1,\lambda}(\mathcal{F}_0)}(s)\,ds\right),\end{aligned} \tag{83}$$

where C does not depend on the inertia.

On the other hand, by classical elliptic theory, one infers from (75) the following estimate, where time is omitted,

$$\big\|u^E\big\|_{C^{1,\lambda}(\mathcal{F}_0)} \leqslant C\big(\big\|\omega^E\big\|_{C^{0,\lambda}(\mathcal{F}_0)} + \big\|\ell^E\big\| + \big\|r^E\big\|\big) \leqslant C\big(\big\|\omega^E\big\|_{C^{0,\lambda}(\mathcal{F}_0)} + \big\|u_0^E\big\|_{L^2(\mathcal{F}_0)}\big), \tag{84}$$

with $C>0$ depending on $\underline{m}>0$ and $\beta>0$.

Plugging (84) in (83) yields

$$\big\|\omega^E(t)\big\|_{C^{0,\lambda}(\mathcal{F}_0)} \leqslant \big\|u_0^E\big\|_{C^{1,\lambda}(\mathcal{F}_0)} \exp\big(C\big\|u_0^E\big\|_{L^2(\mathcal{F}_0)}t\big) \exp\left(C\int_0^t \big\|\omega^E\big\|_{C^{0,\lambda}(\mathcal{F}_0)}(s)\,ds\right),$$

with $C>0$ depending on $\underline{m}>0$ and $\beta>0$, from which one deduces the existence of a small time $T>0$ and an estimate of velocity in $L^\infty(0,T;C^{1,\lambda}(\mathcal{F}_0))$ both uniformly for any $m \geqslant \underline{m}$ and for any symmetric positive 3×3 matrix $\mathcal{J}_0$ with eigenvalues $(\lambda_i)_{i=1,2,3}$ satisfying $\lambda_i \geqslant \beta$.

Let us now prove the second part of Theorem 4 about convergence of the fluid and solid velocities. First (82) yields that ℓ^E and r^E converge to 0 in $L^\infty(0,T;\mathbb{R}^3)$ in the infinite inertia limit. Then, similarly as we proceed in the proof of Theorem 2, by using (77) and (47) we write

$$\begin{bmatrix} \ell^E \\ r^E \end{bmatrix}' = \mathcal{M}^{-1}\big[\mathcal{T}^E_{1,i}\big]_{i\in\{1,\dots,6\}} + \mathcal{M}^{-1}\mathcal{T}^E_2,$$

where

$$\mathcal{T}^E_{1,i} := \int_{\mathcal{F}_0} \big(\big[\big(\big(u^E - u^E_S\big)\cdot\nabla\big)\nabla\Phi_i\big]\cdot u^E - \det\big(r^E_u, u^E, \nabla\Phi_i\big)\big), \quad \text{and} \quad \mathcal{T}^E_2 := \begin{bmatrix} mr^E \wedge \ell^E \\ (\mathcal{J}_0 r^E)\wedge r^E \end{bmatrix}.$$

Thus, by using again the energy bound (82), we deduce that ℓ^E and r^E converge to 0 in $C^1(0,T;\mathbb{R}^3)$.

Finally it remains to prove that $u^E|_{\mathcal{F}_0}$ converges, up to a subsequence, in $L^\infty(0,T;C^{1,\tilde{\lambda}}(\mathcal{F}_0))$, for any $\tilde{\lambda}\in(0,\lambda)$. Since we have some uniform estimates of $u^E|_{\mathcal{F}_0}$ in the space $L^\infty(0,T;C^{1,\lambda}(\mathcal{F}_0))$, it is sufficient to obtain a temporal estimate uniform in infinite inertia limit and to use the Aubin–Lions lemma to conclude that a subsequence is converging in $L^\infty(0,T;C^{1,\tilde{\lambda}}(\mathcal{F}_0))$, for any $\tilde{\lambda}\in(0,\lambda)$. This can be easily achieved by using (73) with some test functions v compactly supported in $\mathcal{F}_0$.

These convergences are sufficient to yield that the limit is a smooth solution of (78)–(81) on $[0,T]$.

Since classical solutions of the Euler equations are unique we finally deduce that the whole sequence is converging. □

5. Inviscid limit

Let us now state the main result of this paper.

Theorem 5. *The following holds true.*

1. *With the notations of Theorems 1 and 3, and assuming that u_0 converges to u_0^E in $\mathcal{H}$ when ν tends to 0 and that $\alpha:=\alpha^\nu$ satisfies $\alpha\nu$ converges to 0 when ν tends to 0, then u converges to u^E in $L^\infty(0,T;\mathcal{H})$, $\sqrt{\nu}\|u|_{\mathcal{F}_0}\|_{L^2(0,T;H^1(\mathcal{F}_0))}$ and $\sqrt{\alpha\nu}\|u-u_{\mathcal{S}}\|_{L^2((0,T)\times\partial\mathcal{S}_0)}$ converge to 0, where $T>0$ is the lifetime of the smooth solution u^E of the inviscid system.*
2. *Moreover (ℓ,r) converges to (ℓ^E,r^E) in $H^1(0,T;\mathbb{R}^6)$.*
3. *Assuming that $u_0=u_0^E$ and that $\alpha>0$ does not depend on ν, then there exists $C>0$ (depending on T) such that*

$$\|u-u^E\|_{L^\infty(0,T;\mathcal{H})}+\sqrt{\nu}\|u-u^E\|_{L^2(0,T;H^1(\mathcal{F}_0))}\leqslant C(1+\alpha)\nu^{3/4}. \tag{85}$$

A few remarks are in order.

Regarding the first part of Theorem 5, let us first mention again Refs. [10,40,1] for the relevance of considering a friction coefficient depending on the viscosity, as a limit of accommodation boundary condition for the Boltzmann equation. The first part of Theorem 5 extends earlier results where a fixed boundary was considered, see [36,45,1, 53]. Let us point out in particular that it could be possible to extend the first part of Theorem 5 to a weaker setting, following the analysis of [1]. This will require to extend P.-L. Lions' definition of dissipative solutions of the Euler incompressible equations to the system (64)–(70), and to modify the proof of the first part of Theorem 5 below following a by now classical method, the so-called relative entropy method or the modulated energy method depending on the context and on the authors. We choose here to consider smooth solutions of the system (64)–(70) in order to keep the unity of the theorem, since the other parts fail in a weaker context.

The second part of Theorem 5 is perhaps the most surprising. It shows that the convergence of the body's dynamics is better than the one implied by the convergence of u to u^E in the energy space $L^\infty(0,T;\mathcal{H})$ stated in the first part. Indeed the latter provides a convergence (ℓ,r) to (ℓ^E,r^E) in $L^\infty(0,T;\mathbb{R}^6)$. This last result relies on the possibility to compute explicitly, in the present case of the Navier slip conditions, a well-known phenomenon in the theory of the systems involving an incompressible flow and a structure, namely the added-mass phenomenon, cf. for instance [6,17].

If we focus on the dependence on the viscosity the estimate (85) says that u converges strongly to u^E in $L^\infty(0,T;L^2(\mathcal{F}_0))$ with a rate of $O(\nu^{3/4})$ and in $L^2(0,T;H^1(\mathcal{F}_0))$ with a rate of $O(\nu^{1/4})$. We therefore recover the optimal rate of convergence, with respect to ν, found in [37] in the case where the Navier conditions are prescribed on a fixed boundary.

Theorem 5 may be useful for controllability issues, in particular for the global controllability of the system. In the case of a fixed boundary a strategy initiated by Coron in [11] proved the global approximate controllability for the 2-D incompressible Navier–Stokes equations with Navier slip boundary conditions. His proof relies on another of his earlier results about the global controllability of the incompressible Euler equations, see [12]. This strategy was used later on by Chapouly in [5] to extend Coron's result into a global null controllability result.

It is therefore possible that a similar strategy could be fruitful in the case of a moving rigid body. Note however that in such a case little is known so far about the controllability of the inviscid system. Let us mention in that direction the result [4] by Chambrion and Munnier in the irrotational case.

6. Proof of Theorem 5

6.1. Two basic observations

Let us observe first that when u converges to u^E in $L^\infty(0,T;\mathcal{H})$ then it follows straightforwardly from (29), (72) and Korn inequality that $\sqrt{\nu}\|u|_{\mathcal{F}_0}\|_{L^2(0,T;H^1(\mathcal{F}_0))}$ and $\sqrt{\alpha\nu}\|u-u_{\mathcal{S}}\|_{L^2((0,T)\times\partial\mathcal{S}_0)}$ converge to 0.

Moreover the second part of Theorem 5 is quite easy to obtain once the first part has been proved. Indeed, using that u converges to u^E in $L^\infty(0,T;\mathcal{H})$, and that $\sqrt{\nu}\|u|_{\mathcal{F}_0}\|_{L^2(0,T;H^1(\mathcal{F}_0))}$ and $\sqrt{\alpha\nu}\|u-u_{\mathcal{S}}\|_{L^2((0,T)\times\partial\mathcal{S}_0)}$ converge to 0 as ν tends to 0, (42), (77) and (47), we obtain that $\mathcal{M}\big[\begin{smallmatrix}\ell\\ r\end{smallmatrix}\big]'$ converges to $\mathcal{M}\big[\begin{smallmatrix}\ell^E\\ r^E\end{smallmatrix}\big]'$ in $L^2(0,T)$. We then infer easily the second part.

6.2. Proof of the first part

Let us now turn to the proof of the first part. In the following, C will denote a constant independent of ν and α that may change from one relation to another.

Let us also introduce the difference

$$w := u - u^E \quad \text{and similarly for the initial condition} \quad w_0 := u_0 - u_0^E.$$

In the solid we will use the notation

$$w_{\mathcal{S}} := u_{\mathcal{S}} - u_{\mathcal{S}}^E.$$

For any $t \in [0,T]$, we have, thanks to (29) and (72),

$$\|w(t,\cdot)\|_{\mathcal{H}}^2 \leqslant \|u_0\|_{\mathcal{H}}^2 + \|u_0^E\|_{\mathcal{H}}^2 - 2\big(u,u^E\big)_{\mathcal{H}}(t) + 4\nu\int_0^t a(u,u).$$

We now apply (22) to $v = u^E$ to get

$$\begin{aligned}\big(u,u^E\big)_{\mathcal{H}}(t) - \big(u_0,u_0^E\big)_{\mathcal{H}} &= \int_0^t \big[\big(u,\partial_t u^E\big)_{\mathcal{H}} + 2\nu a\big(u,u^E\big) + b\big(u,u,u^E\big)\big]\,ds \\ &= \int_0^t \big[2\nu a\big(u,u^E\big) + b\big(u,u,u^E\big) - b\big(u^E,u,u^E\big)\big]\,ds,\end{aligned}$$

using (73) with $v = u$.

Therefore,

$$\begin{aligned}\|w(t,\cdot)\|_{\mathcal{H}}^2 &\leqslant \|w_0\|_{\mathcal{H}}^2 - 2\int_0^t \big[2\nu a\big(u,u^E\big) - 2\nu a(u,u) + b\big(u,u,u^E\big) - b\big(u^E,u,u^E\big)\big]\,ds \\ &= \|w_0\|_{\mathcal{H}}^2 - 2\int_0^t \big[-2\nu a(u,w) + b\big(w,w,u^E\big)\big]\,ds.\end{aligned}$$

This can be recast as follows:

$$\begin{aligned}&\|w(t,\cdot)\|_{\mathcal{H}}^2 + 4\alpha\nu\int_0^t\int_{\partial\mathcal{S}_0} |w-w_{\mathcal{S}}|^2 + 4\nu\int_0^t\int_{\mathcal{F}_0}\big|D(w)\big|^2 \\ &\quad\leqslant \|w_0\|_{\mathcal{H}}^2 - 2\int_0^t\big[-2\nu a\big(u^E,w\big) + b\big(w,w,u^E\big)\big]\,ds. \qquad (86)\end{aligned}$$

Therefore it only remains to use the Cauchy–Schwarz and Young inequalities, (40) and finally the Gronwall Lemma to achieve the first part of the theorem.

6.3. *Proof of the last part*

If one is interested in getting a better rate of convergence (for fixed α) a further treatment of the viscous part of the right-hand side of (86) is needed. The underlying idea is somehow to put more derivatives on the inviscid solution u^E.

We use Lemma 1 to obtain

$$\begin{aligned} a(u^E, w) &= -\alpha \int_{\partial\mathcal{S}_0} (u^E - u^E_{\mathcal{S}}) \cdot (w - w_{\mathcal{S}}) - \int_{\mathcal{F}_0} D(u^E) \cdot D(w) \\ &= -\alpha \int_{\partial\mathcal{S}_0} (u^E - u^E_{\mathcal{S}}) \cdot (w - w_{\mathcal{S}}) + \frac{1}{2} \int_{\mathcal{F}_0} \Delta u^E \cdot w - \int_{\partial\mathcal{S}_0} \big(\big(D(u^E)n\big) \wedge n\big) \cdot \big((w - w_{\mathcal{S}}) \wedge n\big) \\ &\quad - \ell_w \cdot \int_{\partial\mathcal{S}_0} D(u^E) n \, ds - r_w \cdot \int_{\partial\mathcal{S}_0} x \wedge D(u^E) n \, ds. \end{aligned}$$

Thus we infer from (86) the following inequality:

$$\begin{aligned} &\|w(t,\cdot)\|^2_{\mathcal{H}} + 4\alpha\nu \int_0^t \int_{\partial\mathcal{S}_0} |w - w_{\mathcal{S}}|^2 + 4\nu \int_0^t \int_{\mathcal{F}_0} |D(w)|^2 \\ &\quad \leqslant \|w_0\|^2_{\mathcal{H}} - 4\alpha\nu \int_0^t \int_{\partial\mathcal{S}_0} (u^E - u^E_{\mathcal{S}}) \cdot (w - w_{\mathcal{S}}) \\ &\qquad + 2\nu \int_0^t \int_{\mathcal{F}_0} \Delta u^E \cdot w - 4\nu \int_0^t \int_{\partial\mathcal{S}_0} \big(\big(D(u^E)n\big) \wedge n\big) \cdot \big((w - w_{\mathcal{S}}) \wedge n\big) \\ &\qquad - 4\nu \int_0^t \ell_w \cdot \int_{\partial\mathcal{S}_0} D(u^E) n \, ds - 4\nu \int_0^t r_w \cdot \int_{\partial\mathcal{S}_0} x \wedge D(u^E) n \, ds - 2 \int_0^t b(w, w, u^E) \\ &\quad \leqslant \|w_0\|^2_{\mathcal{H}} + \sum_{i=1}^{6} I_i. \end{aligned}$$

To deal with I_1 and I_3 we will use the following lemma:

Lemma 3. *There exists $C > 0$ such that for any $\gamma > 0$ and for any smooth function f divergence free and tangent to the boundary*

$$\|f\|_{L^2(\partial\mathcal{S}_0)} \leqslant C\gamma^{1/3} \|f\|^{2/3}_{L^2(\mathcal{F}_0)} + \frac{1}{4\gamma} \|D(f)\|^2_{L^2(\mathcal{F}_0)} + C\|f\|_{L^2(\mathcal{F}_0)}.$$

Proof. We first apply the following standard trace inequality:

$$\|f\|_{L^2(\partial\mathcal{S}_0)} \leqslant C\|f\|^{1/2}_{L^2(\mathcal{F}_0)} \|f\|^{1/2}_{H^1(\mathcal{F}_0)},$$

then the Korn inequality to find

$$\|f\|_{L^2(\partial\mathcal{S}_0)} \leqslant C\big(\|f\|^{1/2}_{L^2(\mathcal{F}_0)} \|D(f)\|^{1/2}_{L^2(\mathcal{F}_0)} + \|f\|_{L^2(\mathcal{F}_0)}\big)$$

and finally Young's inequality to conclude. □

In order to apply this lemma we are going to substitute to the vector fields $u^E_{\mathcal{S}}$ and $w_{\mathcal{S}}$ another divergence free vector field with the same traces on $\partial\mathcal{S}_0$ but with a better decay at infinity, as we did in Section 3.2. Indeed let χ be a smooth cut-off function defined on $\overline{\mathcal{F}_0}$ such that $\chi = 1$ in Γ_c and $\chi = 0$ in $\mathcal{F}_0 \setminus \Gamma_{2c}$, where

$$\Gamma_c := \{x \in \mathcal{F}_0 \mid d(x) < c\} \quad \text{with } d(x) := \operatorname{dist}(x, \partial\mathcal{S}_0).$$

Let us denote

$$\psi^E_{\mathcal{S}}(t,x) := \frac{1}{2}\big(\ell^E(t) \wedge x - r^E(t)|x|^2\big) \quad \text{and} \quad \tilde{u}^E_{\mathcal{S}} := \operatorname{curl}\big(\chi \psi^E_{\mathcal{S}}\big),$$

and let us define similarly $\tilde{w}_{\mathcal{S}}$. By Cauchy–Schwarz inequality we obtain

$$I_1 \leqslant 4\alpha\nu \int_0^t \big\|u^E - \tilde{u}^E_{\mathcal{S}}\big\|_{L^2(\partial\mathcal{S}_0)} \|w - \tilde{w}_{\mathcal{S}}\|_{L^2(\partial\mathcal{S}_0)} \leqslant \alpha\nu C \int_0^t \|w - \tilde{w}_{\mathcal{S}}\|_{L^2(\partial\mathcal{S}_0)}$$

and then applying Lemma 3 with $\gamma = \frac{\alpha C}{4}$

$$\begin{aligned}
I_1 &\leqslant \nu \int_0^t \big\|D(w - \tilde{w}_{\mathcal{S}})\big\|^2_{L^2(\mathcal{F}_0)} + \alpha^{4/3}\nu C \int_0^t \|w - \tilde{w}_{\mathcal{S}}\|^{2/3}_{L^2(\mathcal{F}_0)} + \alpha\nu C \int_0^t \|w - \tilde{w}_{\mathcal{S}}\|_{L^2(\mathcal{F}_0)} \\
&\leqslant \nu \int_0^t \big\|D(w)\big\|^2_{L^2(\mathcal{F}_0)} + \nu C \int_0^t \|w\|^2_{\mathcal{H}} + \alpha^{4/3}\nu C \int_0^t \|w\|^{2/3}_{\mathcal{H}} + \alpha\nu C \int_0^t \|w\|_{\mathcal{H}}.
\end{aligned}$$

Now, thanks again to Cauchy–Schwarz inequality and applying Lemma 3 but this time with $\gamma = \frac{C}{4}$ we get

$$\begin{aligned}
I_3 &\leqslant 4\nu \int_0^t \big\|D\big(u^E\big)n\big\|_{L^2(\partial\mathcal{S}_0)} \|w - \tilde{w}_{\mathcal{S}}\|_{L^2(\partial\mathcal{S}_0)} \\
&\leqslant \nu C \int_0^t \|w - \tilde{w}_{\mathcal{S}}\|_{L^2(\partial\mathcal{S}_0)} \\
&\leqslant \nu \int_0^t \big\|D(w)\big\|^2_{L^2(\mathcal{F}_0)} + \nu C \int_0^t \|w\|^2_{\mathcal{H}} + \nu C \int_0^t \|w\|^{2/3}_{\mathcal{H}} + \nu C \int_0^t \|w\|_{\mathcal{H}}.
\end{aligned}$$

We simply estimate the following term by

$$I_2 \leqslant 2\nu \int_0^t \big\|\Delta u^E\big\|_{L^2(\mathcal{F}_0)} \|w\|_{L^2(\mathcal{F}_0)} \leqslant \nu C \int_0^t \|w\|_{L^2(\mathcal{F}_0)}.$$

Finally it is straightforward that we can estimate the last terms by

$$I_4 + I_5 \leqslant \nu C \int_0^t \|w\|_{\mathcal{H}} \quad \text{and} \quad I_6 \leqslant C \int_0^t \|w\|^2_{\mathcal{H}}$$

thanks to (40).

We deduce from the above relations that

$$\big\|w(t)\big\|^2_{\mathcal{H}} + 2\nu \int_0^t \big\|D(w)\big\|^2_{L^2(\mathcal{F}_0)} \leqslant \beta(t) + C \int_0^t \|w\|^2_{\mathcal{H}},$$

with

$$\beta(t) := \|w_0\|_{\mathcal{H}}^2 + \nu C \int_0^t \|w\|_{\mathcal{H}}^2 + \left(1+\alpha^{4/3}\right)\nu C \int_0^t \|w\|_{\mathcal{H}}^{2/3} + (1+\alpha)\nu C \int_0^t \|w\|_{\mathcal{H}}.$$

Since $\|w(t)\|_{\mathcal{H}} \leqslant C$ we can deduce that

$$\beta(t) \leqslant \|w_0\|_{\mathcal{H}}^2 + \left(1+\alpha+\alpha^{4/3}\right)\nu t C \|w\|_{L^\infty(0,T;\mathcal{H})}^{2/3},$$

and finally, the Gronwall Lemma implies that

$$\begin{aligned}\|w(t)\|_{\mathcal{H}}^2 &\leqslant \|w_0\|_{\mathcal{H}}^2 + \left(1+\alpha+\alpha^{4/3}\right)\nu t C \|w\|_{L^\infty(0,T;\mathcal{H})}^{2/3} \\ &\quad + C \int_0^t \left(\|w_0\|_{\mathcal{H}}^2 + s\left(1+\alpha+\alpha^{4/3}\right)\nu C \|w\|_{L^\infty(0,T;\mathcal{H})}^{2/3}\right) ds \exp(Ct) \\ &\leqslant C\left(\|w_0\|_{\mathcal{H}}^2 + \left(1+\alpha+\alpha^{4/3}\right)\nu \|w\|_{L^\infty(0,T;\mathcal{H})}^{2/3}\right),\end{aligned}$$

where C is a constant depending on T.

By setting $u_0 = u_0^E$, we obtain estimate (85).

The proof of Theorem 5 is then complete.

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