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On a system of nonlinear Schrödinger equations with quadratic interaction

Nakao Hayashi [a,*], Tohru Ozawa [b], Kazunaga Tanaka [c]

[a] *Department of Mathematics, Graduate School of Science, Osaka University, Osaka, Tokyonaka, Japan*
[b] *Department of Applied Physics, Waseda University, Tokyo 169-8555, Japan*
[c] *Department of Mathematics, Waseda University, Tokyo 169-8555, Japan*

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Dedicated to the memory of Professor Riichi Iino

Abstract

We study a system of nonlinear Schrödinger equations with quadratic interaction in space dimension $n \leqslant 6$. The Cauchy problem is studied in L^2, in H^1, and in the weighted L^2 space $\langle x\rangle^{-1}L^2 = \mathcal{F}(H^1)$ under mass resonance condition, where $\langle x\rangle = (1+|x|^2)^{1/2}$ and $\mathcal{F}$ is the Fourier transform. The existence of ground states is studied by variational methods. Blow-up solutions are presented in an explicit form in terms of ground states under mass resonance condition, which ensures the invariance of the system under pseudo-conformal transformations.

1. Introduction

We study the system of nonlinear Schrödinger equations:

$$\begin{cases} i\partial_t u + \dfrac{1}{2m}\Delta u = \lambda v\bar{u}, \\ i\partial_t v + \dfrac{1}{2M}\Delta v = \mu u^2, \end{cases} \tag{1}$$

where u and v are complex-valued functions of $(t,x) \in \mathbb{R}\times\mathbb{R}^n$, Δ is the Laplacian in $\mathbb{R}^n$, m and M are positive constants, λ and μ are complex constants, and $\bar{u}$ is the complex conjugate of u. Here the interaction terms in the system (1) are quadratic in (u,v). By the standard scaling arguments on (1), the critical function space is $H^{n/2-2}$, where $H^s = (1-\Delta)^{-s/2}L^2$ is the usual Sobolev space of order s (see [3,13,19]). Particularly, L^2 and H^1 are critical spaces for $n=4$ and $n=6$, respectively, from the scaling point of view. Those spaces are also important from the point of view of the invariance under group of motion. L^2 is naturally associated with the conservation of charge,

* Corresponding author.
E-mail address: nhayashi@math.sci.osaka-u.ac.jp (N. Hayashi).

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which follows from invariance under Gauge transform. H^1 is naturally associated with the conservation of energy, which follows from invariance under time-translation.

The system (1) is regarded as a non-relativistic limit of the system of nonlinear Klein–Gordon equations

$$\begin{cases} \frac{1}{2c^2m}\partial_t^2 u - \frac{1}{2m}\Delta u + \frac{mc^2}{2}u = -\lambda v\bar{u}, \\ \frac{1}{2c^2M}\partial_t^2 v - \frac{1}{2M}\Delta v + \frac{Mc^2}{2}v = -\mu u^2, \end{cases} \tag{2}$$

under the mass resonance condition

$$M = 2m \tag{3}$$

since the modulated wave functions $(u_c, v_c) = (e^{itmc^2}u, e^{itMc^2}v)$ satisfy

$$\begin{cases} \frac{1}{2c^2m}\partial_t^2 u_c - i\partial_t u_c - \frac{1}{2m}\Delta u_c = -e^{itc^2(2m-M)}\lambda v_c\overline{u_c}, \\ \frac{1}{2c^2M}\partial_t^2 v_c - i\partial_t v_c - \frac{1}{2M}\Delta v_c = -e^{itc^2(M-2m)}\mu u_c^2, \end{cases} \tag{4}$$

where the phase oscillations on the right hand sides vanish if and only if (3) holds, and under the mass resonance condition (3) the system (4) formally yields (1) as the speed of light c tends to infinity.

The system (2) is closely related to systems studied in [1,7,9] for instance. As regards the non-relativistic limit for the nonlinear Klein–Gordon equations, we refer the reader to [17,18] and references therein. For recent works related to the mass resonance, see [12,24,25].

The Cauchy problem for (1) has been studied from the point of view of small data scattering [10,11]. The purpose of this paper is to study the Cauchy problem for (1) with large data, namely, data which are not necessarily small enough.

The argument in Section 3 is rather standard. We describe it for convenience of readers. Local Cauchy problem is studied in L^2 and in H^1 respectively in Sections 3.1 and 3.2 by a contraction argument based on the Strichartz estimates. To extend local solutions we use a priori estimates, which follow from conservation laws of charge and energy. We show that those conservation laws hold if and only if there exists $c \in \mathbb{R} \setminus \{0\}$ such that $\lambda = c\overline{\mu}$ (Theorems 3.3 and 3.5 below). On the basis of those conservation laws, we prove the existence of unique global solutions in L^2 and in H^1 regardless of the size of the Cauchy data respectively in Sections 3.3 and 3.4. Local Cauchy problem with the data at $t = 0$ in the weighted L^2 space $\langle x\rangle^{-1}L^2 = \mathcal{F}H^1$ is discussed in Section 3.5 under the mass resonance condition, which ensures the invariance of (1) under Galilei transformations. In Section 3.6 we prove the pseudo-conformal identity and apply it to the proof of the existence of unique global solutions with data at $t = 0$ in $\mathcal{F}H^1$. In Section 3.7, we derive the virial identity from the energy and pseudo-conformal identities and apply it to the proof of the non-existence of global solutions of negative energy with data in $H^1 \cap \mathcal{F}H^1$. Section 4 is devoted to the existence of ground states for (1), which are defined as minimizers of action integrals for standing waves for (1) at frequency $(\omega, 2\omega)$ with $\omega > 0$. The method of proof depends on Strauss' compact embedding of the space of radially symmetric H^1 functions into L^3: $H_r^1 \subset L^3$ for $2 \leqslant n \leqslant 5$ and on the concentration-compactness argument for $n = 1$. In Section 5, we prove that the best constant in a Gagliardo–Nirenberg type inequality for $n = 4$ is formulated in a variational setting and characterized by ground states at frequency $(\omega, 2\omega) = (1, 2)$. In Section 6, we prove the existence of threshold on the size of charge of the Cauchy data for which the corresponding solutions to (1) are global in time for $n = 4$. Moreover, the threshold is calculated in terms of the ground states from Section 5. This result is regarded as an analogue to Weinstein's theory in the pseudo-conformal invariant case [26,27]. Under the mass resonance condition (3), we present an explicit representation formula of blow-up solutions at the threshold by means of the ground states from Section 5. In Section 7, we study the inverse condition of mass resonance, namely, $m = 2M$, which reduces the problem of the system (1) to the corresponding problem of a single equation. We characterize the structure of ground states for a quadratic scalar field equation, which could clarify how the inverse mass ratio affects the motion of semitrivial standing waves [4,5,14,15]. Existence of stationary solutions to (1) for $n = 6$ is also discussed in this setting. In Section 8, we study (1) for $n = 1$ in the framework of Lagrangian systems.

2. Preliminaries

In this section we collect basic notation and lemmas which will be used subsequent sections. We refer the reader to [2,22,23] for general information. For any p with $1 \leqslant p \leqslant \infty$, $L^p = L^p(\mathbb{R}^n)$ denotes the Lebesgue space on $\mathbb{R}^n$. The usual scalar product on L^2 or $(L^2)^n$ is denoted by $(\cdot,\cdot)$. For any p with $1 \leqslant p \leqslant \infty$ and any non-negative integer m, W_p^m, denotes the usual Sobolev space of order m built over L^p. If $p = 2$, W_2^m is also written as H^m. For any interval $I \subset \mathbb{R}$ and any Banach space X, we denote by $C(I; X)$ the space of strongly continuous functions from I to X and by $L^p(I; X)$ the space of strongly measurable functions u from I to X such that $\|u(\cdot); X\| \in L^p(I)$. For any p with $1 \leqslant p \leqslant \infty$, p' is the dual exponent defined by $1/p + 1/p' = 1$. For any $a, b \in \mathbb{R}$, $a \vee b = \max(a, b)$. The Cauchy problem for (1) with data $(u(t_0), v(t_0)) = (u_0, v_0)$ given at $t = t_0$ will be treated in the form of the following system of integral equations:

$$\begin{cases} u(t) = U_m(t - t_0)u_0 - i \displaystyle\int_{t_0}^{t} U_m\big(t - t'\big)\lambda v\big(t'\big)\bar{u}\big(t'\big)\,dt', \\ v(t) = U_M(t - t_0)v_0 - i \displaystyle\int_{t_0}^{t} U_M\big(t - t'\big)\mu u^2\big(t'\big)\,dt', \end{cases} \tag{5}$$

where $U_m(t) = \exp(i\frac{t}{2m}\Delta)$ and $U_M(t) = \exp(i\frac{t}{2M}\Delta)$ are free propagators with masses m and M, respectively. A pair of indices (q, r) with $2 \leqslant q, r \leqslant \infty$ is called admissible if $0 \leqslant 2/q = n/2 - n/r \leqslant 1$ with the exception $(n, q, r) = (2, \infty, 2)$.

We use the following Strichartz estimates without particular comments.

Proposition 2.1. *Let $n \geqslant 1$ and let (q, r) and (q_j, r_j) be admissible for $j = 1, 2$. Then the following estimates hold*

$$\big\|U_m(\cdot)\phi; L^q\big(\mathbb{R}; L^r\big)\big\| \leqslant C\big\|\phi; L^2\big\|$$

and

$$\big\|G_{t_0} f; L^{q_2}\big(I; L^{r_2}\big)\big\| \leqslant C\big\|f; L^{q_1'}\big(I; L^{r_1'}\big)\big\|,$$

where $t_0 \in \mathbb{R}$, $I \subset \mathbb{R}$ is an interval with $t_0 \in \bar{I}$, G_{t_0} is the integral operator defined as

$$(G_{t_0} f)(t) = \int_{t_0}^{t} U_m\big(t - t'\big) f\big(t'\big)\,dt', \quad t \in I,$$

and C is a constant independent of t_0, I, and f.

We use Proposition 2.1 to obtain local solutions to (5) by a contraction argument. To be more specific, local solutions to (5) are constructed as a pair of fixed point (u, v) of contraction mapping $(u, v) \mapsto (\Phi(u, v), \Psi(u, v))$, where

$$\begin{cases} \big(\Phi(u, v)\big)(t) = U_m(t - t_0)u_0 - i \displaystyle\int_{t_0}^{t} U_m\big(t - t'\big)\lambda v\big(t'\big)\bar{u}\big(t'\big)\,dt', \\ \big(\Psi(u, v)\big)(t) = U_M(t - t_0)v_0 - i \displaystyle\int_{t_0}^{t} U_M\big(t - t'\big)\mu u^2\big(t'\big)\,dt', \end{cases} \tag{6}$$

on a suitable complete metric space of functions on $I = [t_0 - T, t_0 + T]$ for some $T > 0$.

3. Existence of solutions and non-existence of global solutions

3.1. Local existence of H^1-solutions

In view of the scaling argument and available results on the Cauchy problem for a single nonlinear Schrödinger equation with power nonlinearities, it is natural to treat (5) in L^2 space for $n \leqslant 4$. For any $u_0, v_0 \in L^2$ we solve (5) in the spaces

$$X(I) = (C \cap L^\infty)(I; L^2) \cap L^4(I; L^\infty) \quad \text{for } n = 1,$$
$$X(I) = (C \cap L^\infty)(I; L^2) \cap L^{q_0}(I; L^{r_0}) \quad \text{for } n = 2,$$

where $0 < 2/q_0 = 1 - 2/r_0 < 1$ with r_0 sufficiently large,

$$X(I) = (C \cap L^\infty)(I; L^2) \cap L^2(I; L^{2n/(n-2)}) \quad \text{for } n \geqslant 3$$

on the time interval $I = [t_0 - T, t_0 + T]$ with $T > 0$. The associated norms are defined

$$\|u; X(I)\| = \|u; L^\infty(L^2)\| \vee \|u; L^4(L^\infty)\| \quad \text{for } n = 1,$$
$$\|u; X(I)\| = \|u; L^\infty(L^2)\| \vee \|u; L^{q_0}(L^{r_0})\| \quad \text{for } n = 2,$$
$$\|u; X(I)\| = \|u; L^\infty(L^2)\| \vee \|u; L^2(L^{2n/(n-2)})\| \quad \text{for } n \geqslant 3.$$

Theorem 3.1. *If $n \leqslant 3$, then for any $\rho > 0$ there exists $T(\rho) > 0$ such that for any $(u_0, v_0) \in L^2 \times L^2$ with $\|u_0; L^2\| \vee \|v_0; L^2\| \leqslant \rho$, (5) has a unique pair of solutions $(u, v) \in X(I) \times X(I)$ with $I = [t_0 - T(\rho), t_0 + T(\rho)]$. If $n = 4$, then for any $(u_0, v_0) \in L^2 \times L^2$, there exists $T(u_0, v_0) > 0$ such that (5) has a unique pair of solutions $(u, v) \in X(I) \times X(I)$ with $I = [t_0 - T(u_0, v_0), t_0 + T(u_0, v_0)]$.*

Proof. We first consider the case $n = 1$. We estimate $\Phi(u, v)$ and $\Psi(u, v)$ as

$$\begin{aligned}\|\Phi(u, v); X(I)\| &\leqslant C\|u_0; L^2\| + C\|\bar{u}v; L^1(L^2)\| \\ &\leqslant C\|u_0; L^2\| + CT^{3/4}\|v; L^4(L^\infty)\|\|u; L^\infty(L^2)\|, \\ \|\Psi(u, v); X(I)\| &\leqslant C\|v_0; L^2\| + CT^{3/4}\|u; L^4(L^\infty)\|\|u; L^\infty(L^2)\|.\end{aligned}$$

Similarly,

$$\begin{aligned}\|\Phi(u, v) - \Phi(u', v'); X(I)\| &\leqslant CT^{3/4}(\|u'; L^4(L^\infty)\| + \|v; L^4(L^\infty)\|)(\|u - u'; L^\infty(L^2)\| \\ &\quad + \|v - v'; L^\infty(L^2)\|), \\ \|\Psi(u, v) - \Psi(u', v'); X(I)\| &\leqslant CT^{3/4}(\|u'; L^4(L^\infty)\| + \|u; L^4(L^\infty)\|)\|u - u'; L^\infty(L^2)\|.\end{aligned}$$

We next consider the case $n = 2$. We estimate $\Phi(u, v)$ and $\Psi(u, v)$ as

$$\begin{aligned}\|\Phi(u, v); X(I)\| &\leqslant C\|u_0; L^2\| + C\|\bar{u}v; L^{q_0'}(L^{r_0'})\| \\ &\leqslant C\|u_0; L^2\| + CT^{r_0/(r_0+2)}\|v; L^{r_0}(L^{2r_0/(r_0-2)})\|\|u; L^\infty(L^2)\|, \\ \|\Psi(u, v); X(I)\| &\leqslant C\|v_0; L^2\| + CT^{r_0/(r_0+2)}\|u; L^{r_0}(L^{2r_0/(r_0-2)})\|\|u; L^\infty(L^2)\|.\end{aligned}$$

Similarly,

$$\begin{aligned}\|\Phi(u, v) - \Phi(u', v'); X(I)\| &\leqslant CT^{r_0/(r_0+2)}(\|u'; L^{r_0}(L^{2r_0/(r_0-2)})\| + \|v; L^{r_0}(L^{2r_0/(r_0-2)})\|) \\ &\quad \times (\|u - u'; L^\infty(L^2)\| + \|v - v'; L^\infty(L^2)\|), \\ \|\Psi(u, v) - \Psi(u', v'); X(I)\| &\leqslant CT^{r_0/(r_0+2)}(\|u'; L^{r_0}(L^{2r_0/(r_0-2)})\| \\ &\quad + \|u; L^{r_0}(L^{2r_0/(r_0-2)})\|)\|u - u'; L^\infty(L^2)\|.\end{aligned}$$

Note that $\|u; L^{r_0}(L^{2r_0/(r_0-2)})\| \leqslant \|u; L^{q_0}(L^{r_0})\|^{2/(r_0-2)}\|u; L^\infty(L^2)\|^{(r_0-4)/(r_0-2)}$. We now consider the case $n = 3$. We estimate $\Phi(u, v)$ and $\Psi(u, v)$ as

$$\begin{aligned}\|\Phi(u,v);X(I)\| &\leqslant C\|u_0;L^2\|+C\|\bar{u}v;L^{4/3}(L^{3/2})\|\\&\leqslant C\|u_0;L^2\|+CT^{1/4}\|v;L^2(L^6)\|\|u;L^\infty(L^2)\|,\\\|\Psi(u,v);X(I)\| &\leqslant C\|v_0;L^2\|+CT^{1/4}\|u;L^2(L^6)\|\|u;L^\infty(L^2)\|.\end{aligned}$$

Similarly,

$$\begin{aligned}\|\Phi(u,v)-\Phi(u',v');X(I)\| &\leqslant CT^{1/4}\big(\|u';L^2(L^6)\|+\|v;L^2(L^6)\|\big)\big(\|u-u';L^\infty(L^2)\|\\&\quad+\|v-v';L^\infty(L^2)\|\big),\\\|\Psi(u,v)-\Psi(u',v');X(I)\| &\leqslant CT^{1/4}\big(\|u';L^2(L^6)\|+\|u;L^2(L^6)\|\big)\|u-u';L^\infty(L^2)\|.\end{aligned}$$

Therefore for $n\leqslant 3$ we have obtained the following estimates:

$$\begin{aligned}&\|\Phi(u,v);X(I)\|\leqslant C\|u_0;L^2\|+CT^{1-n/4}\|u;X(I)\|\|v;X(I)\|,\\&\|\Psi(u,v);X(I)\|\leqslant C\|v_0;L^2\|+CT^{1-n/4}\|u;X(I)\|^2,\\&\|\Phi(u,v)-\Phi(u',v');X(I)\|\leqslant CT^{1-n/4}\big(\|u';X(I)\|+\|v;X(I)\|\big)\big(\|u-u';X(I)\|+\|v-v';X(I)\|\big),\\&\|\Psi(u,v)-\Psi(u',v');X(I)\|\leqslant CT^{1-n/4}\big(\|u';X(I)\|+\|u;X(I)\|\big)\|u-u';X(I)\|.\end{aligned}$$

Then the standard contraction argument on $(u,v)\mapsto(\Phi(u,v),\Psi(u,v))$ on a closed ball in $X(I)\times X(I)$ goes through by taking $T>0$ sufficiently small with respect to $\rho>0$ via radius of the ball. This yields the existence and uniqueness of local solutions on $[t_0-T,t_0+T]$ under the size restriction on the radius. The uniqueness of solutions without the size restriction of the radius follows by a similar argument by taking the size of successive time interval sufficiently small.

We finally consider the case $n=4$. We estimate $\Phi(u,v)$ and $\Psi(u,v)$ in $L^2(I;L^4)$ as

$$\begin{aligned}\|\Phi(u,v);L^2(L^4)\| &\leqslant C\|U_m(\cdot)u_0;L^2(L^4)\|+C\|\bar{u}v;L^1(L^2)\|\\&\leqslant C\|U_m(\cdot)u_0;L^2(L^4)\|+C\|v;L^2(L^4)\|\|u;L^2(L^4)\|,\\\|\Psi(u,v);L^2(L^4)\| &\leqslant C\|U_M(\cdot)v_0;L^2(L^4)\|+C\|u;L^2(L^4)\|^2.\end{aligned}$$

Similarly,

$$\begin{aligned}&\|\Phi(u,v)-\Phi(u',v');L^2(L^4)\|\leqslant C\big(\|u';L^2(L^4)\|+\|v;L^2(L^4)\|\big)\big(\|u-u';L^2(L^4)\|+\|v-v';L^2(L^4)\|\big),\\&\|\Psi(u,v)-\Psi(u',v');L^2(L^4)\|\leqslant C\big(\|u';L^2(L^4)\|+\|u;L^2(L^4)\|\big)\|u-u';L^2(L^4)\|.\end{aligned}$$

For u_0, v_0 we know that $U_m(\cdot)u_0, U_M(\cdot)v_0\in L^2(\mathbb{R};L^4)$ and therefore the associated norms may be taken arbitrarily small by taking $T>0$ sufficiently small. Therefore the contraction argument works on a closed ball in $L^2(I;L^4)$ with center at the origin and radius sufficiently small. Then the solution satisfies the integral equations (5) and then belongs to $X(I)$ by the Strichartz estimates. □

3.2. *Local existence of H^1-solutions*

In view of the scaling argument and available results on the Cauchy problem for a single nonlinear Schrödinger equation with power nonlinearities, it is natural to treat (5) in H^1 space for $n\leqslant 6$. For any $u_0,v_0\in H^1$ we solve (5) in the spaces

$$\begin{aligned}&Y(I)=(C\cap L^\infty)(I;H^1)\cap L^4(I;W^1_\infty)\quad\text{for } n=1,\\&Y(I)=(C\cap L^\infty)(I;H^1)\cap L^{q_0}(I;W^1_{r_0})\quad\text{for } n=2,\end{aligned}$$

where $0<2/q_0=1-2/r_0<1$ with r_0 sufficiently large,

$$Y(I)=(C\cap L^\infty)(I;H^1)\cap L^2(I;W^1_{2n/(n-2)})\quad\text{for } n\geqslant 3$$

on the time interval $I=[t_0-T,t_0+T]$ with $T>0$. The associated norms are defined by

$$\|u;Y(I)\| = \|u;L^\infty(H^1)\| \vee \|u;L^4(W^1_\infty)\| \quad \text{for } n=1,$$
$$\|u;Y(I)\| = \|u;L^\infty(H^1)\| \vee \|u;L^{q_0}(W^1_{r_0})\| \quad \text{for } n=2,$$
$$\|u;Y(I)\| = \|u;L^\infty(H^1)\| \vee \|u;L^2(W^1_{2n/(n-2)})\| \quad \text{for } n\geqslant 3.$$

Theorem 3.2. *If $n \leqslant 5$, then for any $\rho > 0$ there exists $T(\rho) > 0$ such that for any $(u_0, v_0) \in H^1 \times H^1$ with $\|u_0; H^1\| \vee \|v_0; H^1\| \leqslant \rho$, (5) has a unique pair of solutions $(u, v) \in Y(I) \times Y(I)$ with $I = [t_0 - T(\rho), t_0 + T(\rho)]$. If $n = 6$, then for any $(u_0, v_0) \in H^1 \times H^1$, there exists $T(u_0, v_0) > 0$ such that (5) has a unique pair of solutions $(u, v) \in Y(I) \times Y(I)$ with $I = [t_0 - T(u_0, v_0), t_0 + T(u_0, v_0)]$.*

Proof. We first consider the case $n \leqslant 3$. The contraction argument in $Y(I)$ works in the same way as in the proof of Theorem 3.1 since necessary estimates are those of first derivatives of $\Phi(u, v)$ and $\Psi(u, v)$, which depend on (u, v) essentially in a bilinear way. To be specific, we obtain

$$\|\Phi(u,v);Y(I)\| \leqslant C\|u_0;H^1\| + CT^{1-n/4}\|u;Y(I)\|\|v;Y(I)\|,$$
$$\|\Psi(u,v);Y(I)\| \leqslant C\|v_0;H^1\| + CT^{1-n/4}\|u;Y(I)\|^2,$$
$$\|\Phi(u,v)-\Phi(u',v');Y(I)\| \leqslant CT^{1-n/4}\big(\|u';Y(I)\| + \|v;Y(I)\|\big)\big(\|u-u';Y(I)\| + \|v-v';Y(I)\|\big),$$
$$\|\Psi(u,v)-\Psi(u',v');Y(I)\| \leqslant CT^{1-n/4}\big(\|u';Y(I)\| + \|u;Y(I)\|\big)\|u-u';Y(I)\|,$$

from which the conclusion follows for $n \leqslant 3$. We next consider the case $4 \leqslant n \leqslant 6$. In this case the pair $(4/(n-4), n/2)$ is admissible and the corresponding dual is given by $(4/(8-n), n/(n-2))$. As for the estimates on the Duhamel terms, the following bilinear estimate plays an essential role:

$$\|uv;L^{4/(8-n)}(I;W^1_{n/(n-2)})\| \leqslant CT^{3/2-n/4}\|u;L^\infty(I;H^1)\|\|v;L^2(I;W^1_{2n/(n-2)})\|.$$

Then the conclusion follows in the same way as in the proof of Theorem 3.1. □

3.3. Global existence of L^2-solutions

Let $n \leqslant 4$ and let $(u, v) \in X(I) \times X(I)$ be the unique pair of local solutions of (5) given in Theorem 3.1. Then in the same way as in [20], we have

$$\|u(t);L^2\|^2 = \|u_0;L^2\|^2 + 2\,\mathrm{Im}\int_{t_0}^{t}\big(\lambda v(t'), u^2(t')\big)\,dt',$$
$$\|v(t);L^2\|^2 = \|v_0;L^2\|^2 + 2\,\mathrm{Im}\int_{t_0}^{t}\big(\mu u^2(t'), v(t')\big)\,dt'$$

for all $t \in I$, where the last integrals of the right hand side are understood to be a duality between $L^q(I;L^r)$ and $L^{q'}(I;L^{r'})$. For the conservation law of total charge it is natural to consider the following condition:

$$\text{There exists a constant } c \in \mathbb{R}\backslash\{0\} \text{ such that } \lambda = c\overline{\mu}. \tag{7}$$

In fact, we have

Theorem 3.3. *Let $n \leqslant 4$ and let λ and μ satisfy (7). Then the unique pair of local solutions $(u, v) \in X(I) \times X(I)$ of (5) given by Theorem 3.1 satisfies the following conservation law for all $t \in I$*

$$\|u(t);L^2\|^2 + c\|v(t);L^2\|^2 = \|u_0;L^2\|^2 + c\|v_0;L^2\|^2.$$

We now state the existence and uniqueness of global L^2 solutions on the basis of the function space $X(\mathbb{R})$:

$$X(\mathbb{R}) = \big(C \cap L^\infty\big)\big(\mathbb{R}; L^2\big) \cap L^4_{loc}\big(\mathbb{R}; L^\infty\big) \quad \text{for } n = 1,$$
$$X(\mathbb{R}) = \big(C \cap L^\infty\big)\big(\mathbb{R}; L^2\big) \cap L^q_{loc}\big(\mathbb{R}; L^r\big) \quad \text{for } n = 2,$$

where $0 < 2/q = 1 - 2/r < 1$ with r sufficiently large,

$$X(\mathbb{R}) = \big(C \cap L^\infty\big)\big(\mathbb{R}; L^2\big) \cap L^2_{loc}\big(\mathbb{R}; L^6\big) \quad \text{for } n = 3.$$

Theorem 3.4. *Let $n \leqslant 3$ and let λ and μ satisfy (7) with $c > 0$. Then for any $(u_0, v_0) \in L^2 \times L^2$, (5) has a unique pair of solutions $(u, v) \in X(\mathbb{R}) \times X(\mathbb{R})$. Moreover,*

$$\big\|u(t); L^2\big\|^2 + c\big\|v(t); L^2\big\|^2 = \big\|u_0; L^2\big\|^2 + c\big\|v_0; L^2\big\|^2$$

for all $t \in \mathbb{R}$.

Proof. The theorem follows from Theorem 3.1 and Theorem 3.3 by the standard continuation argument of local solutions. □

3.4. Global existence of H^1-solutions

Let $n \leqslant 6$ and let $(u, v) \in Y(I) \times Y(I)$ be the unique pair of local solutions of (5) given by Theorem 3.2. Then in the same way as in [20], we have

$$\big\|\nabla u(t); L^2\big\|^2 = \big\|\nabla u_0; L^2\big\|^2 - 2m \operatorname{Re} \int_{t_0}^{t} \big(\lambda v(t'), \partial_t\big(u^2\big)(t')\big)\, dt',$$

$$\big\|\nabla v(t); L^2\big\|^2 = \big\|\nabla v_0; L^2\big\|^2 - 4M \operatorname{Re} \int_{t_0}^{t} \big(\mu u^2(t'), \partial_t v(t')\big)\, dt'$$

for all $t \in I$. Therefore we have

Theorem 3.5. *Let $n \leqslant 6$ and let λ and μ satisfy (7). Then the unique pair of local solutions $(u, v) \in Y(I) \times Y(I)$ of (5) given by Theorem 3.2 satisfies the following conservation law for all $t \in I$*

$$\begin{aligned}
&\frac{1}{2m}\big\|\nabla u(t); L^2\big\|^2 + \frac{c}{4M}\big\|\nabla v(t); L^2\big\|^2 + \operatorname{Re}\big(\lambda\big(v(t), u^2(t)\big)\big) \\
&\quad = \frac{1}{2m}\big\|\nabla u_0; L^2\big\|^2 + \frac{c}{4M}\big\|\nabla v_0; L^2\big\|^2 + \operatorname{Re}\big(\lambda\big(v_0, u_0^2\big)\big).
\end{aligned}$$

We now state the existence and uniqueness of global H^1 solutions on the basis of the function space $Y(\mathbb{R})$:

$$Y(\mathbb{R}) = \big(C \cap L^\infty\big)\big(\mathbb{R}; H^1\big) \cap L^4_{loc}\big(\mathbb{R}; W^1_\infty\big) \quad \text{for } n = 1,$$
$$Y(\mathbb{R}) = \big(C \cap L^\infty\big)\big(\mathbb{R}; H^1\big) \cap L^q_{loc}\big(\mathbb{R}; W^1_r\big) \quad \text{for } n = 2,$$

where $0 < 2/q = 1 - 2/r < 1$ with r sufficiently large,

$$Y(\mathbb{R}) = \big(C \cap L^\infty\big)\big(\mathbb{R}; H^1\big) \cap L^2_{loc}\big(\mathbb{R}; W^1_{2n/(n-2)}\big) \quad \text{for } n \geqslant 3.$$

To obtain an a priori estimate of solutions in $H^1 \times H^1$, it is convenient to introduce the following functionals

$$Q(\phi, \psi) = \big\|\phi; L^2\big\|^2 + c\big\|\psi; L^2\big\|^2,$$
$$K(\phi, \psi) = \frac{1}{2m}\big\|\nabla\phi; L^2\big\|^2 + \frac{c}{4M}\big\|\nabla\psi; L^2\big\|^2,$$
$$P(\phi, \psi) = \operatorname{Re} \int \phi^2 \overline{\psi}\, dx$$

and

$$\alpha_0 = \inf\{J_0(\phi,\psi);\ (\phi,\psi)\in H^1\times H^1\},$$

where

$$J_0(\phi,\psi) = K(\phi,\psi)Q(\phi,\psi)^{1/2}/P(|\phi|,|\psi|).$$

Lemma 3.6. *Let $n = 4$ and let $m, M, c > 0$. Then there exists a constant $C_0 > 0$ such that*

$$P(|\phi|,|\psi|) \leqslant C_0 K(|\phi|,|\psi|)Q(|\phi|,|\psi|)^{1/2} \leqslant C_0 K(\phi,\psi)Q(\phi,\psi)^{1/2}$$

for all $(\phi,\psi)\in H^1\times H^1$.

Proof. By the Gagliardo–Nirenberg inequality:

$$\|\phi; L^3\| \leqslant C\|\nabla\phi; L^2\|^{2/3}\|\phi; L^2\|^{1/3},$$

we obtain

$$\begin{aligned} P(|\phi|,|\psi|) &\leqslant C^3\|\nabla|\phi|; L^2\|^{4/3}\||\phi|; L^2\|^{2/3}\|\nabla|\psi|; L^2\|^{2/3}\||\psi|; L^2\|^{1/3} \\ &\leqslant C^3\big(2mK(|\phi|,|\psi|)\big)^{2/3}Q(\phi,\psi)^{1/3}\left(\frac{4M}{c}K(|\phi|,|\psi|)\right)^{1/3}\left(\frac{1}{c}Q(\phi,\psi)\right)^{1/6} \\ &= C^3\big(16m^2M\big)^{1/3}c^{-1/2}K(|\phi|,|\psi|)Q(\phi,\psi)^{1/2}. \qquad \Box \end{aligned}$$

Theorem 3.7. *Let $n \leqslant 4$ and let λ and μ satisfy (7) with $c > 0$. If $n \leqslant 3$, then for any $(u_0, v_0)\in H^1\times H^1$, (5) has a unique pair of solutions $(u,v)\in Y(\mathbb{R})\times Y(\mathbb{R})$. If $n = 4$, then for any $(u_0,v_0)\in H^1\times H^1$ with*

$$|\lambda|Q(u_0,v_0)^{1/2} < \alpha_0$$

(5) *has a unique pair of solutions $(u,v)\in Y(\mathbb{R})\times Y(\mathbb{R})$.*

Proof. By the standard continuation argument, it suffices to obtain a priori estimates on H^1 norms of u and v. By the following Gagliardo–Nirenberg inequality

$$\|\phi; L^3\| \leqslant C\|\nabla\phi; L^2\|^{n/6}\|\phi; L^2\|^{1-n/6},$$

we estimate the interaction term in the energy as

$$\begin{aligned} \big|\lambda(u^2,v)\big| &\leqslant |\lambda|C^3\|\nabla u; L^2\|^{n/3}\|u; L^2\|^{2-n/3}\|\nabla v; L^2\|^{n/6}\|v; L^2\|^{1-n/6} \\ &\leqslant |\lambda|C^3(2mK)^{n/6}Q^{1-n/6}\left(\frac{4M}{c}K\right)^{n/12}\left(\frac{1}{c}Q\right)^{1/2-n/12} \\ &= |\lambda|C^3\big(16m^2M\big)^{n/12}c^{-1/2}K(u,v)^{n/4}Q(u,v)^{3/2-n/4} \\ &= |\lambda|C^3\big(16m^2M\big)^{n/12}c^{-1/2}Q(u_0,v_0)^{3/2-n/4}K(u,v)^{n/4}, \end{aligned}$$

where we have used the conservation of charge. If $n \leqslant 3$, then $n/4 < 1$ and the interaction term is dominated by an arbitrarily small constant multiple of the kinetic term of the form

$$\varepsilon K(u,v) + C_\varepsilon Q(u_0,v_0)^{(6-n)/(4-n)},$$

which implies the required a priori estimate. If $n = 4$, we estimate

$$\big|\lambda(u^2,v)\big| \leqslant |\lambda|P(|u|,|v|) \leqslant (|\lambda|/\alpha_0)Q(u_0,v_0)^{1/2}K(u,v)$$

by Lemma 3.6 and the required a priori estimate follows if the coefficient to $K(u,v)$ is less than 1. $\Box$

3.5. Galilei invariance of local solutions under mass resonance

Throughout this section we assume that $M = 2m$ and the mass in the second equation is denoted by $2m$. For the free propagator $U_m(t)$, we introduce the standard generator of Galilei transformations as

$$J_m = J_m(t) = U_m(t)xU_m(-t) = x + i\frac{t}{m}\nabla = M_m(t)i\frac{t}{m}\nabla M_m(-t),$$

where $M_m(t) = \exp(i\frac{m}{2t}|x|^2)$, $t \neq 0$. Then we have at least formally

$$\begin{aligned} J_m(v\bar{u}) &= M_m(t)i\frac{t}{m}\nabla\big(M_{2m}(-t)v \cdot \overline{M_m(-t)u}\big) \\ &= \left(M_{2m}(t)i\frac{t}{m}\nabla M_{2m}(-t)v\right)\bar{u} - v\left(\overline{M_m(t)i\frac{t}{m}\nabla M_m(-t)u}\right) \\ &= 2(J_{2m}v)\bar{u} - v\overline{J_m u} \end{aligned}$$

and

$$J_{2m}\big(u^2\big) = M_{2m}(t)i\frac{t}{2m}\nabla\big(M_m(-t)u\big)^2 = uM_m(t)i\frac{t}{m}\nabla M_m(-t)u = uJ_m u.$$

For any $u_0, v_0 \in L^2$ with $J_m(t_0)u_0, J_{2m}(t_0)v_0 \in L^2$ we solve (5) in the space $Z_m(I) \times Z_{2m}(I)$, where

$$Z_m(I) = \big\{u \in X(I);\ J_m u \in X(I)\big\}, \quad I = [t_0 - T, t_0 + T],\ T > 0$$

with norm

$$\|u; Z_m(I)\| = \|u; X(I)\| \vee \|J_m u; X(I)\|.$$

Theorem 3.8. *Let $n \leqslant 6$ and $M = 2m$. If $n \leqslant 5$, then for any $\rho > 0$ there exists $T(\rho) > 0$ such that for any $(u_0, v_0) \in L^2 \times L^2$ with $(J_m(t_0)u_0, J_{2m}(t_0)v_0) \in L^2 \times L^2$ and*

$$\|u_0; L^2\| \vee \|J_m(t_0)u_0; L^2\| \vee \|v_0; L^2\| \vee \|J_{2m}(t_0)v_0; L^2\| \leqslant \rho$$

(5) has a unique pair of solutions $(u, v) \in Z_m(I) \times Z_{2m}(I)$ with $I = [t_0 - T(\rho), t_0 + T(\rho)]$. If $n = 6$, then for any $(u_0, v_0) \in L^2$ with $(J_m(t_0)u_0, J_{2m}(t_0)v_0) \in L^2$ there exists $T(u_0, v_0) > 0$ such that (5) has a unique pair of solutions $(u, v) \in Z_m(I) \times Z_{2m}(I)$ with $I = [t_0 - T(u_0, v_0), t_0 + T(u_0, v_0)]$.

Remark 3.1. Theorem 3.8 ensures the existence of local solutions of (5) which leave the domain of Galilei generators invariant. In the case $t_0 = 0$, the theorem is regarded as a smoothing effect of solutions in terms of Galilei generators.

Proof of Theorem 3.8. Let $(u, v) \in Z_m(I) \times Z_{2m}(I)$ for $I = [t_0 - T, t_0 + T]$ with some $T > 0$. We apply $J_m(t)$ and $J_{2m}(t)$ to $\Phi(u, v)$ and $\Psi(u, v)$, respectively and use

$$J_m(v\bar{u}) = 2(J_{2m}v)\bar{u} - v\overline{J_m u},\ J_{2m}\big(u^2\big) = uJ_m u.$$

Then by a similar argument to that of proof of Theorem 3.1, we prove Theorem 3.8. □

3.6. Galilei invariance of global solutions under mass resonance

As in Section 7, we assume that the mass resonance condition $M = 2m$. Let $n \leqslant 6$ and let $(u, v) \in Z_m(I) \times Z_{2m}(I)$ be the unique pair of local solutions given by Theorem 3.8. Then in the same way as in [20], we have

$$\big\|J_m(t)u(t); L^2\big\|^2 = \big\|J_m(t_0)u_0; L^2\big\|^2 + 2\,\mathrm{Im}\int_{t_0}^{t}\big(\lambda J_m(s)(v\bar{u})(s), J_m(s)u(s)\big)\,ds,$$

$$\big\|J_{2m}(t)v(t); L^2\big\|^2 = \big\|J_{2m}(t_0)v_0; L^2\big\|^2 + 2\,\mathrm{Im}\int_{t_0}^{t}\big(\mu J_{2m}(s)\big(u^2\big)(s), J_{2m}(s)v(s)\big)\,ds$$

for all $t \in I$.

Theorem 3.9. *Let $n \leqslant 6$ and let $M = 2m$. Let λ and μ satisfy (7). Then the unique pair of solutions $(u, v) \in Z_m(I) \times Z_{2m}(I)$ of (5) given by Theorem 3.8 satisfies the following identity for all $t \in I$*

$$\begin{aligned}
&\left\| J_m(t)u(t); L^2\right\|^2 + c\left\| J_{2m}(t)v(t); L^2\right\|^2 + \frac{2}{m}t^2 \operatorname{Re}\bigl(\lambda\bigl(v(t), u^2(t)\bigr)\bigr) \\
&\quad = \left\| J_m(t_0)u_0; L^2\right\|^2 + c\left\| J_{2m}(t_0)v_0; L^2\right\|^2 + \frac{2}{m}t_0^2 \operatorname{Re}\bigl(\lambda\bigl(v_0, u_0^2\bigr)\bigr) + \frac{4-n}{m}\int_{t_0}^{t} s \operatorname{Re}\bigl(\lambda\bigl(v(s), u^2(s)\bigr)\bigr)\, ds.
\end{aligned}$$

Proof. For simplicity, we give a formal calculation for the proof. Actual proof requires a regularization procedure, see [2,8]. We compute by the condition (7)

$$\begin{aligned}
&\operatorname{Im}\lambda\bigl(J_m(v\bar{u}), J_m u\bigr) + c\operatorname{Im}\mu\bigl(J_{2m}\bigl(u^2\bigr), J_{2m}v\bigr) \\
&\quad = 2\operatorname{Im}\lambda(J_{2m}v, uJ_m u) - \operatorname{Im}\lambda\bigl(v, (J_m u)^2\bigr) - c\operatorname{Im}\bar{\mu}(J_{2m}v, uJ_m u) \\
&\quad = \operatorname{Im}\lambda(J_{2m}v, uJ_m u) - \operatorname{Im}\lambda\bigl(v, (J_m u)^2\bigr) = \frac{t}{2m}I + \frac{t^2}{2m^2}II,
\end{aligned}$$

where

$$\begin{aligned}
&I = \operatorname{Re}\bigl(-2\lambda(xv, u\nabla u) + \lambda\bigl(\nabla v, xu^2\bigr) + 4\lambda(v, ux\cdot\nabla u)\bigr), \\
&II = \operatorname{Im}\bigl(\lambda(\nabla v, u\nabla u) + 2\lambda\bigl(v, (\nabla u)^2\bigr)\bigr).
\end{aligned}$$

Then I is written as

$$\begin{aligned}
I &= \operatorname{Re}\bigl(2\lambda(xv, u\nabla u) + \lambda\bigl(\nabla v, xu^2\bigr)\bigr) \\
&= \operatorname{Re}\bigl(\lambda\bigl(xv, \nabla\bigl(u^2\bigr)\bigr) + \lambda\bigl(x\cdot\nabla v, u^2\bigr)\bigr) = -n\operatorname{Re}\lambda\bigl(v, u^2\bigr),
\end{aligned}$$

while II is written as

$$\begin{aligned}
II &= \operatorname{Im}\bigl(\lambda(\nabla v, u\nabla u) - 2\lambda(\nabla v, u\nabla u) - 2\lambda(v, u\Delta u)\bigr) \\
&= \frac{1}{2}\operatorname{Im}\lambda\bigl(\Delta v, u^2\bigr) + 2\operatorname{Im}\bar{\lambda}(\Delta u, \bar{u}v) \\
&= -2m\operatorname{Re}\lambda\bigl(\partial_t v, u^2\bigr) - 4m\operatorname{Re}\bar{\lambda}(\partial_t u, \bar{u}v) = -2m\frac{d}{dt}\operatorname{Re}\lambda\bigl(v, u^2\bigr).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\left\| J_m(t)u(t); L^2\right\|^2 + c\left\| J_{2m}(t)v(t); L^2\right\|^2 - \left\| J_m(t_0)u_0; L^2\right\|^2 - c\left\| J_{2m}(t_0)v_0; L^2\right\|^2 \\
&\quad = \int_{t_0}^{t}\left(\frac{s}{m}I + \frac{s^2}{m^2}II\right) ds \\
&\quad = \frac{1}{m}\int_{t_0}^{t}\left(-\frac{d}{ds}\bigl(2s^2\operatorname{Re}\lambda\bigl(v, u^2\bigr)\bigr) + (4-n)s\operatorname{Re}\lambda\bigl(v, u^2\bigr)\right) ds,
\end{aligned}$$

which is the required identity. □

We now introduce

$$Z_m(\mathbb{R}) = \bigl\{u \in X(\mathbb{R});\; J_m u \in X(\mathbb{R})\bigr\}.$$

Theorem 3.10. *Let $n \leqslant 4$ and let $M = 2m$. Let λ and μ satisfy (7) with $c > 0$. If $n \leqslant 3$, then for any $(u_0, v_0) \in L^2 \times L^2$ with $(J_m(t_0)u_0, J_{2m}(t_0)v_0) \in L^2 \times L^2$, (5) has a unique pair of solutions $(u, v) \in Z_m(\mathbb{R}) \times Z_{2m}(\mathbb{R})$. If $n = 4$, then for any $(u_0, v_0) \in L^2 \times L^2$ with $(J_m(t_0)u_0, J_{2m}(t_0)v_0) \in L^2 \times L^2$ and*

$$|\lambda| Q(u_0, v_0)^{1/2} < \alpha_0$$

(5) has a unique pair of solutions $(u, v) \in Z_m(\mathbb{R}) \times Z_{2m}(\mathbb{R})$.

Proof. By the standard continuation argument, it suffices to obtain a priori estimates on $\|J_m u; L^2\| \vee \|J_{2m} v; L^2\|$. If we notice that

$$\|J_m u; L^2\|^2 + c\|J_{2m} v; L^2\|^2 = \frac{t^2}{m^2}\left(\|\nabla M_m^{-1} u; L^2\|^2 + \frac{c}{4}\|\nabla M_{2m}^{-1} v; L^2\|^2\right) = \frac{2t^2}{m} K\left(M_m^{-1} u, M_{2m}^{-1} v\right),$$

$$\left(v, u^2\right) = \left(M_{2m}^{-1} v, \left(M_m^{-1} u\right)^2\right),$$

then an analogous argument to that of Section 6 implies the theorem. □

3.7. Non-existence of global solutions with negative energy under mass resonance

In this section we assume mass resonance condition $M = 2m$. Let $n \leqslant 6$ and let $(u, v) \in (Z_m(I) \times Z_{2m}(I)) \cap (Y(I) \times Y(I))$ be the unique pair of local solutions given by Theorems 3.2 and 3.8 with data $(u_0, v_0) \in H^1 \times H^1$ at $t = t_0$ satisfying $(J_m(t_0)u_0, J_{2m}(t_0)v_0) \in L^2 \times L^2$, where I is the intersection of time intervals in Theorems 3.2 and 3.8. From now on, we take $t_0 = 0$ for simplicity. The corresponding pair of local solutions (u, v) satisfies the virial identity:

Theorem 3.11. *Let* $n \leqslant 6$ *and let* $M = 2m$. *Let* λ *and* μ *satisfy* (7). *Let* $(u_0, v_0) \in H^1 \times H^1$ *satisfy* $(xu_0, xv_0) \in L^2 \times L^2$ *and let* $(u, v) \in (Z_m(I) \times Z_{2m}(I)) \cap (Y(I) \times Y(I))$ *the corresponding pair of local solutions given by Theorems 3.2 and 3.8 with* $t_0 = 0$. *Then*

$$\|xu(t); L^2\|^2 + c\|xv(t); L^2\|^2 = \|xu_0; L^2\|^2 + c\|xv_0; L^2\|^2 + P_0 t + \frac{n}{2m} E_0 t^2 + \frac{4-n}{m}\int_0^t (t-s)\left(\frac{1}{2m}\|\nabla u(s); L^2\|^2 + \frac{c}{8m}\|\nabla v(s); L^2\|^2\right) ds$$

for all $t \in I$, *where*

$$P_0 = \frac{2}{m}\operatorname{Im}(\nabla u_0, xu_0) + \frac{c}{m}\operatorname{Im}(\nabla v_0, xv_0),$$

$$E_0 = \frac{1}{2m}\|\nabla u_0; L^2\|^2 + \frac{c}{8m}\|\nabla v_0; L^2\|^2 + \operatorname{Re}\lambda\left(v_0, u_0^2\right).$$

Proof. For simplicity, we give a formal calculation for the proof. Actual proof requires a regularization procedure, see [2,8]. We compute

$$\frac{d}{dt}\left(\|xu; L^2\|^2 + c\|xv; L^2\|^2\right) = 2\operatorname{Im}\left(i\partial_t u, |x|^2 u\right) + 2c\operatorname{Im}\left(i\partial_t v, |x|^2 v\right) = \frac{2}{m}\operatorname{Im}(\nabla u, xu) + \frac{c}{m}\operatorname{Im}(\nabla v, xv),$$

where the last two terms are rewritten as

$$\frac{2}{m}\operatorname{Im}(\nabla u, xu) = -\frac{1}{t}\left(\|J_m u; L^2\|^2 - \|xu; L^2\|^2\right) + \frac{t}{m^2}\|\nabla u; L^2\|^2,$$

$$\frac{c}{m}\operatorname{Im}(\nabla v, xv) = -\frac{c}{t}\left(\|J_{2m} v; L^2\|^2 - \|xv; L^2\|^2\right) + \frac{ct}{4m^2}\|\nabla v; L^2\|^2.$$

Therefore we obtain by a direct calculation

$$\frac{d^2}{dt^2}\left(\|xu; L^2\|^2 + c\|xv; L^2\|^2\right) = \frac{n}{m} E_0 + \frac{4-n}{m}\left(\frac{1}{2m}\|\nabla u; L^2\|^2 + \frac{c}{8m}\|\nabla v; L^2\|^2\right),$$

where we have used Theorems 3.5 and 3.9. This proves the theorem. □

Theorem 3.12. *Let $4 \leqslant n \leqslant 6$. Let M, m, λ, μ satisfy $M = 2m$ and (7) with $c > 0$. Let (u_0, v_0) and (u, v) be as in Theorem 3.11. Then the maximal existence time for (u, v) is finite in the following cases*:

$$E_0 < 0, \tag{8}$$

$$E_0 = 0, \qquad P_0 < 0, \tag{9}$$

where E_0 and P_0 are as in Theorem 3.11.

Proof. The theorem follows from the virial identity in Theorem 3.11 in the standard way. □

4. Existence of ground states

In this section we always assume (7) with $c > 0$. The purpose in this and subsequent sections is to study the existence of nontrivial standing wave solutions to (1) and related problems. It is therefore natural to exclude the trivial case $(\lambda, \mu) = (0, 0)$. From now on we always assume that $\lambda = c\overline{\mu}$ with $c > 0$, $\lambda \neq 0$, $\mu \neq 0$. Then it is convenient to rescale (u, v) by introducing new functions $(\tilde{u}, \tilde{v})$ defined by

$$\tilde{u}(t, x) = \sqrt{\frac{c}{2}}|\mu| u\left(t, \sqrt{\frac{1}{2m}}x\right), \qquad \tilde{v}(t, x) = -\frac{\lambda}{2} v\left(t, \sqrt{\frac{1}{2m}}x\right),$$

which satisfy

$$\begin{cases} i\partial_t \tilde{u} + \Delta \tilde{u} = -2\tilde{v}\overline{\tilde{u}}, \\ i\partial_t \tilde{v} + \kappa \Delta \tilde{v} = -\tilde{u}^2, \end{cases} \tag{10}$$

where $\kappa = m/M$ is the mass ratio. We look for standing waves for (10), which are periodic in time and well localized in space. Comparing frequencies in monochromatic wave factors on both sides of (10) we expect that (10) has a pair of solutions of the form

$$\begin{cases} \tilde{u}(t, x) = e^{i\omega t}\phi_\omega(x), \\ \tilde{v}(t, x) = e^{i2\omega t}\psi_\omega(x), \end{cases} \tag{11}$$

where $\omega > 0$ and a pair of real-valued functions in $\mathbb{R}^n$ satisfy

$$\begin{cases} -\Delta \phi_\omega + \omega \phi_\omega = 2\psi_\omega \phi_\omega, \\ -\kappa \Delta \psi_\omega + 2\omega \psi_\omega = \phi_\omega^2. \end{cases} \tag{12}$$

We study the existence of solutions to (12) as ground states that minimize the associated functional I_ω among all non-zero solutions of (12) given by

$$I_\omega(\phi, \psi) = \frac{1}{2}\left(\|\nabla\phi; L^2\|^2 + \kappa\|\nabla\psi; L^2\|^2\right) + \frac{\omega}{2}\left(\|\phi; L^2\|^2 + 2\|\psi; L^2\|^2\right) - \int \phi^2\psi\, dx.$$

To be more specific, we introduce:

Definition 4.1. A pair of real-valued functions $(\phi_0, \psi_0) \in H^1 \times H^1$ is called a ground state for (12) if

$$I_\omega(\phi_0, \psi_0) = \inf\{I_\omega(\phi, \psi);\ (\phi, \psi) \in \mathcal{C}_\omega\},$$

$$\mathcal{C}_\omega = \{(\phi, \psi) \in H^1 \times H^1;\ (\phi, \psi) \text{ is a nontrivial critical point of } I_\omega\}.$$

The set of all ground states for (12) is denoted by $\mathcal{G}_\omega$.

We also introduce the following functionals associated with (12):

$$Q(\phi, \psi) = \|\phi; L^2\|^2 + 2\|\psi; L^2\|^2,$$

$$K(\phi, \psi) = \|\nabla\phi; L^2\|^2 + \kappa\|\nabla\psi; L^2\|^2,$$

$$K_\omega(\phi, \psi) = K(\phi, \psi) + \omega Q(\phi, \psi),$$

$$P(\phi,\psi)=\int \phi^2\psi\,dx,$$
$$R_\omega(\phi,\psi)=K_\omega(\phi,\psi)/P(\phi,\psi)^{2/3},$$
$$J(\phi,\psi)=K(\phi,\psi)Q(\phi,\psi)^{1/2}/P(\phi,\psi).$$

Due to the scale change introduced in this section, the functionals for (12) differ from the corresponding functionals in Section 6 up to constant factors. It causes no confusions as far as the above functionals are used for (12) and the corresponding results for (5) are derived by means of the inverse of the scaling.

Definition 4.2. A pair of real-valued functions $(\phi,\psi)\in H^1\times H^1$ is called a solution of (12) if

$$\int \nabla\phi\cdot\nabla u\,dx+\omega\int \phi u\,dx=2\int \phi\psi u\,dx,$$
$$\kappa\int \nabla\psi\cdot\nabla v\,dx+2\omega\int \psi v\,dx=\int \phi^2 v\,dx$$

for any $u,v\in C_0^\infty(\mathbb{R}^n)$.

Remark 4.1. This is the definition of weak solutions in the sense that those functions satisfy (12) in the distribution sense. By a density argument and the Gagliardo–Nirenberg inequality in Theorem 3.7, it is equivalent to take H^1 as a space of test functions instead of $C_0^\infty(\mathbb{R}^n)$ if $n\leqslant 6$. Weak solutions (ϕ,ψ) satisfy $I'_\omega(\phi,\psi)(u,v)=0$ for any $(u,v)\in C_0^\infty(\mathbb{R}^n)$. By the standard elliptic regularity theory (see [2] for instance), weak solutions satisfy $u,v\in H^m$ for any $m\geqslant 1$ if $\omega>0$ and are regarded as strong solutions.

Theorem 4.1. *Let (ϕ,ψ) be a solution of (12). Then*

$$K_\omega(\phi,\psi)=3P(\phi,\psi), \tag{13}$$
$$P(\phi,\psi)=2I_\omega(\phi,\psi), \tag{14}$$
$$R_\omega(\phi,\psi)=2^{1/3}3I_\omega(\phi,\psi)^{1/3}, \tag{15}$$
$$K(\phi,\psi)=nI_\omega(\phi,\psi), \tag{16}$$
$$\omega Q(\phi,\psi)=(6-n)I_\omega(\phi,\psi). \tag{17}$$

Proof. By the definition of I_ω, we have

$$\frac12 K(\phi,\psi)+\frac12\omega Q(\phi,\psi)-P(\phi,\psi)=I_\omega(\phi,\psi),$$

from which we obtain

$$K(\phi,\psi)+\omega Q(\phi,\psi)-3P(\phi,\psi)=0,$$

since $I'_\omega(\phi,\psi)(\phi,\psi)=0$. The Pohozaev identity

$$\left.\frac{d}{d\lambda}\right|_{\lambda=1} I_\omega(\delta_\lambda\phi,\delta_\lambda\psi)=0$$

with $(\delta_\lambda f)(x)=f(x/\lambda)$ implies

$$\frac{n-2}{2}K(\phi,\psi)+n\left(\frac12\omega Q(\phi,\psi)-P(\phi,\psi)\right)=0.$$

Then the identities of the theorem follow by combining those equalities. □

Corollary 4.2. *Eq.* (12) *has no nontrivial solutions if* $(n-6)\omega\leqslant 0$.

Proof. By (16), we have $I_\omega(\phi,\psi)\geqslant 0$. By (17), we have $Q(\phi,\psi)\leqslant 0$. □

Remark 4.2. By (17) and the definition of ground states for (12), the charge $Q(\phi,\psi)$ of a ground state is independent of choice of ground states. In particular, a solution of (12) with minimal charge is a ground state.

Proposition 4.3. *Let (ϕ_1,ψ_1) be a ground state for (12) with $\omega=1$. Then $(\phi_\omega,\psi_\omega)$ with $\omega>0$ defined by*

$$\big(\phi_\omega(x),\psi_\omega(x)\big)=\big(\omega\phi_1(\sqrt{\omega}x),\omega\psi_1(\sqrt{\omega}x)\big)$$

is a ground state for (12). Moreover,

$$Q(\phi_\omega,\psi_\omega)=\omega^{2-n/2}Q(\phi_1,\psi_1).$$

Proof. The proposition follows by a straightforward calculation. □

Remark 4.3. The charge of a ground state is independent of ω for $n=4$.

We consider the following scaling transformations:

- **Scaling of amplitude:** $(\phi,\psi)\mapsto(a\phi,a\psi)$, $a>0$.
- **Dilation:** $(\phi,\psi)\mapsto(\delta_l\phi,\delta_l\psi)$, $(\delta_l f)(x)=f(x/l)$, $l>0$.
- **Symmetric-decreasing rearrangement:** $(\phi,\psi)\mapsto(\phi^*,\psi^*)$.

The functionals introduced in this section satisfy the following properties under scaling transformations. For general information on the symmetric-decreasing rearrangement, see [16] for instance:

$$\begin{aligned}
&Q(a\phi,a\psi)=a^2Q(\phi,\psi),\\
&K(a\phi,a\psi)=a^2K(\phi,\psi),\\
&K_\omega(a\phi,a\psi)=a^2K_\omega(\phi,\psi),\\
&P(a\phi,a\psi)=a^3P(\phi,\psi); &(18)\\
&Q(\delta_l\phi,\delta_l\psi)=l^nQ(\phi,\psi),\\
&K(\delta_l\phi,\delta_l\psi)=l^{n-2}K(\phi,\psi),\\
&P(\delta_l\phi,\delta_l\psi)=l^nP(\phi,\psi); &(19)\\
&Q\big(\phi^*,\psi^*\big)=Q(\phi,\psi),\\
&K\big(\phi^*,\psi^*\big)\leqslant K(\phi,\psi),\\
&K_\omega\big(\phi^*,\psi^*\big)\leqslant K_\omega(\phi,\psi),\\
&P\big(\phi^*,\psi^*\big)\geqslant P(\phi,\psi); &(20)\\
&R_\omega(a\phi,a\psi)=R_\omega(\phi,\psi),\\
&J(a\phi,a\psi)=J(\phi,\psi); &(21)\\
&J(\delta_l\phi,\delta_l\psi)=l^{n/2-2}J(\phi,\psi); &(22)\\
&R_\omega\big(\phi^*,\psi^*\big)\leqslant R_\omega(\phi,\psi),\\
&J\big(\phi^*,\psi^*\big)\leqslant J(\phi,\psi). &(23)
\end{aligned}$$

Below we use the following characterizations of critical points of R_ω and J:

- (ϕ,ψ) is a critical point of R_ω:
  if and only if $(a\phi,a\psi)$ is a critical point of R_ω for all $a>0$,
  if and only if $(a\phi,a\psi)$ is a critical point of R_ω for some $a>0$.
- When $n=4$, (ϕ,ψ) is a critical point of J:
  if and only if $(a\delta_l\phi,a\delta_l\psi)$ is a critical point of J for all $a,l>0$,
  if and only if $(a\delta_l\phi,a\delta_l\psi)$ is a critical point of J for some $a,l>0$.

To prove the existence of ground states for (12), we prepare

Lemma 4.4.

$$\mathcal{C}_\omega \subset \mathcal{P} \quad \textit{where } \mathcal{P} = \{(\phi,\psi) \in H^1 \times H^1 \setminus \{(0,0)\};\ P(\phi,\psi) > 0\}, \tag{24}$$

$$\beta_\omega \equiv \inf\{R_\omega(\phi,\psi);\ (\phi,\psi) \in \mathcal{P}\} > 0 \quad \textit{for } n \leqslant 6, \tag{25}$$

$$\inf\{I_\omega(\phi,\psi);\ (\phi,\psi) \in \mathcal{C}_\omega\} \geqslant \frac{1}{2}\left(\frac{1}{3}\beta_\omega\right)^3. \tag{26}$$

Proof. *Proof of* (24). Let $(\phi,\psi) \in \mathcal{C}_\omega$. Then $I'_\omega(\phi,\psi)(\phi,\psi) = 0$ is equivalent to $K_\omega(\phi,\psi) - 3P(\phi,\psi) = 0$ and therefore $P(\phi,\psi) = (1/3)K_\omega(\phi,\psi) > 0$.

Proof of (25). By the following Gagliardo–Nirenberg inequality

$$\|\phi; L^3\| \leqslant C\|\nabla\phi; L^2\|^{n/6}\|\phi; L^2\|^{1-n/6}$$

for $n \leqslant 6$, we have

$$\begin{aligned}
P(\phi,\psi) &\leqslant C^3\|\nabla\phi; L^2\|^{n/3}\|\phi; L^2\|^{2-n/3}\|\nabla\psi; L^2\|^{n/6}\|\psi; L^2\|^{1-n/6} \\
&\leqslant C^3 K_\omega(\phi,\psi)^{n/6}\left(\omega^{-1}K_\omega(\phi,\psi)\right)^{1-n/6}\left(\kappa^{-1}K_\omega(\phi,\psi)\right)^{n/12}\left((2\omega)^{-1}K_\omega(\phi,\psi)\right)^{1/2-n/12} \\
&= C^3 2^{n/12-1/2}\omega^{-3/2+5n/12}\kappa^{-n/12}K_\omega(\phi,\psi)^{3/2}
\end{aligned}$$

and therefore by $R_\omega(\phi,\psi) = K_\omega(\phi,\psi)/P(\phi,\psi)^{2/3}$

$$\beta_\omega \geqslant C^{-2}2^{-n/18+1/3}\omega^{1-5n/18}\kappa^{n/18} > 0.$$

Proof of (26). Let $(\phi,\psi) \in \mathcal{P}$. The function

$$(0,\infty) \ni t \mapsto I_\omega(t\phi, t\psi) = \frac{1}{2}K_\omega(\phi,\psi)t^2 - P(\phi,\psi)t^3 \in \mathbb{R}$$

has a critical point

$$t_0 = t_0(\phi,\psi) \equiv \frac{K_\omega(\phi,\psi)}{3P(\phi,\psi)} > 0$$

with critical value

$$I_\omega(t_0\phi, t_0\psi) = \frac{1}{2}\left(\frac{1}{3}R_\omega(\phi,\psi)\right)^3$$

at $(t_0\phi, t_0\psi) \in \mathcal{P}$, which is the only nontrivial critical point of I_ω on $\{(t\phi, t\psi);\ t > 0\}$ if it exists. This implies that

$$\inf\{I_\omega(\phi,\psi);\ (\phi,\psi) \in \mathcal{C}_\omega\} \geqslant \frac{1}{2}\left(\frac{1}{3}\inf\{R_\omega(\phi,\psi);\ (\phi,\psi) \in \mathcal{P}\}\right)^3. \qquad \square$$

Remark 4.4. $t_0(\phi,\psi) = 1$ if $(\phi,\psi) \in \mathcal{C}_\omega$.

Theorem 4.5. *Let $n \leqslant 5$ and $\omega > 0$. Then*:

(1) *There exists a pair of non-negative radially symmetric functions $(\phi_0,\psi_0) \in \mathcal{P}$ such that*

$$R_\omega(\phi_0,\psi_0) = \beta_\omega = \inf\{R_\omega(\phi,\psi);\ (\phi,\psi) \in \mathcal{P}\}.$$

(2) *Let (ϕ_0,ψ_0) be as in Part* (1). *There exists $t_0 > 0$ such that $(\phi,\psi) \equiv (t_0\phi, t_0\psi)$ is a positive solution of* (12). *Moreover, (ϕ,ψ) is a ground state for* (12).

(3) *The set of minimizers of* R_ω *is characterized as*

$$\{(\phi,\psi)\in H^1\times H^1;\ R_\omega(\phi,\psi)=\beta_\omega\}=\{(t\phi,t\psi)\in H^1\times H^1;\ t>0,\ (\phi,\psi)\in\mathcal{G}_\omega\}.$$

Proof. *Proof of Theorem* 4.5 *for* $2\leqslant n\leqslant 5$. Let $\{(\phi_j,\psi_j)\}\subset\mathcal{P}$ be a minimizing sequence for R_ω. By (23) we may assume that ϕ_j,ψ_j are non-negative and radially symmetric functions in H^1. We define

$$\tilde\phi_j=\big(K_\omega(\phi_j,\psi_j)\big)^{-1/2}\phi_j,\qquad \tilde\psi_j=\big(K_\omega(\phi_j,\psi_j)\big)^{-1/2}\psi_j.$$

Then by (18) and (21),

$$K_\omega(\tilde\phi_j,\tilde\psi_j)=1,\qquad R_\omega(\tilde\phi_j,\tilde\psi_j)=R_\omega(\phi_j,\psi_j)\to\beta_\omega>0.$$

In particular,

$$P(\tilde\phi_j,\tilde\psi_j)=R_\omega(\tilde\phi_j,\tilde\psi_j)^{-3/2}\to\beta_\omega^{-3/2}.$$

Since $\{(\tilde\phi_j,\tilde\psi_j)\}$ is bounded in $H^1\times H^1$, there exists a subsequence still denoted by $\{(\tilde\phi_j,\tilde\psi_j)\}$ such that

$$\tilde\phi_j\to\phi_0,\qquad \tilde\psi_j\to\psi_0\quad\text{weakly in } H^1$$

for some $(\phi_0,\psi_0)\in H^1_r\times H^1_r$, where H^1_r denotes the space of radially symmetric H^1 functions. By Strauss' compactness embedding $H^1_r\subset L^3$ for $2\leqslant n\leqslant 5$,

$$\tilde\phi_j\to\phi_0,\qquad \tilde\psi_j\to\psi_0\quad\text{strongly in } L^3.$$

This yields

$$P(\phi_0,\psi_0)=\lim_{j\to\infty}P(\tilde\phi_j,\tilde\psi_j)=\beta_\omega^{-3/2}>0,$$

while weak convergence of $\{(\tilde\phi_j,\tilde\psi_j)\}$ in $H^1\times H^1$ yields

$$K_\omega(\phi_0,\psi_0)\leqslant\lim_{j\to\infty}K_\omega(\tilde\phi_j,\tilde\psi_j)=1.$$

Therefore

$$\beta_\omega\leqslant R_\omega(\phi_0,\psi_0)=\frac{K_\omega(\phi_0,\psi_0)}{P(\phi_0,\psi_0)^{2/3}}\leqslant\beta_\omega.$$

We conclude that $(\phi_0,\psi_0)\in\mathcal{P}\cap(H^1_r\times H^1_r)$ satisfies

$$R_\omega(\phi_0,\psi_0)=\beta_\omega,\qquad K_\omega(\phi_0,\psi_0)=1,\qquad P(\phi_0,\psi_0)=\beta_\omega^{-2/3},$$
$$\tilde\phi_j\to\phi_0,\qquad \tilde\psi_j\to\psi_0\quad\text{strongly in } H^1.$$

This proves Part (1). Since (ϕ_0,ψ_0) is a minimizer of R_ω, it is a critical point. For any $(u,v)\in H^1\times H^1$

$$\frac{d}{ds}\bigg|_{s=0}R_\omega(\phi_0+su,\psi_0+sv)=0,$$

which means that

$$\frac{1}{P(\phi_0,\psi_0)^{2/3}}K'_\omega(\phi_0,\psi_0)(u,v)=\frac{2K_\omega(\phi_0,\psi_0)}{3P(\phi_0,\psi_0)^{5/3}}P'(\phi_0,\psi_0)(u,v).$$

This yields

$$\begin{aligned}K'_\omega(\phi_0,\psi_0)(u,v)&=\frac{2K_\omega(\phi_0,\psi_0)}{3P(\phi_0,\psi_0)}P'(\phi_0,\psi_0)(u,v)\\&=\frac{2}{3}\beta_\omega^{2/3}P'(\phi_0,\psi_0)(u,v),\end{aligned}$$

which means that

$$2\int\big(\nabla\phi_0\cdot\nabla u+\kappa\nabla\psi_0\cdot\nabla v+\omega(\phi_0 u+2\psi_0 v)\big)\,dx=\frac{2}{3}\beta_\omega^{2/3}\int\big(2\phi_0\psi_0 u+\phi_0^2 v\big)\,dx.$$

We now define $(\phi,\psi)=(t_0\phi_0,t_0\psi_0)$ with $t_0=\beta_\omega^{2/3}/3$. Then (ϕ,ψ) is a solution of (12). Since (ϕ_0,ψ_0) is a critical point of R_ω, (ϕ,ψ) is also a critical point of R_ω. By (15), (ϕ,ψ) is a ground state for (12). By the maximum principle, (ϕ,ψ) is a positive solution of (12). This proves Part (2). Part (3) follows by the same argument as above. This proves Theorem 4.5 for $2\leqslant n\leqslant 5$.

Proof of Theorem 4.5 for $n=1$. The argument as above fails for $n=1$ due to the breakdown of compactness on the embedding $H_r^1\subset L^3$. Instead, we employ a concentration-compactness argument on the functional

$$I_\omega=\frac{1}{2}K_\omega-P,$$

since I_ω satisfies assumptions of the mountain pass theorem in the Hilbert space $H^1\times H^1$. We define the mountain pass value b by

$$b=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}I_\omega\big(\gamma(t)\big),$$
$$\Gamma=\big\{\gamma\in C\big([0,1];H^1\times H^1\big);\ I_\omega\big(\gamma(1)\big)<0\text{ and }\gamma(0)=1\big\}.$$

Then, we see that $b>0$ and that I_ω has a Palais–Smale sequence $(PS)_b$ by the Ekeland Principle [6]. Here we give

Definition 4.3. For $c\in\mathbb{R}$ a sequence $\{(\phi_j,\psi_j)\}\subset H^1\times H^1$ is called a $(PS)_c$-sequence of I_ω if

$$I_\omega(\phi_j,\psi_j)\to c,$$
$$\big\|I'_\omega(\phi_j,\psi_j);H^{-1}\times H^{-1}\big\|\to 0.\qquad\square$$

A basic property of $(PS)_c$ sequences of I_ω is given by the following lemma.

Lemma 4.6. *Let $c\in\mathbb{R}$ and let $\{(\phi_j,\psi_j)\}\subset H^1\times H^1$ be a $(PS)_c$-sequence of I_ω. Then*

$$\lim_{j\to\infty}K_\omega(\phi_j,\psi_j)=6c,$$
$$\lim_{j\to\infty}P(\phi_j,\psi_j)=2c.$$

In particular, $c\geqslant 0$ and $c=0$ if and only if $\phi_j\to 0$, $\psi_j\to 0$ in H^1.

Proof. We first note that $K_\omega^{1/2}$ is an equivalent norm on $H^1\times H^1$:

$$\big(\max(1,\kappa,2\omega)\big)^{-1/2}K_\omega(\phi,\psi)^{1/2}\leqslant\big(\|\phi;H^1\|^2+\|\psi;H^1\|^2\big)^{1/2}\leqslant\big(\min(1,\kappa,\omega)\big)^{-1/2}K_\omega(\phi,\psi)^{1/2}.$$

Let $\{(\phi_j,\psi_j)\}$ be a $(PS)_c$-sequence of I_ω. Then

$$I_\omega(\phi_j,\psi_j)\to c,$$
$$\begin{aligned}\big|K_\omega(\phi_j,\psi_j)-3P(\phi_j,\psi_j)\big|&=\big|I'_\omega(\phi_j,\psi_j)(\phi_j,\psi_j)\big|\\&\leqslant\big\|I'_\omega(\phi_j,\psi_j);H^{-1}\times H^{-1}\big\|\big\|(\phi_j,\psi_j);H^1\times H^1\big\|\\&\leqslant\varepsilon_j K_\omega(\phi_j,\psi_j)^{1/2},\end{aligned}$$

where

$$\varepsilon_j\equiv\big(\min(1,\kappa,\omega)\big)^{-1/2}\big\|I'_\omega(\phi_j,\psi_j);H^{-1}\times H^{-1}\big\|.$$

We write $K_\omega(\phi_j,\psi_j)$ as

$$K_\omega(\phi_j,\psi_j)=6\big(I_\omega(\phi_j,\psi_j)-c\big)-2\big(K_\omega(\phi_j,\psi_j)-3P(\phi_j,\psi_j)\big)+6c$$

to estimate

$$\big|K_\omega(\phi_j,\psi_j)\big|\leqslant 6\big|I_\omega(\phi_j,\psi_j)-c\big|+2\varepsilon_j K_\omega(\phi_j,\psi_j)^{1/2}+6|c|,$$

which leads to

$$\big|K_\omega(\phi_j,\psi_j)\big|\leqslant 12\big|I_\omega(\phi_j,\psi_j)-c\big|+4\varepsilon_j^2+12|c|.$$

This implies the boundedness of $\{K_\omega(\phi_j,\psi_j)\}$ in $\mathbb{R}$ as well as that of $\{(\phi_j,\psi_j)\}$ in $H^1\times H^1$. Therefore the lemma follows from

$$\begin{aligned}\big|P(\phi_j,\psi_j)-2c\big|&=\big|\big(3P(\phi_j,\psi_j)-K_\omega(\phi_j,\psi_j)\big)+2\big(I_\omega(\phi_j,\psi_j)-c\big)\big|\\&\leqslant \varepsilon_j K_\omega(\phi_j,\psi_j)^{1/2}+2\big|I_\omega(\phi_j,\psi_j)-c\big|,\\ \big|K_\omega(\phi_j,\psi_j)-6c\big|&=\big|6\big(I_\omega(\phi_j,\psi_j)-c\big)-2\big(K_\omega(\phi_j,\psi_j)-3P(\phi_j,\psi_j)\big)\big|\\&\leqslant 6\big|I_\omega(\phi_j,\psi_j)-c\big|+2\varepsilon_j K_\omega(\phi_j,\psi_j)^{1/2}. \qquad \square\end{aligned}$$

On the basis of the argument given in the proof of Theorem 4.5 for $2\leqslant n\leqslant 5$, it suffices to prove

$$b=\inf\big\{I_\omega(\phi,\psi);\ (\phi,\psi)\in\mathcal{C}_\omega\big\}$$

for $n=1$.

Proof of $b\leqslant\inf\{I_\omega(\phi,\psi);\ (\phi,\psi)\in\mathcal{C}_\omega\}$. Let $(\phi,\psi)\in\mathcal{P}$. Then $\gamma_{(l,\phi,\psi)}$ defined by $\gamma_{(l,\phi,\psi)}(t)=(tl\phi,tl\psi)\in H^1\times H^1$ with $t\in[0,1]$ belongs to Γ for l sufficiently large since

$$I_\omega\big(\gamma_{(l,\phi,\psi)}(t)\big)=\frac{1}{2}K_\omega(\phi,\psi)t^2l^2-P(\phi,\psi)t^3l^3.$$

As in the proof of Lemma 4.4, we have

$$\max_{t\in[0,1]}I_\omega\big(\gamma_{(l,\phi,\psi)}(t)\big)=\frac{1}{2}\left(\frac{1}{3}R_\omega(\phi,\psi)\right)^3$$

and therefore

$$\begin{aligned}b&\leqslant\inf\Big\{\max_{t\in[0,1]}I_\omega\big(\gamma_{(l,\phi,\psi)}(t)\big);\ (\phi,\psi)\in\mathcal{C}_\omega\Big\}\\&=\inf\left\{\frac{1}{2}\left(\frac{1}{3}R_\omega(\phi,\psi)\right)^3;\ (\phi,\psi)\in\mathcal{C}_\omega\right\}\\&=\inf\big\{I_\omega(\phi,\psi);\ (\phi,\psi)\in\mathcal{C}_\omega\big\},\end{aligned}$$

as required.

Proof of $b=\inf\{I_\omega(\phi,\psi);\ (\phi,\psi)\in\mathcal{C}_\omega\}$. Let $\{(\phi_j,\psi_j)\}$ be a $(PS)_b$-sequence of I_ω. The Sobolev embedding ensures that $H^1\subset C_\infty$, where

$$C_\infty=\Big\{u\in C\cap L^\infty;\ \lim_{|x|\to\infty}u(x)=0\Big\}.$$

Therefore for any $j\geqslant 1$ there exists $l_j\in\mathbb{R}$ such that

$$\big|\phi_j(l_j)\big|+\big|\psi_j(l_j)\big|=\big\|(\phi_j,\psi_j);L^\infty\times L^\infty\big\|,$$

where

$$\big\|(u,v);L^\infty\times L^\infty\big\|=\big\|u;L^\infty\big\|+\big\|v;L^\infty\big\|.$$

Then $(\tilde\phi_j,\tilde\psi_j)$ defined by

$$\tilde{\phi}_j(x) = \phi_j(x + l_j), \qquad \tilde{\psi}_j(x) = \psi_j(x + l_j)$$

forms a $(PS)_b$-sequence to I_ω. For simplicity we drop tilde and write (ϕ_j, ψ_j) for $(\tilde{\phi}_j, \tilde{\psi}_j)$. In other words, from now on we take a $(PS)_b$-sequence (ϕ_j, ψ_j) of I_ω with

$$|\phi_j(0)| + |\psi_j(0)| = \|(\phi_j, \psi_j); L^\infty \times L^\infty\|.$$

We estimate $P(\phi_j, \psi_j)$ as

$$\begin{aligned} P(\phi_j, \psi_j) &\leqslant \|\phi_j; L^3\|^2 \|\psi_j; L^3\| \\ &\leqslant \|\phi_j; L^2\|^{4/3} \|\phi_j; L^\infty\|^{2/3} \|\psi_j; L^2\|^{2/3} \|\psi_j; L^\infty\|^{1/3} \\ &\leqslant 2^{-1/3} Q(\phi_j, \psi_j) \|(\phi_j, \psi_j); L^\infty \times L^\infty\| \\ &\leqslant 2^{-1/3} \omega^{-1} K_\omega(\phi_j, \psi_j)\big(|\phi_j(0)| + |\psi_j(0)|\big). \end{aligned}$$

By Lemma 4.6, we have

$$2b \leqslant 2^{-1/3} \omega^{-1} (6b) \liminf_{j\to\infty} \big(|\phi_j(0)| + |\psi_j(0)|\big),$$

which is equivalent to

$$2^{1/3}\omega/3 \leqslant \liminf_{j\to\infty} \big(|\phi_j(0)| + |\psi_j(0)|\big).$$

By the boundedness of $\{(\phi_j, \psi_j)\}$ in $H^1 \times H^1$ we choose a subsequence still denoted by $\{(\phi_j, \psi_j)\}$ and $(\phi_0, \psi_0) \in H^1 \times H^1 \subset C_\infty \times C_\infty$ such that

$$\phi_j \to \phi_0, \qquad \psi_j \to \psi_0 \quad \text{weakly in } H^1.$$

Then, by Rellich's compactness theorem we conclude that

$$\phi_j \to \phi_0, \qquad \psi_j \to \psi_0 \quad \text{locally uniformly in } \mathbb{R}.$$

In particular, $(\phi_0, \psi_0) \neq (0, 0)$ since

$$|\phi_0(0)| + |\psi_0(0)| = \lim_{j\to\infty} \big(|\phi_j(0)| + |\psi_j(0)|\big) \geqslant 2^{1/3}\omega/3 > 0.$$

We now prove that (ϕ_0, ψ_0) is a critical point of I_ω, which implies that $(\phi_0, \psi_0) \in \mathcal{C}_\omega$. By density, it is sufficient to prove that

$$I'_\omega(\phi_0, \psi_0)(u, v) = 0$$

for any $u, v \in C_0^\infty(\mathbb{R})$. By the weak convergence in H^1, we have

$$K'_\omega(\phi_j, \psi_j)(u, v) \to K'_\omega(\phi_0, \psi_0)(u, v),$$

while by the uniform convergence on $\operatorname{supp} u \cup \operatorname{supp} v$, we have

$$P'(\phi_j, \psi_j)(u, v) \to P'(\phi_0, \psi_0)(u, v).$$

Therefore we obtain

$$I'_\omega(\phi_0, \psi_0)(u, v) = \lim_{j\to\infty} I'_\omega(\phi_j, \psi_j)(u, v) = 0,$$

as required.

Let $(\hat{\phi}_j, \hat{\psi}_j) = (\phi_j - \phi_0, \psi_j - \psi_0)$. We prove that $(\hat{\phi}_j, \hat{\psi}_j)$ is a $(PS)_{b - I_\omega(\phi_0, \psi_0)}$-sequence of I_ω. Since $I'_\omega(\phi_0, \psi_0) = 0$,

$$\begin{aligned} I_\omega(\hat{\phi}_j + \phi_0, \hat{\psi}_j + \psi_0) &= I_\omega(\phi_j, \psi_j) \to b \quad \text{in } \mathbb{R}, \\ I'_\omega(\hat{\phi}_j + \phi_0, \hat{\psi}_j + \psi_0) &= I'_\omega(\phi_j, \psi_j) \to 0 \quad \text{in } H^{-1} \times H^{-1}, \end{aligned}$$

it suffices to prove that

$$I_\omega(\hat\phi_j+\phi_0,\hat\psi_j+\psi_0)-I_\omega(\hat\phi_j,\hat\psi_j)-I_\omega(\phi_0,\psi_0)\to 0 \quad \text{in } \mathbb{R},$$
$$I'_\omega(\hat\phi_j+\phi_0,\hat\psi_j+\psi_0)-I'_\omega(\hat\phi_j,\hat\psi_j)-I'_\omega(\phi_0,\psi_0)\to 0 \quad \text{in } H^{-1}\times H^{-1}.$$

We write

$$\begin{aligned}
&I_\omega(\hat\phi_j+\phi_0,\hat\psi_j+\psi_0)-I_\omega(\hat\phi_j,\hat\psi_j)-I_\omega(\phi_0,\psi_0)\\
&\quad=\int \nabla\hat\phi_j\cdot\nabla\phi_0+\omega\hat\phi_j\phi_0+\kappa\nabla\hat\psi_j\cdot\nabla\psi_0+2\omega\hat\psi_j\psi_0\,dx-2\int\hat\phi_j\phi_0\hat\psi_j\,dx-\int\phi_0^2\hat\psi_j\,dx\\
&\qquad-\int\hat\phi_j^2\psi_0\,dx-2\int\hat\phi_j\phi_0\psi_0\,dx
\end{aligned}$$

to see that the first term on the right hand side of the last equality tends to zero by the weak convergence of $\hat\phi_j,\hat\psi_j$ in H^1 and that other terms tend to zero since a sequence $\{u_j\}\subset H^1$ with $u_j\to 0$ weakly in H^1 satisfies $u_jv_0\to 0$ strongly in L^2 for any $v_0\in H^1$. In fact, for any $L>0$

$$\|u_jv_0;L^2\|^2\leqslant\int_{-L}^{L}u_j^2v_0^2\,dx+\sup_k\|u_k;L^2\|^2\cdot\Big(\sup_{|x|\geqslant L}|v_0(x)|\Big)^2,$$

where the first term on the right hand side of the last inequality tends to zero since $u_j\to 0$ uniformly on $[-L,L]$ and the last term tends to zero as $L\to\infty$ since $v_0\in C_\infty$. In the same way we see that

$$\sup\big\{\big|\big(I'_\omega(\hat\phi_j+\phi_0,\hat\psi_j+\psi_0)-I'_\omega(\hat\phi_j,\hat\psi_j)-I'_\omega(\phi_0,\psi_0)\big)(u,v)\big|;\ \|(u,v);H^1\times H^1\|^2\leqslant 1\big\}\to 0.$$

This proves that $(\hat\phi_j,\hat\psi_j)$ is a $(PS)_{b-I_\omega(\phi_0,\psi_0)}$-sequence of I_ω. By Lemma 4.6, $b-I_\omega(\phi_0,\psi_0)\geqslant 0$. By the first part of the proof of the required identity on b, we already know that

$$b\leqslant\inf\{I_\omega(\phi,\psi);\ (\phi,\psi)\in\mathcal{C}_\omega\}\leqslant I_\omega(\phi_0,\psi_0)$$

since $(\phi_0,\psi_0)\in\mathcal{C}_\omega$. We have therefore proved that

$$b=\inf\{I_\omega(\phi,\psi);\ (\phi,\psi)\in\mathcal{C}_\omega\}=I_\omega(\phi_0,\psi_0).$$

This completes the proof of Theorem 4.5 for $n=1$.

Remark 4.5. By Lemma 4.6, $\hat\phi_j\to 0$, $\hat\psi_j\to 0$ strongly in H^1 since $b-I_\omega(\phi_0,\psi_0)=0$. This implies that $\phi_j\to\phi_0$, $\psi_j\to\psi_0$ strongly in H^1.

Remark 4.6. We can see for any $(\phi,\psi)\in\mathcal{G}_\omega$ that ϕ has a constant sign and ψ is positive in $\mathbb{R}^n$.

5. Best constant in an inequality of Gagliardo–Nirenberg type in four space dimensions

In this section we consider the best constant of the Gagliardo–Nirenberg type inequality in $\mathbb{R}^4$

$$P(\phi,\psi)\leqslant CK(\phi,\psi)Q(\phi,\psi)^{1/2}.$$

For that purpose we define

$$\alpha_1=\inf\{J(\phi,\psi);\ (\phi,\psi)\in\mathcal{P}\},$$

where

$$J(\phi,\psi)=K(\phi,\psi)Q(\phi,\psi)^{1/2}/P(\phi,\psi),$$
$$\mathcal{P}=\{(\phi,\psi)\in(H^1\times H^1)\backslash\{(0,0)\};\ P(\phi,\psi)>0\}$$

as in Section 4. By Lemma 3.6, we already know that $\alpha_1>0$.

Theorem 5.1. *Let $n = 4$. Then*:

(1) *There exists a pair of non-negative, radially symmetric functions $(\phi_0, \psi_0) \in \mathcal{P}$ such that*

$$J(\phi_0, \psi_0) = \alpha_1 = \inf\{J(\phi, \psi);\ (\phi, \psi) \in \mathcal{P}\}.$$

(2) *Let (ϕ_0, ψ_0) be as in Part (1). There exist $t_0 > 0$ and $l_0 > 0$ such that $(\phi, \psi) = (t_0\delta_{l_0}\phi_0, t_0\delta_{l_0}\psi_0)$ is a positive solution of (12) with $\omega = 1$. Moreover, (ϕ, ψ) is a ground state for (12) with $\omega = 1$.*

(3) *The set of minimizers of J is characterized as*

$$\{(\phi, \psi) \in H^1 \times H^1;\ J(\phi, \psi) = \alpha_1\} = \{(t\delta_l\phi, t\delta_l\psi) \in H^1 \times H^1;\ t, l > 0,\ (\phi, \psi) \in \mathcal{G}_1\}.$$

For any $(\phi, \psi) \in \mathcal{G}_1$ the following identity holds

$$\alpha_1 = 2Q(\phi, \psi)^{1/2}.$$

Proof. Let $\{(\phi_j, \psi_j)\} \subset \mathcal{P}$ be a minimizing sequence for J. By (23) we may assume that ϕ_j, ψ_j are non-negative and radially symmetric functions in H^1. We define

$$\tilde{\phi}_j = t_j\delta_{l_j}\phi_j, \qquad \tilde{\psi}_j = t_j\delta_{l_j}\psi_j,$$

where

$$t_j = Q(\phi_j, \psi_j)^{1/2}/K(\phi_j, \psi_j), \qquad l_j = K(\phi_j, \psi_j)^{1/2}/Q(\phi_j, \psi_j)^{1/2},$$

so that

$$K(\tilde{\phi}_j, \tilde{\psi}_j) = Q(\tilde{\phi}_j, \tilde{\psi}_j) = 1$$

and

$$1/P(\tilde{\phi}_j, \tilde{\psi}_j) = J(\tilde{\phi}_j, \tilde{\psi}_j) \to \alpha_1.$$

Since $\{(\tilde{\phi}_j, \tilde{\psi}_j)\}$ is bounded in $H^1 \times H^1$, there exists a subsequence still denoted by $\{(\tilde{\phi}_j, \tilde{\psi}_j)\}$ such that

$$\tilde{\phi}_j \to \phi_0, \qquad \tilde{\psi}_j \to \psi_0 \quad \text{weakly in } H^1(\mathbb{R}^4)$$

for some $(\phi_0, \psi_0) \in H_r^1 \times H_r^1$. By Strauss' compact embedding $H_r^1(\mathbb{R}^4) \subset L^3(\mathbb{R}^4)$,

$$\tilde{\phi}_j \to \phi_0, \qquad \tilde{\psi}_j \to \psi_0 \quad \text{strongly in } L^3(\mathbb{R}^4).$$

This yields

$$P(\phi_0, \psi_0) = \lim_{j\to\infty} P(\tilde{\phi}_j, \tilde{\psi}_j) = 1/\alpha_1 > 0,$$

while weak convergence of $\{(\tilde{\phi}_j, \tilde{\psi}_j)\}$ in $H^1 \times H^1$ yields

$$K(\phi_0, \psi_0) \leqslant \lim_{j\to\infty} K(\tilde{\phi}_j, \tilde{\psi}_j) = 1,$$

$$Q(\phi_0, \psi_0) \leqslant \lim_{j\to\infty} Q(\tilde{\phi}_j, \tilde{\psi}_j) = 1.$$

Therefore

$$\alpha_1 \leqslant J(\phi_0, \psi_0) = \frac{K(\phi_0, \psi_0)Q(\phi_0, \psi_0)^{1/2}}{P(\phi_0, \psi_0)} \leqslant \lim_{j\to\infty} J(\tilde{\phi}_j, \tilde{\psi}_j) = \alpha_1.$$

We conclude that $(\phi_0, \psi_0) \in \mathcal{P} \cap (H_r^1 \times H_r^1)$ satisfies

$$J(\phi_0, \psi_0) = \alpha_1, \qquad K(\phi_0, \psi_0) = Q(\phi_0, \psi_0) = 1, \qquad P(\phi_0, \psi_0) = 1/\alpha_1,$$

$$\tilde{\phi}_j \to \phi_0, \qquad \tilde{\psi}_j \to \psi_0 \quad \text{strongly in } H^1(\mathbb{R}^4).$$

Since (ϕ_0, ψ_0) is a minimizer of J, it is a critical point. For any $(u, v) \in H^1 \times H^1$

$$\frac{d}{ds}\bigg|_{s=0} J(\phi_0 + su, \psi_0 + sv) = 0,$$

which means that

$$\frac{Q(\phi_0,\psi_0)^{1/2}}{P(\phi_0,\psi_0)}\left(K'(\phi_0,\psi_0)(u,v) + \frac{K(\phi_0,\psi_0)}{2Q(\phi_0,\psi_0)}Q'(\phi_0,\psi_0)(u,v)\right)$$
$$= \frac{K(\phi_0,\psi_0)Q(\phi_0,\psi_0)^{1/2}}{P(\phi_0,\psi_0)^2}P'(\phi_0,\psi_0)(u,v).$$

This yields

$$K'(\phi_0,\psi_0)(u,v) + \frac{1}{2}Q'(\phi_0,\psi_0)(u,v) = \alpha_1 P'(\phi_0,\psi_0)(u,v),$$

which means that

$$2\int (\nabla\phi_0 \cdot \nabla u + \kappa\nabla\psi_0 \cdot \nabla v)\,dx + \int (\phi_0 u + 2\psi_0 v)\,dx = \alpha_1 \int \left(2\phi_0\psi_0 u + \phi_0^2 v\right)dx.$$

We now define $(\phi, \psi) = (\alpha_1\delta_{1/\sqrt{2}}\phi_0, \alpha_1\delta_{1/\sqrt{2}}\psi_0)$. Then (ϕ, ψ) is a solution of (12) with $\omega = 1$. Since (ϕ_0, ψ_0) is a critical point of J, (ϕ, ψ) is also a critical point of J. By Theorem 4.1, we obtain

$$K(\phi,\psi) = 2P(\phi,\psi)$$

and therefore

$$J(\phi,\psi) = 2Q(\phi,\psi)^{1/2}.$$

By the maximum principle, (ϕ, ψ) is a positive solution of (12) with $\omega = 1$. It follows from Theorem 4.1 that any nontrivial solution (u, v) of (12) with $\omega = 1$ satisfies

$$J(u,v) = 2^{3/2} I_1(u,v)^{1/2}.$$

This implies that any nontrivial solution (u, v) of (12) with $\omega = 1$ that is a minimizer of J is a ground state of (12) with $\omega = 1$. By a similar argument to that of the proof of Theorem 4.5, all the statements of Theorem 5.1 follow. □

6. Global existence and blow-up in four space dimensions

In this section we study the global existence of H^1-solutions and blow-up solutions in four space dimensions on the basis of results in Sections 4 and 5. As in Section 4, we consider the rescaled equations of the form

$$\begin{cases} i\partial_t u + \Delta u = -2v\bar{u}, \\ i\partial_t v + \kappa\Delta v = -u^2, \end{cases} \tag{27}$$

where u and v are complex-valued functions of $(t, x) \in \mathbb{R} \times \mathbb{R}^4$. We have assumed that $\lambda = c\bar{\mu}$ with $c > 0$, $\lambda \neq 0$, $\mu \neq 0$ in (1) and $\kappa = m/M$. As regards the global existence of H^1-solutions of (27), Theorem 3.7 is reformulated as:

Theorem 6.1. *For any* $(u_0, v_0) \in H^1(\mathbb{R}^4) \times H^1(\mathbb{R}^4)$ *with*

$$Q(u_0, v_0) < Q(\phi_0, \psi_0),$$

where $(\phi_0, \psi_0) \in \mathcal{P}$ *is a ground state for* (12) *with* $\omega = 1$*, the system of Eqs.* (27) *has a unique pair of solutions* $(u, v) \in Y(\mathbb{R}) \times Y(\mathbb{R})$ *with* $(u(0), v(0)) = (u_0, v_0)$.

Proof. The theorem follows by the same argument as in the proof of Theorem 3.7 under the condition

$$2\alpha_1^{-1} Q(u_0, v_0)^{1/2} < 1,$$

which is equivalent to

$$Q(u_0, v_0) < \alpha_1^2/4 = Q(\phi_0, \psi_0). \qquad \square$$

We now write down explicit blow-up solutions by means of the pseudo-conformal transformations. For

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R}),$$

we define $C_A(u, v) = (C_A^1 u, C_A^2 v)$ by

$$\bigl(C_A^1 u\bigr)(t, x) = \frac{1}{(a+bt)^2} \exp\biggl(\frac{ib|x|^2}{4(a+bt)}\biggr) u\biggl(\frac{c+dt}{a+bt}, \frac{x}{a+bt}\biggr),$$
$$\bigl(C_A^2 v\bigr)(t, x) = \frac{1}{(a+bt)^2} \exp\biggl(\frac{ib|x|^2}{4\kappa(a+bt)}\biggr) v\biggl(\frac{c+dt}{a+bt}, \frac{x}{a+bt}\biggr).$$

Then by a direct calculation we find:

Proposition 6.2. *Let $n = 4$ and let $\kappa = 1/2$. Then the following statements are equivalent*:

(i) (u, v) *is a solution of* (27).
(ii) $C_A(u, v)$ *is a solution of* (27) *for some* $A \in SL_2(\mathbb{R})$.
(iii) $C_A(u, v)$ *is a solution of* (27) *for all* $A \in SL_2(\mathbb{R})$.

Remark 6.1. The condition $\kappa = 1/2$ is exactly the mass resonance condition $M = 2m$.

It is also standard to verify the following theorem by a straightforward calculation.

Theorem 6.3. *Let $n = 4$ and let $\kappa = 1/2$. Let (ϕ_0, ψ_0) be a ground state for* (12) *with $\omega = 1$ given by Theorem* 5.1. *For $T > 0$, let*

$$u(t, x) = \frac{1}{(T-t)^2} \exp\biggl(-i\frac{|x|^2}{4(T-t)} + i\frac{t}{T(T-t)}\biggr)\phi_0\biggl(\frac{x}{T-t}\biggr),$$
$$v(t, x) = \frac{1}{(T-t)^2} \exp\biggl(-i\frac{|x|^2}{2(T-t)} + i\frac{2t}{T(T-t)}\biggr)\psi_0\biggl(\frac{x}{T-t}\biggr),$$

which are written as $(u, v) = C_A(e^{it}\phi_0, e^{2it}\psi_0)$, where

$$A = \begin{bmatrix} T & -1 \\ 0 & 1/T \end{bmatrix}.$$

Then (u, v) is a solution of (27) *such that*:

(i) $u, v \in C^\infty((-\infty, T); H^\infty(\mathbb{R}^4))$, *where* $H^\infty = \bigcap_{m \geqslant 1} H^m$.
(ii) $(u(0), v(0)) = (\frac{1}{T^2}\exp(-i\frac{|x|^2}{4T})\phi_0(\frac{x}{T}), \frac{1}{T^2}\exp(-i\frac{|x|^2}{2T})\psi_0(\frac{x}{T}))$.
(iii) $Q(u(0), v(0)) = Q(\phi_0, \psi_0) = \alpha_1^2/4$.
(iv) $K(u(t), v(t)) = O((T-t)^{-2})$ *as* $t \uparrow T$.
(v) $\operatorname{Re} P(u(t), v(t)) = \operatorname{Re}\int u(t, x)^2 \overline{v(t, x)}\, dx = O((T-t)^{-2})$ *as* $t \uparrow T$.
(vi) $|u(t)|^2 + 2|v(t)|^2 \to Q(\phi_0, \psi_0)\delta$ *weakly star in* $\mathcal{D}'(\mathbb{R}^4)$ *as* $t \uparrow T$, *where δ is the Dirac delta at the origin.*

Remark 6.2. The condition of $Q(u(0), v(0))$ in Theorem 6.1 is sharp for the global existence.

7. Remarks on the ground states in semitrivial case

If m and M satisfy the inverse condition of mass resonance:

$$m = 2M,$$

or equivalently $\kappa = 2$, then (12) is written as

$$\begin{cases} -\Delta\phi_\omega + \omega\phi_\omega = 2\psi_\omega\phi_\omega, \\ -2\Delta\psi_\omega + 2\omega\psi_\omega = \phi_\omega^2, \end{cases} \tag{28}$$

where the linear part of the equations is governed by a single operator $-\Delta + \omega$. In this case (28) is essentially reduced to the single equation

$$-\Delta\phi_\omega + \omega\phi_\omega = \phi_\omega^2 \tag{29}$$

by setting $\psi_\omega = \phi_\omega/2$. In fact, we have

Theorem 7.1. *Let $1 \leqslant n \leqslant 5$ and let $\kappa = 2$. Then ground states for (28) are unique up to translations. The unique ground state is given by $(\varphi, \varphi/2)$, where φ is a ground state of (29).*

Proof. It is well known that φ is characterized as a minimizer of the functional

$$R_\omega(\phi) = \left(\left\|\nabla\phi; L^2\right\|^2 + \omega\left\|\phi; L^2\right\|^2\right)/\left\|\phi; L^3\right\|^2$$

and is unique up to translations. We now recall that

$$R_\omega(\phi,\psi) = \frac{\|\nabla\phi; L^2\|^2 + 2\|\nabla\psi; L^2\|^2 + \omega(\|\phi; L^2\|^2 + 2\|\psi; L^2\|^2)}{(\int \phi^2\psi\, dx)^{2/3}}.$$

This implies

$$R_\omega\left(\phi, \frac{1}{2}\phi\right) = \frac{3}{2^{1/3}}\frac{\|\nabla\phi; L^2\|^2 + \omega\|\phi; L^2\|^2}{\|\phi; L^3\|^2} = \frac{3}{2^{1/3}}R_\omega(\phi)$$

which in turn implies

$$\beta_\omega \equiv \inf\left\{R_\omega(\phi,\psi);\ (\phi,\psi) \in \mathcal{P}\right\} \leqslant \frac{3}{2^{1/3}}\inf\left\{R_\omega(\phi);\ \phi \in H^1\backslash\{0\}\right\}.$$

We apply the Young inequality

$$ab \leqslant \frac{2}{3}a^{3/2} + \frac{1}{3}b^3$$

for $a, b \geqslant 0$ and the Hölder inequality respectively in the numerator and denominator of $R_\omega(\phi,\psi)$ to obtain

$$\begin{aligned} R_\omega(\phi,\psi) &\geqslant \frac{3}{2^{1/3}}\left(\frac{\|\nabla\phi; L^2\|^2 + \omega\|\phi; L^2\|^2}{\|\phi; L^3\|^2}\right)^{2/3}\left(\frac{\|\nabla\psi; L^2\|^2 + \omega\|\psi; L^2\|^2}{\|\psi; L^3\|^2}\right)^{1/3} \\ &\geqslant \frac{3}{2^{1/3}}\inf\left\{R_\omega(\phi);\ \phi \in H^1\backslash\{0\}\right\}. \end{aligned}$$

Combining those inequalities, we obtain

$$\beta_\omega = \frac{3}{2^{1/3}}\inf\left\{R_\omega(\phi);\ \phi \in H^1\backslash\{0\}\right\}.$$

By Theorem 4.5, β_ω has a minimizer $(\phi_0, \psi_0) \in \mathcal{P}$, which realizes the Young and Hölder inequalities as equalities. Therefore $\phi_0 = 2\psi_0$ and $(\phi_0, \psi_0) = c(\varphi, \varphi/2)$ for some $c \in \mathbb{R}\backslash\{0\}$. This proves the theorem. □

All those results above fail for $n = 6$ as far as $\omega > 0$ by Theorem 4.1. The only case that we could expect nontrivial results is restricted to the case where $\omega = 0$. Here we do not assume $m = 2M$. In this case (12) with $\omega = 0$ is written as

$$\begin{cases} -\Delta\phi = 2\psi\phi, \\ -\kappa\Delta\psi = \phi^2. \end{cases} \tag{30}$$

By changing ϕ by $\sqrt{\kappa/2}\phi$, (30) is rewritten as

$$\begin{cases} -\Delta\phi = 2\psi\phi, \\ -2\Delta\psi = \phi^2, \end{cases} \tag{31}$$

which is understood to be a substitute of (28) with $\omega = 0$ for $n = 6$. New system (31) is essentially reduced to the single equation

$$-\Delta\phi = \phi^2, \tag{32}$$

by setting $\psi = \phi/2$, which is understood to be a substitute (29) with $\omega = 0$ for $n = 6$. In fact, we have

Theorem 7.2. *Let $n = 6$. Then ground states for (31) are unique up to translations and dilations. The unique ground state is given by $(\varphi, \varphi/2)$, where φ is a ground state of (32) in the space*

$$\dot{H}^1 \cap L^3 = \{u \in L^3(\mathbb{R}^6);\ \nabla u \in L^2(\mathbb{R}^6)\}.$$

Proof. It suffices to consider the problem in the space

$$(\dot{H}^1 \cap L^3)_r(\mathbb{R}^6) = \{u \in \dot{H}^1 \cap L^3;\ u(x) = \tilde{u}(|x|)\},$$

where $\tilde{u}$ is a radial function associated with radially symmetric function $u \in \dot{H}^1 \cap L^3$. For $u \in (\dot{H}^1 \cap L^3)_r(\mathbb{R}^6)$ we define Tu by

$$(Tu)(t) = e^{2t}\tilde{u}(e^t), \quad t \in \mathbb{R}.$$

Then we obtain

$$(T\Delta u)(t) = e^{2t}\left[\tilde{u}''(e^t) + 5e^{-t}\tilde{u}'(e^t)\right],$$
$$(Tu)''(t) = 4(Tu)(t) + e^{2t}(T\Delta u)(t).$$

This implies that all solutions (ϕ, ψ) of (31) in $(\dot{H}^1 \cap L^3)_r$ satisfy

$$\begin{aligned} (T\phi)''(t) &= 4(T\phi)(t) + e^{2t}\big(T(-2\psi\phi)\big)(t) \\ &= 4(T\phi)(t) - 2(T\psi)(t)(T\phi)(t), \\ 2(T\psi)''(t) &= 8(T\psi)(t) + e^{2t}\big(T(-\phi^2)\big)(t) \\ &= 8(T\psi)(t) - \big((T\phi)(t)\big)^2. \end{aligned}$$

Namely,

$$\begin{cases} -T\phi'' + 4T\phi = 2T\psi \cdot T\phi, \\ -2T\psi'' + 8T\psi = (T\phi)^2, \end{cases} \tag{33}$$

which is regarded as (28) with $\omega = 4$ for $n = 1$. Therefore, the theorem follows if we can show that the map $T : (\dot{H}^1 \cap L^3)_r(\mathbb{R}^6) \to H^1(\mathbb{R})$ gives a one-to-one correspondence between ground states of (31) and those of (33) which follows in general dimensions $n \geqslant 2$ as

$$T : (\dot{H}^1 \cap L^{2n/(n-2)})_r(\mathbb{R}^n) \to H^1(\mathbb{R}),$$
$$(Tu)(t) = e^{\frac{n-2}{2}t}\tilde{u}(e^t),$$
$$\|\nabla u; L^2(\mathbb{R}^n)\|^2 = \sigma_{n-1}\left(\|(Tu)'; L^2(\mathbb{R})\|^2 + \left(\frac{n-2}{2}\right)^2 \|Tu; L^2(\mathbb{R})\|^2\right),$$

where σ_{n-1} is the surface area of the unit sphere in $\mathbb{R}^n$. □

8. Remarks on one-dimensional problem as a Lagrangian system

In this section we study (12) in one space dimension of the form

$$\begin{cases} -u'' + \omega u = 2vu, \\ -\kappa v'' + 2\omega v = u^2 \end{cases} \tag{34}$$

and regard (34) as a Lagrangian system by taking x as the time variable. We already know the existence of ground states (u, v) for (34). Moreover, u and v are positive and even functions in H^2. Particularly, $(u'(0), v'(0)) = (0, 0)$ and

$$\big(u(x), v(x)\big), \big(u'(x), v'(x)\big) \to (0, 0) \quad \text{as } x \to \pm\infty.$$

Moreover, (34) has the following conserved quantity

$$\frac{1}{2}\big((u')^2 + \kappa (v')^2\big) - \frac{\omega}{2}\big(u^2 + 2v^2\big) + u^2 v$$

with respect to x, which is identically zero since it vanishes at infinity. Since $(u'(0), v'(0)) = (0, 0)$ we have

$$-\frac{\omega}{2}\big(u^2(0) + 2v^2(0)\big) + u^2(0)v(0) = 0.$$

We prove that

$$-\frac{\omega}{2}\big(u^2(x) + 2v^2(x)\big) + u^2(x)v(x) < 0$$

for all $x \neq 0$, or equivalently, $(u(x), v(x)) \in \Omega_\omega$ for all $x \neq 0$, where

$$\Omega_\omega = \left\{(s, t) \in \mathbb{R}^2;\ s, t > 0,\ -\frac{\omega}{2}\big(s^2 + 2t^2\big) + s^2 t < 0\right\}.$$

We also set

$$\begin{aligned} \Gamma_\omega^\pm &= \left\{(s, t) \in \mathbb{R}^2;\ \pm s > 0,\ t > 0,\ -\frac{\omega}{2}\big(s^2 + 2t^2\big) + s^2 t = 0\right\} \\ &= \left\{(s, t) \in \mathbb{R}^2;\ t > \frac{\omega}{2},\ s = \pm\sqrt{\frac{2\omega}{2t - \omega}} t\right\}. \end{aligned}$$

If there exists $x_0 > 0$ such that $(u(x_0), v(x_0)) \in \Gamma_\omega^+$, then $(u'(x_0), v'(x_0)) = (0, 0)$ and

$$u''(x_0) = \omega u(x_0) - 2v(x_0)u(x_0) < 0,$$

since $\omega s - 2st = (\omega - 2t)s < 0$ for any $(s, t) \in \Gamma_\omega^+$. Two pairs of functions $(u_\pm, v_\pm)$ defined by

$$u_\pm(x) = u(\pm x + x_0), \qquad v_\pm(x) = v(\pm x + x_0)$$

satisfy (34) and

$$\big(u_\pm(0), v_\pm(0)\big) = \big(u(x_0), v(x_0)\big) \quad \text{and} \quad \big(u_\pm'(0), v_\pm'(0)\big) = (0, 0).$$

By the uniqueness of solutions to the Cauchy problem of (34) with data at $x = 0$, we have $u_+ = u_- = u$, $v_+ = v_- = v$. Therefore we conclude that (u, v) is a non-constant periodic solution of (34) connecting $(0, 0)$ and $(u(x_0), v(x_0))$, which contradicts that $(u(x), v(x)) \to (0, 0)$ as $x \to \pm\infty$. We have proved that $(u, v) \in \Lambda_\omega$, where

$$\Lambda_\omega = \big\{(u, v) \in H_r^1 \times H_r^1;\ \big(u(0), v(0)\big) \in \Gamma_\omega^+,\ \big(u(x), v(x)\big) \in \Omega_\omega \text{ for all } x \neq 0\big\}.$$

We set

$$\gamma_\omega = \inf\big\{I_\omega(u, v);\ (u, v) \in \Lambda_\omega\big\}.$$

Moreover we have

Theorem 8.1. *Let $n = 1$ and let $\kappa, \omega > 0$. Then*

$$\mathcal{G}_\omega = \{(u, v) \in \Lambda_\omega;\ I_\omega(u, v) = \gamma_\omega\} \cup \{(u, v);\ (-u, v) \in \Lambda_\omega,\ I_\omega(u, v) = \gamma_\omega\}.$$

To show the above result, it suffices to show

$$\gamma_\omega = \inf\{I_\omega(u, v);\ (u, v) \in \mathcal{C}_\omega\}. \tag{35}$$

By the above argument, we have $\gamma_\omega \leqslant \inf\{I_\omega(u, v);\ (u, v) \in \mathcal{C}_\omega\}$. To show the reverse inequality, we will show that γ_ω is attained by a solution of (34).

Before giving a proof of the existence of the minimizer, we remark that $I_\omega(u, v) = \int(\frac{1}{2}((u')^2 + \kappa(v')^2) - V(u, v))\,dx$, $V(x, t) = -\frac{\omega}{2}(s^2 + 2t^2) + s^2 t$ and the integrand is always non-negative for $(u, v) \in \Lambda_\omega$. Moreover we have

Remark 8.1. For $P \in \Gamma_\omega^+$, we consider the following minimizing problem

$$\gamma_{\omega,P} = \inf\{I_\omega(u, v);\ (u, v) \in \Lambda_{\omega,P}\},$$

where

$$\Lambda_{\omega,P} = \{(u, v) \in H_r^1 \times H_r^1;\ (u(0), v(0)) = P,\ (u(x), v(x)) \in \Omega_\omega \text{ for all } x \neq 0\}.$$

Then $\gamma_{\omega,P}$ does not depend on $P \in \Gamma_\omega^+$. In fact, suppose $P_0, P_1 \in \Gamma_\omega^+$ and let $(u, v) \in \Lambda_{\omega,P_1}$. For a curve $(c_1, c_2) \in C^1([0, 1], \Gamma_\omega^+)$ joining P_0 and P_1 and for $n \in \mathbb{N}$, we set $(\tilde{u}_n, \tilde{v}_n)$ by

$$(\tilde{u}_n(x), \tilde{v}_n(x)) = \begin{cases} (c_1(|x|/n), c_2(|x|/n)) & \text{for } |x| \in [0, n], \\ (u(|x| - n), v(|x| - n)) & \text{for } |x| \in (n, \infty). \end{cases}$$

Then we have

$$I_\omega(\tilde{u}_n, \tilde{v}_n) = \frac{1}{n}\int_0^1 ((c_1')^2 + \kappa(c_2')^2)\,dx + I_\omega(u, v) \to I_\omega(u, v) \quad \text{as } n \to \infty.$$

Perturbing $(\tilde{u}_n, \tilde{v}_n)$, we find a (u_n, v_n) such that $(u_n(x), v_n(x)) \in \Omega_\omega$ for all $x \neq 0$, i.e., $(u_n, v_n) \in \Lambda_{\omega,P_0}$, and $I_\omega(u_n, v_n) \to I_\omega(u, v)$ as $n \to \infty$. Thus we have $\gamma_{\omega,P_0} \leqslant I(u, v)$. Since $(u, v) \in \Lambda_{\gamma,P_1}$ is arbitrary, we have $\gamma_{\omega,P_0} \leqslant \gamma_{\omega,P_1}$. Replacing P_0 and P_1, we deduce that $\gamma_{\omega,P_0} = \gamma_{\omega,P_1}$. Thus, $\gamma_{\omega,P}$ does not depend on $P \in \Gamma_\omega^+$.

In particular, we have $\gamma_\omega = \gamma_{\omega,P}$ for all $P \in \Gamma_\omega^+$. We also note that Γ_ω^+ is unbounded and γ_ω has an unbounded minimizing sequence. Indeed, for any sequence $P_n \in \Gamma_\omega^+$, we can find a sequence $(u_n, v_n) \in \Lambda_\omega$ such that $(u_n(0), v_n(0)) = P_n$ and $I_\omega(u_n, v_n) \to \gamma_\omega$. Thus minimizing sequences for γ_ω do not have convergent subsequence in general.

To show the existence of a minimizer of γ_ω, we use an idea from Rabinowitz and Tanaka [21]. We set

$$\widetilde{\Omega}_\omega = \left\{(s, t) \in \mathbb{R}^2;\ -\frac{\omega}{2}(s^2 + 2t^2) + s^2 t < 0\right\}.$$

We easily see that $\partial\widetilde{\Omega}_\omega = \{(0, 0)\} \cup \Gamma_\omega^+ \cup \Gamma_\omega^-$. We introduce the following auxiliary problem for $\lambda \in [0, 1]$:

$$V_\lambda(s, t) = -\frac{\omega + \lambda}{2}(s^2 + 2t^2) + s^2 t,$$

$$J_\lambda(u, v) = \int_{-\infty}^{\infty} \left(\frac{1}{2}((u')^2 + \kappa(v')^2) - V_\lambda(u, v)\right) dx,$$

$$c_\lambda = \inf_{(u,v) \in \widetilde{\Lambda}_\omega} J_\lambda(u, v),$$

where

$$\widetilde{\Lambda}_\omega = \{(u, v) \in H_r^1 \times H_r^1;\ (u(0), v(0)) \in \Gamma_\omega^+ \cup \Gamma_\omega^- \text{ and } (u(x), v(x)) \in \overline{\widetilde{\Omega}_\omega} \text{ for all } x > 0\}.$$

First we remark

Lemma 8.2. *For* $\lambda \in [0,1]$*, we have*:

(i) $c_\lambda = \inf_{(u,v)\in\Lambda_\omega} J_\lambda(u,v)$. *In particular,* $c_0 = \gamma_\omega$.
(ii) *If* c_λ *is attained in* $\widetilde{\Lambda}_\omega$*, then there exists a minimizer* (u,v) *in* Λ_ω*. Moreover it satisfies the following properties in* $(0,\infty)$

$$-u'' + (\omega+\lambda)u = 2uv, \tag{36}$$
$$-\kappa v'' + 2(\omega+\lambda)v = u^2, \tag{37}$$
$$\frac{1}{2}\big((u')^2 + \kappa(v')^2\big) - \frac{\omega+\lambda}{2}\big(u^2+2v^2\big) + u^2 v \equiv 0, \tag{38}$$
$$u(x), v(x) > 0. \tag{39}$$

(iii) *When* $\lambda = 0$*, if* c_0 *is attained, then there exists a minimizer* $(u,v)\in\Lambda_\omega$ *and it satisfies* (36)–(39) *in* $\mathbb{R}$*. That is,* $(u,v)\in\mathcal{C}_\omega$.

Proof. (i) Since $\Lambda_\omega \subset \widetilde{\Lambda}_\omega$, we have $c_\lambda \leqslant \inf_{(u,v)\in\Lambda_\omega} J_\lambda(u,v)$. On the other hand, for any $(u,v)\in\widetilde{\Lambda}_\omega$, we have $(|u|,|v|)\in\overline{\Lambda_\omega}$ and $J_\lambda(|u|,|v|)\leqslant J_\lambda(u,v)$. Thus we have (i).

(ii) Let $(u,v)\in\widetilde{\Lambda}_\omega$ be a minimizer of J_λ in $\widetilde{\Lambda}_\omega$. Since $(|u|,|v|)$ is also a minimizer, we may assume $u(x), v(x)\geqslant 0$ for all $x\in(0,\infty)$. Next we show $(u(x),v(x))\notin\Gamma_\omega^+$ for all $x\in(0,\infty)$. Indeed, if $(u(x_0),v(x_0))\in\Gamma_\omega^+$ for some $x_0\in(0,\infty)$. Then $(\tilde{u},\tilde{v})\in\widetilde{\Lambda}_\omega$ defined by $(\tilde{u}(x),\tilde{v}(x)) = (u(|x|+x_0), v(|x|+x_0))$ satisfies $J_\lambda(\tilde{u},\tilde{v}) < J_\lambda(u,v)$, which is in a contradiction to the minimality of (u,v). Thus, $(u(x),v(x))\in\widetilde{\Omega}_\omega$ for all $x\in(0,\infty)$ and it implies that $J'_\lambda(u,v)(h_1,h_2)=0$ for all $(h_1,h_2)\in C_0^\infty(\mathbb{R})$. Thus (u,v) satisfies (36)–(37). Since $(u(x),v(x))\to(0,0)$ as $x\to\infty$, we also have (38). (39) can be deduced from the fact that $u, v\geqslant 0$ and (36)–(37). In particular, we have $(u,v)\in\Lambda_\omega$.

(iii) When $\lambda=0$, $(u(0),v(0))\in\Gamma_\omega^+$ implies $(u'(0),v'(0))=(0,0)$, from which we deduce that (u,v) satisfies (36)–(39) in $\mathbb{R}$. □

We also have

Corollary 8.3.

(i) $\gamma_\omega \leqslant c_\lambda \leqslant c_1$ *for all* $\lambda\in(0,1]$.
(ii) $\lim_{\lambda\to 0} c_\lambda = \gamma_\omega$.

Next we show that c_λ is attained for $\lambda\in(0,1]$.

Lemma 8.4. *For* $\lambda\in(0,1]$*, there exists* $(u_\lambda,v_\lambda)\in\Lambda_\omega$ *such that* $J_\lambda(u_\lambda,v_\lambda)=c_\lambda$*. Moreover* (u_λ,v_λ) *satisfies* (36)–(39).

Proof. For $\lambda\in(0,1]$, we have $-V_\lambda(s,t)\geqslant\frac{\lambda}{2}(s^2+2t^2)$ in $\widetilde{\Omega}_\omega$. Thus we have for $(u,v)\in\widetilde{\Lambda}_\omega$

$$J_\lambda(u_\lambda,v_\lambda)\geqslant\frac{1}{2}\big(\|u';L^2\|^2+\kappa\|v';L^2\|^2\big)+\frac{\lambda}{2}\big(\|u;L^2\|^2+2\|v;L^2\|^2\big). \tag{40}$$

Let $\{(u_n,v_n)\}$ be a minimizing sequence for c_λ. It follows from (40) that $\{(u_n,v_n)\}$ is bounded in $H^1\times H^1$. Extracting a subsequence if necessary, we may assume that $(u_n,v_n)\to(u_\lambda,v_\lambda)\in\widetilde{\Lambda}_\omega$ weakly in $H^1\times H^1$. By Fatou lemma, we conclude that $J_\lambda(u_\lambda,v_\lambda)\leqslant\liminf_{n\to\infty}J_\lambda(u_n,v_n)=c_\lambda$. Thus (u_λ,v_λ) is a minimizer of J_λ. Applying (ii) of Lemma 8.2, we have Lemma 8.4. □

Next we show that the existence of a uniform $H^1\times H^1$-bound for (u_λ,v_λ).

Lemma 8.5. *Let* $(u_\lambda,v_\lambda)\in\Lambda_\omega$ *be the minimizer given in Lemma* 8.2*. Then* $\|(u_\lambda,v_\lambda);H^1\times H^1\|$ *is bounded as* $\lambda\to 0$.

Proof. Since (u_λ, v_λ) solves (36)–(37) in $(0,\infty)$, by integration by parts we have

$$\int_0^\infty \left((u_\lambda')^2 + \kappa (v_\lambda')^2 + (\omega+\lambda)(u_\lambda^2 + 2v_\lambda^2) - 3u_\lambda^2 v_\lambda\right) dx = -u_\lambda(0)u_\lambda'(0) - \kappa v_\lambda(0) v_\lambda'(0).$$

On the other hand, by (38) and $(u_\lambda(0), v_\lambda(0)) \in \Gamma_\omega^+$

$$\begin{aligned}\frac{1}{2}\left((u_\lambda'(0))^2 + \kappa(v_\lambda'(0))^2\right) &= \frac{\omega+\lambda}{2}\left(u_\lambda(0)^2 + 2v_\lambda(0)^2\right) - u_\lambda(0)^2 v_\lambda(0)\\ &= \frac{\lambda}{2}\left(u_\lambda(0)^2 + 2v_\lambda(0)^2\right).\end{aligned}$$

Thus, for some $C>0$ independent of λ, we have

$$\left|\int_0^\infty \left((u_\lambda')^2 + \kappa (v_\lambda')^2 + (\omega+\lambda)(u_\lambda^2 + 2v_\lambda^2) - 3u_\lambda^2 v_\lambda\right) dx\right| \leqslant C\sqrt{\lambda}\left(u_\lambda(0)^2 + v_\lambda(0)^2\right). \tag{41}$$

Since $J_\lambda(u_\lambda, v_\lambda) = c_\lambda \leqslant c_1$, we have

$$\int_0^\infty \left(\frac{1}{2}\left((u_\lambda')^2 + \kappa(v_\lambda')^2\right) + \frac{\omega+\lambda}{2}\left(u_\lambda^2 + 2v_\lambda^2\right) - u_\lambda^2 v_\lambda\right) dx \leqslant \frac{c_1}{2}.$$

Thus, by (41),

$$\frac{1}{6}\int_0^\infty \left((u_\lambda')^2 + \kappa(v_\lambda')^2 + \omega\left(u_\lambda^2 + 2v_\lambda^2\right)\right) dx \leqslant \frac{c_1}{2} + \frac{C\sqrt{\lambda}}{3}\left(u_\lambda(0)^2 + v_\lambda(0)^2\right). \tag{42}$$

Since $H^1(0,\infty) \subset C([0,\infty))$, there exists $C'>0$ independent of λ

$$u_\lambda(0)^2 + v_\lambda(0)^2 \leqslant C'\int_0^\infty \left((u_\lambda')^2 + \kappa(v_\lambda')^2 + \omega\left(u_\lambda^2 + 2v_\lambda^2\right)\right) dx.$$

Together with (42), we see that (u_λ, v_λ) stays bounded in $H^1 \times H^1$ as $\lambda \to 0$. □

Extracting a subsequence – still denoted by λ – if necessary, we may assume $(u_\lambda, v_\lambda) \to (u_0, v_0)$ weakly in $H^1(0,\infty) \times H^1(0,\infty)$. We can see that

$$\begin{aligned}&\left(u_0(x), v_0(x)\right) \in \overline{\widetilde{\Omega}_\omega} \cap [0,\infty)^2 \quad \text{for all } x \geqslant 0,\\ &\left(u_0(0), v_0(0)\right) \in \Gamma_\omega^+.\end{aligned}$$

We have

Lemma 8.6.

(i) $J_0(u_0, v_0) = \gamma_\omega$.
(ii) $(u_0, v_0) \in \Lambda_\omega$ *and* (u_0, v_0) *satisfies* (36)–(39) *in* $\mathbb{R}$ *with* $\lambda = 0$*, that is,* $(u_0, v_0) \in \mathcal{C}_\omega$.

Proof. (i) We have

$$\gamma_\omega \leqslant J_0(u_\lambda, v_\lambda) \leqslant J_\lambda(u_\lambda, v_\lambda) = c_\lambda.$$

By Fatou's lemma and Corollary 8.3, we have $J_0(u_0, v_0) \leqslant \gamma_\omega$. Since $(u_0, v_0) \in \widetilde{\Lambda}_\omega$, we see that $J_0(u_0, v_0) = \gamma_\omega$ and (u_0, v_0) is a minimizer. (ii) follows from Lemma 8.2. □

End of the proof of Theorem 8.1. By Lemma 8.6, we have γ_ω is attained and (35) holds.

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