

# $L^p$ -maximal regularity of nonlocal parabolic equations and applications

Xicheng Zhang

*School of Mathematics and Statistics, Wuhan University, Wuhan, Hubei 430072, PR China*

Received 18 March 2012; received in revised form 9 September 2012; accepted 8 October 2012

Available online 15 November 2012

## Abstract

By using Fourier's transform and Fefferman–Stein's theorem, we investigate the  $L^p$ -maximal regularity of nonlocal parabolic and elliptic equations with singular and non-symmetric Lévy operators, and obtain the unique strong solvability of the corresponding nonlocal parabolic and elliptic equations, where the probabilistic representation plays an important role. As a consequence, a characterization for the domain of pseudo-differential operators of Lévy type with singular kernels is given in terms of the Bessel potential spaces. As a byproduct, we also show that a large class of non-symmetric Lévy operators generates an analytic semigroup in  $L^p$ -spaces. Moreover, as applications, we prove Krylov's estimate for stochastic differential equations driven by Cauchy processes (i.e. critical diffusion processes), and also obtain the global well-posedness for a class of quasi-linear first order parabolic systems with critical diffusions. In particular, critical Hamilton–Jacobi equations and multidimensional critical Burger's equations are uniquely solvable and the smooth solutions are obtained.

© 2012 Elsevier Masson SAS. All rights reserved.

*Keywords:*  $L^p$ -regularity; Lévy process; Krylov's estimate; Sharp function; Critical Burger's equation

## 1. Introduction

Consider the following Cauchy problem of fractional Laplacian heat equation in the domain  $[0, \infty) \times \mathbb{R}^d$  with  $\alpha \in (0, 2)$  and  $\lambda \geq 0$ :

$$\partial_t u + (-\Delta)^{\frac{\alpha}{2}} u + b \cdot \nabla u + \lambda u = f, \quad u(0) = \varphi, \quad (1.1)$$

where  $b : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a measurable vector field,  $f : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$  and  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$  are two measurable functions, and  $(-\Delta)^{\frac{\alpha}{2}}$  is the fractional Laplacian (also called Lévy operator) defined by

$$(-\Delta)^{\frac{\alpha}{2}} u = \mathcal{F}^{-1}(|\cdot|^\alpha \mathcal{F}(u)), \quad u \in \mathcal{S}(\mathbb{R}^d), \quad (1.2)$$

where  $\mathcal{F}$  (resp.  $\mathcal{F}^{-1}$ ) denotes the Fourier (resp. inverse) transform,  $\mathcal{S}(\mathbb{R}^d)$  is the Schwartz class of smooth real or complex-valued rapidly decreasing functions.

*E-mail address:* [XichengZhang@gmail.com](mailto:XichengZhang@gmail.com).

Let  $(L_t)_{t \leq 0}$  be a symmetric and rotationally invariant  $\alpha$ -stable process. Let  $b, f \in C_b^\infty([0, \infty) \times \mathbb{R}^d)$  and  $X_{t,s}(x)$  solve the following stochastic differential equation (SDE):

$$X_{t,s}(x) = x + \int_t^s b(-r, X_{t,r}(x)) dr + \int_t^s dL_r, \quad t \leq s \leq 0, \quad x \in \mathbb{R}^d.$$

It is well known that for  $\varphi \in C_b^\infty(\mathbb{R}^d)$ , the unique solution of Eq. (1.1) can be represented by the Feynman–Kac formula as (see Theorem 5.2 below):

$$u(t, x) = \mathbb{E}\varphi(X_{-t,0}(x)) + \mathbb{E}\left(\int_{-t}^0 e^{-\lambda(s+t)} f(-s, X_{-t,s}(x)) ds\right), \quad t \geq 0. \quad (1.3)$$

In connection with this representation, the first order term  $b \cdot \nabla u$  is also called the drift term, and the fractional Laplacian term  $(-\Delta)^{\frac{\alpha}{2}} u$  is also called the diffusion term.

Let now  $u(t, x)$  satisfy (1.1). For  $r > 0$  and  $(t, x) \in [0, \infty) \times \mathbb{R}^d$ , define

$$u^r(t, x) := r^{-\alpha} u(r^\alpha t, rx), \quad b^r(t, x) := b(r^\alpha t, rx), \quad f^r(t, x) := f(r^\alpha t, rx),$$

then it is easy to see that  $u^r$  satisfies

$$\partial_t u^r + (-\Delta)^{\frac{\alpha}{2}} u^r + r^{\alpha-1} (b^r \cdot \nabla u^r) + \lambda r^\alpha u^r = f^r. \quad (1.4)$$

If one lets  $r \rightarrow 0$ , this scaling property leads to the following classification:

- (Subcritical case:  $\alpha \in (1, 2)$ .) The drift term is controlled by the diffusion term at small scales.
- (Critical case:  $\alpha = 1$ .) The fractional Laplacian has the same order as the first order term.
- (Supercritical case:  $\alpha \in (0, 1)$ .) The effect of the drift term is stronger than the diffusion term at small scales.

In recent years there is great interest to study the above nonlocal equation, since it has appeared in numerous disciplines, such as quasi-geostrophic fluid dynamics (cf. [10,9]), stochastic control problems (cf. [34]), non-linear filtering with jump (cf. [28]), mathematical finance (cf. [5]), anomalous diffusion in semiconductor growth (cf. [38]), etc. In [12], Droniou and Imbert studied the first order Hamilton–Jacobi equation with the fractional diffusion  $(-\Delta)^{\frac{\alpha}{2}}$  basing upon a “reverse maximal principle”. Therein, when  $\alpha \in (1, 2)$ , the classical solution was obtained; when  $\alpha \in (0, 2)$ , the existence and uniqueness of viscosity solutions in the class of Lipschitz functions was also established. In [9], Caffarelli and Vasseur established the global well-posedness of critical dissipative quasi-geostrophic equations (see also [21] for a simple proof in the periodic and two-dimensional case). On the other hand, Hölder regularity theory for the viscosity solutions of fully non-linear and nonlocal elliptic equations was also developed by Caffarelli and Silvestre [8], and Barles, Chasseigne and Imbert [4] (see also [3] and the series of works of Silvestre [30,31, 33,32], etc.). We emphasize that the arguments in [8] and [4] are different: the former is based on the Alexandorff–Bakelman–Pucci’s (ABP) estimate, and the latter is based on the Ishii–Lions’ simple method. Moreover, in the subcritical case, Kurenok [25] established Krylov’s type estimate for one-dimensional stable processes with drifts (see [39] for multidimensional extension).

The purpose of this paper is to develop an  $L^p$ -regularity theory for nonlocal equations with general Lévy operators. We describe it as follows. Let  $\nu$  be a Lévy measure in  $\mathbb{R}^d$ , i.e., a  $\sigma$ -finite measure satisfying  $\nu(\{0\}) = 0$  and

$$\int_{\mathbb{R}^d} \min(1, |y|^2) \nu(dy) < +\infty.$$

For  $\alpha \in (0, 2)$ , we write

$$y^{(\alpha)} := 1_{\alpha \in (1,2)} y + 1_{\alpha=1} y 1_{|y| \leq 1}.$$

In this article we are mainly concerned with the following pseudo-differential operator of Lévy type:

$$\mathcal{L}^\nu f(x) := \int_{\mathbb{R}^d} [f(x+y) - f(x) - y^{(\alpha)} \cdot \nabla f(x)] \nu(dy), \quad f \in \mathcal{S}(\mathbb{R}^d), \quad (1.5)$$

where  $\nu$  satisfies

$$\nu_1^{(\alpha)}(B) \leq \nu(B) \leq \nu_2^{(\alpha)}(B), \quad B \in \mathcal{B}(\mathbb{R}^d), \tag{1.6}$$

and

$$1_{\alpha=1} \int_{r \leq |y| \leq R} y \nu(dy) = 0, \quad 0 < r < R < +\infty. \tag{1.7}$$

Here,  $\nu_i^{(\alpha)}$ ,  $i = 1, 2$  are the Lévy measures of two  $\alpha$ -stable processes taking the form

$$\nu_i^{(\alpha)}(B) := \int_{\mathbb{S}^{d-1}} \left( \int_0^\infty \frac{1_B(r\theta) dr}{r^{1+\alpha}} \right) \Sigma_i(d\theta), \tag{1.8}$$

where  $\mathbb{S}^{d-1} = \{\theta \in \mathbb{R}^d : |\theta| = 1\}$  is the unit sphere in  $\mathbb{R}^d$ , and  $\Sigma_i$  called the spherical part of  $\nu_i^{(\alpha)}$  is a finite measure on  $\mathbb{S}^{d-1}$ . We remark that condition (1.7) is a common assumption in the critical case (see [27,11]), and is clearly satisfied when  $\nu$  is symmetric. Moreover, in the case of  $\alpha \in (1, 2)$ , for the convenience of proof, we use  $y$  rather than  $y 1_{|y| \leq 1}$  in (1.5). This is not essential since one can always minus a first order term  $(\int_{|y| > 1} y \nu(dy)) \cdot \nabla f(x)$  in (1.5).

One of the aims of the present paper is to determine  $\mathcal{D}^p(\mathcal{L}^\nu)$ , the domain of the Lévy operator  $\mathcal{L}^\nu$  in  $L^p$ -space. We shall prove that under (1.6) and (1.7), if  $\nu_1^{(\alpha)}$  is nondegenerate (see Definition 2.6 below), then for any  $p \in (1, \infty)$ ,

$$\mathcal{D}^p(\mathcal{L}^\nu) = \mathbb{H}^{\alpha,p},$$

where  $\mathbb{H}^{\alpha,p}$  is the  $\alpha$ -order Bessel potential space. When  $\nu(dy) = a(y) dy/|y|^{d+\alpha}$  with  $c_1 \leq |a(y)| \leq c_2$ , this characterization was obtained recently by Dong and Kim [11]. It is remarked that the technique in [4] was used by Dong and Kim to derive some local Hölder estimate for nonlocal elliptic equation in order to prove their characterization. However, the following sum of nonlocal operators is not covered by [11]:

$$\mathcal{L}f(x) = \sum_{i=1}^d \int_{\mathbb{R}} \frac{f(x_1, \dots, x_{i-1}, x_i + y_i, x_{i+1}, \dots, x_d) - f(x) - y_i^{(\alpha)} \cdot \partial_i f(x)}{|y_i|^{1+\alpha}} dy_i,$$

since in this case, the Lévy measure (or the Lévy symbol) is very singular (or non-smooth) (see Remark 2.7). Notice that if the Lévy symbol is smooth and its derivatives satisfy suitable conditions, the above characterization falls into the classical multiplier theorems about pseudo-differential operators (cf. [36,17]). We also mention that Farkas, Jacob and Schilling [13, Theorem 2.1.15] gave another characterization for  $\mathcal{D}^p(\mathcal{L}^\nu)$  in terms of the so called  $\psi$ -Bessel potential space, where  $\psi$  is the symbol of  $\mathcal{L}^\nu$ .

The strategy for proving the above characterization is to prove the following Littlewood–Paley type inequality: for any  $p \in (1, \infty)$ , there exists a  $C > 0$  such that for any  $\lambda \geq 0$ ,  $f \in L^p(\mathbb{R}^+ \times \mathbb{R}^d)$ ,

$$\int_0^\infty \left\| \mathcal{L}^{\nu_2} \int_0^t e^{-\lambda(t-s)} \mathcal{P}_{t-s}^{\nu_1} f(s, \cdot) ds \right\|_p^p dt \leq C \int_0^\infty \|f(t, \cdot)\|_p^p dt,$$

where  $\nu_1, \nu_2$  are two Lévy measures satisfying (1.6) and (1.7), and  $(\mathcal{P}_t^{\nu_1})_{t \geq 0}$  is the semigroup associated with  $\mathcal{L}^{\nu_1}$ . Indeed, this estimate is the key ingredient in  $L^p$ -theory of PDE (see [26,24]), and corresponds to the optimal regularity of nonlocal parabolic equation. Likewise [11], when  $\nu(dy) = a(y) dy/|y|^{d+\alpha}$  with smooth and 0-homogeneous  $a(y)$  and  $c_1 \leq |a(y)| \leq c_2$ , Mikulevicius and Pragarauskas [27] proved this type of estimate by showing some weak (1, 1)-type estimate. In a different way, the proof given here is based on Fourier’s transform and Fefferman–Stein’s theorem about sharp functions (cf. [22,24]). We stress that probabilistic representation (1.3) will play an important role in reducing the general non-homogeneous operator to homogeneous operator (see Step 1 in the proof of Theorem 4.2).

Another aim of this paper is to solve the linear and quasi-linear first order nonlocal parabolic equation with critical diffusions in the  $L^p$ -sense rather than the viscosity sense [12]. The critical case is specially interesting not only

because it appears naturally in quasi-geostrophic flows, but also it is an attractive object in mathematics. In particular, we care about the following multidimensional critical Burger's equation:

$$\partial_t u + (-\Delta)^{\frac{1}{2}} u + u \cdot \nabla u = 0, \quad u(0) = \varphi. \quad (1.9)$$

In one-dimensional case, this equation has a natural variational formulation and admits a unique global smooth solution (see [7,20]) under some regularity assumption on  $\varphi$ . In multidimensional case, the local well-posedness of Burger's equation is relatively easy (cf. [18,40]). However, the global well-posedness of Eq. (1.9) seems to be unknown. The reason lies in two aspects: on one hand, there is no energy inequality and thus, the variational method seems not to be applicable; on the other hand, the first order term has the same order as the diffusion term. In fact, Kiselev, Nazarov and Schterenber [20] have showed the existence of blow up solutions for 1-D supercritical Burger's equation. The idea here is to establish some a priori Hölder estimate for Eq. (1.1) and then use the classical method of freezing coefficients. In [32], Silvestre proved an a priori Hölder estimate for Eq. (1.1) with only bounded measurable  $b$ . This is the key point for us. However, the assumption of scale invariance on Lévy operators seems to be crucial in [32] since the proof is by the iteration of the diminish of oscillation at all scales. As above, we shall use probabilistic representation (1.3) like a perturbation argument to extend Silvestre's estimate to the more general non-homogeneous Lévy operator (see Corollary 6.2).

This paper is organized as follows. In Section 2, we prepare some lemmas and recall some facts for later use. In Section 3, the basic maximum principles for nonlocal parabolic and elliptic equation are proved. In Section 4, we prove the main Theorem 4.2, and give a comparison result between two Lévy operators. In particular, we show that  $(\mathcal{P}_t^\nu)_{t \geq 0}$  forms an analytic semigroup in  $L^p$ -space. In Section 5, we prove the existence of a unique strong solution for the first order nonlocal parabolic equation with critical diffusion and variable coefficients. Here we assume that the first order coefficient is uniformly continuous with respect to the spatial variable since we are working in the critical case, and the non-homogeneous term is in some  $L^p$ -space. As an application, we also prove Krylov's estimate for critical diffusion processes. We mention that in the subcritical case, Krylov's estimate was proved in [25] and [39]. In Section 6, we investigate the quasi-linear first order nonlocal parabolic system, and get the existence of smooth solutions and strong solutions. In particular, the global solvability of Eq. (1.9) is obtained. In this section, the coefficients are assumed to be locally Lipschitz continuous, the zero order term is also required to be linear growth, and the initial value is in some fractional Sobolev spaces. In the whole proofs, basing upon the a priori estimates, we use the mollifying technique in many places.

**Notations.** We collect some frequently used notations below for the reader's convenience.

- $\mathbb{R}^+ := (0, \infty)$ ,  $\mathbb{R}_0^+ := [0, \infty)$ . For a complex number  $z$ ,  $\operatorname{Re}(z)$  ( $\operatorname{Im}(z)$ ): real (image) part of  $z$ .
- $\mathcal{S}(\mathbb{R}^d)$ : the Schwartz class of smooth real or complex-valued rapidly decreasing functions.  $C_b^\infty(\mathbb{R}^d)$  (resp.  $C_b^k(\mathbb{R}^d)$ ,  $C_0^\infty(\mathbb{R}^d)$ ): the space of all bounded smooth functions with bounded derivatives of all orders (resp. up to  $k$ -order, with compact support).
- $\mathcal{F}$  and  $\mathcal{F}^{-1}$ : Fourier's transform and Fourier's inverse transform.
- $\nu$ : Lévy measure;  $\nu^{(\alpha)}$ : the Lévy measure of  $\alpha$ -stable process;  $\Sigma$ : a finite measure on  $\mathbb{S}^{d-1}$ , called the spherical part of  $\nu^{(\alpha)}$ .
- $L_t^\nu$ : the Lévy process associated with Lévy measure  $\nu$ ;  $\mathcal{P}_t^\mu$ : the semigroup associated with  $L_t^\mu$ .  $\mathcal{L}^\nu$ : the generator of  $L_t^\mu$ ,  $\mathcal{L}^{\nu*}$ : the adjoint operator of  $\mathcal{L}^\nu$ ;  $p_t^\nu$ : the heat kernel of  $\mathcal{L}^{\nu*}$ .
- $B_r(x_0) := \{x \in \mathbb{R}^d: |x - x_0| \leq r\}$ ,  $B_r := B_r(0)$ ,  $B_r^c$ : the complement of  $B_r$ .
- $\mathbb{H}^{\alpha,p}$ : Bessel potential space;  $\mathbb{W}^{\alpha,p}$ : Sobolev–Slobodeckij space;  $\mathbb{W}^\infty := \bigcap_{k,p} \mathbb{W}^{k,p}$ .
- $\omega_b$ : the continuous modulus function of  $b$ , i.e.,  $\omega_b(s) := \sup_{|x-y| \leq s} |b(x) - b(y)|$ .
- $\mathcal{H}^\beta$ : the space of Hölder continuous functions with the norm  $\sum_{k=0}^{[\beta]} \|\nabla^k f\|_\infty + \|\nabla^{[\beta]} f\|_{\mathcal{H}^\beta}$ , where  $[\beta]$  denotes the integer part of  $\beta$ , and  $\|\nabla^{[\beta]} f\|_{\mathcal{H}^\beta} := \sup_{|x-y| \leq 1} |\nabla^{[\beta]} f(x) - \nabla^{[\beta]} f(y)|/|x-y|^\beta$ .
- $(\rho_\varepsilon)_{\varepsilon \in (0,1)}$ : a family of mollifiers in  $\mathbb{R}^d$  with  $\rho_\varepsilon(x) = \varepsilon^{-d} \rho(\varepsilon^{-1}x)$ , where  $\rho$  is a nonnegative smooth function with support in  $B_1$  and satisfies  $\int_{\mathbb{R}^d} \rho(x) dx = 1$ .

**Convention.** The letter  $C$  with or without subscripts will denote an unimportant constant. The inner product in Euclidean space is denoted by “ $\cdot$ ”.

## 2. Preliminaries

For  $\alpha \in (0, 2)$ , let  $\nu$  be a Lévy measure in  $\mathbb{R}^d$  and satisfy (1.6) and (1.7). Let  $(L_t^\nu)_{t \geq 0}$  be the  $d$ -dimensional Lévy process, a stationary and independent increment process defined on some probability space  $(\Omega, \mathcal{F}, P)$ , with characteristic function

$$\mathbb{E} e^{i\xi \cdot L_t^\nu} = e^{-t\psi_\nu(\xi)}, \quad \xi \in \mathbb{R}^d, \tag{2.1}$$

where  $\psi_\nu$  is the Lévy exponent with the following form by Lévy–Khintchine’s formula (cf. [2,29]),

$$\psi_\nu(\xi) := \int_{\mathbb{R}^d} (1 + i\xi \cdot y^{(\alpha)} - e^{i\xi \cdot y}) \nu(dy). \tag{2.2}$$

Let  $\nu^{(\alpha)}$  take the form (1.8) and satisfy (1.7). It is well known that  $(L_t^{\nu^{(\alpha)}})_{t \geq 0}$  is a  $d$ -dimensional  $\alpha$ -stable process and has the following self-similarity (cf. [29, Proposition 13.5 and Theorem 14.7]):

$$(L_{rt}^{\nu^{(\alpha)}})_{t \geq 0} \stackrel{(d)}{=} (r^{1/\alpha} L_t^{\nu^{(\alpha)}})_{t \geq 0}, \quad \forall r > 0, \tag{2.3}$$

where  $\stackrel{(d)}{=}$  means that the two processes have the same laws. Moreover, from expression (1.8), it is easy to see that for any  $\beta \in (0, \alpha)$ ,

$$\int_{\mathbb{R}^d} \min(|y|^\beta, |y|^2) \nu^{(\alpha)}(dy) < +\infty, \tag{2.4}$$

and

$$\operatorname{Re}(\psi_{\nu^{(\alpha)}}(\xi)) = \left( \int_0^\infty \frac{1 - \cos r}{r^{1+\alpha}} dr \right) \int_{\mathbb{S}^{d-1}} |\xi \cdot \theta|^\alpha \Sigma(d\theta). \tag{2.5}$$

The Feller semigroup associated with  $(L_t^\nu)_{t \geq 0}$  is defined by

$$\mathcal{P}_t^\nu f(x) := \mathbb{E} f(L_t^\nu + x), \quad f \in \mathcal{S}(\mathbb{R}^d).$$

The generator of  $(\mathcal{P}_t^\nu)_{t \geq 0}$  is then given by (cf. [2, Theorem 3.3.3])

$$\mathcal{L}^\nu f(x) = \int_{\mathbb{R}^d} [f(x + y) - f(x) - y^{(\alpha)} \cdot \nabla f(x)] \nu(dy), \tag{2.6}$$

i.e.,

$$\partial_t \mathcal{P}_t^\nu f(x) = \mathcal{L}^\nu \mathcal{P}_t^\nu f(x) = \mathcal{P}_t^\nu \mathcal{L}^\nu f(x), \quad t > 0. \tag{2.7}$$

Moreover,

$$\mathcal{F}(\mathcal{L}^\nu f)(\xi) = -\psi_\nu(\xi) \cdot \mathcal{F}(f)(\xi),$$

and  $\psi_\nu$  is also called the Lévy symbol of the operator  $\mathcal{L}^\nu$ . From (2.5), one sees that if the spherical part  $\Sigma$  of  $\nu^{(\alpha)}$  is the uniform distribution (equivalently, rotationally invariant) on  $\mathbb{S}^{d-1}$ , then  $\psi_{\nu^{(\alpha)}}(\xi) = c_{d,\alpha} |\xi|^\alpha$  for some constant  $c_{d,\alpha} > 0$ , and thus, by (1.2),

$$-\mathcal{L}^{\nu^{(\alpha)}} f(x) = c_{d,\alpha} (-\Delta)^{\frac{\alpha}{2}} f(x). \tag{2.8}$$

On the other hand, from expression (2.6) and assumption (1.7), it is easy to see that  $\mathcal{L}^\nu$  has the following invariance:

- For  $z \in \mathbb{R}^d$ , define  $f_z(x) := f(z + x)$ , then

$$\mathcal{L}^\nu f_z(x) = \mathcal{L}^\nu f_x(z), \quad \|\mathcal{L}^\nu f_z\|_p = \|\mathcal{L}^\nu f\|_p, \tag{2.9}$$

where  $p \geq 1$  and  $\|\cdot\|_p$  denotes the usual  $L^p$ -norm in  $\mathbb{R}^d$ .

- For  $r > 0$ , define  $f_r(x) := f(rx)$ , then

$$\mathcal{L}^\nu f(rx) = \mathcal{L}^{\nu(r\cdot)} f_r(x) = r^{-\alpha} \mathcal{L}^{r^\alpha \nu(r\cdot)} f_r(x). \tag{2.10}$$

We remark that  $r^\alpha \nu^{(\alpha)}(r\cdot) = \nu^{(\alpha)}$  by (1.8).

- $\mathcal{L}^\nu(C_b^\infty(\mathbb{R}^d)) \subset C_b^\infty(\mathbb{R}^d)$ , and for any  $k \geq 2$ ,  $\mathcal{L}^\nu : C_b^k(\mathbb{R}^d) \rightarrow C_b^{k-2}(\mathbb{R}^d)$  is a continuous linear operator, where  $C_b^\infty(\mathbb{R}^d)$  (resp.  $C_b^k(\mathbb{R}^d)$ ) is the space of all bounded smooth functions with bounded derivatives of all orders (resp. up to  $k$ -order).

The adjoint operator of  $\mathcal{L}^\nu$  is given by

$$\mathcal{L}^{\nu*} f(x) = \int_{\mathbb{R}^d} [f(x-y) - f(x) + y^{(\alpha)} \cdot \nabla f(x)] \nu(dy), \tag{2.11}$$

i.e.,

$$\int_{\mathbb{R}^d} \mathcal{L}^\nu f(x) \cdot g(x) dx = \int_{\mathbb{R}^d} f(x) \cdot \mathcal{L}^{\nu*} g(x) dx, \quad f, g \in \mathcal{S}(\mathbb{R}^d).$$

Clearly,  $\mathcal{L}^{\nu*} = \mathcal{L}^{\nu(-)}$ , where  $\nu(-)$  denotes the Lévy measure  $\nu(-dy)$ .

**Definition 2.1.** Let  $\nu_1$  and  $\nu_2$  be two Borel measures. We say that  $\nu_1$  is less than  $\nu_2$  if

$$\nu_1(B) \leq \nu_2(B), \quad B \in \mathcal{B}(\mathbb{R}^d),$$

and we simply write  $\nu_1 \leq \nu_2$  in this case.

**Lemma 2.2.** Let  $\nu$  be a Lévy measure less than  $\nu^{(\alpha)}$  for some  $\alpha \in (0, 2)$ , where  $\nu^{(\alpha)}$  takes the form (1.8). We also assume (1.7) for  $\nu$ . Then for some  $\kappa_0 > 0$ ,

$$|\psi_\nu(\xi)| \leq \kappa_0 |\xi|^\alpha, \quad \xi \in \mathbb{R}^d. \tag{2.12}$$

**Proof.** Write  $\hat{\xi} := \xi/|\xi|$ . For  $\alpha \in (1, 2)$ , by the definitions of  $\psi_\nu$  and  $\nu^{(\alpha)}$ , we have

$$\begin{aligned} |\text{Im}(\psi_\nu(\xi))| &\stackrel{(2.2)}{\leq} \int_{\mathbb{R}^d} |\xi \cdot y - \sin(\xi \cdot y)| \nu(dy) \leq \int_{\mathbb{R}^d} |\xi \cdot y - \sin(\xi \cdot y)| \nu^{(\alpha)}(dy) \\ &\stackrel{(1.8)}{=} \int_{\mathbb{S}^{d-1}} \int_0^\infty \frac{|\xi \cdot (r\theta) - \sin(\xi \cdot r\theta)|}{r^{1+\alpha}} dr \Sigma(d\theta) \\ &= |\xi|^\alpha \int_{\mathbb{S}^{d-1}} \int_0^\infty \frac{|\hat{\xi} \cdot r\theta - \sin(\hat{\xi} \cdot r\theta)|}{r^{1+\alpha}} dr \Sigma(d\theta) \leq C |\xi|^\alpha. \end{aligned}$$

For  $\alpha = 1$ , by (1.7), we have

$$\begin{aligned} |\text{Im}(\psi_\nu(\xi))| &= \left| \int_{\mathbb{R}^d} (\xi \cdot y 1_{|y| \leq |\xi|^{-1}} - \sin(\xi \cdot y)) \nu(dy) \right| \leq \int_{\mathbb{R}^d} |\xi \cdot y 1_{|y| \leq |\xi|^{-1}} - \sin(\xi \cdot y)| \nu^{(1)}(dy) \\ &= \int_{\mathbb{S}^{d-1}} \int_0^\infty \frac{|\xi \cdot (r\theta) 1_{r \leq |\xi|^{-1}} - \sin(\xi \cdot r\theta)|}{r^2} dr \Sigma(d\theta) \\ &= |\xi| \int_{\mathbb{S}^{d-1}} \int_0^\infty \frac{|\hat{\xi} \cdot r\theta 1_{r \leq 1} - \sin(\hat{\xi} \cdot r\theta)|}{r^2} dr \Sigma(d\theta) \leq C |\xi|. \end{aligned}$$

For  $\alpha \in (0, 1)$ , we have

$$\begin{aligned} |\operatorname{Im}(\psi_\nu(\xi))| &\leq \int_{\mathbb{R}^d} |\sin(\xi \cdot y)| \nu(dy) \leq \int_{\mathbb{R}^d} |\sin(\xi \cdot y)| \nu^{(\alpha)}(dy) \\ &= |\xi|^\alpha \int_{\mathbb{S}^{d-1}} \int_0^\infty \frac{|\sin(\hat{\xi} \cdot r\theta)|}{r^{1+\alpha}} dr \Sigma(d\theta) \leq C |\xi|^\alpha. \end{aligned}$$

Thus, combining with (2.5), we obtain (2.12).  $\square$

For  $k \in \mathbb{N}$  and  $p \in [1, \infty]$ , let  $\mathbb{W}^{k,p}$  be the usual Sobolev space with the norm

$$\|f\|_{k,p} := \sum_{j=0}^k \|\nabla^j f\|_p,$$

where  $\nabla^j$  denotes the  $j$ -order gradient.

We need the following simple interpolation result.

**Lemma 2.3.** *Let  $p \in [1, \infty]$  and  $\beta \in [0, 1]$ . For any  $f \in \mathbb{W}^{1,p}$  and  $y \in \mathbb{R}^d$ , we have*

$$\|f(\cdot + y) - f(\cdot)\|_p \leq (2\|f\|_p)^{1-\beta} (\|\nabla f\|_p |y|)^\beta. \tag{2.13}$$

**Proof.** Observing that for  $f \in \mathcal{S}(\mathbb{R}^d)$ ,

$$|f(x + y) - f(x)| \leq |y| \int_0^1 |\nabla f|(x + sy) ds,$$

by a density argument, we have for any  $f \in \mathbb{W}^{1,p}$ ,

$$\|f(\cdot + y) - f(\cdot)\|_p \leq \|\nabla f\|_p |y|.$$

Thus, for any  $\beta \in [0, 1]$ ,

$$\|f(\cdot + y) - f(\cdot)\|_p \leq (2\|f\|_p) \wedge (\|\nabla f\|_p |y|) \leq (2\|f\|_p)^{1-\beta} (\|\nabla f\|_p |y|)^\beta.$$

The result follows.  $\square$

The following lemma will be used to derive some asymptotic estimate of large time for the heat kernel of Lévy operator (see Corollary 2.9 below).

**Lemma 2.4.** *Assume that Lévy measure  $\nu$  is less than  $\nu^{(\alpha)}$  for some  $\alpha \in (0, 2)$ , where  $\nu^{(\alpha)}$  takes the form (1.8). Then for any  $p \in [1, \infty]$  and  $f \in \mathbb{W}^{2,p}$ , we have*

$$\|\mathcal{L}^\nu f\|_p \leq C \begin{cases} \|\nabla f\|_p^{1-\gamma} \|\nabla^2 f\|_p^\gamma + \|\nabla f\|_p^{1-\beta} \|\nabla^2 f\|_p^\beta, & \alpha \in (1, 2), \gamma \in (\alpha - 1, 1], \beta \in [0, \alpha - 1), \\ \|\nabla f\|_p^{1-\gamma} \|\nabla^2 f\|_p^\gamma + \|f\|_p^{1-\beta} \|\nabla f\|_p^\beta, & \alpha = 1, \gamma \in (0, 1], \beta \in [0, 1), \\ \|f\|_p^{1-\gamma} \|\nabla f\|_p^\gamma + \|f\|_p^{1-\beta} \|\nabla f\|_p^\beta, & \alpha \in (0, 1), \gamma \in (\alpha, 1], \beta \in [0, \alpha), \end{cases}$$

where the constant  $C$  depends only on  $\alpha, \beta, \gamma$  and the Lévy measure  $\nu^{(\alpha)}$ .

**Proof.** Let us first look at the case of  $\alpha \in (1, 2)$ . In this case, we have

$$\mathcal{L}^\nu f(x) = \int_{\mathbb{R}^d} y \cdot \left( \int_0^1 [\nabla f(x + sy) - \nabla f(x)] ds \right) \nu(dy).$$

Since  $\nu$  is bounded by  $\nu^{(\alpha)}$ , by Minkowski’s inequality and Lemma 2.3, we have for  $\gamma \in (\alpha - 1, 1]$  and  $\beta \in [0, \alpha - 1)$ ,

$$\|\mathcal{L}^\nu f\|_p \leq (2\|\nabla f\|_p)^{1-\gamma} \|\nabla^2 f\|_p^\gamma \int_{|y|\leq 1} |y|^{1+\gamma} \nu^{(\alpha)}(dy) + (2\|\nabla f\|_p)^{1-\beta} \|\nabla^2 f\|_p^\beta \int_{|y|>1} |y|^{1+\beta} \nu^{(\alpha)}(dy).$$

In the case of  $\alpha = 1$ , we similarly have for  $\gamma \in (0, 1]$  and  $\beta \in [0, 1)$ ,

$$\|\mathcal{L}^\nu f\|_p \leq (2\|\nabla f\|_p)^{1-\gamma} \|\nabla^2 f\|_p^\gamma \int_{|y|\leq 1} |y|^{1+\gamma} \nu^{(1)}(dy) + (2\|f\|_p)^{1-\beta} \|\nabla f\|_p^\beta \int_{|y|>1} |y|^\beta \nu^{(1)}(dy).$$

In the case of  $\alpha \in (0, 1)$ , we have for  $\gamma \in (\alpha, 1]$  and  $\beta \in [0, \alpha)$ ,

$$\|\mathcal{L}^\nu f\|_p \leq (2\|f\|_p)^{1-\gamma} \|\nabla f\|_p^\gamma \int_{|y|\leq 1} |y|^\gamma \nu^{(\alpha)}(dy) + (2\|f\|_p)^{1-\beta} \|\nabla f\|_p^\beta \int_{|y|>1} |y|^\beta \nu^{(\alpha)}(dy).$$

The proof is complete by (2.4).  $\square$

We also need the following estimate, which will be used frequently in localizing the nonlocal equation.

**Lemma 2.5.** Assume that Lévy measure  $\nu$  is less than  $\nu^{(\alpha)}$  for some  $\alpha \in (0, 2)$ , where  $\nu^{(\alpha)}$  takes the form (1.8). Let  $\zeta \in \mathcal{S}(\mathbb{R}^d)$  and set  $\zeta_z(x) := \zeta(x - z)$  for  $z \in \mathbb{R}^d$ .

1. For any  $\beta \in (0 \vee (\alpha - 1), 1)$  and  $p \in [1, \infty)$ , there exists a constant  $C = C(\nu^{(\alpha)}, \beta, p, d) > 0$  such that for all  $f \in \mathbb{W}^{1,p}$ ,

$$\left( \int_{\mathbb{R}^d} \|\mathcal{L}^\nu(f\zeta_z) - (\mathcal{L}^\nu f)\zeta_z\|_p^p dz \right)^{1/p} \leq C \|\zeta\|_{2,p} \|f\|_p^{1-\beta} \|f\|_{1,p}^\beta. \tag{2.14}$$

2. For any  $\beta \in (0 \vee (\alpha - 1), 1)$  and  $\gamma \in [0, \alpha)$ , there exists a constant  $C = C(\nu^{(\alpha)}, \beta, \gamma, d) > 0$  such that for any  $p \in [1, \infty]$  and  $f \in \mathcal{H}^\beta$ ,

$$\|\mathcal{L}^\nu(f\zeta) - (\mathcal{L}^\nu f)\zeta\|_p \leq C((\|\mathcal{L}^\nu \zeta\|_p + \|\zeta\|_p^{1-\gamma} \|\nabla \zeta\|_p^\gamma) \|f\|_\infty + \|\nabla \zeta\|_p \|f\|_{\mathcal{H}^\beta}), \tag{2.15}$$

where  $\|f\|_{\mathcal{H}^\beta} := \sup_{x \neq y, |x-y| \leq 1} |f(x) - f(y)|/|x - y|^\beta$ , and for any  $p \in [1, \infty]$  and  $f \in \mathbb{W}^{1,p}$ ,

$$\|\mathcal{L}^\nu(f\zeta) - (\mathcal{L}^\nu f)\zeta\|_p \leq C((\|\mathcal{L}^\nu \zeta\|_\infty + \|\zeta\|_\infty^{1-\gamma} \|\nabla \zeta\|_\infty^\gamma) \|f\|_p + \|\nabla \zeta\|_\infty \|f\|_p^{1-\beta} \|\nabla f\|_p^\beta). \tag{2.16}$$

**Proof.** (i) By formula (2.6), we have

$$\begin{aligned} & \mathcal{L}^\nu(f\zeta_z)(x) - \mathcal{L}^\nu f(x) \cdot \zeta_z(x) - f(x) \cdot \mathcal{L}^\nu \zeta_z(x) \\ &= \int_{\mathbb{R}^d} [f(x+y) - f(x)][\zeta_z(x+y) - \zeta_z(x)] \nu(dy) \\ &= \int_{|y|\leq 1} [f(x+y) - f(x)][\zeta_z(x+y) - \zeta_z(x)] \nu(dy) + \int_{|y|>1} [f(x+y) - f(x)][\zeta_z(x+y) - \zeta_z(x)] \nu(dy) \\ &=: I_z^{(1)}(x) + I_z^{(2)}(x). \end{aligned} \tag{2.17}$$

For  $I_z^{(1)}(x)$ , by Fubini’s theorem, Minkowski’s inequality and Lemma 2.3, we have

$$\int_{\mathbb{R}^d} \|I_z^{(1)}\|_p^p dz \leq \int_{\mathbb{R}^d} \left\| \int_{|y|\leq 1} |f(\cdot + y) - f(\cdot)| \left( \int_0^1 |\nabla \zeta_z|(\cdot + sy) ds \right) |y| \nu(dy) \right\|_p^p dz$$



$$\begin{aligned} &\leq \|\nabla \zeta\|_p^p \int_{\mathbb{R}^d} \left( \int_{|y| \leq 1} |f(x+y) - f(x)| \cdot |y| \nu(dy) \right)^p dx \\ &\leq \|\nabla \zeta\|_p^p \left( \int_{|y| \leq 1} \|f(\cdot + y) - f(\cdot)\|_p \cdot |y| \nu(dy) \right)^p \\ &\leq \|\nabla \zeta\|_p^p (2\|f\|_p)^{p(1-\beta)} \|\nabla f\|_p^{p\beta} \left( \int_{|y| \leq 1} |y|^{1+\beta} \nu^{(\alpha)}(dy) \right)^p. \end{aligned}$$

For  $I_z^{(2)}(x)$ , we similarly have

$$\int_{\mathbb{R}^d} \|I_z^{(2)}\|_p^p dz \leq 4^p (v^{(\alpha)}(B_1^c))^p \|\zeta\|_p^p \|f\|_p^p.$$

Moreover, by (2.9) and Lemma 2.4, we also have

$$\int_{\mathbb{R}^d} \|f \mathcal{L}^\nu \zeta\|_p^p dz = \|\mathcal{L}^\nu \zeta\|_p^p \|f\|_p^p \leq C \|\zeta\|_{2,p}^p \|f\|_p^p.$$

Combining the above calculations, we obtain (2.14).

(ii) We have

$$\|I_0^{(1)}\|_p \leq \|f\|_{\mathcal{H}^\beta} \|\nabla \zeta\|_p \int_{|y| \leq 1} |y|^{1+\beta} \nu(dy) \leq \|f\|_{\mathcal{H}^\beta} \|\nabla \zeta\|_p \int_{|y| \leq 1} |y|^{1+\beta} \nu^{(\alpha)}(dy),$$

and by Lemma 2.3,

$$\|I_0^{(2)}\|_p \leq \|f\|_\infty (2\|\zeta\|_p)^{1-\gamma} \|\nabla \zeta\|_p^\gamma \int_{|y| > 1} |y|^\gamma \nu(dy) \leq \|f\|_\infty (2\|\zeta\|_p)^{1-\gamma} \|\nabla \zeta\|_p^\gamma \int_{|y| > 1} |y|^\gamma \nu^{(\alpha)}(dy).$$

Estimate (2.15) follows by (2.17) and  $\|f \mathcal{L}^\nu \zeta\|_p \leq \|f\|_\infty \|\mathcal{L}^\nu \zeta\|_p$ . As for (2.16), it is similar.  $\square$

We introduce the following notion about the nondegeneracy of  $\nu^{(\alpha)}$ .

**Definition 2.6.** Let  $\nu^{(\alpha)}$  be a Lévy measure with the form (1.8). We say that  $\nu^{(\alpha)}$  is nondegenerate if the spherical part  $\Sigma$  of  $\nu^{(\alpha)}$  satisfies

$$\int_{\mathbb{S}^{d-1}} |\theta_0 \cdot \theta|^\alpha \Sigma(d\theta) \neq 0, \quad \forall \theta_0 \in \mathbb{S}^{d-1}. \tag{2.18}$$

By the compactness of  $\mathbb{S}^{d-1}$  and (2.5), the above condition is equivalent to saying that for some constant  $\kappa_1 > 0$ ,

$$\text{Re}(\psi_{\nu^{(\alpha)}}(\xi)) \geq \kappa_1 |\xi|^\alpha, \quad \xi \in \mathbb{R}^d. \tag{2.19}$$

**Remark 2.7.** Let  $L_t^1, \dots, L_t^n$  be  $n$ -independent copies of Lévy process  $L_t^\nu$ . Write

$$\mathbf{L}_t = (L_t^1, \dots, L_t^n).$$

Then  $\mathbf{L}_t$  is an  $nd$ -dimensional Lévy process and the characteristic function of  $\mathbf{L}_1$  is given by  $\boldsymbol{\psi}(\vec{\xi}) = \psi_\nu(\xi^1) + \dots + \psi_\nu(\xi^n)$ , where  $\vec{\xi} = (\xi^1, \dots, \xi^n) \in \mathbb{R}^{nd}$  with  $\xi^i \in \mathbb{R}^d$ . Clearly, if

$$\text{Re}(\psi_\nu(\xi)) \geq \kappa_1 |\xi|^\alpha, \quad \xi \in \mathbb{R}^d,$$

then

$$\text{Re}(\boldsymbol{\psi}(\vec{\xi})) \geq \kappa_1 |\vec{\xi}|^\alpha, \quad \vec{\xi} \in \mathbb{R}^{nd}.$$

It should be noticed that the Lévy measure  $\nu$  of  $L_t$  is very singular and has the expression

$$\nu(d\vec{x}) = \nu(dx^1)\epsilon_0(dx^2, \dots, dx^n) + \dots + \epsilon_0(dx^1, \dots, dx^{n-1})\nu(dx^n),$$

where  $\vec{x} = (x^1, \dots, x^n) \in \mathbb{R}^{nd}$  with  $x^i \in \mathbb{R}^d$ ,  $\epsilon_0$  denotes the Dirac measure in  $\mathbb{R}^{(n-1)d}$ , and the generator of  $L_t$  is given by

$$\mathcal{L}f(\vec{x}) = \sum_{i=1}^n \int_{\mathbb{R}^d} [f(x^1, \dots, x^i + y, \dots, x^n) - f(\vec{x}) - y^{(\alpha)} \cdot \nabla_{x^i} f(\vec{x})] \nu(dy). \tag{2.20}$$

We need the following simple result about the smoothness of the distribution density of Lévy process (see [16, Lemma 3.1] for the symmetric case).

**Proposition 2.8.** *Let  $\psi_\nu$  be defined by (2.2) and satisfy*

$$\operatorname{Re}(\psi_\nu(\xi)) \geq \kappa_1 |\xi|^\alpha, \quad \xi \in \mathbb{R}^d. \tag{2.21}$$

*Then for each  $t > 0$ , the law of  $L_t^\nu$  in  $\mathbb{R}^d$  has a  $C^\infty$ -density  $p_t^\nu$  with respect to the Lebesgue measure, and  $p_t^\nu \in \bigcap_{k \in \mathbb{N}} \mathbb{W}^{k,1}$ . In particular, by (2.7),*

$$\partial_t p_t^\nu(x) = \mathcal{L}^{\nu*} p_t^\nu(x), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d, \tag{2.22}$$

*where  $\mathcal{L}^{\nu*}$  is defined by (2.11), and  $p_t^\nu(x)$  is also called the heat kernel of  $\mathcal{L}^{\nu*}$ .*

**Proof.** By (2.21) and [29, p. 190, Proposition 28.1],  $L_t^\nu$  has a smooth density  $p_t^\nu$ . Let us now prove that for each  $n \in \mathbb{N}$ ,  $\nabla^n p_t^\nu \in L^1(\mathbb{R}^d)$ . By Fourier’s transform (2.1), one sees that

$$p_t^\nu(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} e^{-t\psi_\nu(\xi)} d\xi.$$

Set

$$\phi(\xi) := \int_{|y| \leq 1} (1 + i\xi \cdot y - e^{i\xi \cdot y}) \nu(dy).$$

It is easy to see that  $\phi$  is a smooth complex-valued function, and by (2.21), for any  $n \in \mathbb{N}$  and  $j_1, \dots, j_n \in \{1, \dots, d\}$ ,

$$\xi \rightarrow \xi_{j_1} \cdots \xi_{j_n} e^{-t\phi(\xi)} \in \mathcal{S}(\mathbb{R}^d),$$

where  $\xi = (\xi_1, \dots, \xi_d)$ . Since Fourier’s transform  $\mathcal{F}$  is a bijective and continuous linear operator from  $\mathcal{S}(\mathbb{R}^d)$  onto itself, there is a function  $f \in \mathcal{S}(\mathbb{R}^d)$  such that

$$\hat{f}(\xi) := \mathcal{F}(f)(\xi) = \xi_{j_1} \cdots \xi_{j_n} e^{-t\phi(\xi)}.$$

On the other hand, by Lévy–Khinchine’s representation theorem (cf. [2, Theorem 1.2.14]), there is a probability measure  $\mu$  on  $\mathbb{R}^d$  such that

$$\hat{\mu}(\xi) := \int_{\mathbb{R}^d} e^{i\xi \cdot y} \mu(dy) = e^{-t(\psi_\nu - \phi)(\xi)}.$$

Thus, by the property of Fourier’s transform, we have

$$\begin{aligned} \partial_{x_{j_1}} \cdots \partial_{x_{j_n}} p_t^\nu(x) &= \frac{(-i)^n}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} (\xi_{j_1} \cdots \xi_{j_n} e^{-t\phi(\xi)}) e^{-t(\psi_\nu - \phi)(\xi)} d\xi \\ &= \frac{(-i)^n}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} \hat{f}(\xi) \hat{\mu}(\xi) d\xi = (-i)^n \int_{\mathbb{R}^d} f(x - y) \mu(dy). \end{aligned}$$

From this, we immediately deduce that  $\nabla^n p_t^\nu \in L^1(\mathbb{R}^d)$ .  $\square$

Using Proposition 2.8 and Lemma 2.4, we have the following useful estimates about the heat kernel.

**Corollary 2.9.** *Let  $\nu_i^{(\alpha)}$ ,  $i = 1, 2$  be two Lévy measures with the form (1.8), where  $\nu_1^{(\alpha)}$  is nondegenerate. Let  $\nu$  be another Lévy measure less than  $\nu_2^{(\alpha)}$ . Then, there are two indexes  $\delta_1, \delta_2 > 1$  (depending only on  $\alpha$ ) and constants  $C_1, C_2 > 0$  (depending only on  $d, \alpha, \nu_i^{(\alpha)}$  and not on  $\nu$ ) such that for all  $t \geq 1$ ,*

$$\|\nabla \mathcal{L}^\nu p_t^{\nu_1^{(\alpha)}}\|_1 \leq C_1 t^{-\delta_1}, \tag{2.23}$$

$$\|\partial_t \mathcal{L}^\nu p_t^{\nu_1^{(\alpha)}}\|_1 \leq C_2 t^{-\delta_2}. \tag{2.24}$$

**Proof.** First of all, by the scaling property (2.3) and Proposition 2.8, we have

$$p_t^{\nu_1^{(\alpha)}}(x) = t^{-d/\alpha} p_1^{\nu_1^{(\alpha)}}(t^{-1/\alpha}x),$$

and for each  $n \in \mathbb{N}$ ,

$$\int_{\mathbb{R}^d} |\nabla^n p_t^{\nu_1^{(\alpha)}}|(x) dx = t^{-n/\alpha} \int_{\mathbb{R}^d} |\nabla^n p_1^{\nu_1^{(\alpha)}}|(x) dx \leq C t^{-n/\alpha}. \tag{2.25}$$

Estimate (2.23) follows from Lemma 2.4 by suitable choices of  $\beta$  and  $\gamma$ . Notice that by (2.22),

$$\partial_t \mathcal{L}^\nu p_t^{\nu_1^{(\alpha)}}(x) = \mathcal{L}^\nu \mathcal{L}^{\nu_1^{(\alpha)*}} p_t^{\nu_1^{(\alpha)}}(x).$$

Estimate (2.24) follows by using Lemma 2.4 twice.  $\square$

Now we turn to recall the classical Fefferman–Stein’s theorem. Fix  $\alpha \in (0, 2)$ . Let  $\mathbb{Q}^{(\alpha)}$  be the collection of all parabolic cylinders

$$Q_r := (t_0, t_0 + r^\alpha) \times \{x \in \mathbb{R}^d : |x - x_0| \leq r\}.$$

For  $f \in L^1_{loc}(\mathbb{R}^{d+1})$ , define the Hardy–Littlewood maximal function by

$$\mathcal{M}f(t, x) := \sup_{Q \in \mathbb{Q}^{(\alpha)}, (t,x) \in Q} \int_Q |f(s, y)| dy ds,$$

and the sharp function by

$$f^\sharp(t, x) := \sup_{Q \in \mathbb{Q}^{(\alpha)}, (t,x) \in Q} \int_Q |f(s, y) - f_Q| dy ds,$$

where  $f_Q := \int_Q f(s, y) dy ds = \frac{1}{|Q|} \int_Q f(s, y) dy ds$  and  $|Q|$  is the Lebesgue measure of  $Q$ . One says that  $f \in BMO(\mathbb{R}^{d+1})$  if  $f^\sharp \in L^\infty(\mathbb{R}^{d+1})$ . Clearly,  $f \in BMO(\mathbb{R}^{d+1})$  if and only if there exists a constant  $C > 0$  such that for any  $Q \in \mathbb{Q}^{(\alpha)}$ , and for some  $a_Q \in \mathbb{R}$ ,

$$\int_Q |f(s, y) - a_Q| dy ds \leq C.$$

The following theorem is taken from [24, Chapter 3] (see also [36, p. 148, Theorem 2]).

**Theorem 2.10** (Fefferman–Stein’s theorem). *For  $p \in (1, \infty)$ , there exists a constant  $C = C(p, d, \alpha)$  such that for all  $f \in L^p(\mathbb{R}^{d+1})$ ,*

$$\|f\|_p \leq C \|f^\sharp\|_p. \tag{2.26}$$

Using this theorem, we have

**Theorem 2.11.** For  $q \in (1, \infty)$ , let  $\mathcal{T}$  be a bounded linear operator from  $L^q(\mathbb{R}^{d+1})$  to  $L^q(\mathbb{R}^{d+1})$  and also from  $L^\infty(\mathbb{R}^{d+1})$  to  $BMO(\mathbb{R}^{d+1})$ . Then for any  $p \in [q, \infty)$  and  $f \in L^p(\mathbb{R}^{d+1})$ ,

$$\|\mathcal{T}f\|_p \leq C\|f\|_p,$$

where the constant  $C$  depends only on  $d, p, q, \alpha$  and the norms  $\|\mathcal{T}\|_{L^q \rightarrow L^q}$  and  $\|\mathcal{T}\|_{L^\infty \rightarrow BMO}$ .

**Proof.** Since by [35, p. 13, Theorem 1],

$$\|(\mathcal{T}f)^\sharp\|_q \leq 2\|\mathcal{M}\mathcal{T}f\|_q \leq C\|\mathcal{T}f\|_q \leq C\|\mathcal{T}\|_{L^q \rightarrow L^q}\|f\|_q$$

and

$$\|(\mathcal{T}f)^\sharp\|_\infty \leq \|\mathcal{T}\|_{L^\infty \rightarrow BMO}\|f\|_\infty,$$

by the classical Marcinkiewicz’s interpolation theorem (cf. [35]), we have

$$\|\mathcal{T}f\|_p \stackrel{(2.26)}{\leq} C\|(\mathcal{T}f)^\sharp\|_p \leq C\|f\|_p,$$

where  $p \in [q, \infty)$ .  $\square$

### 3. A maximum principle of nonlocal parabolic equation

In this section we fix a Lévy measure  $\nu$  less than  $\nu^{(\alpha)}$  for some  $\alpha \in (0, 2)$ , where  $\nu^{(\alpha)}$  takes the form (1.8), and prove basic maximum principles for nonlocal parabolic and elliptic equations for later use.

**Lemma 3.1** (Maximum principle). For  $T > -\infty$ , let  $b(t, x)$  be a bounded measurable vector field on  $[T, \infty) \times \mathbb{R}^d$  and  $u \in C([T, \infty); C_b^2(\mathbb{R}^d))$ . Assume that for all  $(t, x) \in [T, \infty) \times \mathbb{R}^d$ ,  $u$  satisfies

$$u(t, x) = u(T, x) + \int_T^t \mathcal{L}^\nu u(s, x) ds + \int_T^t (b \cdot \nabla u)(s, x) ds + \int_T^t f(s, x) ds. \tag{3.1}$$

If  $f \leq 0$ , then

$$\sup_{t \geq T} \sup_{x \in \mathbb{R}^d} u(t, x) \leq \sup_{x \in \mathbb{R}^d} u(T, x).$$

In particular, the above equation admits at most one solution  $u \in C([T, \infty); C_b^2(\mathbb{R}^d))$ .

**Proof.** Let  $\chi(x) \in [0, 1]$  be a nonnegative smooth function with  $\chi(x) = 1$  for  $|x| \leq 1$ , and  $\chi(x) = 0$  for  $|x| \geq 2$ . Set for  $R > 0$ ,

$$\chi_R(x) := \chi(R^{-1}x),$$

and for  $\delta > 0$ ,

$$w_R^\delta(t, x) := \chi_R(x)u(t, x) - \delta(t - T).$$

By (3.1), one sees that for all  $(t, x) \in [T, \infty) \times \mathbb{R}^d$ ,

$$w_R^\delta(t, x) = w_R^\delta(T, x) + \int_T^t \mathcal{L}^\nu w_R^\delta(s, x) ds + \int_T^t (b \cdot \nabla w_R^\delta)(s, x) ds + \int_T^t g_R(s, x) ds - \delta(t - T),$$

where

$$g_R := \chi_R \mathcal{L}^\nu u - \mathcal{L}^\nu w_R - ub \cdot \nabla \chi_R + f \chi_R. \tag{3.2}$$

For fixed  $S > T$  and  $\delta > 0$ , we want to show that for large  $R$ ,

$$\sup_{t \in [T, S]} \sup_{x \in \mathbb{R}^d} w_R^\delta(t, x) \leq \sup_{x \in \mathbb{R}^d} w_R^\delta(T, x) \leq \sup_{x \in \mathbb{R}^d} u(T, x). \tag{3.3}$$

If this is proven, then the result follows by first letting  $R \rightarrow \infty$  and then  $\delta \rightarrow 0$ .

Below, for simplicity of notation, we drop the indexes  $R$  and  $\delta$ . Suppose that (3.3) does not hold, then there exists a time  $t_0 \in (T, S]$  and  $x_0 \in \mathbb{R}^d$  such that  $w$  achieves its maximum at point  $(t_0, x_0)$ . Thus,

$$\nabla w(t_0, x_0) = 0, \tag{3.4}$$

and

$$\begin{aligned} 0 &\leq \overline{\lim}_{h \downarrow 0} \frac{1}{h} (w(t_0, x_0) - w(t_0 - h, x_0)) \\ &\leq \overline{\lim}_{h \downarrow 0} \frac{1}{h} \int_{t_0-h}^{t_0} \mathcal{L}^v w(s, x_0) \, ds + \overline{\lim}_{h \downarrow 0} \frac{1}{h} \int_{t_0-h}^{t_0} (b \cdot \nabla w)(s, x_0) \, ds + \overline{\lim}_{h \downarrow 0} \frac{1}{h} \int_{t_0-h}^{t_0} g(s, x_0) \, ds - \delta \\ &=: I_1 + I_2 + I_3 - \delta. \end{aligned} \tag{3.5}$$

Since for all  $y \in \mathbb{R}^d$ ,

$$w(t_0, x_0 + y) \leq w(t_0, x_0),$$

in view of  $w \in C([T, S]; C_b^2(\mathbb{R}^d))$  and by (3.4), we have

$$I_1 = \overline{\lim}_{h \downarrow 0} \frac{1}{h} \int_{t_0-h}^{t_0} [\mathcal{L}^v w(s, x_0) - \mathcal{L}^v w(t_0, x_0)] \, ds + \mathcal{L}^v w(t_0, x_0) \leq 0.$$

Similarly, for  $I_2$ , we have

$$I_2 = \overline{\lim}_{h \downarrow 0} \frac{1}{h} \int_{t_0-h}^{t_0} b(s, x_0) \cdot (\nabla w(s, x_0) - \nabla w(t_0, x_0)) \, ds = 0.$$

For  $I_3$ , recalling (3.2) and  $f \leq 0$ , by (ii) of Lemma 2.5 and Lemma 2.4, we have for some  $\gamma \in (0, 1)$ ,

$$\begin{aligned} I_3 &\leq \|\chi_R \mathcal{L}^v u - \mathcal{L}^v(\chi_R u)\|_\infty + \frac{\|u\|_\infty \|b\|_\infty \|\nabla \chi\|_\infty}{R} \\ &\leq \frac{C(\|u\|_\infty + \|\nabla u\|_\infty)}{R^\gamma} + \frac{\|u\|_\infty \|b\|_\infty \|\nabla \chi\|_\infty}{R}, \end{aligned}$$

where  $C$  is independent of  $R$ . Choosing  $R$  to be sufficiently large, we obtain

$$I_1 + I_2 + I_3 - \delta < 0,$$

a contradiction with (3.5). Thus, we conclude the proof of (3.3).  $\square$

Similarly, we also have the following maximum principle.

**Lemma 3.2** (Maximum principle). Assume  $\lambda > 0$  and  $b$  is a bounded measurable vector field. Let  $u \in C_b^2(\mathbb{R}^{d+1})$  (resp.  $u \in C_b^2(\mathbb{R}^d)$ ) satisfy

$$\mathcal{L}_{b,\lambda}^v u := \partial_t u - \mathcal{L}^v u + (b \cdot \nabla)u + \lambda u \leq 0 \quad (\text{resp. } (\lambda - \mathcal{L}^v)u \leq 0).$$

Then  $u \leq 0$ . In particular,  $\mathcal{L}_{b,\lambda}^v u = 0$  (resp.  $(\lambda - \mathcal{L}^v)u = 0$ ) admits at most one solution in  $C_b^2(\mathbb{R}^{d+1})$  (resp.  $C_b^2(\mathbb{R}^d)$ ).

**Corollary 3.3.** Let  $\vartheta \in \mathbb{R}^d$  and  $\lambda > 0$ . Then for any  $p > 1$ ,  $(\partial_t - \mathcal{L}^v + \vartheta \cdot \nabla + \lambda)(C_0^\infty(\mathbb{R}^{d+1}))$  (resp.  $(\lambda - \mathcal{L}^v)(C_0^\infty(\mathbb{R}^d))$ ) is dense in  $L^p(\mathbb{R}^{d+1})$  (resp.  $L^p(\mathbb{R}^d)$ ).

**Proof.** Let  $g \in L^{p/(p-1)}(\mathbb{R}^{d+1})$ . By Hahn–Banach’s theorem, it is enough to prove that if for all  $u \in C_0^\infty(\mathbb{R}^{d+1})$ ,

$$\int_{\mathbb{R}^{d+1}} g(t, x) \cdot (\partial_t - \mathcal{L}^\nu + \vartheta \cdot \nabla + \lambda)u(t, x) \, dx \, dt = 0,$$

then  $g = 0$ . Since for any  $(s, y) \in \mathbb{R}^{d+1}$ , the mapping  $(t, x) \mapsto u(s + t, y + x)$  belongs to  $C_0^\infty(\mathbb{R}^{d+1})$ . Thus, we have

$$(\partial_t - \mathcal{L}^\nu + \vartheta \cdot \nabla + \lambda)(g \star u) = 0,$$

where  $g \star u$  stands for  $(s, y) \mapsto \int_{\mathbb{R}^{d+1}} g(t, x)u(s + t, y + x) \, dy \, dt$ . By Lemma 3.2,  $g \star u = 0$  for all  $u \in C_0^\infty(\mathbb{R}^{d+1})$ , which yields that  $g = 0$ .  $\square$

#### 4. $L^q(\mathbb{R}; L^p(\mathbb{R}^d))$ -maximal regularity for nonlocal parabolic equation

Let  $\vartheta \in C_b^\infty(\mathbb{R}; \mathbb{R}^d)$  be a time dependent vector field. For  $s < t$ , set

$$\Theta_{t,s} := \int_s^t \vartheta(r) \, dr.$$

Let  $\nu$  be a Lévy measure and satisfy (2.21). For  $f \in \mathcal{S}(\mathbb{R}^d)$ , define

$$\mathcal{T}_{t,s}^\nu f(x) := \mathbb{E}f(x - \Theta_{t,s} + L_{t-s}^\nu) = \mathcal{P}_{t-s}^\nu f(x - \Theta_{t,s}) = \int_{\mathbb{R}^d} f(y)p_{t-s}^\nu(y - x + \Theta_{t,s}) \, dy. \tag{4.1}$$

By (2.22), one has

$$\begin{aligned} \partial_t \mathcal{T}_{t,s}^\nu f(x) &= \int_{\mathbb{R}^d} f(y)\partial_t p_{t-s}^\nu(y - x + \Theta_{t,s}) \, dy + \int_{\mathbb{R}^d} f(y)(\vartheta_t \cdot \nabla p_{t-s}^\nu)(y - x + \Theta_{t,s}) \, dy \\ &= \int_{\mathbb{R}^d} f(y)(\mathcal{L}^{\nu*} p_{t-s}^\nu)(y - x + \Theta_{t,s}) \, dy - \vartheta_t \cdot \nabla \int_{\mathbb{R}^d} f(y)p_{t-s}^\nu(y - x + \Theta_{t,s}) \, dy \\ &= \mathcal{L}^\nu \mathcal{T}_{t,s}^\nu f(x) - \vartheta_t \cdot \nabla \mathcal{T}_{t,s}^\nu f(x). \end{aligned} \tag{4.2}$$

For  $\lambda \geq 0$  and  $f \in \mathcal{S}(\mathbb{R}^{d+1})$ , define

$$u(t, x) := \int_{-\infty}^t e^{-\lambda(t-s)} \mathcal{T}_{t,s}^\nu f(s, x) \, ds,$$

then it is easy to check by (4.2) that  $u \in C_b^\infty(\mathbb{R}^{d+1})$  and uniquely solves

$$\partial_t u - \mathcal{L}^\nu u + \vartheta \cdot \nabla u + \lambda u = f. \tag{4.3}$$

**Remark 4.1.** Let  $\nu_1$  and  $\nu_2$  be two Lévy measures. Let  $(L_t^{\nu_1})_{t \in \mathbb{R}}$  and  $(L_t^{\nu_2})_{t \in \mathbb{R}}$  be two independent Lévy processes associated with  $\nu_1$  and  $\nu_2$  respectively. Then it is clear that

$$(L_t^{\nu_1 + \nu_2})_{t \in \mathbb{R}} \stackrel{(d)}{=} (L_t^{\nu_1} + L_t^{\nu_2})_{t \in \mathbb{R}}.$$

Thus, we have

$$\mathcal{T}_{t,s}^{\nu_1 + \nu_2} f(x) = \mathcal{P}_{t-s}^{\nu_1} \mathcal{P}_{t,s}^{\nu_2} f(x - \Theta_{t,s}) = \mathbb{E}(\mathcal{P}_{t,s}^{\nu_2} f(x + (L_t^{\nu_1} - \Theta_{t,0}) - (L_s^{\nu_1} - \Theta_{s,0}))). \tag{4.4}$$

The main aim of this section is to prove the following  $L^q(\mathbb{R}; L^p(\mathbb{R}^d))$ -regularity estimate to the above  $u$  when  $f \in L^q(\mathbb{R}; L^p(\mathbb{R}^d))$ .

**Theorem 4.2.** For  $\alpha \in (0, 2)$ , let  $\nu_i^{(\alpha)}$ ,  $i = 1, 2$  be two Lévy measures with the form (1.8), where  $\nu_1^{(\alpha)}$  is nondegenerate in the sense of Definition 2.6. Let  $\nu_1$  and  $\nu_2$  be two Lévy measures and satisfy that

$$\nu_1 \geq \nu_1^{(\alpha)}, \quad \nu_2 \leq \nu_2^{(\alpha)},$$

and for all  $0 < r < R < +\infty$ ,

$$1_{\alpha=1} \int_{r \leq |y| \leq R} y \nu_2(dy) = 0.$$

Let  $\vartheta : \mathbb{R} \rightarrow \mathbb{R}^d$  be a locally integrable function, and  $\mathcal{T}_{t,s}^{\nu_1}$  be defined by (4.1). Then for any  $p, q \in (1, \infty)$ , there exists a constant  $C = C(\nu_1^{(\alpha)}, \nu_2^{(\alpha)}, \alpha, p, q, d) > 0$  such that for any  $-\infty \leq T < S \leq \infty$ ,  $f \in L^q((T, S); L^p(\mathbb{R}^d))$  and  $\lambda \geq 0$ ,

$$\int_T^S \left\| \mathcal{L}^{\nu_2} \int_T^t e^{-\lambda(t-s)} \mathcal{T}_{t,s}^{\nu_1} f(s, \cdot) ds \right\|_p^q dt \leq C \int_T^S \|f(t, \cdot)\|_p^q dt. \tag{4.5}$$

**Proof.** By replacing  $f(t, x)$  by  $f(t, x)1_{(T,S)}(t)$ , it is enough to prove that

$$\int_{-\infty}^{\infty} \left\| \mathcal{L}^{\nu_2} \int_{-\infty}^t e^{-\lambda(t-s)} \mathcal{T}_{t,s}^{\nu_1} f(s, \cdot) ds \right\|_p^q dt \leq C \int_{-\infty}^{\infty} \|f(t, \cdot)\|_p^q dt. \tag{4.6}$$

We divide the proof into seven steps.

**Step 1.** Let  $(L_t^{\nu_1 - \nu_1^{(\alpha)}})_{t \in \mathbb{R}}$  be a  $d$ -dimensional Lévy process associated with the Lévy measure  $\nu_1 - \nu_1^{(\alpha)}$ . By (4.4), we have

$$\int_{-\infty}^t e^{-\lambda(t-s)} \mathcal{T}_{t,s}^{\nu_1} f(s, x) ds = \int_{-\infty}^t e^{-\lambda(t-s)} \mathcal{P}_{t-s}^{\nu_1 - \nu_1^{(\alpha)}} \mathcal{T}_{t,s}^{\nu_1^{(\alpha)}} f(s, x) ds = \mathbb{E}u(t, x + L_t^{\nu_1 - \nu_1^{(\alpha)}} - \Theta_{t,0}),$$

where

$$u(t, x) := \int_{-\infty}^t e^{-\lambda(t-s)} \mathcal{P}_{t-s}^{\nu_1^{(\alpha)}} f(s, x - L_s^{\nu_1 - \nu_1^{(\alpha)}} + \Theta_{s,0}) ds.$$

Suppose that (4.6) has been proven for  $\nu_1 = \nu_1^{(\alpha)}$  and  $\vartheta = 0$ . By Fubini’s theorem and Minkowski’s inequality, we have for  $f \in \mathcal{S}(\mathbb{R}^{d+1})$ ,

$$\begin{aligned} & \int_{-\infty}^{\infty} \left\| \mathcal{L}^{\nu_2} \int_{-\infty}^t e^{-\lambda(t-s)} \mathcal{T}_{t,s}^{\nu_1} f(s, \cdot) ds \right\|_p^q dt = \int_{-\infty}^{\infty} \left\| \mathbb{E} \mathcal{L}^{\nu_2} u(t, \cdot + L_t^{\nu_1 - \nu_1^{(\alpha)}} - \Theta_{t,0}) \right\|_p^q dt \\ & \leq \int_{-\infty}^{\infty} \mathbb{E} \left\| \mathcal{L}^{\nu_2} u(t, \cdot + L_t^{\nu_1 - \nu_1^{(\alpha)}} - \Theta_{t,0}) \right\|_p^q dt \stackrel{(2.9)}{=} \int_{-\infty}^{\infty} \left\| \mathcal{L}^{\nu_2} u(t, \cdot) \right\|_p^q dt \\ & \leq C \mathbb{E} \int_{-\infty}^{\infty} \left\| f(s, \cdot - L_s^{\nu_1 - \nu_1^{(\alpha)}} + \Theta_{s,0}) \right\|_p^q ds = C \int_{-\infty}^{\infty} \|f(s, \cdot)\|_p^q ds. \end{aligned}$$

Hence, we need only to prove (4.6) for  $\nu_1 = \nu_1^{(\alpha)}$  and  $\vartheta = 0$ . Below, for simplicity of notation, we write

$$\mathcal{L} := \mathcal{L}^{\nu_2}, \quad \mathcal{L} := \mathcal{L}^{\nu_1^{(\alpha)}}, \quad \mathcal{P}_t := \mathcal{P}_t^{\nu_1^{(\alpha)}}, \quad \psi_1 = \psi_{\nu_1^{(\alpha)}}, \quad \psi_2 = \psi_{\nu_2}.$$

**Step 2.** Let us first prove (4.6) for  $p = q = 2$ . For  $f \in \mathcal{S}(\mathbb{R}^{d+1})$ , let  $\hat{f}(s, \cdot) = \mathcal{F}f(s, \cdot)$ . By (2.1), the Fourier transform of  $\mathcal{P}_t f$  is clearly given by

$$\widehat{\mathcal{P}_t f}(\xi) = e^{-\psi_1(\xi)t} \hat{f}(\xi).$$

By Parseval’s identity and Minkowski’s inequality, we have

$$\begin{aligned} & \int_{-\infty}^{\infty} \left\| \mathcal{L} \int_{-\infty}^t e^{-\lambda(t-s)} \mathcal{P}_{t-s} f(s, \cdot) ds \right\|_2^2 dt \\ &= \int_{-\infty}^{\infty} \int_{\mathbb{R}^d} \left| \psi_2(\xi) \int_{-\infty}^t e^{-\lambda(t-s) - \psi_1(\xi)(t-s)} \hat{f}(s, \xi) ds \right|^2 d\xi dt \\ &\stackrel{(2.12)}{\leq} \kappa_0^2 \int_{-\infty}^{\infty} \int_{\mathbb{R}^d} \left( |\xi|^\alpha \int_{-\infty}^t e^{-\operatorname{Re}(\psi_1(\xi))(t-s)} |\hat{f}(s, \xi)| ds \right)^2 d\xi dt \\ &\stackrel{(2.19)}{\leq} \kappa_0^2 \int_{-\infty}^{\infty} \int_{\mathbb{R}^d} \left( |\xi|^\alpha \int_{-\infty}^t e^{-\kappa_1 |\xi|^\alpha (t-s)} |\hat{f}(s, \xi)| ds \right)^2 d\xi dt \\ &= \kappa_0^2 \int_{-\infty}^{\infty} \int_{\mathbb{R}^d} \left( \int_0^\infty |\xi|^\alpha e^{-\kappa_1 |\xi|^\alpha s} |\hat{f}(t-s, \xi)| ds \right)^2 d\xi dt \\ &\leq \kappa_0^2 \int_{\mathbb{R}^d} \left( \int_0^\infty |\xi|^\alpha e^{-\kappa_1 |\xi|^\alpha s} \left( \int_{-\infty}^\infty |\hat{f}(t-s, \xi)|^2 dt \right)^{1/2} ds \right)^2 d\xi \\ &= \frac{\kappa_0^2}{\kappa_1^2} \int_{\mathbb{R}^d} \int_{-\infty}^\infty |\hat{f}(t, \xi)|^2 dt d\xi = \frac{\kappa_0^2}{\kappa_1^2} \int_{-\infty}^\infty \|f(t)\|_2^2 dt. \end{aligned}$$

Since  $\mathcal{S}(\mathbb{R}^{d+1})$  is dense in  $L^2(\mathbb{R}^{d+1})$ , (4.6) follows for  $p = q = 2$ .

**Step 3.** For  $f \in L^\infty(\mathbb{R}^{d+1})$ , define

$$\mathcal{T}f(t, x) := \left( \mathcal{L} \int_{-\infty}^t e^{-\lambda(t-s)} \mathcal{P}_{t-s} f(s, \cdot) ds \right)(x).$$

We want to show that

$$\mathcal{T} : L^\infty(\mathbb{R}^{d+1}) \rightarrow BMO(\mathbb{R}^{d+1}) \quad \text{is a bounded linear operator.} \tag{4.7}$$

More precisely, we want to prove that there is a constant  $C > 0$  independent of  $\lambda$  such that for any  $f \in L^\infty(\mathbb{R}^{d+1})$  with  $\|f\|_\infty \leq 1$ , and any parabolic cylinder  $Q = (t_0, t_0 + r^\alpha) \times B_r(x_0)$ ,

$$\frac{1}{|Q|} \int_Q |\mathcal{T}f(t, x) - a_Q|^2 dx dt \leq C, \tag{4.8}$$

where  $a_Q$  is a constant depending on  $Q$ .

By shifting the origin, we may assume  $t_0 = 0, x_0 = 0$ . On the other hand, by the scaling properties (1.4) and (2.10), if one makes the following change in (4.8):

$$v_2(B) \rightarrow r^\alpha v_2(rB), \quad f(t, x) \rightarrow f(r^\alpha t, rx), \quad \lambda \rightarrow \lambda r^\alpha,$$



then we may further assume  $r = 1$ . Thus, it suffices to prove that for any  $f \in L^\infty(\mathbb{R}^{d+1})$  with  $\|f\|_\infty \leq 1$ ,

$$\int_{Q_1} |\mathcal{T}f(t, x) - a_{Q_1}|^2 dx dt \leq C,$$

where  $Q_1 = (0, 1) \times B_1$  and  $C = C(v_1^{(\alpha)}, v_2^{(\alpha)}, \alpha, d)$  is independent of  $v_2$  and  $\lambda$ .

Following Krylov [22], we now split  $\mathcal{T}f$  into two parts:

$$\mathcal{T}f(t, x) = \mathcal{T}_1f(t, x) + \mathcal{T}_2f(t, x),$$

where for  $(t, x) \in (0, 1) \times B_1$ ,

$$\begin{aligned} \mathcal{T}_1f(t, x) &:= \mathcal{L} \left( \int_{-1}^t e^{-\lambda(t-s)} \mathcal{P}_{t-s} f(s, \cdot) ds \right) (x), \\ \mathcal{T}_2f(t, x) &:= \mathcal{L} \left( \int_{-\infty}^{-1} e^{-\lambda(t-s)} \mathcal{P}_{t-s} f(s, \cdot) ds \right) (x). \end{aligned}$$

**Step 4.** In this step, we treat  $\mathcal{T}_1f$ . Let  $f_\varepsilon(t, x) := f * \rho_\varepsilon(t, x)$  be the mollifying approximation of  $f$ , where  $\rho_\varepsilon$  is the usual mollifier in  $\mathbb{R}^{d+1}$ . Define

$$\begin{aligned} u_\varepsilon(t, x) &:= \int_{-1}^t e^{-\lambda(t-s)} \mathcal{P}_{t-s} f_\varepsilon(s, x) ds, \\ u(t, x) &:= \int_{-1}^t e^{-\lambda(t-s)} \mathcal{P}_{t-s} f(s, x) ds. \end{aligned}$$

By definition (4.1) and  $\|f\|_\infty \leq 1$ , we have

$$|u_\varepsilon(t, x)| \leq 2, \quad \forall (t, x) \in [-1, 1] \times \mathbb{R}^d, \tag{4.9}$$

and by the dominated convergence theorem,

$$\lim_{\varepsilon \rightarrow 0} \int_0^1 \int_{B_1} |u_\varepsilon(t, x) - u(t, x)|^2 dx dt = 0. \tag{4.10}$$

On the other hand, by Lemma 2.3, for any  $\beta \in [0, \alpha \wedge 1)$ , we have for all  $t \in [-1, 1], x, x' \in \mathbb{R}^d$ ,

$$\begin{aligned} |u_\varepsilon(t, x) - u_\varepsilon(t, x')| &\leq \int_{-1}^t \int_{\mathbb{R}^d} |p_{t-s}(y-x) - p_{t-s}(y-x')| dy ds \\ &\stackrel{(2.13)}{\leq} 2^{1-\beta} \int_{-1}^t \left( |x-x'| \int_{\mathbb{R}^d} |\nabla p_{t-s}(y)| dy \right)^\beta ds \\ &\stackrel{(2.25)}{\leq} C|x-x'|^\beta \int_{-1}^t (t-s)^{-\beta/\alpha} ds \leq C|x-x'|^\beta. \end{aligned} \tag{4.11}$$

Moreover, as in the beginning of this section, since  $f_\varepsilon \in C_b^\infty(\mathbb{R}^{d+1})$ , by (4.2) and Lemma 3.1, one sees that  $u_\varepsilon \in C_b^\infty([0, \infty) \times \mathbb{R}^{d+1})$  uniquely solves

$$\partial_t u_\varepsilon - \mathcal{L}u_\varepsilon + \lambda u_\varepsilon = f_\varepsilon, \quad u_\varepsilon(-1, x) = 0.$$

Let  $\chi$  be a nonnegative smooth function with  $\chi(x) = 1$  for  $|x| \leq 1$  and  $\chi(x) = 0$  for  $|x| \geq 2$ . Multiplying the above equation by  $\chi$ , we obtain

$$\partial_t(u_\varepsilon \chi) = (\mathcal{L}u_\varepsilon)\chi - \lambda u_\varepsilon \chi + f_\varepsilon \chi = \mathcal{L}(u_\varepsilon \chi) - \lambda(u_\varepsilon \chi) + g_\varepsilon^\chi,$$

where

$$g_\varepsilon^\chi := \chi \mathcal{L}u_\varepsilon - \mathcal{L}(u_\varepsilon \chi) + f_\varepsilon \chi.$$

Since  $\chi$  has compact support, we have for each  $t \in [0, 1]$ ,

$$g_\varepsilon^\chi(t, \cdot) \in C_b^\infty(\mathbb{R}^d).$$

Thus, by Lemma 3.1 again, one has the representation

$$(u_\varepsilon \chi)(t, x) = \int_{-1}^t e^{-\lambda(t-s)} \mathcal{P}_{t-s} g_\varepsilon^\chi(s, x) ds.$$

Moreover, by (4.9), (4.11) and (ii) of Lemma 2.5,

$$\int_{-1}^1 \|g_\varepsilon^\chi(t, \cdot)\|_2^2 dt \leq C \left( \int_{-1}^1 \|\chi \mathcal{L}u_\varepsilon(t) - \mathcal{L}(u_\varepsilon(t)\chi)\|_2^2 dt + \|\chi\|_2^2 \right) \leq C.$$

Here and below, the constant  $C$  is independent of  $\varepsilon$  and  $\lambda$ .

As in Step 2, by Fourier’s transform again, we have

$$\begin{aligned} \int_0^1 \int_{\mathbb{R}^d} |\mathcal{L}(u_\varepsilon \chi)(t, x)|^2 dx dt &\leq \kappa_0^2 \int_0^1 \int_{\mathbb{R}^d} \left| \int_0^{t+1} |\xi|^\alpha e^{-\kappa_1 |\xi|^\alpha s} |\hat{g}_\varepsilon^\chi(t-s, \xi)| ds \right|^2 d\xi dt \\ &\leq \kappa_0^2 \int_{\mathbb{R}^d} \left( \int_0^1 |\xi|^\alpha e^{-\kappa_1 |\xi|^\alpha s} \left( \int_{s-1}^1 |\hat{g}_\varepsilon^\chi(t-s, \xi)|^2 dt \right)^{1/2} ds \right)^2 d\xi \\ &\leq \kappa_0^2 \int_{\mathbb{R}^d} \left( \int_0^1 |\xi|^\alpha e^{-\kappa_1 |\xi|^\alpha s} \left( \int_{-1}^1 |\hat{g}_\varepsilon^\chi(t, \xi)|^2 dt \right)^{1/2} ds \right)^2 d\xi \\ &\leq C \int_{\mathbb{R}^d} \int_{-1}^1 |\hat{g}_\varepsilon^\chi(t, \xi)|^2 dt d\xi = C \int_{-1}^1 \|g_\varepsilon^\chi(t, \cdot)\|_2^2 dt \leq C. \end{aligned}$$

Thus, by (4.9), (4.10), (4.11) and (ii) of Lemma 2.5 again, we arrive at

$$\begin{aligned} \int_{Q_1} |\mathcal{F}_1 f(t, x)|^2 dx dt &= \int_{Q_1} |\mathcal{L}u(t, x)|^2 dx dt \leq \sup_{\varepsilon \in (0,1)} \int_0^1 \int_{B_1} |\mathcal{L}u_\varepsilon(t, x)|^2 dx dt \\ &\leq \sup_{\varepsilon \in (0,1)} \int_0^1 \int_{\mathbb{R}^d} |\mathcal{L}u_\varepsilon(t, x) \cdot \chi(x)|^2 dx dt \leq C. \end{aligned}$$

**Step 5.** In this step, we treat  $\mathcal{F}_2 f$  and prove that for some  $a_{Q_1} \in \mathbb{R}$  and some constant  $C > 0$  independent of  $\lambda$ ,

$$\int_{Q_1} |\mathcal{F}_2 f(t, x) - a_{Q_1}|^2 dx dt \leq C. \tag{4.12}$$

Note that by (4.1),

$$e^{\lambda t} \mathcal{T}_2 f(t, x) = \int_{-\infty}^{-1} e^{\lambda s} \int_{\mathbb{R}^d} f(s, y) \mathcal{L}^* p_{t-s}(y-x) dy ds =: \mathcal{T}_3 f(t, x).$$

In view of  $\lambda \geq 0$  and  $\|f\|_\infty \leq 1$ , by (2.23), we have for some  $\delta_1 > 1$  and any  $(t, x) \in [0, 1] \times \mathbb{R}^d$ ,

$$|\nabla \mathcal{T}_3 f(t, x)| \leq \int_{-\infty}^{-1} \int_{\mathbb{R}^d} |\nabla \mathcal{L}^* p_{t-s}(y)| dy ds \leq C \int_{-\infty}^{-1} (t-s)^{-\delta_1} ds \leq C,$$

and by (2.24), for some  $\delta_2 > 1$  and any  $t \in [0, 1]$ ,

$$\begin{aligned} |\mathcal{T}_3 f(t, 0) - \mathcal{T}_3 f(0, 0)| &\leq \int_{-\infty}^{-1} \int_{\mathbb{R}^d} |\mathcal{L}^* p_{t-s}(y) - \mathcal{L}^* p_{-s}(y)| dy ds \\ &\leq \int_{-\infty}^{-1} \int_{\mathbb{R}^d} \int_0^t |\partial_r \mathcal{L}^* p_{r-s}(y)| dr dy ds \\ &\leq C \int_{-\infty}^{-1} \int_0^t (r-s)^{-\delta_2} dr ds \leq C. \end{aligned}$$

Hence,

$$|\mathcal{T}_3 f(t, x) - \mathcal{T}_3 f(0, 0)| \leq C, \quad \forall (t, x) \in [0, 1] \times B_1,$$

and

$$\int_{Q_1} |\mathcal{T}_2 f(t, x) - e^{-\lambda t} \mathcal{T}_3 f(0, 0)|^2 dx dt \leq C.$$

If  $\lambda = 0$ , we immediately have (4.12). Now let us assume  $\lambda > 0$ . In this case, by Lemma 2.4 and (2.25), we have

$$|\mathcal{T}_3 f(0, 0)| \leq \int_{-\infty}^{-1} e^{\lambda s} \left( \int_{\mathbb{R}^d} |\mathcal{L}^* p_{-s}(y)| dy \right) ds \leq C \int_{-\infty}^{-1} e^{\lambda s} ds = C e^{-\lambda} / \lambda,$$

where  $C$  is independent of  $\lambda$  and  $f$ . So,

$$\int_{Q_1} |(1 - e^{-\lambda t}) \mathcal{T}_3 f(0, 0)|^2 dx dt \leq \frac{C}{\lambda^2} \int_0^1 (1 - e^{-\lambda t})^2 dt \leq \frac{C}{3},$$

where we have used that  $1 - e^{-s} \leq s$  for all  $s \geq 0$ . Thus, we obtain (4.12) with  $a_{Q_1} = \mathcal{T}_3 f(0, 0)$ .

**Step 6.** Combining the above Steps 3–5, we have proven (4.7). By Step 2 and Theorem 2.11, we get (4.6) for  $p = q \in [2, \infty)$ . As for  $p = q \in (1, 2)$ , it follows by the following duality: Let  $g \in C_0^\infty(\mathbb{R}^{d+1})$ . By the integration by parts formula and the change of variables, we have

$$\begin{aligned} &\int_{-\infty}^\infty \int_{\mathbb{R}^d} \left( \mathcal{L} \int_{-\infty}^t e^{-\lambda(t-s)} \mathcal{P}_{t-s} f(s, \cdot) ds \right) (x) \cdot g(t, x) dx dt \\ &= \int_{-\infty}^\infty \int_{\mathbb{R}^d} f(t, x) \left( \mathcal{L}^* \int_{-\infty}^t e^{-\lambda(t-s)} \mathcal{P}_{t-s}^* g(s, \cdot) ds \right) (x) dx dt, \end{aligned}$$

where  $\mathcal{L}^*$  is the adjoint operator of  $\mathcal{L}$  and  $\mathcal{P}_t^* g(s, x) := \mathbb{E} g(s, x - L_t^{v^{(\alpha)}})$ .

**Step 7.** For  $q \neq p \in (1, \infty)$ , we use a trick due to Krylov [23]. Clearly, it suffices to prove that for any  $T > -\infty$  and  $f \in C_0^\infty([T, \infty) \times \mathbb{R}^d)$ ,

$$\int_T^\infty \left\| \mathcal{L} \int_T^t e^{-\lambda(t-s)} \mathcal{P}_{t-s} f(s, \cdot) ds \right\|_p^q dt \leq C \int_T^\infty \|f(t, \cdot)\|_p^q dt, \quad (4.13)$$

where  $C$  is independent of  $T$ .

Set

$$u(t, x) := \int_T^t e^{-\lambda(t-s)} \mathcal{P}_{t-s} f(s, x) ds, \quad w(t, x) := \mathcal{L}u(t, x).$$

By (4.2), one can verify that  $w \in C([T, \infty); C_b^\infty(\mathbb{R}^d))$  and uniquely solves

$$\partial_t w - \mathcal{L}w + \lambda w = \mathcal{L}f, \quad w(T, x) = 0.$$

For  $\vec{x} = (x^1, \dots, x^n) \in \mathbb{R}^{nd}$  with  $x^i = (x_1^i, \dots, x_d^i) \in \mathbb{R}^d$ , define

$$U(t, \vec{x}) := w(t, x^1) \cdots w(t, x^n).$$

Then

$$\partial_t U - \mathcal{L}U + n\lambda U = F, \quad U(T, \vec{x}) = 0,$$

where  $\mathcal{L}$  is defined by (2.20) and

$$F(t, \vec{x}) = \sum_{i=1}^n \mathcal{L}_{x^i} G^i(t, \vec{x}), \quad G^i(t, \vec{x}) = f(t, x^i) \prod_{k \neq i} w(t, x^k).$$

Here  $\mathcal{L}_{x^i}$  means that  $\mathcal{L}$  acts on the component  $x^i$  of  $\vec{x}$ . By the maximum principle, the unique solution  $U$  can be represented by

$$U(t, \vec{x}) = \int_T^t e^{-n\lambda(t-s)} \mathcal{P}_{t-s} F(s, \vec{x}) ds = \sum_{i=1}^n \mathcal{L}_{x^i} \int_T^t e^{-n\lambda(t-s)} \mathcal{P}_{t-s} G^i(s, \vec{x}) ds,$$

where  $(\mathcal{P}_t)_{t \geq 0}$  is the semigroup associated with  $\mathcal{L}$ .

Thus, by Step 6 and Minkowski's inequality, we have

$$\begin{aligned} \int_T^\infty \|\mathcal{L}u(t)\|_p^{np} dt &= \int_T^\infty \|w(t)\|_p^{np} dt = \int_T^\infty \int_{\mathbb{R}^{nd}} |U(t, \vec{x})|^p d\vec{x} dt \\ &\leq \left( \sum_{i=1}^n \left( \int_T^\infty \int_{\mathbb{R}^{nd}} \left| \mathcal{L}_{x^i} \int_T^t e^{-n\lambda(t-s)} \mathcal{P}_{t-s} G^i(s, \vec{x}) ds \right|^p d\vec{x} dt \right)^{\frac{1}{p}} \right)^p \\ &\leq C \left( \sum_{i=1}^n \left( \int_T^\infty \int_{\mathbb{R}^{nd}} |G^i(t, \vec{x})|^p d\vec{x} dt \right)^{\frac{1}{p}} \right)^p \\ &= Cn \int_T^\infty \|f(t)\|_p^p \|\mathcal{L}u(t)\|_p^{(n-1)p} dt \\ &\leq Cn \left( \int_T^\infty \|f(t)\|_p^{np} dt \right)^{\frac{1}{n}} \left( \int_T^\infty \|\mathcal{L}u(t)\|_p^{np} dt \right)^{1-\frac{1}{n}}. \end{aligned}$$

From this, we get that for any  $n \in \mathbb{N}$  and  $p > 1$ ,

$$\int_T^\infty \|\mathcal{L}u(t)\|_p^{np} dt \leq (Cn)^n \int_T^\infty \|f(t)\|_p^{np} dt.$$

Thus, by Marcinkiewicz’s interpolation theorem (cf. [35]), we get (4.13) for any  $q \geq p$ . The case  $q \leq p$  follows by duality as in Step 6. The whole proof is complete.  $\square$

We have the following important comparison result between two different Lévy operators.

**Theorem 4.3.** *Keep the same assumptions as in Theorem 4.2. For any  $p \in (1, \infty)$ , there exists a constant  $C > 0$  such that for all  $u \in \mathcal{S}(\mathbb{R}^d)$  and  $\lambda_1, \lambda_2 > 0$ ,*

$$\|(\mathcal{L}^{v_2} - \lambda_2)u\|_p \leq C \left(1 + \frac{\lambda_2}{\lambda_1}\right) \|(\mathcal{L}^{v_1} - \lambda_1)u\|_p. \tag{4.14}$$

In particular,

$$\|\mathcal{L}^{v_2}u\|_p \leq C \|\mathcal{L}^{v_1}u\|_p. \tag{4.15}$$

**Proof.** For  $u \in \mathcal{S}(\mathbb{R}^d)$ , set

$$f := (\mathcal{L}^{v_1} - \lambda_1)u.$$

By Fourier’s transform, it is easy to see that

$$u(x) = \int_0^\infty e^{-\lambda_1 t} \mathcal{P}_t^{v_1} f(x) dt.$$

Define

$$u_T(x) := \frac{1}{T} \int_0^T \int_0^t e^{-\lambda_1(t-s)} \mathcal{P}_{t-s}^{v_1} f(x) ds dt = \int_0^T \frac{T-t}{T} e^{-\lambda_1 t} \mathcal{P}_t^{v_1} f(x) dt.$$

Then

$$u(x) - u_T(x) = \int_T^\infty e^{-\lambda_1 t} \mathcal{P}_t^{v_1} f(x) dt + \frac{1}{T} \int_0^T t e^{-\lambda_1 t} \mathcal{P}_t^{v_1} f(x) dt.$$

In view of  $\|\mathcal{P}_t^{v_1} f\|_p \leq \|f\|_p$ , we have

$$\|u - u_T\|_p \leq \|f\|_p \left( \int_T^\infty e^{-\lambda_1 t} dt + \frac{1}{T} \int_0^T t e^{-\lambda_1 t} dt \right) = \|f\|_p (\lambda_1^{-1} e^{-\lambda_1 T} + \lambda_1^{-2} T^{-1}). \tag{4.16}$$

On the other hand, by (4.5) we have

$$\begin{aligned} \|(\mathcal{L}^{v_2} - \lambda_2)u_T\|_p^p &\leq \frac{1}{T} \int_0^T \left\| (\mathcal{L}^{v_2} - \lambda_2) \int_0^t e^{-\lambda_1(t-s)} \mathcal{P}_{t-s}^{v_1} f(\cdot) ds \right\|_p^p dt \\ &\leq C \|f\|_p^p + \frac{2^{p-1}}{T} \int_0^T \left( \lambda_2 \int_0^t e^{-\lambda_1(t-s)} \|f\|_p ds \right)^p dt \\ &\leq C \left(1 + \frac{\lambda_2^p}{\lambda_1^p}\right) \|f\|_p^p = C \left(1 + \frac{\lambda_2^p}{\lambda_1^p}\right) \|(\mathcal{L}^{v_1} - \lambda_1)u\|_p^p, \end{aligned}$$

which together with (4.16) yields (4.14). As for (4.15), it follows by firstly letting  $\lambda_2 \downarrow 0$  and then  $\lambda_1 \downarrow 0$ .  $\square$

In the remaining part of this paper, we make the following assumption:

**(H<sub>v</sub><sup>(α)</sup>)** Let  $\nu_i^{(\alpha)}$ ,  $i = 1, 2$  be two Lévy measures with the form (1.8), where  $\nu_1^{(\alpha)}$  is nondegenerate in the sense of Definition 2.6. Let  $\nu$  be a Lévy measure satisfying (1.7) and

$$\nu_1^{(\alpha)} \leq \nu \leq \nu_2^{(\alpha)}.$$

Let  $\mathcal{D}^p(\mathcal{L}^\nu)$  be the domain of  $\mathcal{L}^\nu$  in  $L^p$ -space, i.e.,

$$\mathcal{D}^p(\mathcal{L}^\nu) := \{u \in L^p(\mathbb{R}^d) : \|\mathcal{L}^\nu u\|_p < +\infty\}.$$

For  $\alpha \geq 0$  and  $p \geq 1$ , the Bessel potential space  $\mathbb{H}^{\alpha,p}$  is defined as the completion of  $\mathcal{S}(\mathbb{R}^d)$  with respect to the norm:

$$\|f\|_{\tilde{\alpha},p} := \|(I - \Delta)^{\frac{\alpha}{2}} u\|_p \simeq \|u\|_p + \|(-\Delta)^{\frac{\alpha}{2}} u\|_p.$$

Notice that for  $k \in \mathbb{N}$  and  $p > 1$ ,  $\mathbb{H}^{k,p} = \mathbb{W}^{k,p}$  (see [35, p. 135, Theorem 3]).

**Corollary 4.4.** Assume **(H<sub>v</sub><sup>(α)</sup>)** with  $\alpha \in (0, 2)$ . For any  $p > 1$ ,  $f \in L^p(\mathbb{R}^d)$  and  $\lambda > 0$ , the equation  $(\mathcal{L}^\nu - \lambda)u = f$  admits a unique strong solution  $u \in \mathbb{H}^{\alpha,p}$ . In particular, for any  $p > 1$ ,  $\mathcal{D}^p(\mathcal{L}^\nu) = \mathbb{H}^{\alpha,p}$  and

$$\|\mathcal{L}^\nu u\|_p \simeq \|(-\Delta)^{\frac{\alpha}{2}} u\|_p, \tag{4.17}$$

and if  $\alpha = 1$ , then

$$\|\mathcal{L}^\nu u\|_p \simeq \|\nabla u\|_p. \tag{4.18}$$

**Proof.** Let  $\nu_0^{(\alpha)}$  be the Lévy measure associated with  $(-\Delta)^{\frac{\alpha}{2}}$  (see (2.8)). In Theorem 4.3, let us take  $\nu_1 = \nu_0^{(\alpha)}$ ,  $\nu_2 = \nu$  and  $\nu_1 = \nu$ ,  $\nu_2 = \nu_0^{(\alpha)}$  respectively, then there exist  $C_1, C_2 > 0$  such that for any  $u \in \mathcal{S}(\mathbb{R}^d)$  and  $\lambda_1, \lambda_2 > 0$ ,

$$\|((-\Delta)^{\frac{\alpha}{2}} + \lambda_2)u\|_p \leq C_1 \left(1 + \frac{\lambda_2}{\lambda_1}\right) \|(\mathcal{L}^\nu - \lambda_1)u\|_p, \tag{4.19}$$

$$\|(\mathcal{L}^\nu - \lambda_1)u\|_p \leq C_2 \left(1 + \frac{\lambda_1}{\lambda_2}\right) \|((-\Delta)^{\frac{\alpha}{2}} + \lambda_2)u\|_p. \tag{4.20}$$

For  $\lambda > 0$  and  $f \in L^p(\mathbb{R}^d)$ , by Corollary 3.3, there exists a sequence  $u_n \in C_0^\infty(\mathbb{R}^d)$  such that

$$(\mathcal{L}^\nu - \lambda)u_n \xrightarrow{L^p} f.$$

By (4.19),  $u_n$  is a Cauchy sequence in  $\mathbb{H}^{\alpha,p}$ . Let  $u \in \mathbb{H}^{\alpha,p}$  be the limit point. By (4.20), one finds that  $(\mathcal{L}^\nu - \lambda)u = f$ . As for (4.17), it follows by (4.15), and (4.18) follows by the boundedness of Riesz transform in  $L^p$ -space (cf. [35, Chapter III]). □

**Corollary 4.5.** Assume **(H<sub>v</sub><sup>(α)</sup>)** with  $\alpha \in (0, 2)$ . Then for any  $p > 1$ ,  $(\mathcal{P}_t^\nu)_{t \geq 0}$  forms an analytic semigroup in  $L^p$ -space.

**Proof.** By [15, Theorem 5.2], it suffices to prove that

$$\|\mathcal{L}^\nu \mathcal{P}_t^\nu f\|_p \leq Ct^{-1} \|f\|_p, \quad t > 0, \quad f \in L^p(\mathbb{R}^d).$$

By (4.4), we have for any  $f \in \mathcal{S}(\mathbb{R}^d)$ ,

$$\mathcal{P}_t^\nu f = \mathcal{P}_t^{\nu_1^{(\alpha)}} \mathcal{P}_t^{\nu - \nu_1^{(\alpha)}} f.$$

Thus, by (2.25), we have

$$\|\Delta \mathcal{P}_t^\nu f\|_p \leq Ct^{-\frac{2}{\alpha}} \|\mathcal{P}_t^{\nu - \nu_1^{(\alpha)}} f\|_p \leq Ct^{-\frac{2}{\alpha}} \|f\|_p.$$

Since  $\mathcal{S}(\mathbb{R}^d)$  is dense in  $L^p(\mathbb{R}^d)$ , we further have for any  $f \in L^p(\mathbb{R}^d)$ ,

$$\|\Delta \mathcal{P}_t^\nu f\|_p \leq C t^{-\frac{\alpha}{2}} \|f\|_p.$$

Now, by (4.18) and the Gagliardo–Nirenberg’s inequality (cf. [6, p. 168]), we have

$$\|\mathcal{L}^\nu \mathcal{P}_t^\nu f\|_p \leq C \|(-\Delta)^{\frac{\alpha}{2}} \mathcal{P}_t^\nu f\|_p \leq C \|\mathcal{P}_t^\nu f\|_p^{1-\frac{\alpha}{2}} \|\Delta \mathcal{P}_t^\nu f\|_p^{\frac{\alpha}{2}} \leq C t^{-1} \|f\|_p,$$

where  $C$  is independent of  $t$  and  $f$ .  $\square$

### 5. Critical nonlocal parabolic equation with variable coefficients

In this section we assume  $(\mathbf{H}_\nu^{(1)})$  with critical index  $\alpha = 1$ . For simplicity of notation, we write

$$\mathcal{L} = \mathcal{L}^\nu.$$

Consider the following Cauchy problem of the first order critical parabolic system:

$$\partial_t u = \mathcal{L}u + b \cdot \nabla u + f, \quad u(0) = \varphi, \tag{5.1}$$

where  $u = (u^1, \dots, u^m)$ ,  $f : \mathbb{R}^+ \times \mathbb{R}^d \rightarrow \mathbb{R}^m$ ,  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^m$  are measurable functions, and  $b : \mathbb{R}^+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a bounded measurable vector field and satisfies

$$|b(t, x) - b(t, y)| \leq \omega_b(|x - y|), \tag{5.2}$$

where  $\omega_b : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is an increasing function with  $\lim_{s \downarrow 0} \omega_b(s) = 0$ .

For obtaining the optimal regularity about the initial value, we need the following real interpolation space: for  $p > 1$  and  $\beta \in (0, 1)$ , let  $\mathbb{W}^{\beta,p}$  be the real interpolation space (called Sobolev–Slobodeckij space) between  $L^p$  and  $\mathbb{W}^{1,p}$ . By [37, p. 190, (15)], an equivalent norm in  $\mathbb{W}^{\beta,p}$  is given by

$$\|f\|_{\beta,p} := \|f\|_p + \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x) - f(y)|^p}{|x - y|^{d+\beta p}} dx dy \right)^{1/p}. \tag{5.3}$$

We remark that for  $p \geq 2$ ,  $\mathbb{H}^{\beta,p} \subset \mathbb{W}^{\beta,p}$ , and for  $p \leq 2$ ,  $\mathbb{W}^{\beta,p} \subset \mathbb{H}^{\beta,p}$  (cf. [35, p. 155, Theorem 5(A) and (C)]). Moreover, by Sobolev’s embedding theorem (see [37, p. 203, (5)]), if  $\beta p > d$  and  $\beta - \frac{d}{p}$  is not an integer, then

$$\mathbb{W}^{\beta,p} \hookrightarrow \mathcal{H}^{\beta - \frac{d}{p}}, \tag{5.4}$$

where for  $\gamma > 0$ ,  $\mathcal{H}^\gamma$  is the usual Hölder space.

Let us first prove the following important a priori estimate by using the classical method of freezing coefficients (cf. [24]).

**Lemma 5.1.** *For given  $p \in (1, \infty)$ , let  $f \in L^p_{loc}(\mathbb{R}^+; L^p(\mathbb{R}^d; \mathbb{R}^m))$  and*

$$u \in C(\mathbb{R}_0^+; \mathbb{W}^{1-\frac{1}{p},p}(\mathbb{R}^d; \mathbb{R}^m)) \cap L^p_{loc}(\mathbb{R}_0^+; \mathbb{W}^{1,p}(\mathbb{R}^d; \mathbb{R}^m)).$$

Assume that  $(\mathbf{H}_\nu^{(1)})$  and (5.2) hold, and  $u$  satisfies

$$\partial_t u(t, x) = \mathcal{L}u(t, x) + b(t, x) \cdot \nabla u(t, x) + f(t, x), \quad a.e. (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d. \tag{5.5}$$

Then for any  $T > 0$ ,

$$\sup_{t \in [0, T]} \|u(t)\|_{1-\frac{1}{p},p}^p + \int_0^T \|\nabla u(t)\|_p^p dt \leq C(1 + T^p) e^{CT^{p-1}} \left( \|u(0)\|_{1-\frac{1}{p},p}^p + \int_0^T \|f(t)\|_p^p dt \right), \tag{5.6}$$

where the constant  $C$  depends only on  $p, d, \|b\|_\infty$ , the modulus function  $\omega_b$  and the Lévy measures  $\nu_i^{(1)}$ ,  $i = 1, 2$ . Moreover,  $u$  also satisfies the following integral equation: for all  $t \geq 0$  and Lebesgue-almost all  $x \in \mathbb{R}^d$ ,

$$u(t, x) = \mathcal{P}_t u(0, x) + \int_0^t \mathcal{P}_{t-s} (b(s) \cdot \nabla u(s))(x) \, ds + \int_0^t \mathcal{P}_{t-s} f(s, x) \, ds, \tag{5.7}$$

where  $\mathcal{P}_t$  is the heat semigroup associated with  $\mathcal{L}$ .

**Proof.** Let  $(\rho_\varepsilon)_{\varepsilon \in (0,1)}$  be a family of mollifiers in  $\mathbb{R}^d$ . Define

$$u_\varepsilon(t) := u(t) * \rho_\varepsilon, \quad b_\varepsilon(t) := b(t) * \rho_\varepsilon, \quad f_\varepsilon(t) := f(t) * \rho_\varepsilon.$$

Taking convolutions for both sides of (5.5), we obtain

$$\partial_t u_\varepsilon(t, x) = \mathcal{L}u_\varepsilon(t, x) + b_\varepsilon(t, x) \cdot \nabla u_\varepsilon(t, x) + F_\varepsilon(t, x), \tag{5.8}$$

where

$$F_\varepsilon(t, x) := [(b(t) \cdot \nabla u(t)) * \rho_\varepsilon](x) - b_\varepsilon(t, x) \cdot \nabla u_\varepsilon(t, x) + f_\varepsilon(t, x).$$

Moreover, by Duhamel’s formula, one sees that

$$u_\varepsilon(t, x) = \mathcal{P}_t u_\varepsilon(0, x) + \int_0^t \mathcal{P}_{t-s} (b_\varepsilon(s) \cdot \nabla u_\varepsilon(s))(x) \, ds + \int_0^t \mathcal{P}_{t-s} F_\varepsilon(s, x) \, ds. \tag{5.9}$$

By the assumptions, it is easy to see that for all  $\varepsilon \in (0, 1)$ ,

$$|b_\varepsilon(t, x) - b_\varepsilon(t, y)| \leq \omega_b(|x - y|), \quad |b_\varepsilon(t, x) - b(t, x)| \leq \omega_b(\varepsilon),$$

and

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \|F_\varepsilon(t) - f(t)\|_p^p \, dt = 0.$$

Taking limits for both sides of (5.9), one finds that (5.7) holds. Below, we use the method of freezing the coefficients to prove

$$\sup_{t \in [0, T]} \|u_\varepsilon(t)\|_p^p + \int_0^T \|\nabla u_\varepsilon(t)\|_p^p \, dt \leq C(1 + T^p) e^{CT^{p-1}} \left( \|u_\varepsilon(0)\|_{1-\frac{1}{p}, p}^p + C \int_0^T \|F_\varepsilon(t)\|_p^p \, dt \right), \tag{5.10}$$

where the constant  $C$  is independent of  $\varepsilon$  and  $T$ .

For simplicity of notation, we drop the subscript  $\varepsilon$  below. Fix  $\delta > 0$  being small enough, whose value will be determined below. Let  $\zeta$  be a smooth function with support in  $B_\delta$  and  $\|\zeta\|_p = 1$ . For  $z \in \mathbb{R}^d$ , set

$$\zeta_z(x) := \zeta(x - z).$$

Multiplying both sides of (5.8) by  $\zeta_z$ , we obtain

$$\partial_t (u \zeta_z) = (\mathcal{L}u) \zeta_z + (b \cdot \nabla u) \zeta_z + F \zeta_z = \mathcal{L}(u \zeta_z) + \vartheta_z^b \cdot \nabla (u \zeta_z) + g_z^\zeta,$$

where  $\vartheta_z^b(t) := b(t, z)$  and

$$g_z^\zeta := (b - \vartheta_z^b) \cdot \nabla (u \zeta_z) - ub \cdot \nabla \zeta_z + (\mathcal{L}u) \zeta_z - \mathcal{L}(u \zeta_z) + F \zeta_z.$$

By Lemma 3.1,  $u \zeta_z$  can be uniquely written as

$$u \zeta_z(t, x) = \mathcal{T}_{t,0}^{\vartheta_z^b} (u(0) \zeta_z)(x) + \int_0^t \mathcal{T}_{t,s}^{\vartheta_z^b} g_z^\zeta(s, x) \, ds,$$

where  $\mathcal{T}_{t,s}^{\vartheta_z^b}$  is defined by (4.1) through  $\vartheta_z^b$ . Thus, we have



$$\int_0^T \|\nabla(u\xi_z)(t, \cdot)\|_p^p dt \leq 2^{p-1} \int_0^T \|\nabla\mathcal{T}_{t,0}^{\vartheta_z^b}(u(0)\xi_z)\|_p^p dt + 2^{p-1} \int_0^T \left\| \nabla \int_0^t \mathcal{T}_{t,s}^{\vartheta_z^b} g_z^\xi(s, \cdot) ds \right\|_p^p dt$$

$$=: I_1(T, z) + I_2(T, z).$$

For  $I_1(T, z)$ , by Corollary 4.5 and [37, p. 96, Theorem 1.14.5], we have

$$\int_0^T \|\nabla\mathcal{T}_{t,0}^{\vartheta_z^b}(u(0)\xi_z)\|_p^p dt \stackrel{(4.1)}{=} \int_0^T \left\| \nabla\mathcal{P}_t(u(0)\xi_z) \left( \cdot - \int_0^t \vartheta_z^b(s) ds \right) \right\|_p^p dt = \int_0^T \|\nabla\mathcal{P}_t(u(0)\xi_z)\|_p^p dt$$

$$\stackrel{(4.18)}{\leq} C \int_0^T \|\mathcal{L}\mathcal{P}_t(u(0)\xi_z)\|_p^p dt \leq C \|u(0)\xi_z\|_{1-\frac{1}{p},p}^p. \tag{5.11}$$

Here and below,  $C$  is independent of  $T$ . Thus, by definition (5.3), it is easy to see that

$$\int_{\mathbb{R}^d} I_1(T, z) dz \leq C \int_{\mathbb{R}^d} \|u(0)\xi_z\|_{1-\frac{1}{p},p}^p dz \leq C (\|u(0)\|_{1-\frac{1}{p},p}^p \|\xi\|_p^p + \|u(0)\|_p^p \|\xi\|_{1-\frac{1}{p},p}^p).$$

For  $I_2(T, z)$ , by (4.18) and Theorem 4.2, we have

$$I_2(T, z) \leq C \int_0^T \|g_z^\xi(s, \cdot)\|_p^p ds$$

$$\leq C \int_0^T \|((b - \vartheta_z^b) \cdot \nabla(u\xi_z))(s, \cdot)\|_p^p ds + C \int_0^T \|(ub \cdot \nabla\xi_z)(s, \cdot)\|_p^p ds + C \int_0^T \|F\xi_z(s, \cdot)\|_p^p ds$$

$$+ C \int_0^T \|((\mathcal{L}u)\xi_z - \mathcal{L}(u\xi_z))(s, \cdot)\|_p^p ds$$

$$=: I_{21}(T, z) + I_{22}(T, z) + I_{23}(T, z) + I_{24}(T, z).$$

For  $I_{21}(T, z)$ , by (5.2) and  $\|\xi\|_p = 1$ , we have

$$\int_{\mathbb{R}^d} I_{21}(T, z) dz \stackrel{(5.2)}{\leq} C\omega_b^p(\delta) \int_0^T \int_{\mathbb{R}^d} \|\nabla(u\xi_z)(s, \cdot)\|_p^p dz ds$$

$$\leq C\omega_b^p(\delta) \int_0^T \|\nabla u(s)\|_p^p ds + C\omega_b^p(\delta) \|\nabla\xi\|_p^p \int_0^T \|u(s)\|_p^p ds.$$

For  $I_{24}(T, z)$ , by (i) of Lemma 2.5, we have

$$\int_{\mathbb{R}^d} I_{24}(T, z) dz \leq C \int_0^T \|u(s)\|_p^p ds + C \int_0^T \|u(s)\|_p^{p/2} \|\nabla u(s)\|_p^{p/2} ds.$$

Moreover, it is easy to see that

$$\int_{\mathbb{R}^d} I_{22}(T, z) dz \leq C \|b\|_\infty^p \|\nabla\xi\|_p^p \int_0^T \|u(s)\|_p^p ds,$$

$$\int_{\mathbb{R}^d} I_{23}(T, z) dz \leq C \int_0^T \|F(s)\|_p^p ds.$$

Combining the above calculations, we get

$$\begin{aligned} \int_0^T \|\nabla u(s)\|_p^p ds &= \int_0^T \int_{\mathbb{R}^d} \|\nabla u(s) \cdot \zeta_z\|_p^p dz ds \\ &\leq 2^{p-1} \int_0^T \int_{\mathbb{R}^d} \|\nabla(u\zeta_z)(s)\|_p^p dz ds + 2^{p-1} \|\nabla \zeta\|_p^p \int_0^T \|u(s)\|_p^p ds \\ &\leq C \|u(0)\|_{1-\frac{1}{p},p}^p + C \omega_b^p(\delta) \int_0^T \|\nabla u(s)\|_p^p ds + C \int_0^T \|u(s)\|_p^p ds \\ &\quad + C \int_0^T \|u(s)\|_p^{p/2} \|\nabla u(s)\|_p^{p/2} ds + C \int_0^T \|F(s)\|_p^p ds. \end{aligned}$$

Using Young’s inequality and letting  $\delta$  be small enough so that  $C\omega_b^p(\delta) \leq \frac{1}{4}$ , we arrive at

$$\int_0^T \|\nabla u(s)\|_p^p ds \leq C \|u(0)\|_{1-\frac{1}{p},p}^p + C \int_0^T \|u(s)\|_p^p ds + C \int_0^T \|F(s)\|_p^p ds. \tag{5.12}$$

On the other hand, by (5.9), it is easy to see that

$$\|u(t)\|_p^p \leq C \|u(0)\|_p^p + Ct^{p-1} \|b\|_\infty^p \int_0^t \|\nabla u(s)\|_p^p ds + Ct^{p-1} \int_0^t \|F(s)\|_p^p ds,$$

which together with (5.12) and Gronwall’s inequality yields that for any  $T > 0$ ,

$$\sup_{t \in [0, T]} \|u(t)\|_p^p + \int_0^T \|\nabla u(s)\|_p^p ds \leq C(1 + T^p) e^{CT^{p-1}} \left( \|u(0)\|_{1-\frac{1}{p},p}^p + \int_0^T \|F(s)\|_p^p ds \right).$$

Thus, we conclude the proof of (5.10), and therefore,

$$\int_0^T \|\nabla u(s)\|_p^p ds \leq C(1 + T^p) e^{CT^{p-1}} \left( \|u(0)\|_{1-\frac{1}{p},p}^p + \int_0^T \|f(s)\|_p^p ds \right). \tag{5.13}$$

Lastly, we show (5.6). From Eq. (5.5) and using estimate (5.13), we have

$$\begin{aligned} \int_0^T \|\partial_t u(t)\|_p^p dt &\leq C \left( \int_0^T \|\mathcal{L}u(t)\|_p^p dt + \|b\|_\infty^p \int_0^T \|\nabla u(t)\|_p^p dt + \int_0^T \|f(t)\|_p^p dt \right) \\ &\stackrel{(4.18)}{\leq} C \left( (1 + \|b\|_\infty^p) \int_0^T \|\nabla u(t)\|_p^p dt + \int_0^T \|f(t)\|_p^p dt \right) \\ &\leq C(1 + T^p) e^{CT^{p-1}} \left( \|u(0)\|_{1-\frac{1}{p},p}^p + \int_0^T \|f(s)\|_p^p ds \right). \end{aligned}$$

Noticing the following embedding relation (cf. [1, p. 180, Theorem III, 4.10.2])

$$L^p([0, T], \mathbb{W}^{1,p}) \cap \mathbb{W}^{1,p}([0, T], L^p) \hookrightarrow C([0, T]; \mathbb{W}^{1-\frac{1}{p},p}),$$

we have

$$\begin{aligned} \sup_{t \in [0, T]} \|u(t)\|_{1-\frac{1}{p}, p}^p &\leq C \left( \int_0^T \|\partial_t u(t)\|_p^p dt + \int_0^T \|u(t)\|_{1, p}^p dt \right) \\ &\leq C(1 + T^p) e^{CT^{p-1}} \left( \|u(0)\|_{1-\frac{1}{p}, p}^p + \int_0^T \|f(s)\|_p^p ds \right), \end{aligned}$$

which together with (5.13) yields (5.6).  $\square$

Before proving the existence of strong solutions to Eq. (5.1), we recall a well-known fact (cf. [14,40]).

**Theorem 5.2** (Feyman–Kac formula). *Let  $\nu$  be a Lévy measure and  $b \in L_{loc}^\infty(\mathbb{R}^+; C_b^\infty(\mathbb{R}^d; \mathbb{R}^d))$ ,  $f \in L_{loc}^1(\mathbb{R}^+; \mathbb{W}^\infty(\mathbb{R}^d; \mathbb{R}^m))$ . For any  $\varphi \in \mathbb{W}^\infty(\mathbb{R}^d; \mathbb{R}^m)$ , there exists a unique  $u \in C(\mathbb{R}_0^+; \mathbb{W}^\infty(\mathbb{R}^d; \mathbb{R}^m))$  satisfying*

$$u(t, x) = \varphi(x) + \int_0^t \mathcal{L}^\nu u(s, x) ds + \int_0^t (b \cdot \nabla u)(s, x) ds + \int_0^t f(s, x) ds.$$

Moreover,  $u(t, x)$  can be represented by

$$u(t, x) := \mathbb{E}\varphi(X_{-t,0}(x)) + \mathbb{E}\left(\int_{-t}^0 f(-s, X_{-t,s}(x)) ds\right), \quad t \geq 0, \tag{5.14}$$

where  $\{X_{t,s}(x), t \leq s \leq 0, x \in \mathbb{R}^d\}$  is defined by the following SDE:

$$X_{t,s}(x) = x + \int_t^s b(-r, X_{t,r}(x)) dr + \int_t^s dL_r^\nu, \quad t \leq s \leq 0.$$

We are now in a position to prove

**Theorem 5.3.** *Assume  $(\mathbf{H}_\nu^{(1)})$  and (5.2). Let  $p \in (1, \infty)$  and*

$$\varphi \in \mathbb{W}^{1-\frac{1}{p}, p}(\mathbb{R}^d; \mathbb{R}^m), \quad f \in L_{loc}^p(\mathbb{R}_0^+; L^p(\mathbb{R}^d; \mathbb{R}^m)).$$

*Then there exists a unique  $u \in C(\mathbb{R}_0^+; \mathbb{W}^{1-\frac{1}{p}, p}(\mathbb{R}^d; \mathbb{R}^m)) \cap L_{loc}^p(\mathbb{R}_0^+; \mathbb{W}^{1, p}(\mathbb{R}^d; \mathbb{R}^m))$  satisfying Eq. (5.5).*

**Proof.** Let  $b_\varepsilon, f_\varepsilon$  and  $\varphi_\varepsilon$  be the mollifying approximations of  $b, f$  and  $\varphi$ :

$$b_\varepsilon(t, x) := b(t) * \rho_\varepsilon(x), \quad f_\varepsilon(t, x) := f(t) * \rho_\varepsilon(x), \quad \varphi_\varepsilon(x) := \varphi * \rho_\varepsilon(x).$$

By Theorem 5.2, there exists a unique  $u_\varepsilon \in C(\mathbb{R}_0^+; \mathbb{W}^\infty(\mathbb{R}^d; \mathbb{R}^m))$  satisfying the following equation:

$$u_\varepsilon(t, x) = \varphi_\varepsilon(x) + \int_0^t \mathcal{L} u_\varepsilon(s, x) ds + \int_0^t b_\varepsilon(s, x) \cdot \nabla u_\varepsilon(s, x) ds + \int_0^t f_\varepsilon(s, x) ds. \tag{5.15}$$

First of all, by Lemma 5.1, we have the following uniform estimate: for any  $T > 0$ ,

$$\sup_{t \in [0, T]} \|u_\varepsilon(t)\|_{1-\frac{1}{p}, p}^p + \int_0^T \|\nabla u_\varepsilon(t)\|_p^p dt \leq C \left( \|\varphi\|_{1-\frac{1}{p}, p}^p + \int_0^T \|f(t)\|_p^p dt \right),$$

where  $C$  is independent of  $\varepsilon$ .

Noticing that  $w_{\varepsilon, \varepsilon'} := u_\varepsilon - u_{\varepsilon'}$  satisfies

$$\partial_t w_{\varepsilon, \varepsilon'} = \mathcal{L}w_{\varepsilon, \varepsilon'} + b_\varepsilon \cdot \nabla w_{\varepsilon, \varepsilon'} + (b_\varepsilon - b_{\varepsilon'}) \cdot \nabla u_{\varepsilon'} + f_\varepsilon - f_{\varepsilon'}, \quad w_{\varepsilon, \varepsilon'}(0) = \varphi_\varepsilon - \varphi_{\varepsilon'},$$

by Lemma 5.1 again, we also have

$$\begin{aligned} & \sup_{t \in [0, T]} \|w_{\varepsilon, \varepsilon'}(t)\|_{1-\frac{1}{p}, p}^p + \int_0^T \|\nabla w_{\varepsilon, \varepsilon'}(s)\|_p^p \, ds \\ & \leq C \|\varphi_\varepsilon - \varphi_{\varepsilon'}\|_{1-\frac{1}{p}, p}^p + C \int_0^T \|f_\varepsilon(s) - f_{\varepsilon'}(s)\|_p^p \, ds + C \sup_{s \in [0, T]} \|b_\varepsilon(s) - b_{\varepsilon'}(s)\|_\infty^p \int_0^T \|\nabla u_{\varepsilon'}(s)\|_p^p \, ds. \end{aligned}$$

On the other hand, by (5.2), it is easy to see that

$$\sup_{s \geq 0} \|b_\varepsilon(s) - b_{\varepsilon'}(s)\|_\infty \leq \omega_b(\varepsilon) + \omega_b(\varepsilon').$$

So, for any  $T > 0$ ,

$$\lim_{\varepsilon, \varepsilon' \rightarrow 0} \left( \sup_{t \in [0, T]} \|w_{\varepsilon, \varepsilon'}(t)\|_{1-\frac{1}{p}, p}^p + \int_0^T \|\nabla w_{\varepsilon, \varepsilon'}(s)\|_p^p \, ds \right) = 0,$$

and there exists a  $u \in C(\mathbb{R}_0^+; \mathbb{W}^{1-\frac{1}{p}, p}(\mathbb{R}^d; \mathbb{R}^m)) \cap L_{loc}^p(\mathbb{R}_0^+; \mathbb{W}^{1, p}(\mathbb{R}^d; \mathbb{R}^m))$  such that for any  $T > 0$ ,

$$\lim_{\varepsilon \rightarrow 0} \left( \sup_{t \in [0, T]} \|u_\varepsilon(t) - u(t)\|_{1-\frac{1}{p}, p}^p + \int_0^T \|\nabla u_\varepsilon(s) - \nabla u(s)\|_p^p \, ds \right) = 0.$$

By taking limits in  $L^p$ -space for (5.15), one finds that for all  $t \geq 0$  and almost all  $x \in \mathbb{R}^d$ ,

$$u(t, x) = \varphi(x) + \int_0^t \mathcal{L}u(s, x) \, ds + \int_0^t b(s, x) \cdot \nabla u(s, x) \, ds + \int_0^t f(s, x) \, ds.$$

The existence follows. As for the uniqueness, it follows from Lemma 5.1.  $\square$

Now we present an application by proving Krylov’s estimate for critical diffusion process:

$$X_t = X_0 + \int_0^t b(s, X_s) \, ds + L_t. \tag{5.16}$$

**Theorem 5.4.** Assume  $(\mathbf{H}_v^{(1)})$  and (5.2). Then there exists a solution to SDE (5.16) such that for fixed  $T_0 > 0$  and any  $p > d + 1$ , stopping time  $\tau$ ,  $0 \leq T \leq S \leq T_0$  and  $f \in L^p([T, S] \times \mathbb{R}^d)$ ,

$$\mathbb{E} \left( \int_{T \wedge \tau}^{S \wedge \tau} f(s, X_s) \, ds \middle| \mathcal{F}_{T \wedge \tau} \right) \leq C \|f\|_{L^p([T, S] \times \mathbb{R}^d)}, \tag{5.17}$$

where  $C$  is independent of  $f$  and  $\tau$ . Here, a solution to Eq. (5.16) means that there exists a probability space  $(\Omega, \mathcal{F}, P)$  and two càdlàg stochastic processes  $X_t$  and  $L_t$  defined on it such that (5.16) is satisfied, and  $L_t$  is a Lévy process with respect to the completed filtration  $\mathcal{F}_t := \sigma^P\{X_s, L_s, s \leq t\}$ , and whose Lévy measure is given by  $\nu$ .

**Proof.** Let  $b_\varepsilon(t, x) := b(t) * \rho_\varepsilon(x)$  be the mollifying approximation of  $b$  and let  $X_t^\varepsilon$  solve the following SDE:

$$X_t^\varepsilon = X_0 + \int_0^t b_\varepsilon(s, X_s^\varepsilon) \, ds + L_t. \tag{5.18}$$

It is by now standard to prove that the laws of  $\{(X_t^\varepsilon, L_t)_{t \geq 0}, \varepsilon \in (0, 1)\}$  are tight in the space of all càdlàg functions (for example, see [39]). Thus, by Skorohod’s representation theorem (cf. [19, Theorem 3.30]), there exist a probability space still denoted by  $(\Omega, \mathcal{F}, P)$  and càdlàg stochastic processes  $(X_t^\varepsilon, L_t^\varepsilon)_{t \geq 0}$  and  $(X_t, L_t)_{t \geq 0}$  such that  $(X_t^\varepsilon, L_t^\varepsilon)$  almost surely converges to  $(X_t, L_t)$  for each  $t \geq 0$ , and

$$X_t^\varepsilon = X_0^\varepsilon + \int_0^t b_\varepsilon(s, X_s^\varepsilon) ds + L_t^\varepsilon.$$

By taking limits for Eq. (5.18), it is easy to see that  $(X_t, L_t)$  is a solution of SDE (5.16).

Fix  $f \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$  and  $T_0 > 0$ . Let  $u_\varepsilon(t, x) \in C(\mathbb{R}_0^+; C_b^\infty(\mathbb{R}^d))$  solve the following PDE

$$\partial_t u_\varepsilon - \mathcal{L}u_\varepsilon - b_\varepsilon(T_0 - \cdot, \cdot) \cdot \nabla u_\varepsilon = -f(T_0 - \cdot, \cdot), \quad u_\varepsilon(0) = 0.$$

Set

$$w_\varepsilon(t, x) = u_\varepsilon(T_0 - t, x).$$

Then

$$\partial_t w_\varepsilon + \mathcal{L}w_\varepsilon + b \cdot \nabla w_\varepsilon = f, \quad w(T_0, x) = 0.$$

Let  $\tau$  be any stopping time. By Ito’s formula (cf. [2, Theorem 4.4.7]), we have

$$\begin{aligned} w_\varepsilon(t, X_t^\varepsilon) &= w(T \wedge \tau, X_{T \wedge \tau}^\varepsilon) + \int_{T \wedge \tau}^t (\partial_s w_\varepsilon(s) + \mathcal{L}w_\varepsilon(s) + b_\varepsilon(s) \cdot \nabla w_\varepsilon(s))(X_s^\varepsilon) ds + \text{a martingale} \\ &= w(T \wedge \tau, X_{T \wedge \tau}^\varepsilon) + \int_{T \wedge \tau}^t f(s, X_s^\varepsilon) ds + \text{a martingale.} \end{aligned}$$

Taking the conditional expectations with respect to  $\mathcal{F}_{T \wedge \tau}$  and by the optional theorem (cf. [19, Theorem 6.12]), we obtain

$$\mathbb{E} \left( \int_{T \wedge \tau}^{S \wedge \tau} f(s, X_s^\varepsilon) ds \middle| \mathcal{F}_{T \wedge \tau} \right) = \mathbb{E}(w(S \wedge \tau, X_{S \wedge \tau}^\varepsilon) | \mathcal{F}_{T \wedge \tau}) - w(T \wedge \tau, X_{T \wedge \tau}^\varepsilon).$$

On the other hand, since

$$|b_\varepsilon(t, x) - b_\varepsilon(t, y)| \leq \omega_b(|x - y|),$$

by (5.4) and (5.6), we have

$$\sup_{t \in [T, S]} \|u_\varepsilon\|_\infty \leq C \sup_{t \in [T, S]} \|u_\varepsilon(t)\|_{1-\frac{1}{p}, p} \leq C \|f\|_{L^p([T, S] \times \mathbb{R}^d)},$$

where the constant  $C$  is independent of  $\varepsilon$ . Hence,

$$\mathbb{E} \left( \int_{T \wedge \tau}^{S \wedge \tau} f(s, X_s^\varepsilon) ds \middle| \mathcal{F}_{T \wedge \tau} \right) \leq C \|f\|_{L^p([T, S] \times \mathbb{R}^d)}.$$

Since  $f \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ , estimate (5.17) now follows by taking limit  $\varepsilon \rightarrow 0$ . For general  $f \in L^p([T, S] \times \mathbb{R}^d)$ , it follows by a standard density argument.  $\square$

### 6. Quasi-linear first order parabolic system with critical diffusion

In this section we study the solvability of quasi-linear first order parabolic system with critical diffusions. Let us first recall and extend a result of Silvestre [32] about the Hölder estimate of advection fractional diffusion equations.

**Theorem 6.1.** (See Silvestre [32].) Assume that  $b \in L^\infty([0, 1]; C_b^\infty(\mathbb{R}^d; \mathbb{R}^d))$  and  $f \in L^\infty([0, 1]; C_b^\infty(\mathbb{R}^d))$ . For given  $a > 0$ , let  $u \in C([0, 1]; C_b^\infty(\mathbb{R}^d))$  satisfy that for all  $(t, x) \in [0, 1] \times \mathbb{R}^d$ ,

$$u(t, x) = u(0, x) - a \int_0^t (-\Delta)^{\frac{1}{2}} u(s, x) \, ds + \int_0^t b(s, x) \cdot \nabla u(s, x) \, ds + \int_0^t f(s, x) \, ds. \tag{6.1}$$

Then for any  $\gamma \in (0, 1)$ , there exist  $\beta \in (0, 1)$  and  $C$  depending only on  $d, a, \gamma$  and  $\|b\|_\infty$  such that

$$\sup_{t \in [0, 1]} \|u(t)\|_{\mathcal{H}^\beta} \leq C(\|u\|_\infty + \|f\|_\infty + \|u(0)\|_{\mathcal{H}^\gamma}), \tag{6.2}$$

where  $\|u\|_{\mathcal{H}^\beta} := \sup_{|x-y| \leq 1} |u(x) - u(y)|/|x - y|^\beta$ .

**Proof.** By [32, Theorem 1.1], there exist  $\beta_0 \in (0, 1)$  and  $C > 0$  depending only on  $d, a$  and  $\|b\|_\infty$  such that

$$\|u(t)\|_{\mathcal{H}^{\beta_0}} \leq Ct^{-\beta_0}(\|u\|_\infty + \|f\|_\infty), \quad t \in (0, 1]. \tag{6.3}$$

Recall the following probabilistic representation of  $u(t, x)$  (see Theorem 5.2):

$$u(t, x) = \mathbb{E}u(0, X_{-t,0}(x)) + \mathbb{E}\left(\int_{-t}^0 f(-s, X_{-t,s}(x)) \, ds\right), \quad t \in [0, 1], \tag{6.4}$$

where  $\{X_{t,s}(x), -1 \leq t \leq s \leq 0, x \in \mathbb{R}^d\}$  is defined by the following SDE:

$$X_{t,s}(x) = x + \int_t^s b(-r, X_{t,r}(x)) \, dr + \int_t^s dL_r, \quad -1 \leq t \leq s \leq 0, \tag{6.5}$$

where  $(L_t)_{t \leq 0}$  is the Lévy process associated with  $(-\Delta)^{\frac{1}{2}}$ .

By (6.4) and (6.5), we have

$$\begin{aligned} |u(t, x) - u(0, x)| &\leq \|u(0)\|_{\mathcal{H}^\gamma} \mathbb{E}\|X_{-t,0}(x) - x\|^\gamma + t\|f\|_\infty \\ &\leq \|u(0)\|_{\mathcal{H}^\gamma} (t^\gamma \|b\|_\infty + \mathbb{E}\|L_{-t}\|^\gamma) + t\|f\|_\infty \\ &\stackrel{(2.3)}{=} \|u(0)\|_{\mathcal{H}^\gamma} (t^\gamma \|b\|_\infty + t^\gamma \mathbb{E}\|L_{-1}\|^\gamma) + t\|f\|_\infty \\ &\leq t^\gamma (\|u(0)\|_{\mathcal{H}^\gamma} (\|b\|_\infty + \mathbb{E}\|L_{-1}\|^\gamma) + \|f\|_\infty). \end{aligned} \tag{6.6}$$

For given  $x, y \in \mathbb{R}^d$  and  $t \in (0, 1]$ , if  $t > |x - y|^{\frac{1}{2}}$ , then by (6.3) we have

$$|u(t, x) - u(t, y)| \leq C|x - y|^{\beta_0/2}(\|u\|_\infty + \|f\|_\infty);$$

if  $t \leq |x - y|^{\frac{1}{2}}$ , then by (6.6) we have

$$\begin{aligned} |u(t, x) - u(t, y)| &\leq |u(t, x) - u(0, x)| + |u(t, y) - u(0, y)| + |u(0, x) - u(0, y)| \\ &\leq 2|x - y|^{\gamma/2}(\|u(0)\|_{\mathcal{H}^\gamma} (\|b\|_\infty + \mathbb{E}\|L_{-1}\|^\gamma) + \|f\|_\infty) + |x - y|^\gamma \|u(0)\|_{\mathcal{H}^\gamma}. \end{aligned}$$

Estimate (6.2) now follows by taking  $\beta = \min(\gamma, \beta_0)/2$ .  $\square$

Notice that the proof of Silvestre [32] seems strongly depend on the scale invariance of  $(-\Delta)^{\frac{1}{2}}$ . Below, we use probabilistic representation (6.4) again to extend Silvestre’s Hölder estimate to the more general Lévy operator (not necessarily homogeneous and symmetric). Consider the following Lévy measure

$$\nu(dy) = \frac{a(y)}{|y|^{d+1}} \, dy,$$

where  $a(y)$  is a measurable function on  $\mathbb{R}^d$ . Let  $\mathcal{L}^\nu$  be the Lévy operator associated to  $\nu$ . We have

**Corollary 6.2.** Assume that  $b \in L^\infty([0, 1]; C_b^\infty(\mathbb{R}^d; \mathbb{R}^d))$  and  $f \in L^\infty([0, 1]; C_b^\infty(\mathbb{R}^d))$ . For given  $\varphi \in C_b^\infty(\mathbb{R}^d)$ , let  $u \in C([0, 1]; C_b^\infty(\mathbb{R}^d))$  satisfy that for all  $(t, x) \in [0, 1] \times \mathbb{R}^d$ ,

$$u(t, x) = \varphi(x) + \int_0^t \mathcal{L}^\nu u(s, x) \, ds + \int_0^t b(s, x) \cdot \nabla u(s, x) \, ds + \int_0^t f(s, x) \, ds. \tag{6.7}$$

If  $a(y)$  is bounded from below by  $c_1 > 0$ , then for any  $\gamma \in (0, 1)$ , there exist  $\beta \in (0, 1)$  and  $C$  depending only on  $d, c_1, \gamma$  and  $\|b\|_\infty$  such that

$$\sup_{t \in [0, 1]} \|u(t)\|_{\mathcal{H}^\beta} \leq C(\|f\|_\infty + \|\varphi\|_\infty + \|\varphi\|_{\mathcal{H}^\gamma}). \tag{6.8}$$

**Proof.** Define

$$\nu_0(dy) := c_1 \, dy/|y|^{d+1}, \quad \nu_1(dy) := \nu(dy) - \nu_0(dy) = (a(y) - c_1) \, dy/|y|^{d+1}.$$

Let  $(L_t^{\nu_0})_{t \leq 0}$  and  $(L_t^{\nu_1})_{t \leq 0}$  be two independent  $d$ -dimensional Lévy processes with the Lévy measures  $\nu_0$  and  $\nu_1$ . Then we have

$$(L_t^\nu)_{t \leq 0} \stackrel{(d)}{=} (L_t^{\nu_0} + L_t^{\nu_1})_{t \leq 0}.$$

Recall the probabilistic representation (6.4) of  $u(t, x)$ , where  $\{X_{t,s}(x), -1 \leq t \leq s \leq 0, x \in \mathbb{R}^d\}$  is defined by the following SDE:

$$X_{t,s}(x) = x + \int_t^s b(-r, X_{t,r}(x)) \, dr + \int_t^s dL_r^{\nu_0} + \int_t^s dL_r^{\nu_1}, \quad -1 \leq t \leq s \leq 0.$$

Let  $\mathbb{D}([-1, 0])$  be the space of all càdlàg functions  $\ell : [-1, 0] \rightarrow \mathbb{R}^d$ . Below, we fix  $t_0 \in [0, 1]$  and a path  $\ell \in \mathbb{D}([-1, 0])$ . Let  $Y_{t,s}(x, \ell)$  solve the following SDE:

$$Y_{t,s}(x, \ell) = x + \int_t^s b(-r, Y_{t,r}(x, \ell) + \ell_r - \ell_{-t_0}) \, dr + \int_t^s dL_r^{\nu_0}, \quad -1 \leq t \leq s \leq 0.$$

By the uniqueness of solutions to SDEs, it is easy to see that

$$X_{-t_0,s}(x) = Y_{-t_0,s}(x, L^{\nu_1}) + L_s^{\nu_1} - L_{-t_0}^{\nu_1}, \quad -t_0 \leq s \leq 0.$$

Substituting this into (6.4), we get

$$u(t_0, x) = \mathbb{E}\varphi(Y_{-t_0,0}(x, L^{\nu_1}) + L_0^{\nu_1} - L_{-t_0}^{\nu_1}) + \mathbb{E}\left(\int_{-t_0}^0 f(-s, Y_{-t_0,s}(x, L^{\nu_1}) + L_s^{\nu_1} - L_{-t_0}^{\nu_1}) \, ds\right). \tag{6.9}$$

Now let us define

$$w(t, x, \ell) := \mathbb{E}\varphi(Y_{-t,0}(x, \ell) + \ell_0 - \ell_{-t_0}) + \mathbb{E}\left(\int_{-t}^0 f(-s, Y_{-t,s}(x, \ell) + \ell_s - \ell_{-t_0}) \, ds\right). \tag{6.10}$$

Using Theorem 5.2 again, one sees that  $w(t, x, \ell)$  satisfies

$$\begin{aligned} w(t, x, \ell) &= \varphi(x + \ell_0 - \ell_{-t_0}) + \int_0^t \mathcal{L}^{\nu_0} w(s, x, \ell) \, ds + \int_0^t b(s, x + \ell_{-s} - \ell_{-t_0}) \cdot \nabla w(s, x, \ell) \, ds \\ &\quad + \int_0^t f(s, x + \ell_{-s} - \ell_{-t_0}) \, ds, \end{aligned}$$

where for some  $a > 0$ ,  $\mathcal{L}^{v_0} = -a(-\Delta)^{\frac{1}{2}}$  is the Lévy operator associated with  $v_0$  (see (2.8)). Thus, by Theorem 6.1, there exist  $\beta \in (0, 1)$  and  $C$  depending only on  $d, a, \gamma$  and  $\|b\|_\infty$  such that

$$\begin{aligned} \sup_{t \in [0, 1]} \|w(t, \cdot, \ell.)\|_{\mathcal{H}^\beta} &\leq C(\|w\|_\infty + \|f\|_\infty + \|\varphi\|_{\mathcal{H}^\gamma}) \\ &\stackrel{(6.10)}{\leq} C(\|f\|_\infty + \|\varphi\|_\infty + \|\varphi\|_{\mathcal{H}^\gamma}). \end{aligned} \tag{6.11}$$

On the other hand, since  $(L_t^{v_0})_{t \leq 0}$  and  $(L_t^{v_1})_{t \leq 0}$  are independent, by (6.9) and (6.10), we have

$$u(t_0, x) = \mathbb{E}w(t_0, x, L^{v_1}).$$

Estimate (6.8) now follows by (6.11).  $\square$

Below, we assume that  $a$  satisfies that

$$c_1 \leq a(y) \leq c_2,$$

and for all  $0 < r < R < +\infty$ ,

$$\int_{r \leq |y| \leq R} \frac{ya(y)}{|y|^{d+1}} dy = 0.$$

For the sake of simplicity, we write

$$\mathcal{L} = \mathcal{L}^v.$$

Consider the following Cauchy problem of semilinear first order parabolic system:

$$\partial_t u = \mathcal{L}u + b(u) \cdot \nabla u + f(u), \quad u(0) = \varphi, \tag{6.12}$$

where  $u(t, x) = (u^1(t, x), \dots, u^m(t, x))$ , and  $\varphi(x) : \mathbb{R}^d \rightarrow \mathbb{R}^m$ ,

$$\begin{aligned} b(t, x, u) &: [0, 1] \times \mathbb{R}^d \times \mathbb{R}^m \rightarrow \mathbb{R}^d, \\ f(t, x, u) &: [0, 1] \times \mathbb{R}^d \times \mathbb{R}^m \rightarrow \mathbb{R}^m \end{aligned}$$

are Borel measurable functions.

We introduce the following notion about the strong solution for Eq. (6.12).

**Definition 6.3.** Let  $p > 1$  and  $\varphi \in \mathbb{W}^{1-\frac{1}{p}, p}(\mathbb{R}^d; \mathbb{R}^m)$ . A function

$$u \in C([0, 1]; \mathbb{W}^{1-\frac{1}{p}, p}(\mathbb{R}^d; \mathbb{R}^m)) \cap L^p([0, 1]; \mathbb{W}^{1, p}(\mathbb{R}^d; \mathbb{R}^m))$$

is called a strong solution of Eq. (6.12) if for all  $t \in [0, 1]$  and almost all  $x \in \mathbb{R}^d$ ,

$$u(t, x) = \varphi(x) + \int_0^t \mathcal{L}u(s, x) ds + \int_0^t b(s, x, u(s, x)) \cdot \nabla u(s, x) ds + \int_0^t f(s, x, u(s, x)) ds.$$

We firstly prove the following uniqueness of strong solutions to Eq. (6.12).

**Lemma 6.4.** Suppose that for any  $R > 0$ , there are two constants  $C_{f,R}, C_{b,R} > 0$  such that for all  $t \in [0, 1]$ ,  $x, y \in \mathbb{R}^d$  and  $u, u' \in \mathbb{R}^m$  with  $|u|, |u'| \leq R$ ,

$$\begin{aligned} |f(t, x, u) - f(t, x, u')| &\leq C_{f,R}|u - u'|, \\ |b(t, x, u) - b(t, y, u')| &\leq \omega_{b,R}(|x - y|) + C_{b,R}|u - u'|, \end{aligned}$$

where  $\omega_{b,R} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is an increasing function with  $\lim_{s \downarrow 0} \omega_{b,R}(s) = 0$ . Then there exists at most one strong solution in the sense of Definition 6.3 provided  $p > d + 1$ .



**Proof.** Let  $\varphi \in \mathbb{W}^{1-\frac{1}{p},p}(\mathbb{R}^d; \mathbb{R}^m)$  and

$$u, \tilde{u} \in C([0, 1]; \mathbb{W}^{1-\frac{1}{p},p}(\mathbb{R}^d; \mathbb{R}^m)) \cap L^p([0, 1]; \mathbb{W}^{1,p}(\mathbb{R}^d; \mathbb{R}^m))$$

be two strong solutions of Eq. (6.12) with the same initial value  $\varphi$ . Let

$$w(t, x) := u(t, x) - \tilde{u}(t, x).$$

Then for all  $t \in [0, 1]$  and almost all  $x \in \mathbb{R}^d$ ,

$$w(t, x) = \int_0^t \mathcal{L}w(s, x) \, ds + \int_0^t b(s, x, u(s, x)) \cdot \nabla w(s, x) \, ds + \int_0^t g(s, x) \, ds,$$

where

$$g(t, x) := (b(t, x, u(t, x)) - b(t, x, \tilde{u}(t, x))) \cdot \nabla \tilde{u}(t, x) + f(t, x, u(t, x)) - f(t, x, \tilde{u}(t, x)).$$

Since  $u, \tilde{u} \in C([0, 1]; \mathbb{W}^{1-\frac{1}{p},p}(\mathbb{R}^d; \mathbb{R}^m))$ , by Sobolev’s embedding (5.4), for some  $C > 0$ ,

$$\sup_{t \in [0,1]} \|u(t)\|_\infty \leq C \sup_{t \in [0,1]} \|u(t)\|_{1-\frac{1}{p},p}, \quad \sup_{t \in [0,1]} \|\tilde{u}(t)\|_\infty \leq C \sup_{t \in [0,1]} \|\tilde{u}(t)\|_{1-\frac{1}{p},p}.$$

Let

$$R := C \sup_{t \in [0,1]} \|u(t)\|_{1-\frac{1}{p},p} + C \sup_{t \in [0,1]} \|\tilde{u}(t)\|_{1-\frac{1}{p},p},$$

then by the assumptions, we have for all  $t \in [0, 1]$  and  $x, y \in \mathbb{R}^d$ ,

$$\begin{aligned} |b(t, x, u(t, x)) - b(t, y, u(t, y))| &\leq \omega_{b,R}(|x - y|) + C_{b,R} |u(t, x) - u(t, y)| \\ &\stackrel{(5.4)}{\leq} \omega_{b,R}(|x - y|) + C \sup_{t \in [0,1]} \|u(t)\|_{1-\frac{1}{p},p} |x - y|^{1-\frac{d+1}{p}}. \end{aligned}$$

Thus, by Lemma 5.1 and the assumptions, for all  $t \in [0, 1]$ , we have

$$\begin{aligned} \|w(t)\|_{1-\frac{1}{p},p}^p &\leq C \int_0^t \|g(s)\|_p^p \, ds \\ &\leq C \int_0^t (C_{b,R}^p \|\nabla \tilde{u}(s)\|_p^p \|w(s)\|_\infty^p + C_{f,R}^p \|w(s)\|_p^p) \, ds \\ &\leq C \int_0^t (\|\nabla \tilde{u}(s)\|_p^p + 1) \|w(s)\|_{1-\frac{1}{p},p}^p \, ds. \end{aligned} \tag{6.13}$$

The uniqueness follows by Gronwall’s inequality.  $\square$

We have the following existence and uniqueness of smooth solutions for Eq. (6.12).

**Theorem 6.5.** *Suppose that for all  $R > 0$  and  $j, k = 0, 1, 2, \dots$ , there exist  $C_{b,j,k,R}, C_{f,j,k,R} > 0$  such that for all  $(t, x) \in [0, 1] \times \mathbb{R}^d$  and  $u \in \mathbb{R}^m$  with  $|u| \leq R$ ,*

$$|\nabla_x^j \nabla_u^k b(t, x, u)| \leq C_{b,j,k,R}, \quad |\nabla_x^j \nabla_u^k f(t, x, u)| \leq C_{f,j,k,R}, \tag{6.14}$$

and there exist  $\gamma_j \in \mathbb{N}, C_{f,j} > 0$  and  $h_j \in (L^1 \cap L^\infty)(\mathbb{R}^d)$  such that for all  $(t, x, u) \in [0, 1] \times \mathbb{R}^d \times \mathbb{R}^m$ ,

$$|\nabla_x^j f(t, x, u)| \leq C_{f,j} |u|^{\gamma_j} + h_j(x), \tag{6.15}$$

where  $\gamma_0 = 1$ . Then for any  $\varphi \in \mathbb{W}^\infty(\mathbb{R}^d; \mathbb{R}^m)$ , there exists a unique solution

$$u \in C([0, 1]; \mathbb{W}^\infty(\mathbb{R}^d; \mathbb{R}^m))$$

to Eq. (6.12) with initial value  $\varphi$ . Moreover,

$$\sup_{t \in [0, 1]} \|u(t)\|_\infty \leq e^{C_{f,0}} (\|\varphi\|_\infty + \|h_0\|_\infty), \tag{6.16}$$

and for any  $p > d + 1$ ,

$$\sup_{t \in [0, 1]} \|u(t)\|_{1-\frac{1}{p}, p}^p + \int_0^1 \|\nabla u(t)\|_p^p dt \leq K_p, \tag{6.17}$$

where the constant  $K_p$  depends only on  $p, d, \nu$  and  $\|\varphi\|_{1-\frac{1}{p}, p}, C_{f,0}, \|h_0\|_\infty, \|h_0\|_p, C_{b,0,0,R}, C_{b,0,1,R}$  and the function

$$\omega_{b,R}(s) := \sup_{|x-y| \leq s} \sup_{t \in [0, 1]} \sup_{|u| \leq R} |b(t, x, u) - b(t, y, u)|, \quad s > 0. \tag{6.18}$$

**Proof.** We construct Picard’s approximation for Eq. (6.12) as follows. Set  $u_0(t, x) \equiv 0$ . Since for any  $u \in C([0, 1]; \mathbb{W}^\infty(\mathbb{R}^d; \mathbb{R}^m))$ , by (6.14), (6.15) and the chain rules,

$$\begin{aligned} (t, x) &\mapsto b(t, x, u(t, x)) \in L^\infty([0, 1]; C_b^\infty(\mathbb{R}^d; \mathbb{R}^m)), \\ (t, x) &\mapsto f(t, x, u(t, x)) \in L^\infty([0, 1]; \mathbb{W}^\infty(\mathbb{R}^d; \mathbb{R}^m)), \end{aligned}$$

by Theorem 5.2, for each  $n \in \mathbb{N}$ , there exists a unique  $u_n \in C([0, 1]; \mathbb{W}^\infty(\mathbb{R}^d; \mathbb{R}^m))$  solving the following linear equation:

$$\partial_t u_n = \mathcal{L}u_n + b(u_{n-1}) \cdot \nabla u_n + f(u_{n-1}), \quad u_n(0) = \varphi. \tag{6.19}$$

Set

$$\tilde{u}_n(t, x) := u_n(t, x) - \int_0^t \|f(s, \cdot, u_{n-1}(s, \cdot))\|_\infty ds,$$

then for each  $j = 1, 2, \dots, m$ ,

$$\partial_t \tilde{u}_n^j - \mathcal{L}\tilde{u}_n^j - b(u_{n-1}) \cdot \nabla \tilde{u}_n^j = f^j(u_{n-1}) - \|f_n(u_{n-1})\|_\infty \leq 0.$$

By Lemma 3.1 and (6.15), in view of  $\gamma_0 = 1$ , we have

$$\begin{aligned} \|u_n(t)\|_\infty &\leq \|\tilde{u}_n(t)\|_\infty + \int_0^t \|f(s, \cdot, u_{n-1}(s, \cdot))\|_\infty ds \\ &\leq \|\tilde{u}_n(0)\|_\infty + \int_0^t (C_{f,0} \|u_{n-1}(s)\|_\infty + \|h_0\|_\infty) ds \\ &\leq \|\varphi\|_\infty + \|h_0\|_\infty + C_{f,0} \int_0^t \|u_{n-1}(s)\|_\infty ds, \end{aligned}$$

which yields by Gronwall’s inequality that

$$\sup_{t \in [0, 1]} \|u_n(t)\|_\infty \leq e^{C_{f,0}} (\|\varphi\|_\infty + \|h_0\|_\infty) =: K_0. \tag{6.20}$$

We mention that this  $L^\infty$ -estimate can be also derived by representation formula (5.14).

Since

$$|b(t, x, u_{n-1}(t, x))| \leq C_{b,0,0,K_0} =: K_1,$$

by Corollary 6.2, there exist  $\beta \in (0, 1)$  and  $C$  depending only on  $d, \nu, p$  and  $K_1$  such that

$$\begin{aligned} \sup_{t \in [0,1]} \|u_n(t)\|_{\mathcal{H}^\beta} &\leq C(\|f(u_{n-1})\|_\infty + \|\varphi\|_\infty + \|\varphi\|_{\mathcal{H}^{1-\frac{d+1}{p}}}) \\ &\stackrel{(6.15), (6.20), (5.4)}{\leq} C(C_{f,0}K_0 + \|h_0\|_\infty + \|\varphi\|_\infty + \|\varphi\|_{1-\frac{1}{p},p}) =: K_2. \end{aligned} \tag{6.21}$$

Thus, letting  $\omega_{b,K_0}$  be defined by (6.18) with  $R = K_0$  and using (6.14), (6.20), we have

$$|b(t, x, u_{n-1}(t, x)) - b(t, y, u_{n-1}(t, y))| \leq \omega_{b,K_0}(|x - y|) + C_{b,0,1,K_0}K_2|x - y|^\beta. \tag{6.22}$$

Hence, we can use Lemma 5.1 to derive that for any  $p > 1$ ,

$$\begin{aligned} \|u_n(t)\|_{1-\frac{1}{p},p}^p + \int_0^t \|\nabla u_n(s)\|_p^p ds &\leq C\left(\|\varphi\|_{1-\frac{1}{p},p}^p + \int_0^t \|f(s, u_{n-1}(s))\|_p^p ds\right) \\ &\leq C_1\left(\|\varphi\|_{1-\frac{1}{p},p}^p + \int_0^t (C_{f,0}^p \|u_{n-1}(s)\|_p^p + \|h_0\|_p^p) ds\right), \end{aligned} \tag{6.23}$$

where  $C_1 \geq 1$  depends only on  $p, d, \nu, K_1, K_2, \omega_{b,K_0}$  and  $C_{b,0,1,K_0}$ . In particular, for any  $t \in [0, 1]$ ,

$$\|u_n(t)\|_p^p \leq C_1(\|\varphi\|_{1-\frac{1}{p},p}^p + \|h_0\|_p^p) + C_1 C_{f,0}^p \int_0^t \|u_{n-1}(s)\|_p^p ds,$$

and by Gronwall’s inequality,

$$\sup_{t \in [0,1]} \|u_n(t)\|_p^p \leq C_1(\|\varphi\|_{1-\frac{1}{p},p}^p + \|h_0\|_p^p)e^{C_1 C_{f,0}^p}.$$

Substituting this into (6.23), we obtain

$$\sup_{t \in [0,1]} \|u_n(t)\|_{1-\frac{1}{p},p}^p + \int_0^1 \|\nabla u_n(t)\|_p^p dt \leq C_1\left(\|\varphi\|_{1-\frac{1}{p},p}^p + \int_0^1 \|f(s, u_{n-1}(s))\|_p^p ds\right) \leq K_3, \tag{6.24}$$

where  $K_3$  depends only on  $p, C_1, \|\varphi\|_{1-\frac{1}{p},p}, C_{f,0}, \|h_0\|_p$ .

Let us now estimate the higher order derivatives of  $u_n$ . For given  $k \in \mathbb{N}$ , set

$$w_n^{(k)}(t, x) := \nabla^k u_n(t, x).$$

By Eq. (6.19) and the chain rules, one sees that

$$\partial_t w_n^{(k)} = \mathcal{L}w_n^{(k)} + b(u_{n-1}) \cdot \nabla w_n^{(k)} + g_n^{(k)},$$

where

$$g_n^{(k)}(t, x) := \nabla^k (f(t, \cdot, u_{n-1}(t, \cdot)))(x) + \sum_{j=1}^k \frac{k!}{(k-j)!j!} \nabla^j (b(t, \cdot, u_{n-1}(t, \cdot)))(x) \cdot \nabla^{k-j} \nabla u_n(t, x).$$

By (6.22) and Lemma 5.1, for any  $p > 1$ , we have

$$\sup_{t \in [0,1]} \|w_n^{(k)}(t)\|_{1-\frac{1}{p},p}^p + \int_0^1 \|\nabla w_n^{(k)}(s)\|_p^p ds \leq C\left(\|\nabla^k \varphi\|_{1-\frac{1}{p},p}^p + \int_0^1 \|g_n^{(k)}(s)\|_p^p ds\right).$$

Since  $g_n^{(k)}(s)$  contains at most  $k$ -order derivatives of  $u_n(s)$  and the powers of lower order derivatives of  $u_n(s)$ , by induction method, it is easy to see that for any  $k \in \mathbb{N}$  and  $p > 1$ ,

$$\sup_{t \in [0,1]} \|w_n^{(k)}(t)\|_{1-\frac{1}{p},p}^p + \int_0^1 \|\nabla w_n^{(k)}(s)\|_p^p ds \leq K_{p,k}, \tag{6.25}$$

where  $K_{p,k}$  is independent of  $n$ .

Define

$$w_{n,m}(t, x) := u_n(t, x) - u_m(t, x).$$

Then

$$\partial_t w_{n,m} = \mathcal{L}w_{n,m} + b(u_{n-1}) \cdot \nabla w_{n,m} + (G_{1,n,m} + G_{2,n,m})w_{n-1,m-1},$$

subject to  $w_{n,m}(0) = 0$ , where

$$G_{1,n,m}^{ki}(t, x) := \sum_j \int_0^1 \partial_{u^i} b^j(t, x, u_{n-1}(t, x) + r(u_{n-1} - u_{m-1})(t, x)) dr \cdot \partial_j u_m^k(t, x),$$

$$G_{2,n,m}^{ki}(t, x) := \int_0^1 \partial_{u^i} f^k(t, x, u_{n-1}(t, x) + r(u_{n-1} - u_{m-1})(t, x)) dr.$$

By (6.22) and Lemma 5.1 again, we have

$$\|w_{n,m}(t)\|_{1-\frac{1}{p},p}^p \leq C \int_0^t \|(G_{1,n,m}(s) + G_{2,n,m}(s))w_{n-1,m-1}(s)\|_p^p ds.$$

By (6.14) and as in estimating (6.13), we further have

$$\begin{aligned} \|w_{n,m}(t)\|_{1-\frac{1}{p},p}^p &\leq C \int_0^t (\|\nabla u_m(s)\|_p^p + 1) \|w_{n-1,m-1}(s)\|_{1-\frac{1}{p},p}^p ds \\ &\stackrel{(6.25)}{\leq} C(K_{p,1} + 1) \int_0^t \|w_{n-1,m-1}(s)\|_{1-\frac{1}{p},p}^p ds. \end{aligned}$$

Taking super-limit for both sides and by Fatou’s lemma, we obtain

$$\overline{\lim}_{n,m \rightarrow \infty} \sup_{s \in [0,t]} \|w_{n,m}(s)\|_{1-\frac{1}{p},p}^p \leq C(K_{p,1} + 1) \int_0^t \overline{\lim}_{n,m \rightarrow \infty} \sup_{s \in [0,r]} \|w_{n-1,m-1}(s)\|_{1-\frac{1}{p},p}^p dr.$$

Thus, by Gronwall’s inequality, we get

$$\overline{\lim}_{n,m \rightarrow \infty} \sup_{t \in [0,1]} \|w_{n,m}(t)\|_{1-\frac{1}{p},p}^p = 0,$$

which together with (6.25) and the interpolation inequality yields that for any  $k \in \mathbb{N}$ ,

$$\lim_{n,m \rightarrow \infty} \sup_{t \in [0,1]} \|u_n(t) - u_m(t)\|_{k,p}^p = 0.$$

Hence, there exists a  $u \in C([0, 1]; \mathbb{W}^\infty(\mathbb{R}^d; \mathbb{R}^m))$  such that for any  $k \in \mathbb{N}$ ,

$$\lim_{n \rightarrow \infty} \sup_{t \in [0,1]} \|u_n(t) - u(t)\|_{k,p}^p = 0.$$

The proof is finished by taking limits for Eq. (6.19).  $\square$

Next we show the well-posedness of Eq. (6.12) under less regularity conditions on  $b, f$ .

**Theorem 6.6.** Let  $p > d + 1$ . Suppose that there exist  $C_f > 0$  and  $h \in (L^p \cap L^\infty)(\mathbb{R}^d)$  such that for all  $(t, x, u) \in [0, 1] \times \mathbb{R}^d \times \mathbb{R}^m$ ,

$$|f(t, x, u)| \leq C_f |u| + h(x); \tag{6.26}$$

and for any  $R > 0$ , there are three constants  $C_{f,R}, C_{b,0,R}, C_{b,1,R} > 0$  such that for all  $t \in [0, 1]$ ,  $x, y \in \mathbb{R}^d$  and  $u, u' \in \mathbb{R}^m$  with  $|u|, |u'| \leq R$ ,

$$\begin{cases} |f(t, x, u) - f(t, x, u')| \leq C_{f,R} |u - u'|, & |b(t, x, u)| \leq C_{b,0,R}, \\ |b(t, x, u) - b(t, y, u')| \leq \omega_{b,R}(|x - y|) + C_{b,1,R} |u - u'|, \end{cases} \tag{6.27}$$

where  $\omega_{b,R} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is an increasing function with  $\lim_{s \downarrow 0} \omega_{b,R}(s) = 0$ . Then for any  $\varphi \in \mathbb{W}^{1-\frac{1}{p},p}(\mathbb{R}^d; \mathbb{R}^m)$ , there exists a unique strong solution  $u$  in the sense of Definition 6.3. Moreover,

$$\sup_{t \in [0,1]} \|u(t)\|_\infty \leq e^{C_f} (\|\varphi\|_\infty + \|h\|_\infty). \tag{6.28}$$

**Proof.** We divide the proof into three steps.

**Step 1.** Let  $\chi(x) \in [0, 1]$  be a nonnegative smooth function with  $\chi(x) = 1$  for  $|x| \leq 1$  and  $\chi(x) = 0$  for  $|x| > 2$ . Let  $(\rho_\varepsilon^x)_{\varepsilon \in (0,1)}$  and  $(\rho_\varepsilon^u)_{\varepsilon \in (0,1)}$  be the mollifiers in  $\mathbb{R}^d$  and  $\mathbb{R}^m$ . Define

$$b_\varepsilon(t, x, u) := b(t, \cdot, \cdot) * (\rho_\varepsilon^x \rho_\varepsilon^u)(x, u), \quad \varphi_\varepsilon(x) := \varphi * \rho_\varepsilon^x(x),$$

and

$$f_\varepsilon(t, x, u) := f(t, \cdot, \cdot) * (\rho_\varepsilon^x \rho_\varepsilon^u)(x, u) \chi(\varepsilon x).$$

By (6.26) and (6.27), one sees that (6.14) and (6.15) are satisfied for  $b_\varepsilon$  and  $f_\varepsilon$ , and

$$\begin{aligned} |f_\varepsilon(t, x, u)| &\leq (C_f(|u| + \varepsilon) + h * \rho_\varepsilon^x(x)) \chi(\varepsilon x) \\ &\leq C_f |u| + C_f \varepsilon \chi(\varepsilon x) + h * \rho_\varepsilon^x(x), \end{aligned} \tag{6.29}$$

and for any  $R > 0$  and all  $t \in [0, 1]$ ,  $x, y \in \mathbb{R}^d$  and  $u, u' \in \mathbb{R}^m$  with  $|u|, |u'| \leq R$ ,

$$\begin{cases} |f_\varepsilon(t, x, u) - f_\varepsilon(t, x, u')| \leq C_{f,R+1} |u - u'|, & |b_\varepsilon(t, x, u)| \leq C_{b,R+1}, \\ |b_\varepsilon(t, x, u) - b_\varepsilon(t, y, u')| \leq \omega_{b,R+1}(|x - y|) + C_{b,R+1} |u - u'|. \end{cases} \tag{6.30}$$

Moreover, by definition (5.3),

$$\|\varphi_\varepsilon\|_{1-\frac{1}{p},p} \leq \|\varphi\|_{1-\frac{1}{p},p}. \tag{6.31}$$

By Theorem 6.5, let  $u_\varepsilon \in C([0, 1]; \mathbb{W}^\infty(\mathbb{R}^d; \mathbb{R}^m))$  solve the following equation

$$\partial_t u_\varepsilon = \mathcal{L}u_\varepsilon + b_\varepsilon(u_\varepsilon) \cdot \nabla u_\varepsilon + f_\varepsilon(u_\varepsilon), \quad u_\varepsilon(0) = \varphi_\varepsilon. \tag{6.32}$$

By (6.16) and (6.29), we have

$$\sup_{t \in [0,1]} \|u_\varepsilon(t)\|_\infty \leq e^{C_f} (\|\varphi\|_\infty + C_f \varepsilon + \|h\|_\infty), \tag{6.33}$$

and by (6.29), (6.30), (6.31) and (6.17),

$$\sup_{\varepsilon \in (0,1)} \left( \sup_{t \in [0,1]} \|u_\varepsilon(t)\|_{1-\frac{1}{p},p}^p + \int_0^1 \|\nabla u_\varepsilon(t)\|_p^p dt \right) \leq K, \tag{6.34}$$

where we have particularly used that for  $p > d + 1$ ,

$$\|C_f \varepsilon \chi(\varepsilon \cdot) + h * \rho_\varepsilon^x\|_p \leq C_f \varepsilon^{1-d/p} \|\chi\|_p + \|h\|_p \leq C_f \|\chi\|_p + \|h\|_p.$$

**Step 2.** In this step we want to show that

$$\lim_{N \rightarrow \infty} \sup_{\varepsilon \in (0,1)} \sup_{t \in [0,1]} \int_{|x| \geq N} |u_\varepsilon(t, x)|^p dx = 0. \tag{6.35}$$

Let  $\zeta_N(x) := 1 - \chi(N^{-1}x)$ . Multiplying both sides of Eq. (6.32) by  $\zeta_N(x)$ , we have

$$\partial_t(u_\varepsilon \zeta_N) = \mathcal{L}(u_\varepsilon \zeta_N) + b_\varepsilon(u_\varepsilon) \cdot \nabla(u_\varepsilon \zeta_N) + g_{N,\varepsilon},$$

where

$$g_{N,\varepsilon} := \zeta_N \mathcal{L}u_\varepsilon - \mathcal{L}(u_\varepsilon \zeta_N) - u_\varepsilon b_\varepsilon(u_\varepsilon) \cdot \nabla \zeta_N + f_\varepsilon(u_\varepsilon) \zeta_N.$$

Let

$$R := e^{C_f} (\|\varphi\|_\infty + C_f + \|h\|_\infty).$$

Since

$$\begin{aligned} |b_\varepsilon(t, x, u_\varepsilon(t, x)) - b_\varepsilon(t, y, u_\varepsilon(t, y))| &\stackrel{(6.30)}{\leq} \omega_{b,R+1}(|x - y|) + C_{b,R+1} |u_\varepsilon(t, x) - u_\varepsilon(t, y)| \\ &\stackrel{(5.4)}{\leq} \omega_{b,R+1}(|x - y|) + C \sup_{t \in [0,1]} \|u_\varepsilon(t)\|_{1-\frac{1}{p},p} |x - y|^{1-\frac{d+1}{p}} \\ &\stackrel{(6.34)}{\leq} \omega_{b,R+1}(|x - y|) + CK^{\frac{1}{p}} |x - y|^{1-\frac{d+1}{p}}, \end{aligned} \tag{6.36}$$

here and below, the constant  $C$  is independent of  $N$  and  $\varepsilon$ , by Lemma 5.1, we have

$$\|u_\varepsilon(t)\zeta_N\|_{1-\frac{1}{p},p}^p \leq C \|\varphi_\varepsilon \zeta_N\|_{1-\frac{1}{p},p}^p + C \int_0^t \|g_{N,\varepsilon}(s)\|_p^p ds. \tag{6.37}$$

Clearly,

$$\|\varphi_\varepsilon \zeta_N\|_{1-\frac{1}{p},p}^p \leq C \|\varphi_\varepsilon \zeta_N\|_{1,p}^p \leq C \|\varphi \zeta_N\|_p^p + C \|\nabla \varphi \zeta_N\|_p^p + C \|\varphi \nabla \zeta_N\|_p^p \rightarrow 0, \quad N \rightarrow \infty.$$

By (2.16) and (6.29), we have

$$\begin{aligned} \|g_{N,\varepsilon}\|_p &\leq \|\zeta_N \mathcal{L}u_\varepsilon - \mathcal{L}(u_\varepsilon \zeta_N)\|_p + \|u_\varepsilon b_\varepsilon(u_\varepsilon) \cdot \nabla \zeta_N\|_p + \|f_\varepsilon(u_\varepsilon) \zeta_N\|_p \\ &\leq C \left( (\|\mathcal{L}\zeta_N\|_\infty + \|\zeta_N\|_\infty^{\frac{1}{2}} \|\nabla \zeta_N\|_\infty^{\frac{1}{2}}) \|u_\varepsilon\|_p + \|\nabla \zeta_N\|_\infty \|u_\varepsilon\|_p^{\frac{1}{2}} \|\nabla u_\varepsilon\|_p^{\frac{1}{2}} \right) \\ &\quad + \|u_\varepsilon\|_p \|b_\varepsilon(u_\varepsilon)\|_\infty \|\nabla \zeta_N\|_\infty + C_f \|u_\varepsilon \zeta_N\|_p + C_f \varepsilon \|\chi(\varepsilon \cdot) \zeta_N\|_p + \|(h * \rho_\varepsilon^x) \zeta_N\|_p. \end{aligned}$$

Noticing that

$$\varepsilon^p \|\chi(\varepsilon \cdot) \zeta_N\|_p^p = \varepsilon^{p-d} \int_{\mathbb{R}^d} |\chi(x)(1 - \chi(N^{-1}\varepsilon^{-1}x))|^p dx \leq \left(\frac{2}{N}\right)^{p-d} \int_{\mathbb{R}^d} |\chi(x)|^p dx$$

and

$$\|(h * \rho_\varepsilon^x) \zeta_N\|_p^p \leq \int_{B_{N-1}^c} |h(x)|^p dx,$$

by Lemma 2.4 and (6.34), we have

$$\int_0^t \|g_{N,\varepsilon}(s)\|_p^p ds \leq \frac{C}{N^{\frac{p}{2}}} + C \int_0^t \|u_\varepsilon(s)\zeta_N\|_p^p ds + \frac{C}{N^{p-d}} \int_{\mathbb{R}^d} |\chi(x)|^p dx + C \int_{B_{N-1}^c} |h(x)|^p dx.$$

Substituting this into (6.37) and using Gronwall’s inequality, we obtain

$$\lim_{N \rightarrow \infty} \sup_{\varepsilon \in (0,1)} \sup_{t \in [0,1]} \|u_\varepsilon(t)\zeta_N\|_p^p = 0.$$

This clearly implies (6.35).

**Step 3.** For fixed  $\varepsilon, \varepsilon' \in (0, 1)$ , let us define

$$w_{\varepsilon,\varepsilon'}(t, x) := u_\varepsilon(t, x) - u_{\varepsilon'}(t, x).$$

Then

$$\partial_t w_{\varepsilon,\varepsilon'} = \mathcal{L}w_{\varepsilon,\varepsilon'} + b_\varepsilon(u_\varepsilon) \cdot \nabla w_{\varepsilon,\varepsilon'} + (G_{1,\varepsilon,\varepsilon'} + G_{2,\varepsilon,\varepsilon'})w_{\varepsilon,\varepsilon'} + F_{1,\varepsilon,\varepsilon'} + F_{2,\varepsilon,\varepsilon'},$$

subject to  $w_{\varepsilon,\varepsilon'}(0) = \varphi_\varepsilon - \varphi_{\varepsilon'}$ , where

$$G_{1,\varepsilon,\varepsilon'}^{ki}(t, x) := \sum_j \int_0^1 \partial_{u^i} b_\varepsilon^j(t, x, u_\varepsilon(t, x) + r(u_\varepsilon - u_{\varepsilon'})(t, x)) \, dr \cdot \partial_j u_{\varepsilon'}^k(t, x),$$

$$G_{2,\varepsilon,\varepsilon'}^{ki}(t, x) := \int_0^1 \partial_{u^i} f_\varepsilon^k(t, x, u_\varepsilon(t, x) + r(u_\varepsilon - u_{\varepsilon'})(t, x)) \, dr,$$

$$F_{1,\varepsilon,\varepsilon'}(t, x) := (b_\varepsilon(t, x, u_{\varepsilon'}(t, x)) - b_{\varepsilon'}(t, x, u_{\varepsilon'}(t, x))) \cdot \nabla u_{\varepsilon'}(t, x),$$

$$F_{2,\varepsilon,\varepsilon'}(t, x) := f_\varepsilon(t, x, u_{\varepsilon'}(t, x)) - f_{\varepsilon'}(t, x, u_{\varepsilon'}(t, x)).$$

By (6.36) and Lemma 5.1 again, we have

$$\|w_{\varepsilon,\varepsilon'}(t)\|_{1-\frac{1}{p},p}^p + \int_0^t \|\nabla w_{\varepsilon,\varepsilon'}(s)\|_p^p \, ds \leq h_{\varepsilon,\varepsilon'} + C \int_0^t \|(G_{1,\varepsilon,\varepsilon'}(s) + G_{2,\varepsilon,\varepsilon'}(s))w_{\varepsilon,\varepsilon'}(s)\|_p^p \, ds,$$

where

$$h_{\varepsilon,\varepsilon'} := C \|w_{\varepsilon,\varepsilon'}(0)\|_{1-\frac{1}{p},p}^p + C \int_0^1 \|F_{1,\varepsilon,\varepsilon'}(s) + F_{2,\varepsilon,\varepsilon'}(s)\|_p^p \, ds.$$

By (6.30) and as in estimating (6.13), we further have

$$\|w_{\varepsilon,\varepsilon'}(t)\|_{1-\frac{1}{p},p}^p + \int_0^t \|\nabla w_{\varepsilon,\varepsilon'}(s)\|_p^p \, ds \leq h_{\varepsilon,\varepsilon'} + C \int_0^t (\|\nabla u_{\varepsilon'}(s)\|_p^p + 1) \|w_{\varepsilon,\varepsilon'}(s)\|_{1-\frac{1}{p},p}^p \, ds.$$

By Gronwall’s inequality and (6.34), one sees that

$$\sup_{s \in [0,1]} \|w_{\varepsilon,\varepsilon'}(s)\|_{1-\frac{1}{p},p}^p + \int_0^1 \|\nabla w_{\varepsilon,\varepsilon'}(s)\|_p^p \, ds \leq Ch_{\varepsilon,\varepsilon'}. \tag{6.38}$$

Now it is standard to show that

$$\lim_{\varepsilon,\varepsilon' \rightarrow 0} \|w_{\varepsilon,\varepsilon'}(0)\|_{1-\frac{1}{p},p}^p \leq C \lim_{\varepsilon,\varepsilon' \rightarrow 0} \|w_{\varepsilon,\varepsilon'}(0)\|_{1,p}^p = 0,$$

and by (6.27) and (6.34),

$$\lim_{\varepsilon,\varepsilon' \rightarrow 0} \int_0^1 \|F_{1,\varepsilon,\varepsilon'}(s)\|_p^p \, ds \leq K \lim_{\varepsilon,\varepsilon' \rightarrow 0} (\omega_{b,R+1}(\varepsilon) + C_{b,1,R+1}\varepsilon + \omega_{b,R+1}(\varepsilon') + C_{b,1,R+1}\varepsilon')^p = 0.$$

We now look at  $F_{2,\varepsilon,\varepsilon'}$ . For any  $N > 0$ , we write

$$\int_0^1 \int_{\mathbb{R}^d} |F_{1,\varepsilon,\varepsilon'}(s, x)|_p^p dx ds = \int_0^1 \int_{B_N^c} |F_{1,\varepsilon,\varepsilon'}(s, x)|_p^p dx ds + \int_0^1 \int_{B_N} |F_{1,\varepsilon,\varepsilon'}(s, x)|_p^p dx ds =: I_1 + I_2.$$

For  $I_1$ , by (6.29) we have

$$\begin{aligned} I_1 &\leq \int_0^1 \int_{B_N^c} (2C_f |u_{\varepsilon'}(s, x)| + \varepsilon \chi(\varepsilon x) + h * \rho_\varepsilon(x) + \varepsilon' \chi(\varepsilon' x) + h * \rho_{\varepsilon'}(x))^p dx ds \\ &\leq C \sup_{s \in [0,1]} \int_{B_N^c} |u_{\varepsilon'}(s, x)|^p dx + \frac{C}{N^{p-d}} \int_{\mathbb{R}^d} |\chi(x)|^p dx + C \int_{B_{N-1}^c} |h(x)|^p dx, \end{aligned}$$

which converges to zero uniformly in  $\varepsilon' \in (0, 1)$  by (6.35) as  $N \rightarrow \infty$ .

For  $I_2$  and for fixed  $N > 0$ , by the dominated convergence theorem, (6.30) and the approximation of the identity (cf. [36, p. 23, (16)]), we have

$$I_2 \leq \int_0^1 \int_{B_N} \sup_{u \in B_R} |f_\varepsilon(t, x, u) - f_{\varepsilon'}(t, x, u)|^p dx dt \rightarrow 0, \quad \varepsilon, \varepsilon' \rightarrow 0.$$

Combining the above calculations and letting  $\varepsilon, \varepsilon' \downarrow 0$  for (6.38), we obtain

$$\overline{\lim}_{\varepsilon, \varepsilon' \downarrow 0} \sup_{s \in [0,1]} \|w_{\varepsilon, \varepsilon'}(s)\|_{1-\frac{1}{p}, p}^p = 0, \quad \overline{\lim}_{\varepsilon, \varepsilon' \downarrow 0} \int_0^1 \|\nabla w_{\varepsilon, \varepsilon'}(s)\|_p^p ds = 0.$$

Hence, there exists a  $u \in C([0, 1]; \mathbb{W}^{1-\frac{1}{p}, p}(\mathbb{R}^d; \mathbb{R}^m)) \cap L^p([0, 1]; \mathbb{W}^{1, p}(\mathbb{R}^d; \mathbb{R}^m))$  such that

$$\lim_{\varepsilon \downarrow 0} \sup_{s \in [0,1]} \|u_\varepsilon(s) - u(s)\|_{1-\frac{1}{p}, p}^p = 0, \quad \lim_{\varepsilon \downarrow 0} \int_0^1 \|\nabla u_\varepsilon(s) - \nabla u(s)\|_p^p ds = 0.$$

Taking limits in  $L^p$ -space for Eq. (6.32), it is easy to see that  $u$  solves Eq. (6.12).  $\square$

**Remark 6.7.** In this remark, we explain how to use the above results to the critical Hamilton–Jacobi equation (cf. [12,31]). Let

$$H(t, x, u, q) : [0, 1] \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{M}_{m \times d} \rightarrow \mathbb{R}^m$$

be a measurable and smooth function in  $x, u, q$ , where  $\mathbb{M}_{m \times d}$  denotes the set of all real valued  $m \times d$ -matrices. Consider the following Hamilton–Jacobi equation

$$\partial_t u = \mathcal{L}u + H(t, x, u, \nabla u), \quad u(0) = \varphi. \tag{6.39}$$

Formally, taking the gradient we obtain

$$\partial_t \nabla u = \mathcal{L} \nabla u + \nabla_x H(t, x, u, \nabla u) + \nabla_u H(t, x, u, \nabla u) \cdot \nabla u + \nabla_q H(t, x, u, \nabla u) \cdot \nabla \nabla u.$$

If we let

$$w(t, x) := (u(t, x), \nabla u(t, x))^t,$$

then

$$\partial_t w = \mathcal{L}w + b(w) \cdot \nabla w + f(w), \quad w(0) = (\varphi, \nabla \varphi)^t,$$



where for  $w = (u, q)$ ,

$$b(t, x, w) := (0, \nabla_q H(t, x, u, q))$$

and

$$f(t, x, w) := (H(t, x, u, q), \nabla_x H(t, x, u, q) + \nabla_u H(t, x, u, q) \cdot q)^t.$$

Thus, we can use Theorems 6.5 and 6.6 to uniquely solve Eq. (6.39) under some assumptions on  $H$  and  $\varphi$ .

## Acknowledgements

The author would like to thank the referee for his/her very useful suggestions. This work is supported by NSFs of China (Nos. 10971076, 11271294) and Program for New Century Excellent Talents in University (NCET-10-0654).

## References

- [1] H. Amann, Linear and Quasilinear Parabolic Problems, vol. I. Abstract Linear Theory, Monogr. Math., vol. 89, Birkhäuser, Boston, MA, 1995.
- [2] D. Applebaum, Lévy Processes and Stochastic Calculus, Camb. Stud. Adv. Math., vol. 93, Cambridge University Press, 2004.
- [3] G. Barles, E. Chasseigne, A. Ciomaga, C. Imbert, Lipschitz regularity of solutions for mixed integro-differential equations, arXiv: 1107.3228v1.
- [4] G. Barles, E. Chasseigne, C. Imbert, Hölder continuity of solutions of second-order non-linear elliptic integro-differential equations, J. Eur. Math. Soc. 13 (2011) 1–26.
- [5] F.E. Benth, K.H. Karlsen, K. Reikvam, Optimal portfolio selection with consumption and nonlinear integro-differential equations with gradient constraint: a viscosity solution approach, Finance Stoch. 5 (3) (2001) 275–303.
- [6] J. Bergh, J. Löfström, Interpolation Spaces, Grundlehren Math. Wiss., vol. 223, Springer-Verlag, 1976.
- [7] P. Biler, T. Funaki, W.A. Woyczynski, Fractal Burgers equations, J. Differ. Equ. 148 (1998) 9–46.
- [8] L. Caffarelli, L. Silvestre, Regularity theory for fully nonlinear integro-differential equations, Commun. Pure Appl. Math. 62 (5) (2009) 597–638.
- [9] L. Caffarelli, A. Vasseur, Drift diffusion equations with fractional diffusion and the quasigeostrophic equation, Ann. Math. 171 (3) (2010) 1903–1930.
- [10] P. Constantin, Euler equations, Navier–Stokes equations and turbulence, in: Mathematical Foundation of Turbulent Flows, in: Lect. Notes Math., vol. 1871, 2006, pp. 1–43.
- [11] H. Dong, D. Kim, On  $L_p$ -estimates for a class of nonlocal elliptic equations, J. Funct. Anal. 262 (3) (2012) 1166–1199.
- [12] J. Droniou, C. Imbert, Fractal first order partial equations, Arch. Ration. Mech. Anal. 182 (2) (2006) 299–331.
- [13] W. Farkas, N. Jacob, R. Schilling, Function spaces related to continuous negative definite functions:  $\psi$ -Bessel potential spaces, Diss. Math. 393 (2001).
- [14] A. Friedman, Stochastic Differential Equations and Applications, vol. 1, Academic Press, New York, 1975.
- [15] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Appl. Math. Sci., vol. 44, Springer-Verlag, 1983.
- [16] E. Priola, Pathwise uniqueness for singular SDEs driven by stable processes, Osaka J. Math. 49 (2) (2012) 421–447.
- [17] N. Jacob, Pseudo Differential Operators, Markov Processes, vol. I. Fourier Analysis and Semigroups, Imperial College Press, World Scientific Publishing, Singapore, 2001.
- [18] N. Jacob, A. Potrykus, J.L. Wu, Solving a nonlinear pseudo-differential equation of Burgers type, Stoch. Dyn. 8 (4) (2008) 613–624.
- [19] O. Kallenberg, Foundations of Modern Probability, Springer-Verlag, New York, 1997.
- [20] A. Kiselev, F. Nazarov, R. Schterenberg, Blow up and regularity for fractal Burgers equation, Dyn. Partial Differ. Equ. 5 (3) (2008) 211–240.
- [21] A. Kiselev, F. Nazarov, A. Volberg, Global well-posedness for the critical 2D dissipative quasi-geostrophic equation, Invent. Math. 167 (2007) 445–453.
- [22] N.V. Krylov, A generalization of the Littlewood–Paley inequality and some other results related to stochastic partial differential equations, Ulam Quarterly 2 (4) (1994) 16–26.
- [23] N.V. Krylov, The heat equation in  $L_q((0, T), L_p)$ -spaces with weights, SIAM J. Math. Anal. 32 (5) (2001) 1117–1141.
- [24] N.V. Krylov, Lectures on Elliptic and Parabolic Equations in Sobolev Spaces, Grad. Stud. Math., vol. 96, AMS, 2008.
- [25] V.P. Kurenok, A note on  $L_2$ -estimates for stable integrals with drift, Trans. Am. Math. Soc. 360 (2008) 925–938.
- [26] O.A. Ladyzhenskaya, V.A. Solonnikov, N.N. Uraltceva, Linear and Quasi-Linear Parabolic Equations, Nauka, Moscow, 1967 (in Russian); English translation: AMS, 1968.
- [27] R. Mikulevicius, H. Pragarauskas, On the Cauchy problem for certain integro-differential operators in Sobolev and Hölder spaces, Lith. Math. J. 32 (2) (1992).
- [28] R. Mikulevicius, H. Pragarauskas, On  $L_p$  theory for Zakai equation with discontinuous observation process, arXiv:1012.5816v1.
- [29] K. Sato, Lévy Processes and Infinitely Divisible Distributions, Cambridge University Press, 1999.
- [30] L. Silvestre, Hölder estimates for solutions of integro-differential equations like the fractional Laplace, Indiana Univ. Math. J. 55 (3) (2006) 1155–1174.

- [31] L. Silvestre, On the differentiability of the solution to the Hamilton–Jacobi equation with critical fractional diffusion, *Adv. Math.* 226 (2) (2011) 2020–2039.
- [32] L. Silvestre, Hölder estimates for advection fractional-diffusion equations, arXiv:1009.5723v2, 2011, *Ann. Sc. Norm. Super. Pisa, Cl. Sci.* (2011).
- [33] L. Silvestre, On the differentiability of the solution to an equation with drift and fractional-diffusion, arXiv:1012.2401, 2011, *Indiana Univ. Math. J.* (2011).
- [34] H.M. Soner, Optimal control with state-space constraint II, *SIAM J. Control Optim.* 24 (6) (2006) 1110–1122.
- [35] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, NJ, 1970.
- [36] E.M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton University Press, 1993.
- [37] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland Publishing Company, Amsterdam, 1978.
- [38] W.A. Woyczyński, Lévy processes in the physical sciences, in: *Lévy Processes*, Birkhäuser, Boston, MA, 2001, pp. 241–266.
- [39] X. Zhang, Stochastic differential equations with Sobolev drifts and driven by  $\alpha$ -stable processes, *Ann. Inst. Henri Poincaré B, Probab. Stat.*, <http://dx.doi.org/10.1214/12-AIHP476>.
- [40] X. Zhang, Stochastic functional differential equations driven by Lévy processes and quasi-linear partial integro-differential equations, *Ann. Appl. Probab.*, <http://dx.doi.org/10.1214/12-AAP851>.