

# Energy and local energy bounds for the 1-d cubic NLS equation in $H^{-\frac{1}{4}}$

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## Abstract

We consider the cubic nonlinear Schrödinger equation (NLS) in one space dimension, either focusing or defocusing. We prove that the solutions satisfy a priori local in time  $H^s$  bounds in terms of the  $H^s$  size of the initial data for  $s \geq -\frac{1}{4}$ . This improves earlier results of Christ, Colliander and Tao [3] and of the authors (Koch and Tataru, 2007 [13]). The new ingredients are a localization in space and local energy decay, which we hope to be of independent interest.

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## 1. Introduction

We consider the cubic nonlinear Schrödinger equation (NLS)

$$iu_t - u_{xx} \pm u|u|^2 = 0, \quad u(0) = u_0, \quad (1)$$

in one space dimension, either focusing or defocusing. This problem is invariant with respect to the scaling

$$u(x, t) \rightarrow \lambda u(\lambda x, \lambda^2 t)$$

as is the Sobolev space  $\dot{H}^{-\frac{1}{2}}$ , which one may view as the critical Sobolev space. The NLS equation (1) is also invariant under the Galilean transformation

$$u(x, t) \rightarrow e^{icx - ic^2 t} u(x + 2ct, t)$$

which corresponds to a shift in the frequency space. However the space  $\dot{H}^{-\frac{1}{2}}$  is not Galilean invariant.

This problem is globally well-posed for initial data  $u_0 \in L^2$ , and the  $L^2$  norm of the solution is conserved along the flow. Furthermore, the solution has a Lipschitz dependence on the initial data, uniformly for time in a compact set and data in bounded sets in  $L^2$ . Precisely, if  $u$  and  $v$  are two solutions for (1) with initial data  $u_0$ , respectively  $v_0$ , then we have

$$\|u(t) - v(t)\|_{L^2} \lesssim \|u_0 - v_0\|_{L^2}, \quad |t| < 1, \quad \|u_0\|_{L^2}, \|v_0\|_{L^2} \leq 1.$$

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By scaling and reiteration this implies a global in time bound

$$\|u(t) - v(t)\|_{L^2} \lesssim e^{C|t|(\|u_0\|_{L^2} + \|v_0\|_{L^2})^4} \|u_0 - v_0\|_{L^2}. \tag{2}$$

A natural question to ask is whether local well-posedness also holds in negative Sobolev spaces between  $H^{-1/2}$  and  $L^2$ . As a consequence of the Galilean invariance, the map from initial data to the solution at time 1 cannot be uniformly continuous in the unit ball in  $H^s$  with  $s < 0$  (see [11,4]). However, it is not implausible that one may have well-posedness with only continuous dependence on the initial data. This problem seems to be closely related to that of relaxing the exponential bound in (2) to a polynomial bound. Choosing the focusing or the defocusing problem may also make a difference.

At this point we are unable to tackle the question of uniqueness or continuous dependence on the initial data in  $L^2$  in the  $H^s$  norm for  $s < 0$ . This remains a fundamental open problem, whose answer may depend also on the focusing or defocusing character of the equation.

The problem of obtaining a priori estimates in negative Sobolev spaces was previously considered by Christ, Colliander and Tao [3] ( $s \geq -1/12$ ) and by the authors [13] ( $s \geq -1/6$ ), see also the discussion by Oh and Sulem [14] on the NLS below  $L^2$ . One key idea was that one can bootstrap suitable Strichartz type norms of the solution but only on frequency dependent time-scales. Another idea was to use the  $I$ -method to construct better almost conserved  $H^s$  type norms for the problem.

In this article we introduce another ingredient into the mix, namely local energy bounds. By establishing separately that the solutions satisfy local energy bounds on the unit time scale we are able to weaken the interval summation losses and obtain a better result with more a priori bounds on the solutions.

As in our previous work [13], here we focus on the question of a priori bounds in negative Sobolev spaces. In the process, we also establish certain space–time bounds for the solution, as well as for the nonlinearity in the equation; these bounds insure that the equation is satisfied in the sense of distributions even for weak limits, and hence we also obtain existence of global weak solutions for initial data in  $H^s$  for  $-1/4 \leq s < 0$ . It is likely that  $-1/4$  is not optimal. Our main result is as follows:

**Theorem 1.** *There exists  $\varepsilon > 0$  such that the following is true. Let*

$$-\frac{1}{4} \leq s < 0, \quad \Lambda \geq 1$$

and assume that the initial data  $u_0 \in L^2$  satisfies

$$\|u_0\|_{H^s_\Lambda}^2 := \int (\Lambda^2 + \xi^2)^s |\hat{u}_0|^2 d\xi < \varepsilon^2.$$

Then the solution  $u$  to (1) satisfies

$$\sup_{0 \leq t \leq 1} \|u(t)\|_{H^s_\Lambda} \leq 2\|u_0\|_{H^s_\Lambda}. \tag{3}$$

The norms  $H^s_\Lambda$  are equivalent for all  $\Lambda$ . In  $H^s_\Lambda$  all frequencies below  $\Lambda$  carry the same weight. Beyond  $\Lambda$  we essentially obtain the standard  $H^s$  norm. The constant  $\Lambda$  describes the minimal localization scale in frequency. The reader will find a more detailed discussion of the role of  $\Lambda$  below.

As a byproduct of our analysis, in addition to the uniform bound (4), we also establish space–time bounds for the solution  $u$  as well as for the nonlinearity  $|u|^2u$  in the time interval  $[0, 1]$ , namely

$$\|X_{[0,1]}u\|_{X^s_\Lambda \cap X^s_{\Lambda,le}} \lesssim \|u_0\|_{H^s_\Lambda}, \quad \|X_{[0,1]}|u|^2u\|_{Y^s_\Lambda \cap Y^s_{\Lambda,le}} \lesssim \|u_0\|_{H^s_\Lambda} \tag{4}$$

where the spaces  $X^s_\Lambda$ ,  $X^s_{\Lambda,le}$ ,  $Y^s_\Lambda$  and  $Y^s_{\Lambda,le}$  are defined in the next section.

The above theorem captures most of the technical contents of our analysis. However, it is not scale invariant, so taking scaling into account we obtain further bounds. Indeed, rescaling

$$u_\mu(x, t) = \mu u(\mu x, \mu^2 t)$$

we have

$$\|u_\mu(0)\|_{H^s_{\mu\Lambda}} = \mu^{\frac{2s+1}{2}} \|u(0)\|_{H^s_\Lambda}.$$

Applying the above theorem to  $u_\mu$  for  $s = \frac{1}{4}$  we obtain the case  $s = \frac{1}{4}$  of the following

**Corollary 1.1.** *Let  $-\frac{1}{4} \leq s \leq 0$ . Suppose that  $M > 0$  and  $\Lambda > 0$  satisfy  $\Lambda \gg M^4$ . Let  $u$  be a solution to (1) with initial data  $u_0 \in L^2$  so that*

$$\|u_0\|_{H^{-\frac{1}{4}}_\Lambda} \leq M.$$

Then  $u$  satisfies

$$\sup_{|t| \leq T} \|u(t)\|_{H^s_\Lambda} \lesssim \|u_0\|_{H^s_\Lambda}, \quad T \ll M^{-8}. \tag{5}$$

The general case follows from the  $s = \frac{1}{4}$  case due to the following equivalence:

$$\|v\|_{H^s_\Lambda}^2 \approx \sum_{\lambda \geq \Lambda} \lambda^{\frac{1}{2}+2s} \|v\|_{H^{-\frac{1}{4}}_\lambda}^2. \tag{6}$$

Here and below all the  $\lambda$  summations are dyadic.

Applying the above corollary to a given solution for increasing values of  $\Lambda$  yields global in time bounds. Consider first the case when  $1/4 < s < 0$ . Given  $M \geq 1$  and an initial data  $u_0$  so that  $\|u_0\|_{H^s} \leq M$  we have

$$\|u_0\|_{H^{-\frac{1}{4}}_\Lambda} \lesssim \Lambda^{-s-\frac{1}{4}} M, \quad \Lambda \geq 1.$$

By the above corollary this yields

$$\sup_{0 \leq t \leq T} \|u(t)\|_{H^s_\Lambda} \lesssim \|u_0\|_{H^s_\Lambda}, \quad \Lambda \gg \max\{T^{\frac{1}{8s+2}} M^{\frac{4}{4s+1}}, M^{\frac{2}{2s+1}}\}.$$

Hence we have proved

**Corollary 1.2.** *Let  $-\frac{1}{4} < s < 0$  and  $M \geq 1$ . Let  $u$  be a solution to (1) with initial data  $u_0 \in L^2$  so that*

$$\|u_0\|_{H^s} \leq M.$$

Then for all  $T > 0$  the function  $u$  satisfies

$$\sup_{|t| \leq T} \|u(t)\|_{H^s_{\Lambda(T)}} \lesssim M, \quad \Lambda(T) = \max\{T^{\frac{1}{8s+2}} M^{\frac{4}{4s+1}}, M^{\frac{2}{2s+1}}\}. \tag{7}$$

Here it is only the principle that matters. The exact exponents here are less important since it is very unlikely that the  $s = -\frac{1}{4}$  result is sharp.

The case  $s = -\frac{1}{4}$  is more delicate. There all we can say is that

$$\lim_{\Lambda \rightarrow \infty} \|u_0\|_{H^{-\frac{1}{4}}_\Lambda} = 0 \quad \text{for } u_0 \in H^{-\frac{1}{4}}. \tag{8}$$

Thus we obtain

**Corollary 1.3.** *Let  $u$  be a solution to (1) with initial data  $u_0 \in L^2$ . Then for all  $T > 0$  the function  $u$  satisfies*

$$\sup_{|t| \leq T} \|u(t)\|_{H^{-\frac{1}{4}}_{\Lambda(T)}} \leq 1 \tag{9}$$

for some increasing function  $\Lambda(T)$  which depends only on the function

$$\Lambda \rightarrow \|u_0\|_{H^\Lambda}^{-\frac{1}{4}}$$

in (8).

The a priori estimates suffice to construct global weak solutions. Using the uniform bounds (4) one may prove the following statement.

**Theorem 2.** *Suppose that  $u_0 \in H^s$ ,  $s \geq -\frac{1}{4}$ . Then there exists a weak solution  $u \in C(\mathbb{R}, H^s)$ , so that for all  $T > 0$  we have<sup>1</sup>  $\chi_T u \in X^s \cap X_{le}^s$  and*

$$\sup_{-T \leq t \leq T} \|u(t)\|_{H^s} + \|\chi_{[-T, T]} u\|_{X^s \cap X_{le}^s} + \|\chi_{[-T, T]} |u|^2 u\|_{Y^s \cap Y_{le}^s} \leq C \tag{10}$$

with  $C$  depending on  $T$  and on the  $H^{-\frac{1}{4}}$  frequency envelope of  $u_0$ .

1.1. Some heuristic considerations

The nonlinear Schrödinger equation is completely integrable. Depending on whether we look at the focusing or the defocusing problem, we expect two possible types of behavior for frequency localized data.

In the defocusing case, we expect the solutions to disperse spatially. However, in frequency there should only be a limited spreading, to a range below the dyadic scale, which depends only on the  $L^2$  size of the data. Precisely energy estimates show that for frequency localized data with  $L^2$  norm  $\lambda$ , frequency spreading occurs at most up to scale  $\lambda$ .

In the focusing case, the expected long time behavior (or short time for large data) is a resolution into a number of solitons (possibly infinitely many) plus a dispersive part. The situation is somewhat complicated by the fact that some of these solitons may have the same speed, and thus considerable overlapping. The inverse scattering formalism provides formulas for such solutions with many interacting solitons. Nevertheless it is instructive to consider first the case of a single soliton, which in the simplest case has the form

$$u(x, t) = e^{-it} \operatorname{sech}(2^{-1/2}x).$$

Rescaling we get a soliton with  $L^2$  norm  $\lambda$ , namely

$$u^\lambda(x, t) = e^{-it\lambda^4} \lambda^2 \operatorname{sech}(2^{-1/2}\lambda^2 x).$$

More soliton solutions can be obtained due to the Galilean invariance. However, our function spaces here break the Galilean invariance, so our worst enemies are the zero speed solitons.

The above solution is constant in time, up to a phase factor. It is essentially localized to an interval of size  $\lambda^{-2}$  in  $x$ , and of size  $\lambda^2$  in frequency. It also saturates our local energy estimates in (4) for  $s = -\frac{1}{4}$ , exactly when  $\Lambda = \lambda^4$ .

In many cases error estimates for a nonlinear semiclassical ansatz for solutions are available. An example is the initial data

$$u_0(x) = \lambda \operatorname{sech}(2^{-1/2}x)$$

where a semiclassical ansatz for an approximate solution is given by

$$u(t, x) = \lambda A(x, t) e^{-i\lambda S(t, x)}$$

where  $\rho = A^2$  and  $\mu = A^2 \partial_x S$  satisfy the Whitham equations

$$\rho_t + \lambda \partial_x \mu = 0, \quad \partial_t \mu + \lambda \partial_x (\mu^2 / \rho \pm \rho^2 / 2) = 0$$

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<sup>1</sup> Here we drop the subscript  $\Lambda = 1$  from the notation for the space–time norms.

with + in defocusing and – in the focusing case. The Whitham equations are hyperbolic for the defocusing case and they can be solved up to a time  $T \sim \lambda^{-1}$ , when singularities corresponding to caustics occur. Grenier [6] has justified this ansatz up to the time when caustics occur.

The Whitham equations are elliptic in the focusing case. Akhmanov, Khokhlov and Sukhorukov [1] realized that the implicit equation

$$\mu = -2\lambda t \rho^2 \tanh\left(\frac{\rho x - \mu \lambda t}{\rho}\right), \quad \rho = (1 + \lambda^2 t^2 \rho^2) \operatorname{sech}^2(\rho x - \mu t) \tag{11}$$

defines a solution to the Whitham equation with the  $\lambda$  sech initial data. The semiclassical ansatz for small semiclassical times has been studied by Thomann [17].

The direct scattering problem has been solved by Satsuma and Yajima [15]. In particular, if  $\lambda$  is an integer one obtains a pure soliton solution with  $\lambda$  solitons with velocity 0. In this case the solution is periodic with period 2. Formula (11) seems to indicate that the solution remains concentrated in a spatial area for size  $\sim \ln(1 + \lambda)$ . The semiclassical limit has been worked in a number of problems, see Jin, Levermore and McLaughlin [8], Kamvissis [9], Deift and Zhou [5], and Kamvissis, McLaughlin and Miller [10].

These examples indicate that energy may spread over a large frequency interval even if the energy is concentrated at frequencies  $\lesssim 1$  initially, and there are solutions with energy distributed over a large frequency interval with velocity zero. For the proof of our main result we use localization in frequency and space. These examples provide natural limits for the localization. This is reflected in the estimates and the definition of the function spaces.

### 1.2. An overview of the proof

We begin with a dyadic Littlewood–Paley frequency decomposition of the solution  $u$ ,

$$u = \sum_{\lambda \geq \Lambda} u_\lambda, \quad u_\lambda = P_\lambda u$$

where  $\lambda$  takes dyadic values not smaller than  $\Lambda$ , and  $u_\lambda$  contains all frequencies up to size  $\lambda$ . Here the multipliers  $P_\lambda$  are standard Littlewood–Paley projectors. For each such  $\lambda$  we also use a spatial partition of unity on the  $\lambda^{1+4s}$  scale,

$$1 = \sum_{j \in \mathbb{Z}} \chi_j^\lambda(x), \quad \chi_j^\lambda(x) = \chi(\lambda^{-1-4s}x - j) \tag{12}$$

with  $\chi \in C_0^\infty(-1, 1)$ . To prove the theorem we will use:

- (i) Two energy spaces, namely a standard energy norm

$$\|u\|_{l^2 L^\infty H_\Lambda^s}^2 = \sum_{\lambda \geq \Lambda} \lambda^{2s} \|u_\lambda\|_{L_t^\infty L_x^2}^2 \tag{13}$$

and a local energy norm<sup>2</sup> adapted to the  $\lambda^{1+4s}$  spatial scale,

$$\|u\|_{l^2 l^\infty L^2 H_\Lambda^{-s}}^2 = \sum_{\lambda \geq \Lambda} \lambda^{-2s-2} \sup_{j \in \mathbb{Z}} \|\chi_j^\lambda \partial_x u_\lambda\|_{L_t^2 L_x^2}^2. \tag{14}$$

Here  $l^2$  refers to the dyadic summation with respect to  $\lambda$ ,  $l^\infty$  to the sup with respect to  $j$ , the  $L^2$  is with respect to time and  $H_\Lambda^{-s}$  refers to  $x$ . We suppress the coordinates  $t$  and  $x$  in the notation of the spaces often in the sequel.

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<sup>2</sup> For  $s = -\frac{1}{4}$  the spatial scale is one and this corresponds to the familiar gain of one half of a derivative. It may seem more natural to remove the  $\partial_x$  derivative and appropriately adjust the power of  $\lambda$ . This would be equivalent for all frequencies  $\lambda > \Lambda$ . However, in  $u_\lambda$  we are including all lower frequencies, which correspond to waves with lower group velocities and to a worse local energy bound, should the operator  $\partial_x$  not be present here. Based on the standard form of the local energy bounds for the linear Schrödinger equation one may still expect to be able to relax the  $\partial_x$  operator almost to  $\partial_x^{\frac{1}{2}}$ . At least in the focusing case this is not possible; indeed, if  $s = -\frac{1}{4}$  then the local energy component of the bound (19) below is saturated by the frequency  $\Lambda^{\frac{1}{2}}$  soliton.

- (ii) Two Banach spaces  $X_\Lambda^s$  and  $X_{\Lambda,le}^s$  measuring the space–time regularity of the solution  $u$ . The first one measures the dyadic parts of  $u$  on small frequency dependent timescales, and is mostly similar to the spaces introduced in [3,13]. The second one is new, and measures the spatially localized size of the solution on the unit time scale. These spaces are defined in the next section. The index  $le$  refers to ‘local energy’.
- (iii) Two corresponding Banach spaces  $Y_\Lambda^s$  and  $Y_{\Lambda,le}^s$  measuring the regularity of the nonlinear term  $|u|^2u$ . These are also defined in the next section.

The linear part of the argument is a straightforward consequence of our definition of the spaces, and is given by

**Proposition 1.4.** *The following estimates hold for solutions to (1):*

$$\|u\|_{X_\Lambda^s} \lesssim \|u\|_{l^2L^\infty H_\Lambda^s} + \|(i\partial_t - \Delta)u\|_{Y_\Lambda^s} \tag{15}$$

respectively

$$\|u\|_{X_{\Lambda,le}^s} \lesssim \|u\|_{l^2l^\infty L^2 H_\Lambda^{-s}} + \|(i\partial_t - \Delta)u\|_{Y_{\Lambda,le}^s}. \tag{16}$$

To estimate the nonlinearity we need a cubic bound,

**Proposition 1.5.** *Let  $u \in X_\Lambda^s \cap X_{\Lambda,le}^s$ . Then  $|u|^2u \in Y_\Lambda^s \cap Y_{\Lambda,le}^s$  and*

$$\||u|^2u\|_{Y_\Lambda^s \cap Y_{\Lambda,le}^s} \lesssim \|u\|_{X_\Lambda^s \cap X_{\Lambda,le}^s}^3. \tag{17}$$

Finally, to close the argument we need to propagate the energy norms:

**Proposition 1.6.** *Let  $u$  be a solution to (1) with*

$$\|u\|_{l^2L^\infty H_\Lambda^s} \ll 1.$$

*Then we have the energy bound*

$$\|u\|_{l^2L^\infty H_\Lambda^s} \lesssim \|u_0\|_{H_\Lambda^s} + \|u\|_{X_\Lambda^s \cap X_{\Lambda,le}^s}^3, \tag{18}$$

*respectively the local energy decay*

$$\|u\|_{l^2l^\infty L^2 H_\Lambda^{-s}} \lesssim \|u_0\|_{H_\Lambda^s} + \|u\|_{X_\Lambda^s \cap X_{\Lambda,le}^s}^3. \tag{19}$$

The bootstrap argument which leads from Propositions 1.4, 1.5 and 1.6 to Theorem 1 is straightforward and thus omitted. Instead we refer the reader to the similar argument in [13].

We remark that in our set-up  $s = -\frac{1}{4}$  is the actual threshold in the energy estimates in Proposition 1.6, though not in the bounds for the cubic nonlinearity in Proposition 1.5. In principle the former can be improved by adding further corrections to the energy functional; we choose not to pursue this here.

The plan of the paper is as follows. In the next section we motivate and introduce the spaces  $X_\Lambda^s$ ,  $X_{\Lambda,le}^s$ ,  $Y_\Lambda^s$  and  $Y_{\Lambda,le}^s$ , as well as establish the linear mapping properties in Proposition 1.4. In Section 4 we discuss the linear and bilinear Strichartz estimates for solutions to the linear equation.

The trilinear estimate in Proposition 1.5 is proved in Section 5. In the last section we use a variation of the I-method to construct a quasi-conserved energy functional and compute its behavior along the flow, thus proving the first bound (18) in Proposition 1.6. A modification of the same idea leads to the local energy decay estimate (19).

## 2. The function spaces

To understand what to expect in terms of the regularity of  $u$  we begin with some heuristic considerations. If the initial data  $u_0$  to (1) satisfies  $\|u_0\|_{L^2} \ll 1$  then the equation can be solved iteratively using the Strichartz estimates on a unit time interval. We obtain essentially linear dynamics, by which we mean that the difference between the solution

to the linear Schrödinger equation and NLS is small, and the solution  $u$  belongs to the space  $X^{0,1}[0, 1]$  associated to the Schrödinger equation (see the definition in (20) below).

Let  $s < 0$ . Consider now NLS with initial data  $u_0 \in H^s$ , localized at frequency  $\lambda$ . Then the initial data satisfies  $\|u_0\|_{L^2} \lesssim \lambda^{-s}$ . By rescaling the small  $L^2$  data result we conclude that the evolution is still described by linear dynamics up to the shorter time  $\lambda^{4s}$ .

We expect the frequency localization of the solution to be somewhat robust. Then it is natural to consider a decomposition of the solution  $u$  into its dyadic components  $u_\lambda = P_\lambda u$  and to measure the  $u_\lambda$  component uniformly in  $\lambda^{4s}$  time intervals.

Linear waves with frequency  $\lambda$  travel with group velocity  $2\lambda$ , therefore they cover a distance of about  $\lambda^{1+4s}$  within a  $\lambda^{4s}$  time interval. Hence we can naturally partition frequency  $\lambda$  waves with respect to a grid of size

$$\delta t_\lambda = \lambda^{4s}, \quad \delta x_\lambda = \lambda^{1+4s}.$$

Correspondingly we have the spatial partition of unity (12). We remark that the spatial scale of this partition increases with  $\lambda$  for  $s > -1/4$ , and decreases with  $s$ . It is independent of  $\lambda$  exactly for  $s = -1/4$ , which makes the threshold  $s = -1/4$  very convenient technically.

Now we introduce the function spaces for the solutions  $u$ . Following an idea of M. Christ [2], given an interval  $I = [t_0, t_1]$  we define the space

$$\|\phi\|_{X^{0,1}[I]}^2 = \|\phi(t_0)\|_{L^2}^2 + |I| \|(i\partial_t - \Delta)\phi\|_{L^2[I]}^2 \tag{20}$$

where  $L^2[I]$  denotes  $L^2(I \times \mathbb{R})$ . We suppress the dependence in the  $x$  variable often in the sequel. Ideally we would like to place the dyadic pieces  $u_\lambda$  of  $u$  in such a space on the  $\delta t_\lambda$  scale. However, this does not quite work and we have to introduce a slightly larger space

$$X_\lambda[I] = X^{0,1}[I] + \lambda^{-1-2s} U_\Delta^2[I].$$

The  $U^p$  and  $V^p$  spaces are a refinement of the Fourier restriction spaces of Bourgain. We refer to the next section and to [12,7] for a discussion of them. They are related to the Bourgain spaces through the embeddings

$$X^{0, \frac{1}{2}, 1} \subset U_\Delta^2 \subset X^{0, \frac{1}{2}, \infty} \tag{21}$$

where the above  $X^{s,b}$  type norms are defined by

$$\|u\|_{X^{0, \frac{1}{2}, 1}} = \sum_\mu \mu^{\frac{1}{2}} \|Q_\mu u\|_{L^2}, \quad \|u\|_{X^{0, \frac{1}{2}, \infty}} = \sup_\mu \mu^{\frac{1}{2}} \|Q_\mu u\|_{L^2} \tag{22}$$

and the modulation localization multipliers  $Q_\mu$  select the dyadic region  $\{|\tau + \xi^2| \sim \mu\}$ .

Let us compare the two parts of the  $X_\lambda$  norms. First, Hölder’s inequality implies

$$\|(i\partial_t - \Delta)\phi\|_{L^1(I, L^2)} \leq |I|^{\frac{1}{2}} \|(i\partial_t - \Delta)\phi\|_{L^2[I]}$$

and hence (again referring to the next section for a discussion of the  $U^p$  spaces)

$$\|u\|_{U_\Delta^1(I)} \leq \|u\|_{X^{0,1}(I)}$$

and by the embedding properties of the  $U^p$  spaces

$$\|u_\lambda\|_{U_\Delta^2(I)} \leq \|u_\lambda\|_{X^{0,1}(I)}.$$

Thus we obtain

$$\|u_\lambda\|_{U_\Delta^2(I)} \lesssim \|u_\lambda\|_{X_\lambda[I]}. \tag{23}$$

This bound will suffice for most of our estimates.

On the other hand, the structure of the  $X_\lambda$  norms is so that we expect to have better bounds at high modulations ( $\gtrsim \lambda^2$ , e.g.). However, some care is required in order to make this precise, because modulation localizations do not commute with interval localizations. To address this issue we introduce extension operators  $E_I$  which take a function

$u \in X_\lambda[I]$  to its extension  $E_I u$  solving the homogeneous Schrödinger equation outside  $I$  with matching data at the two endpoints of  $I$ . By definition we have

$$\|(i\partial_t - \Delta)E_I u_\lambda\|_{\lambda^{-2s}L^2 + \lambda^{-1-2s}DU_\Delta^2} \lesssim \|u_\lambda\|_{X_\lambda[I]}, \quad |I| = \lambda^{4s}.$$

This implies the high modulation bound

$$\|Q_{\geq \sigma} E_I u_\lambda\|_{L^2} \lesssim \min\{\lambda^{-2s}\sigma^{-1}, \lambda^{-1-2s}\sigma^{-\frac{1}{2}}\} \|u_\lambda\|_{X_\lambda[I]}. \tag{24}$$

We remark that the balance between the norms of the two component spaces in  $X_\lambda$  is achieved at modulation  $\lambda^2$ . Since the  $X_\lambda$  norm is only used on frequency  $\lambda$  functions, it follows that the  $U_\Delta^2$  component of the space  $X_\lambda$  is only relevant in the elliptic region  $\{|\tau - \xi^2| \sim |\tau| + |\xi|^2\}$ .

Now we can define the  $X_\lambda^s$  norm in a time interval  $I$  by

$$\|u\|_{X_\lambda^s[I]}^2 = \sum_{\lambda \geq \Lambda} \lambda^{2s} \sup_{|J|=\lambda^{4s}, J \subset I} \|u_\lambda\|_{X_\lambda[J]}^2. \tag{25}$$

In the sequel we will mostly drop the interval  $I = [0, 1]$  from the notation. We remark that within each interval  $J$  we have square summability on the  $\lambda^{1+4s}$  spatial scale as well as on any larger scale,

$$\sum_{j \in \mathbb{Z}} \|\chi_j^\mu u_\lambda\|_{X_\lambda[J]}^2 \lesssim \|u_\lambda\|_{X_\lambda[J]}^2, \quad |J| = \lambda^{4s}, \quad \mu \gtrsim \lambda. \tag{26}$$

This can be viewed as a consequence of the fact that frequency  $\lambda$  waves travel with speed  $\lambda$ . We refer to Lemma 3.1 in the next section for more details.

Next we introduce the related local smoothing space  $X_{\lambda,le}^s$ , where the above summation with respect to spatial intervals is replaced by a summation with respect to time intervals:

$$\|u\|_{X_{\lambda,le}^s[I]}^2 = \sum_{\lambda} \lambda^{2s-2} \sup_{j \in \mathbb{Z}} \sum_{|J|=\lambda^{4s}, J \subset I} \|\chi_j^\lambda \partial_x u_\lambda\|_{X_\lambda[J]}^2. \tag{27}$$

Here and below the  $J$  summation is understood to be over a partition of  $I$  into intervals  $J$  of the indicated size.

To measure the regularity of the nonlinear term we begin with

$$Y_\lambda[I] = |I|^{-\frac{1}{2}}L^2 + \lambda^{-1-2s}DU_\Delta^2[I]$$

which is exactly the output of the linear Schrödinger operator  $i\partial_t - \Delta$  applied to  $X_\lambda[I]$  functions (see next section for a discussion of  $DU^2$ ). We use it to define the  $Y_\lambda^s$  norm

$$\|f\|_{Y_\lambda^s[I]}^2 = \sum_{\lambda \geq \Lambda} \lambda^{2s} \sup_{|J|=\lambda^{4s}, J \subset I} \|\chi_J f_\lambda\|_{Y_\lambda}^2 \tag{28}$$

as well as its local energy counterpart

$$\|f\|_{Y_{\lambda,le}^s[I]}^2 = \sum_{\lambda \geq \Lambda} \lambda^{2s-2} \sup_{j \in \mathbb{Z}} \sum_{|J|=\lambda^{4s}, J \subset I} \|\chi_J \chi_j^\lambda \partial_x f_\lambda\|_{Y_\lambda}^2. \tag{29}$$

### 3. $U^p$ and $V^p$ spaces

We sketch the construction of the spaces  $U^p$  and  $V^p$  and their properties and refer to [12,7] for more details.

Both  $U^p$  and  $V^p$  are spaces of functions in  $\mathbb{R}$  which take values in a Hilbert space,  $L^2(\mathbb{R})$  in our case. To define them we first introduce the class  $\mathcal{P}$  of finite partitions of  $\mathbb{R}$  into intervals. A partition  $\sigma \in \mathcal{P}$  is determined by the endpoints of the intervals, which are identified with a finite increasing sequence  $(t_n)_{n=0, N(\sigma)}$  with  $t_0 = -\infty$  and  $t_N(\sigma) = \infty$ .



Let  $1 \leq p < \infty$ . A  $U^p$  atom is a right continuous piecewise constant function

$$a = \sum_{n=2}^{N(\sigma)} 1_{[t_{n-1}, t_n)} a_n, \quad \sum \|a_n\|^p = 1,$$

on the real line associated to a partition  $\sigma = (t_n) \in \mathcal{P}$  of the real line. We define  $U^p$  as the atomic space consisting of all functions for which the following norm is finite:

$$\|u\|_{U^p} = \inf \left\{ \sum_{k=1}^{\infty} \lambda_k : f = \sum \lambda_k a_k, a_k \text{ atoms, } \lambda_k \geq 0 \right\}.$$

This is a Banach space of bounded right continuous functions which have limit zero as  $t$  goes to  $-\infty$ .

The space of bounded  $p$  variation functions  $V^p$  consists of all functions on  $\mathbb{R}$  for which the following norm is finite,

$$\|u\|_{V^p}^p = \sup_{\sigma \in \mathcal{P}} \sum_{n=2}^{N-1} \|u(t_n) - u(t_{n-1})\|^p.$$

In this formula we set  $u(\infty) = 0$ . This is a Banach space of bounded functions. The functions in  $V^p$  have lateral limits everywhere. By  $V_{rc}^p$  we denote the subspace of right continuous functions in  $V^p$  which have limit zero as  $t$  goes to  $\infty$ .

Both spaces are invariant under monotone reparametrizations of  $\mathbb{R}$  and can therefore be easily defined for intervals. Given any partition  $\sigma = (t_n) \in \mathcal{P}$  of the real line we also have the interval summability bounds

$$\|u\|_{U^p}^p \lesssim \sum_{n=1}^{N(\sigma)} \|1_{[t_{n-1}, t_n)} u\|_{U^p}^p. \tag{30}$$

Clearly, if  $1 \leq p < q$  then

$$U^p \subset U^q, \quad V^p \subset V^q, \quad U^p \subset V^p.$$

We also have the nontrivial relation

$$V_{rc}^p \subset U^q, \quad 1 \leq p < q.$$

More precisely, as proved in [7], there exists  $\delta > 0$  such that for each  $v \in V_{rc}^p$  and  $M > 1$  there exists  $u \in U^p, w \in U^q$  such that  $v = u + w$  and

$$M^{-1} \|u\|_{U^p} + e^{\delta M} \|w\|_{U^q} \lesssim \|v\|_{V_{rc}^p}. \tag{31}$$

The relation of  $U^p$  and  $V^p$  to Besov spaces is as follows:

$$\dot{B}_1^{1/p, p} \subset U^p \subset V_{rc}^p \subset \dot{B}_\infty^{1/p, p}. \tag{32}$$

In particular the norms of  $u$  in  $U^p$  and  $V_{rc}^p$  are equivalent if  $\hat{u}$  is supported in a fixed dyadic frequency interval. Moreover if  $Q_\mu$  denotes the projection to a dyadic frequency range we have

$$\|Q_\mu u\|_{L^p} \leq c \mu^{-1/p} \|u\|_{V^p}.$$

There is also a duality relation: Let  $1 < p, q < \infty$  be dual exponents. Then

$$U^p \times V^q \ni (u, v) \rightarrow B(u, v) = \int u v_t dt \tag{33}$$

defines an isometry  $V^q \rightarrow (U^p)^*$ . The notation in (33) is formal, and making it rigorous requires considerable care, for which we refer to [7]. We use the spaces  $DU^p$  and  $DV_{rc}^p$  as distributional time derivatives of functions in  $U^p$  and  $DV_{rc}^p$ . This is possible since for  $1/p + 1/q = 1$

$$\|u\|_{U^p} = \sup \{ B(u, v) : v \in C_0^\infty : \|v\|_{V^q} = 1 \}$$

and

$$\|v\|_{V_{rc}^q} = \sup\{B(u, v) : u \in C_0^\infty : \|u\|_{U^p} = 1\}.$$

All these constructions apply to functions with values in Hilbert spaces. Of particular interest is the Hilbert space  $l^2$ . A short reflection shows that

$$U^2(l^2) \subset l^2U^2, \tag{34}$$

where on the left we have  $l^2$  sequences with values in  $U^2$ , and on the right  $l^2$ -valued functions in  $U^2$ . Similarly  $l^2V^2 \subset V^2l^2$ .

We use Bourgain’s recipe to adapt the function spaces to the Schrödinger equation

$$\|u\|_{U_\Delta^p} = \|e^{-it\Delta}u(t)\|_{U^p}$$

and

$$\|v\|_{V_\Delta^p} = \|e^{-it\Delta}v(t)\|_{V^p}.$$

We will always consider right continuous functions and we drop  $rc$  from the notation.

The relation to the  $X^{s,b}$  spaces can be seen from estimate (32), which also implies the high modulation estimate

$$\|Q_\mu u\|_{L^2} \leq c\mu^{-1/2}\|u\|_{V_\Delta^2}$$

where, as before,  $Q_\mu$  is the projection to modulations of size  $\mu$ , namely the frequency region  $\{\tau + \xi^2 \approx \mu\}$ .

The next lemma, combined with a rescaling argument, proves the bound (26).

**Lemma 3.1.** *Let  $\lambda > 0$ , and  $I$  be an interval with  $|I|\lambda \leq 1$ . If  $u$  is frequency localized in  $[-\lambda, \lambda]$  then the following estimates hold<sup>3</sup>:*

$$\sum \|\chi_j u\|_{U_\Delta^2[I]}^2 \lesssim \|u\|_{U_\Delta^2[I]}^2, \quad \sum \|\chi_j f\|_{DU_\Delta^2[I]}^2 \lesssim \|f\|_{DU_\Delta^2[I]}^2, \tag{35}$$

$$\|u\|_{V_\Delta^2[I]}^2 \lesssim \sum \|\chi_j u\|_{V_\Delta^2[I]}^2, \quad \|f\|_{DV_\Delta^2[I]}^2 \lesssim \sum \|\chi_j f\|_{DV_\Delta^2[I]}^2. \tag{36}$$

**Proof.** It suffices to verify the first inequality for  $U^p$  atoms. Furthermore, due to the first bound in (30), we only need to prove it for each step in an atom. Thus consider a solution  $u$  to the homogeneous Schrödinger equation in a subinterval  $J = [a, b) \subset I$ . For each  $j \in \mathbb{Z}$  we write an equation for  $\chi_j u$ , namely

$$(i\partial_t - \partial_x^2)(\chi_j u) = f_j$$

where the right hand sides  $f_j$  are given by

$$f_j = -\partial_x^2 \chi_j u - 2\partial_x \chi_j \partial_x u \tag{37}$$

and can be estimated as follows:

$$\sum_j \|f_j\|_{L_t^1 L_x^2}^2 \lesssim |J| \sum_j \|f_j\|_{L_{t,x}^2}^2 \lesssim |J|(\|u\|_{L_{t,x}^2}^2 + \|\partial_x u\|_{L_{t,x}^2}^2) \lesssim \lambda^2 |J|^2 \|u\|_{L_t^\infty L_x^2}^2.$$

Then, using  $\lambda|J| \leq 1$ , we have

$$\sum_j \|\chi_j u\|_{U^2[J]}^2 \lesssim \sum_j \|\chi_j u\|_{U^1[J]}^2 \lesssim \sum_j \|\chi_j u(a)\|_{L^2}^2 + \|f_j\|_{L_t^1 L_x^2}^2 \lesssim \|u(a)\|_{L^2}^2.$$

The proof of the first bound in (35) is completed by summing over the intervals  $J$ .

<sup>3</sup> Here  $\chi_j = \chi_j^1$ ; the superscript 1 is omitted.

For the second bound we consider  $f$  of the form  $f = (i\partial_t - \partial_x^2)u$  with  $u \in U_{\Delta}^2[I]$ . Then we can write  $\chi_j f$  as

$$\chi_j f = (i\partial_t - \partial_x^2)(\chi_j u) - f_j$$

with  $f_j$  as in (37). For the first term we use the first bound in (35), and for the second we bound  $f_j$  in  $L^1 L^2$  as above.

Next we consider the first bound in (36). For a partition  $\sigma = (t_n)$  of the interval  $I$  we need to estimate the sum:

$$S = \sum_n \|e^{i(t_{n+1}-t_n)\Delta} u(t_n) - u(t_{n+1})\|_{L^2}^2.$$

We have

$$\begin{aligned} S &\approx \sum_j \sum_n \|\chi_j (e^{i(t_{n+1}-t_n)\Delta} u(t_n) - u(t_{n+1}))\|_{L^2}^2 \\ &\lesssim \sum_j \sum_n \|e^{i(t_{n+1}-t_n)\Delta} (\chi_j u(t_n)) - \chi_j u(t_{n+1})\|_{L^2}^2 + \sum_j \sum_n \|[\chi_j, e^{i(t_{n+1}-t_n)\Delta}] u(t_n)\|_{L^2}^2 \end{aligned}$$

and the first term is directly estimated in terms of the right hand side in (36). For the second term we will establish a stronger bound, namely

$$\sum_j \|[\chi_j, e^{iT\Delta}] u_0\|_{L^2}^2 \lesssim (T\lambda)^2 \|u_0\|_{L^2}^2$$

whenever  $u_0$  is frequency localized in  $[-\lambda, \lambda]$ . This suffices since we have  $\sum (t_{n+1} - t_n)^2 \lesssim |I|^2$ . Denoting  $u(t) = e^{it\Delta} u_0$  we write

$$(i\partial_t - \partial_x^2)[\chi_j, e^{iT\Delta}] u_0 = f_j$$

with  $f_j$  again as in (37). Hence

$$\sum_j \|[\chi_j, e^{iT\Delta}] u_0\|_{L^2}^2 \lesssim \sum_j \|f_j\|_{L^1 L^2}^2$$

and the bound for  $f_j$  in  $L^1 L^2$  is the same as above. The proof of the first part of (36) is concluded.

For the second part of (36) we write  $f$  in the form  $f = (i\partial_t - \partial_x^2)u$  in  $I = [a, b]$  with  $u(a) = 0$ . Arguing by duality (see (33)), from the first part of (35) applied to solutions for the homogeneous equation we obtain the uniform energy bound

$$\|u\|_{L^\infty L^2}^2 \lesssim \sum \|\chi_j f\|_{DV_{\Delta}^2[I]}^2.$$

The rest of the argument is similar to the one for the second part of (35).  $\square$

We conclude this section with the proof of Proposition 1.4. For  $t \geq 0$  we consider the solution  $u$  to the inhomogeneous equation

$$i\partial_t u + \Delta u = f, \quad u(0) = u_0.$$

We set  $u(t) = 0$  for  $t < 0$ . Then from the definitions we immediately obtain the linear bound

$$\|u\|_{U^p} \leq \|u_0\|_{L^2} + \|f\|_{DU^p}. \tag{38}$$

We now consider the bound (15). The frequency localization commutes with the Schrödinger operator, and it suffices to verify (15) for a fixed dyadic frequency range  $\lambda$ . We can also restrict our attention to a time interval  $J = [a, b]$  with  $|J| = \lambda^{4s}$ . There we need to show that

$$\|u_\lambda\|_{X_\lambda[J]} \lesssim \|u_\lambda(a)\|_{L^2} + \|f_\lambda\|_{Y_\lambda[J]}, \quad (i\partial_t - \Delta)u_\lambda = f_\lambda, \tag{39}$$

which follows directly from (38) and the definitions of the norms.

For the second estimate (16) we again localize to a dyadic frequency  $\lambda$ . Let us first consider  $\lambda > \Lambda$ ; there it takes the form

$$\sup_{j \in \mathbb{Z}} \sum_{J \subset I}^{|J|=\lambda^{4s}} \|\chi_j^\lambda u_\lambda\|_{X_\lambda[J]}^2 \lesssim \sup_{j \in \mathbb{Z}} \sum_{J \subset I}^{|J|=\lambda^{4s}} (\lambda^{-4s} \|\chi_j^\lambda u_\lambda\|_{L^2[J]}^2 + \|\chi_j^\lambda f_\lambda\|_{Y_\lambda[J]}^2).$$

This in turn follows after integration over  $t \in J$ ,  $J$  summation and  $k$  summation from the next estimate:

$$\|\chi_j^\lambda u_\lambda\|_{X_\lambda[J]} \lesssim \sum_{k \in \mathbb{Z}} \langle j - k \rangle^{-N} (\|\chi_k^\lambda u_\lambda(t)\|_{L^2} + \|\chi_k^\lambda f_\lambda\|_{Y_\lambda[J]}).$$

This is equivalent to considering an inhomogeneous Cauchy problem in an interval  $J = [a, b]$  with  $|J| = \lambda^{4s}$ ,

$$(i \partial_t - \Delta) u_\lambda = P_\lambda \chi_k^\lambda f, \quad u_\lambda(a) = \chi_k^\lambda u_0^\lambda$$

and proving that

$$\|\chi_j^\lambda u_\lambda\|_{X_\lambda[J]} \lesssim \langle j - k \rangle^{-N} (\|u_0\|_{L^2} + \|f_\lambda\|_{Y_\lambda[J]}).$$

For  $j = k + O(1)$  this is a direct consequence of (39). For  $j$  away from  $k$  this follows from favorable bounds on the kernel  $K_{jk}$  of  $\chi_j^\lambda e^{it\Delta} P_\lambda \chi_j$ , which satisfies the rapid decay bounds

$$|\partial_x^\alpha \partial_y^\beta \partial_t^\gamma K_{j,k}(t, x, y)| \leq c_{\alpha\beta\gamma} \lambda^{-N} \langle j - k \rangle^{-N}, \quad |t| \leq \delta t_\lambda, \quad |j - k| \gg 1.$$

If  $\lambda = \Lambda$  we apply the same argument to  $\partial_x u_\Lambda$ .

#### 4. Linear and bilinear estimates

Solutions to the homogeneous equation,

$$i v_t - \Delta v = 0, \quad v(0) = v_0 \tag{40}$$

satisfy the Strichartz estimates:

**Proposition 4.1.** *Let  $p, q$  be indices satisfying*

$$\frac{2}{p} + \frac{1}{q} = \frac{1}{2}, \quad 4 \leq p \leq \infty. \tag{41}$$

*Then the solution  $u$  to (40) satisfies*

$$\|v\|_{L_t^p L_x^q} \lesssim \|v_0\|_{L^2}.$$

In particular we note the pairs of indices  $(\infty, 2)$ ,  $(6, 6)$  and  $(4, \infty)$ . As a straightforward consequence we have

**Corollary 4.2.** *Let  $p, q$  be indices satisfying (41). Then*

$$\|v\|_{L_t^p L_x^q} \lesssim \|v\|_{U^p}.$$

The proof is straightforward, since it suffices to do it for atoms. By duality we also obtain

**Corollary 4.3.** *Let  $p, q$  be indices satisfying (41). Then*

$$\|v\|_{DV^{p'}} \lesssim \|v\|_{L_t^{p'} L_x^{q'}}.$$

The second type of estimates we use are bilinear:

**Proposition 4.4.** *Let  $\lambda > 0$ . Assume that  $u, v$  are solutions to the homogeneous Schrödinger equation (40). Then*

$$\|P_{>\lambda}(u\bar{v})\|_{L^2} \lesssim \lambda^{-\frac{1}{2}} \|u_0\|_{L^2} \|v_0\|_{L^2}. \tag{42}$$

**Proof.** In the Fourier space we have

$$\hat{u}(\tau, \xi) = \hat{u}_0(\xi)\delta_{\tau-\xi^2}, \quad \hat{v}(\tau, \xi) = \hat{v}_0(\xi)\delta_{\tau-\xi^2}.$$

Then

$$\widehat{u\bar{v}}(\tau, \xi) = \int_{\xi_1 - \xi_2 = \xi} \hat{u}_0(\xi_1)\hat{v}_0(\xi_2)\delta_{\tau-\xi_1^2-\xi_2^2} d\xi_1$$

which gives

$$\widehat{u\bar{v}}(\tau, \xi) = \frac{1}{2|\xi_1 - \xi_2|} (\hat{u}_0(\xi_1)\hat{v}_0(\xi_2) + \hat{u}_0(\xi_2)\hat{v}_0(\xi_1))$$

where  $\xi_1$  and  $\xi_2$  are the solutions to

$$\xi_1^2 + \xi_2^2 = \tau, \quad \xi_1 - \xi_2 = \xi.$$

We have

$$d\tau d\xi = 2|\xi_1 - \xi_2| d\xi_1 d\xi_2$$

therefore we obtain

$$\|P_{>\lambda}(u\bar{v})\|_{L^2}^2 \leq \int_{|\xi_1 - \xi_2| \gtrsim \lambda} |\hat{u}_0(\xi_1)|^2 |\hat{v}_0(\xi_2)|^2 |\xi_1 - \xi_2|^{-1} d\xi_1 d\xi_2.$$

The conclusion follows.  $\square$

As a consequence we obtain

**Corollary 4.5.** *The following estimates hold:*

$$\|P_\lambda(uv)\|_{L^2} \lesssim \lambda^{-1/2} \|u\|_{U_\Delta^2} \|v\|_{U_\Delta^2}. \tag{43}$$

$$\|u_\lambda v_\mu\|_{L^2} \lesssim \mu^{-\frac{1}{2}} \|u_\lambda\|_{U_\Delta^2} \|v_\mu\|_{U_\Delta^2}, \quad \lambda \ll \mu. \tag{44}$$

Again it suffices to prove these estimates for atoms, and then for solutions to the homogeneous Schrödinger equation. But this follows from the bilinear estimate of Proposition 4.4.

### 5. The cubic nonlinearity

In this section we prove Proposition 1.5. For a dyadic frequency  $\lambda$  we estimate the nonlinearity  $|u|^2u$  at frequency  $\lambda$  in a time interval  $I$  of length  $\lambda^{4s}$  in  $DU_\Delta^2$ . By duality this leads to a study of a quadrilinear form of the type

$$J = \int \chi_I u_{\lambda_1} \bar{u}_{\lambda_2} u_{\lambda_3} \bar{u}_{\lambda_4} dx dt.$$

The position of the complex conjugates is of little importance in the sequel. We will assume that  $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$ ; some of the constants in the next lemma improve if the complex conjugates are placed differently, but this plays no role in our subsequent analysis.

**Lemma 5.1.** *Let  $I$  be any compact interval. Then the following estimates hold:*

**A).** *If  $\lambda_1 \sim \lambda_2 \sim \lambda_3 \sim \lambda_4$  then*

$$|J| \lesssim \|u_{\lambda_1}\|_{L^2} \|u_{\lambda_2}\|_{V_\Delta^2} \|u_{\lambda_3}\|_{V_\Delta^2} \|u_{\lambda_4}\|_{V_\Delta^2}, \tag{45}$$

$$|J| \lesssim |I|^{\frac{1}{2}} \|u_{\lambda_1}\|_{V_\Delta^2} \|u_{\lambda_2}\|_{V_\Delta^2} \|u_{\lambda_3}\|_{V_\Delta^2} \|u_{\lambda_4}\|_{V_\Delta^2}. \tag{46}$$

**B).** *If  $\lambda_1 \sim \lambda_2 \ll \lambda_3 \sim \lambda_4$  then*

$$|J| \lesssim \lambda_1^{\frac{1}{2}} \lambda_4^{-\frac{1}{2}} \|u_{\lambda_1}\|_{L^2} \|u_{\lambda_2}\|_{U_\Delta^2} \|u_{\lambda_3}\|_{U_\Delta^2} \|u_{\lambda_4}\|_{U_\Delta^2}, \tag{47}$$

$$|J| \lesssim \lambda_1^{\frac{1}{2}} \lambda_4^{-\frac{1}{2}} \|u_{\lambda_1}\|_{U_\Delta^2} \|u_{\lambda_2}\|_{U_\Delta^2} \|u_{\lambda_3}\|_{L^2} \|u_{\lambda_4}\|_{U_\Delta^2}, \tag{48}$$

$$|J| \lesssim \lambda_4^{-1} \|u_{\lambda_1}\|_{U_\Delta^2} \|u_{\lambda_2}\|_{U_\Delta^2} \|u_{\lambda_3}\|_{U_\Delta^2} \|u_{\lambda_4}\|_{U_\Delta^2}. \tag{49}$$

**C).** *If  $\lambda_1 \ll \lambda_2 \ll \lambda_3 \sim \lambda_4$  then*

$$|J| \lesssim \lambda_1^{\frac{1}{2}} \lambda_4^{-\frac{1}{2}} \|u_{\lambda_1}\|_{L^2} \|u_{\lambda_2}\|_{U_\Delta^2} \|u_{\lambda_3}\|_{U_\Delta^2} \|u_{\lambda_4}\|_{U_\Delta^2}, \tag{50}$$

$$|J| \lesssim \min\{\lambda_1^{\frac{1}{2}} \lambda_2^{-1} \lambda_4^{1/2}, 1\} \lambda_2^{\frac{1}{2}} \lambda_4^{-\frac{1}{2}} \|u_{\lambda_1}\|_{U_\Delta^2} \|u_{\lambda_2}\|_{L^2} \|u_{\lambda_3}\|_{U_\Delta^2} \|u_{\lambda_4}\|_{U_\Delta^2}, \tag{51}$$

$$|J| \lesssim \lambda_1^{\frac{1}{2}} \lambda_4^{-\frac{1}{2}} \|u_{\lambda_1}\|_{U_\Delta^2} \|u_{\lambda_2}\|_{U_\Delta^2} \|u_{\lambda_3}\|_{L^2} \|u_{\lambda_4}\|_{U_\Delta^2}, \tag{52}$$

$$|J| \lesssim \min\{\lambda_1^{\frac{1}{2}} \lambda_2^{-1} \lambda_4^{1/2}, 1\} \lambda_4^{-1} \|u_{\lambda_1}\|_{U_\Delta^2} \|u_{\lambda_2}\|_{U_\Delta^2} \|u_{\lambda_3}\|_{U_\Delta^2} \|u_{\lambda_4}\|_{U_\Delta^2}. \tag{53}$$

**D).** *If  $\lambda_1 \ll \lambda_2 \sim \lambda_3 \sim \lambda_4$  then*

$$|J| \lesssim \lambda_1^{\frac{1}{2}} \lambda_4^{-\frac{1}{2}} \|u_{\lambda_1}\|_{L^2} \|u_{\lambda_2}\|_{U_\Delta^2} \|u_{\lambda_3}\|_{U_\Delta^2} \|u_{\lambda_4}\|_{U_\Delta^2}, \tag{54}$$

$$|J| \lesssim \lambda_1^{\frac{1}{2}} \lambda_4^{-\frac{1}{2}} \|u_{\lambda_1}\|_{U_\Delta^2} \|u_{\lambda_2}\|_{L^2} \|u_{\lambda_3}\|_{U_\Delta^2} \|u_{\lambda_4}\|_{U_\Delta^2}, \tag{55}$$

$$|J| \lesssim \lambda_1^{\frac{1}{4}} \lambda_4^{-\frac{1}{4}} \|u_{\lambda_1}\|_{U_\Delta^2} \|u_{\lambda_2}\|_{U_\Delta^2} \|u_{\lambda_3}\|_{L^2} \|u_{\lambda_4}\|_{U_\Delta^2}, \tag{56}$$

$$|J| \lesssim \lambda_1^{\frac{1}{4}} \lambda_4^{-\frac{1}{4}} \|u_{\lambda_1}\|_{U_\Delta^2} \|u_{\lambda_2}\|_{U_\Delta^2} \|u_{\lambda_3}\|_{U_\Delta^2} \|u_{\lambda_4}\|_{U_\Delta^2}. \tag{57}$$

It is worth noting that the length of the interval enters only in (46). In all other cases the cutoff  $\chi_I$  can be safely discarded. The bounds in parts B and C improve if the none or both of the high frequency factors have complex conjugates. We also note the weaker bound (56) when the complex conjugates fall on the first and third factor; this directly leads to the weaker bound in (57), and causes some small difficulties later on.

We also remark that combining the results in (45), (47), (48), (50), (51) (54), and (55) we obtain by duality

**Corollary 5.2.** *Suppose that  $\lambda_1 \leq \lambda_2 \leq \lambda_3$  and  $\lambda_0 \lesssim \lambda_2$ . Then*

$$\|P_{\lambda_0}(u_{\lambda_1} \bar{u}_{\lambda_2} u_{\lambda_3})\|_{L^2} \lesssim \lambda_0^{\frac{1}{2}} \lambda_3^{-\frac{1}{2}} \|u_{\lambda_1}\|_{U_\Delta^2} \|u_{\lambda_2}\|_{U_\Delta^2} \|u_{\lambda_3}\|_{U_\Delta^2}. \tag{58}$$

**Proof.** **A.** For (45) we use the  $L^6$  Strichartz estimate. For (46) we use the  $L^8 L^4$  Strichartz estimate and Hölder’s inequality to obtain

$$J \lesssim |I|^{\frac{1}{2}} \prod_{j=1}^4 \|\chi_I u_{\lambda_j}\|_{L^8 L^4} \lesssim |I|^{\frac{1}{2}} \prod_{j=1}^4 \|\chi_I u_{\lambda_j}\|_{U^8} \lesssim |I|^{\frac{1}{2}} \prod_{j=1}^4 \|\chi_I u_{\lambda_j}\|_{V^2}.$$

**B.** For both (47) and (48) we use an  $L^2$  bilinear estimate for  $u_{\lambda_2} u_{\lambda_4}$ , and the remaining two factors are estimated in  $L^2$  respectively  $L^\infty L^2$  with an added Bernstein inequality for  $u_{\lambda_1}$ . For (49) we use two  $L^2$  bilinear estimates for  $u_{\lambda_1} u_{\lambda_3}$ , respectively  $u_{\lambda_2} u_{\lambda_4}$ .

C. The bounds (50) and (52) are similar to (47) and (48). The same argument also yields the  $\lambda_2^{\frac{1}{2}}\lambda_4^{-\frac{1}{2}}$  factor in (51). To complete the proof of (51) we observe that we can harmlessly insert a projector  $\tilde{P}_{\lambda_2}(u_{\lambda_3}\bar{u}_{\lambda_4})$  in the last product. Here and later,  $\tilde{P}_\lambda$  denotes a wider frequency  $\lambda$  projector, for instance  $\tilde{P}_\lambda = \sum_{\mu \sim \lambda} P_\mu$ . By the  $L^2$  bilinear estimate we have

$$\|\tilde{P}_{\lambda_2}(u_{\lambda_3}\bar{u}_{\lambda_4})\|_{L^2} \lesssim \lambda_2^{-\frac{1}{2}} \|u_{\lambda_3}\|_{U_\Delta^2} \|u_{\lambda_4}\|_{U_\Delta^2}$$

and using the  $L^\infty$  bound for  $u_{\lambda_1}$  we obtain the second desired factor  $\lambda_1^{\frac{1}{2}}\lambda_2^{-\frac{1}{2}}$  in (51).

To prove (53) we decompose each factor into a term with modulation  $\gtrsim \lambda_2\lambda_4$ , and one with smaller modulation,

$$u_{\lambda_j} = u_{\lambda_j}^h + u_{\lambda_j}^l.$$

If all four modulations are low, then a simple frequency-modulation analysis shows that  $J = 0$ . Hence we assume without any restriction in generality that one of the factors is at high modulation. For that factor we have an  $L^2$  bound, see (21) and (22),

$$\|u_{\lambda_j}^h\|_{L^2} \lesssim (\lambda_2\lambda_4)^{-\frac{1}{2}} \|u_{\lambda_j}\|_{U_\Delta^2}.$$

Hence the constant in (53) is obtained by adding a  $(\lambda_2\lambda_4)^{-\frac{1}{2}}$  factor to each of the constants in (50)–(52) and summing them up.

D. Case D is similar to Case C except for the inequality (56), where we only obtain the worse factor  $\lambda_1^{\frac{1}{4}}\lambda_4^{-\frac{1}{4}}$ . The difference there is that  $\xi_2$  and  $\xi_4$  no longer need to have dyadic separation so we cannot use directly the bilinear  $L^2$  bound. To address this issue we split the problem into two cases by writing

$$u_{\lambda_2}u_{\lambda_4} = P_{>\lambda}(u_{\lambda_2}u_{\lambda_4}) + P_{<\lambda}(u_{\lambda_2}u_{\lambda_4}), \quad \lambda = (\lambda_1\lambda_4)^{\frac{1}{2}}.$$

For the first term we use the bilinear  $L^2$  bound to obtain

$$\|P_{>\lambda}(u_{\lambda_2}u_{\lambda_4})\|_{L^2} \lesssim (\lambda_1\lambda_4)^{-\frac{1}{4}} \|u_{\lambda_2}\|_{U_\Delta^2} \|u_{\lambda_4}\|_{U_\Delta^2}$$

and conclude with the pointwise bound for  $u_{\lambda_1}$ .

For the second term we have orthogonality with respect to frequency intervals of size  $\lambda$ , therefore the problem reduces to the case when  $u_{\lambda_2}$  and  $u_{\lambda_4}$  are frequency localized in  $\lambda$  sized intervals. Then we use the  $L^2$  bilinear bound for  $u_{\lambda_1}u_{\lambda_4}$  gaining a  $\lambda_4^{-\frac{1}{2}}$  factor, and then use Bernstein for  $u_{\lambda_2}$  (now localized on the  $(\lambda_1\lambda_4)^{\frac{1}{2}}$  scale) for a loss of  $(\lambda_1\lambda_4)^{\frac{1}{4}}$ .  $\square$

We continue with the proof of Proposition 1.5.

**Proof of Proposition 1.5.** For the  $Y_{\Lambda}^s$  bound we need to estimate the trilinear expression

$$P_\lambda(u_{\lambda_1}\bar{u}_{\lambda_2}u_{\lambda_3})$$

in  $Y_\lambda[I]$  over intervals of size  $|I| = \lambda^{4s}$ , and then square sum with respect to all frequencies  $\lambda, \lambda_1, \lambda_2, \lambda_3$ . In the case of the  $Y_{\Lambda,le}^s$  bound we have to estimate the better, localized, trilinear expression

$$\chi_j^\lambda P_\lambda(u_{\lambda_1}\bar{u}_{\lambda_2}u_{\lambda_3})$$

in  $Y_\lambda[I]$ , but we need to perform an additional summation with respect to the time interval  $I$ .

We separately consider several cases depending on the relative size of all  $\lambda$ 's. In order for the output to be nonzero we must be in one of the following two cases:

1.  $\max\{\lambda_1, \lambda_2, \lambda_3\} \sim \lambda$ .
2.  $\{\lambda_1, \lambda_2, \lambda_3\} = \{\alpha, \mu, \mu\}$  with  $\lambda \ll \mu, \alpha \leq \mu$ .

Here we allow for a slight abuse of notation, as the two highest  $\lambda_j$ 's need not be equal but merely comparable. We will consider these two cases separately. In the second case we will subdivide into further cases depending on the relative size of  $\alpha$  and  $\lambda$ .

The space  $Y_\lambda[I]$  is a weighted sum of an  $L^2$  space and a  $DU_\Delta^2$  space. In Case 1 we will estimate the cubic term only in  $L^2$ . In Case 2 we will estimate the cubic term in both  $L^2$  and  $DU_\Delta^2$ . The estimates in either spaces are good enough to complete the argument, and hence there is some redundancy. Nevertheless we find it instructive to do the extra work, as it shows that this argument does not break at  $s = -1/4$ .

We remark, though, that in order to continue it below  $s = -1/4$  some extra care is required as the balance of the spatial scales changes. We also remark that this difficulty disappears exactly at  $s = -1/4$ , when all spatial scales coincide.

**Case 1:**  $\max\{\lambda_1, \lambda_2, \lambda_3\} \sim \lambda$ . This case imposes no restrictions on  $s$  beyond  $s \geq -1/2$ . Instead it makes the arguments for the length of the time intervals in our  $X_\Lambda^s, X_{\Lambda,le}^s, Y_\Lambda^s$  and  $Y_{\Lambda,le}^s$  norms precise. In this case we restrict ourselves to the  $L^2$  bound. Using duality and (45) we obtain

$$\begin{aligned} \|\chi_I u_{\lambda_1} u_{\lambda_2} u_{\lambda_3}\|_{L^2} &\lesssim \|u_{\lambda_1}\|_{V^2[I]} \|u_{\lambda_2}\|_{V^2[I]} \|u_{\lambda_3}\|_{V^2[I]} \\ &\lesssim (\lambda_1 \lambda_2 \lambda_3)^{-s} \|u_{\lambda_1}\|_{X_\Lambda^s} \|u_{\lambda_2}\|_{X_\Lambda^s} \|u_{\lambda_3}\|_{X_\Lambda^s}. \end{aligned}$$

For intervals  $I$  of size  $|I| = \lambda^{4s}$  this gives

$$\|P_\lambda(u_{\lambda_1} u_{\lambda_2} u_{\lambda_3})\|_{Y_\Lambda^s[I]} \lesssim (\lambda_1 \lambda_2 \lambda_3)^{-s} \lambda^{3s} \|u_{\lambda_1}\|_{X_\Lambda^s} \|u_{\lambda_2}\|_{X_\Lambda^s} \|u_{\lambda_3}\|_{X_\Lambda^s}$$

where the  $\lambda_{1,2,3}$  summations are straightforward.

For the estimate in  $Y_{\Lambda,le}^s$  we observe that

$$\|\chi_j^\lambda \chi_I P_\lambda(u_{\lambda_1} u_{\lambda_2} u_{\lambda_3})\|_{L^2} \lesssim \sum_l \langle j-l \rangle^{-N} \|\chi_l^\lambda \chi_I u_{\lambda_1} u_{\lambda_2} u_{\lambda_3}\|_{L^2}.$$

If  $\lambda \gg \Lambda$  then the same argument as above applies since the square summability with respect to time intervals is inherited from  $u_{\lambda_{max}}$ . If  $\lambda \sim \Lambda$  then we apply the argument to  $\partial_x(u_{\lambda_1} u_{\lambda_2} u_{\lambda_3})$ ; then the square summability with respect to  $I$  is inherited from the differentiated factor.

**Case 2:**  $\{\lambda_1, \lambda_2, \lambda_3\} = \{\alpha, \mu, \mu\}$  with  $\lambda \ll \mu$ . We subdivide this as follows:

**Case 2(a):**  $\lambda \sim \alpha \ll \mu$ . This case and the next is where we gain most from the local energy decay bounds. This case requires no explicit restriction on  $s$  beyond  $s \geq -1/2$ ; instead it determines the power of  $\lambda$  in the  $DU_\Delta^2$  component of  $Y^s$  and  $Y_{le}^s$ . We remark however that for the argument below it is important to know that the frequency  $\mu$  spatial scale is larger than the frequency  $\lambda$  spatial scale; this breaks down for  $s < -1/4$  therefore the above mentioned power of  $\lambda$  would have to be adjusted for such  $s$ .

The placement of the complex conjugates is irrelevant here, therefore we let, say,  $\lambda_1 = \alpha$ . We decompose each of the factors as

$$u_{\lambda_i} = \sum_{j \in \mathbb{Z}} u_{\lambda_i, j}, \quad u_{\lambda_i, j} = \tilde{P}_{\lambda_i}(\chi_j^{\lambda_i} u_{\lambda_i})$$

preserving the frequency localizations. The spatial localizations are not preserved but the tails are negligible,

$$|\chi_k^{\lambda_i} u_{\lambda_i, j}| \lesssim |k-j|^{-N} \lambda_i^{-N} \|\chi_j^{\lambda_i} u_{\lambda_i}\|_{L^\infty L^2}, \quad |k-j| \gg 1. \tag{59}$$

For  $j \in \mathbb{Z}$  and  $|I| = \lambda^{4s}$  we define the localized trilinear expressions

$$f_{I, j} := \chi_I u_{\lambda_1, j} \bar{u}_{\lambda_2} u_{\lambda_3} = \chi_I u_{\lambda_1, j} \sum_{I' \subset I} \sum_{k_2, k_3}^{|I'|=\mu^{4s}} \chi_{I'} \bar{u}_{\lambda_2, k_2} \cdot \chi_{I'} u_{\lambda_3, k_3}.$$

Due to the above mentioned ordering of spatial scales there is some unique (up to  $O(1)$ )  $k = k(j)$  so that the supports of  $\chi_j^{\lambda_1}$  and  $\chi_k^{\lambda_{2,3}}$  overlap. We will first bound  $P_\lambda f_{I, j}$  in  $L^2$ . Using (47) and duality we obtain



$$\|P_\lambda f_{I,j}\|_{L^2} \lesssim \frac{\lambda^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \|u_{\lambda_1,j}\|_{U_\Delta^2[I]} \sum_{k_2,k_3}^{|I'|=\mu^{4s}} \sum_{I' \subset I} \frac{\|u_{\lambda_2,k_2}\|_{U_\Delta^2[I']}}{\langle k_2 - k(j) \rangle^N} \frac{\|u_{\lambda_3,k_3}\|_{U_\Delta^2[I']}}{\langle k_3 - k(j) \rangle^N}$$

where the rapid decay away from when  $k_{2,3} = k(j)$  is due to (59). Next we use Cauchy–Schwarz with respect to  $I'$  and then sum with respect to  $k_2, k_3$  to get

$$\|P_\lambda f_{I,j}\|_{L^2} \lesssim \lambda^{\frac{1}{2}} \mu^{-\frac{1}{2}-2s} \|\chi_j^\lambda u_{\lambda_1}\|_{U_\Delta^2[I]} \|u_{\lambda_2}\|_{X_{\Lambda,le}^s} \|u_{\lambda_3}\|_{X_{\Lambda,le}^s}. \tag{60}$$

The square summability with respect to space or time intervals is inherited from  $u_{\lambda_1}$ , so we conclude that

$$\|P_\lambda(u_{\lambda_1} \bar{u}_{\lambda_2} u_{\lambda_3})\|_{Y_\Lambda^s} \lesssim \lambda^{\frac{1}{2}+2s} \mu^{-\frac{1}{2}-2s} \|u_{\lambda_1}\|_{X_\Lambda^s} \|u_{\lambda_2}\|_{X_{\Lambda,le}^s} \|u_{\lambda_3}\|_{X_{\Lambda,le}^s} \tag{61}$$

and similarly for the  $Y_{\Lambda,le}^s$  norm. The summation with respect to  $\lambda$  and  $\mu$  is straightforward.

We remark that this approach gives a better bound for high modulations but works only when  $s \geq -\frac{1}{4}$ . However, the estimate in  $DU^2$  shows that there is some room beyond  $s = -1/4$ .

From (49) by duality and (59) for the tails we obtain

$$\|P_\lambda f_{I,j}\|_{DV_\Delta^2[I]} \lesssim \mu^{-1} \|\chi_j^\lambda u_{\lambda_1}\|_{U_\Delta^2[I]} \sum_{k_2,k_3}^{|I'|=\mu^{4s}} \sum_{I' \subset I} \frac{\|u_{\lambda_2,k_2}\|_{U_\Delta^2[I']}}{\langle k_2 - k(j) \rangle^N} \frac{\|u_{\lambda_3,k_3}\|_{U_\Delta^2[I']}}{\langle k_3 - k(j) \rangle^N}.$$

After Cauchy–Schwarz with respect to  $I'$  and  $k_2, k_3$  summation we obtain

$$\|P_\lambda f_{I,j}\|_{DV_\Delta^2[I]} \lesssim \mu^{-1-2s} \|\chi_j^\lambda u_{\lambda_1}\|_{U_\Delta^2[I]} \|u_{\lambda_2}\|_{X_{\Lambda,le}^s} \|u_{\lambda_3}\|_{X_{\Lambda,le}^s}. \tag{62}$$

Comparing this with (60) we see that (60) is stronger at modulations  $\geq \lambda\mu$  while (62) is stronger at modulations  $\leq \lambda\mu$ . Precisely, from (60) we obtain

$$\begin{aligned} \|Q_{>\lambda\mu} P_\lambda f_{I,j}\|_{Y_\Lambda[I]} &\lesssim \lambda^{1+2s} \|Q_{>\lambda\mu} P_\lambda f_{I,j}\|_{DU_\Delta^2} \\ &\lesssim \lambda^{1/2+2s} \mu^{-1/2} \|P_\lambda f_{I,j}\|_{L^2} \\ &\lesssim \lambda^{1+2s} \mu^{-1-2s} \|\chi_j^\lambda u_{\lambda_1}\|_{U_\Delta^2[I]} \|u_{\lambda_2}\|_{X_{\Lambda,le}^s} \|u_{\lambda_3}\|_{X_{\Lambda,le}^s} \\ &\lesssim \lambda^{1+3s} \mu^{-1-2s} \|\chi_j^\lambda u_{\lambda_1}\|_{X_\Lambda^s[I]} \|u_{\lambda_2}\|_{X_{\Lambda,le}^s} \|u_{\lambda_3}\|_{X_{\Lambda,le}^s}, \end{aligned}$$

while from (62) we have

$$\|Q_{<\lambda\mu} P_\lambda f_{I,j}\|_{Y_\Lambda} \lesssim \log(\mu/\lambda) \lambda^{1+3s} \mu^{-1-2s} \|\chi_j^\lambda u_{\lambda_1}\|_{X_\Lambda^s[I]} \|u_{\lambda_2}\|_{X_{\Lambda,le}^s} \|u_{\lambda_3}\|_{X_{\Lambda,le}^s}$$

where the logarithmic loss is due to the number of dyadic regions between modulations  $\lambda^2$  and  $\lambda\mu$  arising in the conversion of the  $DV^2$  norm into a  $DU^2$  norm, see (32).

Again the square summability with respect to space<sup>4</sup> or time intervals is inherited from  $u_{\lambda_1}$ , so we obtain an improved form of (61), namely

$$\|P_\lambda(u_{\lambda_1} \bar{u}_{\lambda_2} u_{\lambda_3})\|_{Y_\Lambda^s} \lesssim \left(\frac{\lambda}{\mu}\right)^{1+2s} \log\left(\frac{\mu}{\lambda}\right) \|u_{\lambda_1}\|_{X_\Lambda^s} \|u_{\lambda_2}\|_{X_{\Lambda,le}^s} \|u_{\lambda_3}\|_{X_{\Lambda,le}^s} \tag{63}$$

and similarly for the  $Y_{\Lambda,le}^s$  norm. Thus  $s \geq -\frac{1}{4}$  is more than enough.

**Case 2(b):**  $\alpha \ll \lambda \ll \mu$ . This case is similar to the previous case in that the local energy norms give a crucial gain in the estimates. This case is also different from the previous case in a fundamental way, namely that the interaction is nonresonant. Precisely, either the output or at least one of the inputs must have modulation at least  $\lambda\mu$ . In the latter case, there is a further gain due to our definition of the  $X^s$  respectively  $X_{le}^s$  spaces. Unfortunately, this gain disappears as  $\alpha$  gets small, so we cannot take good advantage of it, and instead we end up repeating the arguments of Case 2(a).

<sup>4</sup> Here, as well as in all the other cases, we want to use the second bound in (36) rather than (35), so the spatial summation must precede the  $DV^2$  to  $DU^2$  conversion.

Again the placement of the complex conjugates is irrelevant, so we let  $\lambda_1 = \alpha$ . However we readjust the definition of  $f_{I,j}$  to

$$f_{I,j} = \chi_I \tilde{P}_{\lambda_1} (\chi_j^\lambda u_{\lambda_1}) u_{\lambda_2} u_{\lambda_3}$$

using the larger spatial scale  $\delta x_\lambda$  instead of  $\delta x_\alpha$  for the cutoffs. Using a dual form of (51) as well as (59) for off-diagonal tails we obtain the trilinear  $L^2$  bound

$$\|P_\lambda f_{I,j}\|_{L^2} \lesssim \min\{\alpha^{\frac{1}{2}} \lambda^{-1} \mu^{\frac{1}{2}}, 1\} \lambda^{\frac{1}{2}} \mu^{-\frac{1}{2}} \|\chi_j^\lambda u_{\lambda_1}\|_{U_\Delta^2[I]} \sum_{k_2, k_3} \sum_{|I'|=\mu^{4s}} \frac{\|u_{\lambda_2, k_2}\|_{U_\Delta^2[I']}}{\langle k_2 - k(j) \rangle^N} \frac{\|u_{\lambda_3, k_3}\|_{U_\Delta^2[I']}}{\langle k_3 - k(j) \rangle^N}.$$

On the other hand from (53) by duality we obtain

$$\|P_\lambda f_{I,j}\|_{DV^2[I]} \lesssim \min\{\alpha^{\frac{1}{2}} \lambda^{-1} \mu^{\frac{1}{2}}, 1\} \mu^{-1} \|\chi_j^\lambda u_{\lambda_1}\|_{U_\Delta^2[I]} \sum_{k_2, k_3} \sum_{|I'|=\mu^{4s}} \frac{\|u_{\lambda_2, k_2}\|_{U_\Delta^2[I']}}{\langle k_2 - k(j) \rangle^N} \frac{\|u_{\lambda_3, k_3}\|_{U_\Delta^2[I']}}{\langle k_3 - k(j) \rangle^N}.$$

Using the former for modulations  $\geq \lambda\mu$  and the latter for smaller modulations we obtain

$$\begin{aligned} \|P_\lambda f_{I,j}\|_{Y_\lambda[I]} &\lesssim \log(\lambda/\mu) \min\{\alpha^{\frac{1}{2}} \lambda^{-1} \mu^{\frac{1}{2}}, 1\} \lambda^{1+2s} \mu^{-1} \\ &\quad \times \|\chi_j^\lambda u_{\lambda_1}\|_{U_\Delta^2[I]} \sum_{k_2, k_3} \sum_{|I'|=\mu^{4s}} \frac{\|u_{\lambda_2, k_2}\|_{U_\Delta^2[I']}}{\langle k_2 - k(j) \rangle^N} \frac{\|u_{\lambda_3, k_3}\|_{U_\Delta^2[I']}}{\langle k_3 - k(j) \rangle^N}. \end{aligned} \tag{64}$$

After Cauchy–Schwarz with respect to  $I'$  this gives

$$\|P_\lambda f_{I,j}\|_{Y_\lambda[I]} \lesssim \log(\lambda/\mu) \min\{\alpha^{\frac{1}{2}} \lambda^{-1} \mu^{\frac{1}{2}}, 1\} \lambda^{1+2s} \mu^{-1-2s} \|\chi_j^\lambda u_{\lambda_1}\|_{U_\Delta^2[I]} \|u_{\lambda_2}\|_{X_{\Lambda,le}^s} \|u_{\lambda_3}\|_{X_{\Lambda,le}^s}.$$

The square summability with respect to spatial intervals is inherited from  $u_{\lambda_1}$ , so we conclude that

$$\|P_\lambda (u_{\lambda_1} \bar{u}_{\lambda_2} u_{\lambda_3})\|_{Y_\lambda^s} \lesssim \log(\lambda/\mu) \min\{\alpha^{\frac{1}{2}} \lambda^{-1} \mu^{\frac{1}{2}}, 1\} \alpha^{-s} \lambda^{1+3s} \mu^{-1-2s} \|u_{\lambda_1}\|_{X_\lambda^s} \|u_{\lambda_2}\|_{X_{\Lambda,le}^s} \|u_{\lambda_3}\|_{X_{\Lambda,le}^s}. \tag{65}$$

The summation with respect to  $\alpha$  and  $\mu$  is straightforward.

The local energy  $Y_{\Lambda,le}^s$  estimate does not pose additional difficulties. We first estimate the frequency  $\alpha$  factor in (64) by

$$\|\chi_j^\lambda u_{\lambda_1}\|_{U_\Delta^2[I]} \lesssim \alpha^{-s} \|u_{\lambda_1}\|_{X_\lambda^s}.$$

Then we sum over all  $I \subset [0, 1]$  and use Cauchy–Schwarz with respect to  $I' \subset [0, 1]$ . Thus the time interval summation is inherited from the highest frequencies, and we obtain the same constants as in (65).

**Case 2(c):  $\lambda \ll \alpha \ll \mu$ .** In this case, as  $\alpha$  increases, the usefulness of the local energy norms decreases. The interaction is still nonresonant, i.e. either the output or at least one of the inputs must have modulation at least  $\lambda\mu$ . However, this time we can fully exploit the gain coming from the better bounds for high modulation inputs. On the other hand if the output has high modulation then we get to use the better constant in (50) (compared to (51)).

Again the placement of complex conjugates does not matter, so we let  $\alpha = \lambda_1$  but we return to our original notation in Case 2(a),

$$f_{I,j} := u_{\lambda_1, j} \bar{u}_{\lambda_2} u_{\lambda_3}.$$

We claim that the following bound holds for  $|I| = \lambda^{4s}$ :

$$\|P_\lambda f_{I,j}\|_{Y_\lambda[I]} \lesssim C \sup_{I' \subset I, |I'|=\alpha^{4s}} \|\chi_j^\alpha u_{\lambda_1}\|_{X_\alpha[I']} \|u_{\lambda_2}\|_{X_{\Lambda,le}^s} \|u_{\lambda_3}\|_{X_{\Lambda,le}^s} \tag{66}$$

where  $C = C(\lambda, \alpha, \mu)$  is given by

$$C(\lambda, \alpha, \mu) = \log(\mu/\lambda) \lambda^{1+2s} \mu^{-1-2s} (\lambda^{\frac{1}{2}} \alpha^{-\frac{1}{2}} + \alpha^{-1-2s} \min\{1, \lambda^{\frac{1}{2}} \alpha^{-1} \mu^{\frac{1}{2}}\}).$$

Here the second term is at most as large as the first if  $s \geq -1/4$  and could be omitted. Using (66) we conclude the proof in this case. For the local energy decay norm  $Y_{\Lambda,le}^s$  we square sum over  $I \subset [0, 1]$  and inherit the square summability from  $u_{\lambda_1}$ , so there is no further loss; we obtain

$$\|P_\lambda(u_{\lambda_1}\bar{u}_{\lambda_2}u_{\lambda_3})\|_{Y_{\Lambda,le}^s} \lesssim \log(\mu/\lambda) \frac{\lambda^{\frac{3}{2}+3s}}{\alpha^{\frac{1}{2}+s}\mu^{1+2s}} \|u_{\lambda_1}\|_{X_{\Lambda,le}^s} \|u_{\lambda_2}\|_{X_{\Lambda,le}^s} \|u_{\lambda_3}\|_{X_{\Lambda,le}^s}.$$

For the  $Y_\Lambda^s$  norm we square sum over  $j$  in (66). The  $j$  summation is inherited from  $u_{\lambda_1}$ ; however we cannot interchange the  $I'$  supremum with the square summation in  $j$ . Instead we relax the supremum to an  $l^2$  norm, which allows us to interchange norms but causes an  $(\alpha/\lambda)^{-2s}$  loss in the  $I'$  summation. This yields the worse bound

$$\|P_\lambda(u_{\lambda_1}\bar{u}_{\lambda_2}u_{\lambda_3})\|_{Y_\Lambda^s} \lesssim \log(\mu/\lambda) \frac{\lambda^{\frac{3}{2}+5s}}{\alpha^{\frac{1}{2}+3s}\mu^{1+2s}} \|u_{\lambda_1}\|_{X_\Lambda^s} \|u_{\lambda_2}\|_{X_{\Lambda,le}^s} \|u_{\lambda_3}\|_{X_{\Lambda,le}^s}.$$

It is easy to check that the  $\alpha$  and  $\mu$  summation is favorable since  $s > -3/10$ .

We now prove (66). We begin by writing

$$f_{I,j} := \sum_{k_2,k_3} \chi_I u_{\lambda_1,j} \bar{u}_{\lambda_2,k_2} u_{\lambda_3,k_3}.$$

The off-diagonal terms where  $k_2$  or  $k_3$  are away from  $k(j)$  are estimated directly as in Case 2(a), (b) using the rapid decay in (59). It remains to consider the diagonal contribution which we write as

$$f_{I,j}^d := \chi_I u_{\lambda_1,j} \bar{u}_{\lambda_2,k} u_{\lambda_3,k}, \quad k = k(j).$$

We actually obtain a finite sum of such terms, which we suppress in the notation.

We consider a time interval decomposition of  $f_{I,j}^d$ ,

$$f_{I,j}^d = \sum_{I' \subset I}^{|I'|=\alpha^{4s}} \chi_{I'} u_{\lambda_1,j} \sum_{I'' \subset I'}^{|I''|=\mu^{4s}} \chi_{I''} \bar{u}_{\lambda_2,k} \cdot \chi_{I''} \bar{u}_{\lambda_3,k} := \sum_{I'' \subset I}^{|I''|=\mu^{4s}} f_{I'',j}^d.$$

Since we will use modulation truncations which are nonlocal in time, for the rest of the argument we extend each of the three factors above to solutions to the homogeneous Schrödinger equation outside  $I'$  respectively  $I''$ . A key reason for working with these extensions is that they satisfy better high modulation bounds than the original interval localized functions, see (24). Recalling that  $E_J$  is the extension operator for the interval  $J$  we define the extension  $f_{I'',j}^e$  of  $f_{I'',j}^d$  by

$$f_{I'',j}^e = E_{I'}(u_{\lambda_1,j}) \overline{E_{I''}(u_{\lambda_2,k})} E_{I''}(u_{\lambda_3,k}).$$

For  $f_{I'',j}^e$  we establish a global  $L^2$  bound, as well as a stronger low modulation  $DV_\Delta^2$  estimate; the balance between these two bounds is at modulation  $\alpha\mu$ . For the  $L^2$  bound we recall that (50) holds on the whole real line. Hence we obtain the  $L^2$  estimate

$$\|P_\lambda f_{I'',j}^e\|_{L^2} \lesssim \lambda^{\frac{1}{2}} \mu^{-\frac{1}{2}} \|u_{\lambda_1,j}\|_{U^2[I']} \|u_{\lambda_2,k}\|_{U^2[I'']} \|u_{\lambda_3,k}\|_{U^2[I'']}. \tag{67}$$

Next we consider the low modulations  $Q_{\ll\alpha\mu} P_\lambda f_{I'',j}^e$ . Since we are in a nonresonant case, this is nonzero only if one of the three factors has high modulation ( $\gtrsim \alpha\mu$ ). But for the high modulations we have better bounds. We write

$$Q_{\ll\alpha\mu} P_\lambda f_{I'',j}^e = Q_{\ll\alpha\mu} P_\lambda (g_{I'',j}^1 + g_{I'',j}^2 + g_{I'',j}^3)$$

where

$$\begin{aligned} g_{I'',j}^1 &= Q_{\gtrsim\alpha\mu} E_{I'} u_{\lambda_1,j} \cdot \overline{E_{I''} u_{\lambda_2,k}} \cdot E_{I''} u_{\lambda_3,k}, \\ g_{I'',j}^2 &= Q_{\ll\alpha\mu} E_{I'} u_{\lambda_1,j} \cdot \overline{Q_{\gtrsim\alpha\mu} E_{I''} u_{\lambda_2,k}} \cdot E_{I''} u_{\lambda_3,k}, \\ g_{I'',j}^3 &= Q_{\ll\alpha\mu} E_{I'} u_{\lambda_1,j} \cdot \overline{Q_{\ll\alpha\mu} E_{I''} u_{\lambda_2,k}} \cdot Q_{\gtrsim\alpha\mu} E_{I''} u_{\lambda_3,k}. \end{aligned}$$

We will only consider the first term  $g_{I''}^1$ ; the analysis for the other two terms is similar but the result is better since the high modulation gain of  $\alpha^{-1-2s}$  is replaced by  $\mu^{-1-2s}$ .

For the high modulation truncation we use the  $L^2$  bound in (24). Then by (51) and duality we estimate  $P_\lambda g_{I''}^1$  as

$$\begin{aligned} \|P_\lambda g_{I''}^1\|_{DV_\Delta^2} &\lesssim C_1 \|Q_{\gtrsim \alpha \mu} E_I u_{\lambda_1, j}\|_{L^2} \|u_{\lambda_2, k}\|_{U^2[I'']} \|u_{\lambda_3, k}\|_{U^2[I'']} \\ &\lesssim C_1 \alpha^{-1-2s} (\alpha \mu)^{-\frac{1}{2}} \|u_{\lambda_1, j}\|_{X_\lambda[I']} \|u_{\lambda_2, k}\|_{U^2[I'']} \|u_{\lambda_3, k}\|_{U^2[I'']} \end{aligned}$$

with  $C_1 = \min\{\mu^{\frac{1}{2}} \alpha^{-1} \lambda^{\frac{1}{2}}, 1\} \mu^{-\frac{1}{2}} \alpha^{\frac{1}{2}}$ . Adding the similar bounds for  $P_\lambda g_{I''}^2$  and  $P_\lambda g_{I''}^3$  we obtain

$$\|Q_{\ll \alpha \mu} P_\lambda f_{I''}^e\|_{DV_\Delta^2} \lesssim C_1 \alpha^{-\frac{3}{2}-2s} \mu^{-\frac{1}{2}} \|u_{\lambda_1, j}\|_{X_\lambda[I']} \|u_{\lambda_2, k}\|_{X_\lambda[I'']} \|u_{\lambda_3, k}\|_{X_\lambda[I'']}.$$

We combine this with the high modulation control derived from (67) to conclude that

$$\|P_\lambda f_{I''}^e\|_{DV_\Delta^2} \lesssim C_2 \|u_{\lambda_1, j}\|_{X_\lambda[I']} \|u_{\lambda_2, k}\|_{X_\lambda[I'']} \|u_{\lambda_3, k}\|_{X_\lambda[I'']} \tag{68}$$

where

$$C_2 = C_1 \alpha^{-\frac{3}{2}-2s} \mu^{-\frac{1}{2}} + \lambda^{\frac{1}{2}} \mu^{-1} \alpha^{-\frac{1}{2}}.$$

Since neither (67) nor (68) contain modulation localizations, we can truncate both of them to the interval  $I''$  and obtain the similar bounds for  $f_{I''}^d$ . Given the definition of the  $Y_\lambda$  space, from (67) nor (68) for  $f_{I''}^d$  we obtain

$$\|f_{I''}^d\|_{Y_\lambda} \lesssim \log(\mu/\lambda) \lambda^{1+2s} C_2 \|u_{\lambda_1, j}\|_{X_\lambda[I']} \|u_{\lambda_2, k}\|_{X_\lambda[I'']} \|u_{\lambda_3, k}\|_{X_\lambda[I'']} \tag{69}$$

where the logarithmic factor counts the number of dyadic regions between modulations  $\lambda^2$  and  $\alpha \mu$  arising in the transition from  $DV_\Delta^2$  to  $DU_\Delta^2$ . Since  $C = \log(\mu/\lambda) \lambda^{1+2s} \mu^{-2s} C_2$ , the bound (66) follows from (69) after summation over  $I'' \subset I$ .

**Case 2(d):**  $\lambda \ll \alpha \sim \mu$ . For the most part this case is identical to Case 2(c). The interaction is still nonresonant, i.e. either the output or at least one of the inputs must have modulation at least  $\mu^2$ . The high modulation output bound is unchanged. We have singled out this case because of a peculiarity which occurs when the middle (conjugated) factor is at high modulation (see (56) and (57)). This leads to a constant  $C_1 = (\lambda/\mu)^{\frac{1}{4}}$  in the bound for  $g_{I''}^2$ , which in turn yields a constant  $C$  in the counterpart of (66) of the form

$$C(\lambda, \mu, \mu) = \log(\mu/\lambda) (\lambda^{\frac{1}{2}} \mu^{-\frac{3}{2}-2s} + \mu^{-1-2s} \mu^{-1-2s} (\lambda/\mu)^{\frac{1}{4}}).$$

The first part is as in Case 2(c), but for the second we need to sum up with respect to  $\mu$  in the expression

$$\log(\mu/\lambda) \lambda^{1+3s} \mu^{-s} (\lambda/\mu)^{2s} \mu^{-1-2s} \mu^{-1-2s} (\lambda/\mu)^{\frac{1}{4}} = \log(\mu/\lambda) \mu^{7s+\frac{9}{4}} \lambda^{\frac{5}{4}+5s}$$

which is favorable since  $s > -9/28$ . We note that this is the worst among all cases we have considered.  $\square$

### 6. The energy conservation

In this section we study the weighted energy conservation for solutions  $u$  to (1). In order to keep the notations and the exposition as simple as possible, here and in the next section we will restrict ourselves to the endpoint case  $s = -\frac{1}{4}$ . This suffices in order to obtain the  $H^s$  energy estimates and to fully prove Theorem 1, but not the space–time estimates (4) for  $s > -\frac{1}{4}$ . The arguments here can easily be adapted to all larger  $s$ . The main result here is Proposition 6.2, which will be used in the last section of the paper to prove the first part of Proposition 1.6, namely the bound (18).

Given a positive multiplier  $a$  we set

$$E_0(u) = \langle A(D)u, u \rangle := \|u\|_{H^a}^2.$$

For the straight  $H_A^s$  energy conservation it suffices to take

$$a(\xi) = (\Lambda^2 + \xi^2)^s.$$

Here and in the sequel we denote the symbol of a Fourier multiplier  $A(D)$  by  $a(\xi)$ . However, as in [13], in order to gain the uniformity in  $t$  required by (25) we need to allow a slightly larger class of symbols.

**Definition 6.1.** (a) Let  $\Lambda \geq 1$ . Then  $S_\Lambda$  is the class of spherically symmetric symbols with the following properties:

(i) symbol regularity,

$$|\partial^\alpha a(\xi)| \lesssim a(\xi)(\Lambda^2 + \xi^2)^{-\alpha/2},$$

(ii) decay at infinity,

$$a(\xi) \geq (\Lambda^2 + \xi^2)^{-1/2},$$

and

$$[0, \infty) \ni \xi \rightarrow a(\xi)(\Lambda^2 + \xi^2)^{1/2}$$

is nondecreasing.

(b) If  $a$  satisfies (i) and (ii) then we say that  $d$  is dominated by  $a$ ,  $d \in S(a)$ , if

$$|\partial^\alpha d| \lesssim a(\Lambda^2 + \xi^2)^{-\alpha/2}$$

with constant depending only on  $\alpha$ .

For such symbols  $a$  we denote by  $X^a$  respectively  $X_{le}^a$  the spaces defined as  $X_\Lambda^{-\frac{1}{4}}$  respectively  $X_{\Lambda,le}^{-\frac{1}{4}}$  but with the symbol  $(\Lambda^2 + \lambda^2)^{-\frac{1}{4}}$  replaced by  $a(\lambda)$ . Here the spatial and temporal scales are the ones corresponding to  $s = -\frac{1}{4}$ , namely  $\delta x = 1, \delta t = \lambda^{-1}$ .

We compute the derivative of  $E_0$  along the flow,

$$\frac{d}{dt} E_0(u) = R_4(u) = 2\Re\{i A(D)u, |u|^2 u\}.$$

We write  $R_4$  as a multilinear operator in the Fourier space,

$$R_4(u) = 2\Re \int_{P_4} i a(\xi_0) \hat{u}(\xi_0) \hat{u}(\xi_1) \overline{\hat{u}(\xi_2) \hat{u}(\xi_3)} d\sigma$$

where

$$P_4 = \{\xi_0 + \xi_1 - \xi_2 - \xi_3 = 0\}.$$

This can be symmetrized,

$$R_4(u) = \frac{1}{2} \Re \int_{P_4} i (a(\xi_0) + a(\xi_1) - a(\xi_2) - a(\xi_3)) \hat{u}(\xi_0) \hat{u}(\xi_1) \overline{\hat{u}(\xi_2) \hat{u}(\xi_3)} d\sigma.$$

Following a variation of the  $I$ -method, see Tao [16] and references therein, we seek to cancel this term by perturbing the energy, namely by

$$E_1(u) = \int_{P_4} b_4(\xi_0, \xi_1, \xi_2, \xi_3) \hat{u}(\xi_0) \hat{u}(\xi_1) \overline{\hat{u}(\xi_2) \hat{u}(\xi_3)} d\sigma.$$

To determine the best choice for  $b_4$  we compute

$$\frac{d}{dt} E_1(u) = \int_{P_4} i b_4(\xi_0, \xi_1, \xi_2, \xi_3) (\xi_0^2 + \xi_1^2 - \xi_2^2 - \xi_3^2) \hat{u}(\xi_0) \hat{u}(\xi_1) \overline{\hat{u}(\xi_2) \hat{u}(\xi_3)} d\sigma + R_6(u)$$

where  $R_6(u)$  is given by

$$R_6(u) = 4\Re \int_{\xi_0 + \xi_1 - \xi_2 - \xi_3 = 0} i b_4(\xi_0, \xi_1, \xi_2, \xi_3) \widehat{|u|^2 u}(\xi_0) \hat{u}(\xi_1) \overline{\hat{u}(\xi_2) \hat{u}(\xi_3)} d\sigma.$$

To achieve the cancellation of the quadrilinear form we define  $b_4$  by

$$b_4(\xi_0, \xi_1, \xi_2, \xi_3) = -\frac{a(\xi_0) + a(\xi_1) - a(\xi_2) - a(\xi_3)}{\xi_0^2 + \xi_1^2 - \xi_2^2 - \xi_3^2} \quad \text{for } (\xi_0, \xi_1, \xi_2, \xi_3) \in P_4. \tag{70}$$

We will see in Proposition 6.2 that this is well defined. Summing up the result of our computation, we obtain

$$\frac{d}{dt}(E_0(u) + E_1(u)) = R_6(u). \tag{71}$$

We integrate this relation to estimate  $E_0(u) = \|u\|_{H^a}^2$  uniformly in time:

**Proposition 6.2.** *Let  $a \in S_\Lambda$ . Then for any  $L^2$  solution  $u$  to (1) in the time interval  $[0, 1]$  we have*

$$\|u\|_{L^\infty H^a}^2 \lesssim \|u_0\|_{H^a}^2 + \|u\|_{L^\infty H^a}^2 \|u\|_{L^\infty H_\Lambda^{-\frac{1}{2}}}^2 + \|u\|_{X^a \cap X_{le}^a}^2 \|u\|_{X_\Lambda^{-\frac{1}{4}} \cap X_{\Lambda, le}^{-\frac{1}{4}}}^4. \tag{72}$$

The proposition follows directly from the bounds for  $E_1(u)$  and of  $R_6(u)$  in Lemmas 6.4, 6.5 below. The aim of the rest of this section is to prove these two lemmas.

In order to estimate the size of  $E_1(u)$  and of  $R_6(u)$  we need to understand the size and regularity of  $b$ . A priori  $b$  is only defined on the diagonal  $P_4$ . However, in order to separate variables it is convenient to extend it off diagonal in a favorable way. The next lemma is a more precise version of a similar result in [13]; the additional information is needed for the proof of the local energy decay estimates in the next section.

**Proposition 6.3.** *Assume that  $a \in S_\Lambda$  and  $d \in S(a)$ . Then there exist functions  $b_4$  and  $c_4$  such that*

$$d(\xi_0) + d(\xi_1) - d(\xi_2) - d(\xi_3) = b_4(\xi_0, \xi_1, \xi_2, \xi_3)(\xi_0^2 + \xi_1^2 - \xi_2^2 - \xi_3^2) + c_4(\xi_0, \xi_1, \xi_2, \xi_3)(\xi_0 + \xi_1 - \xi_2 - \xi_3)$$

which for each dyadic

$$\lambda \leq \alpha \leq \mu, \quad \xi_0 \sim \lambda, \quad \xi_2 \sim \alpha, \quad \xi_1, \xi_3 \sim \mu \tag{73}$$

satisfy the size and regularity conditions

$$\begin{aligned} |\partial_0^{\beta_0} \partial_1^{\beta_1} \partial_2^{\beta_2} \partial_3^{\beta_3} b_4(\xi_0, \xi_1, \xi_2, \xi_3)| &\lesssim a(\lambda) \alpha^{-1} \mu^{-1} \lambda^{-\beta_0} \alpha^{-\beta_2} \mu^{-\beta_1 - \beta_3}, \\ |\partial_0^{\beta_0} \partial_1^{\beta_1} \partial_2^{\beta_2} \partial_3^{\beta_3} c_4(\xi_0, \xi_1, \xi_2, \xi_3)| &\lesssim a(\lambda) \alpha^{-1} \lambda^{-\beta_0} \alpha^{-\beta_2} \mu^{-\beta_1 - \beta_3} \end{aligned} \tag{74}$$

with implicit constants dependent on the  $\beta_j$ 's but independent of  $\lambda, \alpha, \mu$ .

**Proof.** We first note that we have the formula

$$\xi_0^2 + \xi_1^2 - \xi_2^2 - \xi_3^2 = 2(\xi_0 - \xi_2)(\xi_0 - \xi_3) - (\xi_0 + \xi_1 - \xi_2 - \xi_3)(\xi_0 - \xi_1 - \xi_2 - \xi_3). \tag{75}$$

In particular we obtain the factorization

$$\xi_0^2 + \xi_1^2 - \xi_2^2 - \xi_3^2 = 2(\xi_0 - \xi_2)(\xi_0 - \xi_3) \quad \text{on } P_4$$

along with all versions of it due to the symmetries. It suffices to construct  $b_4$  and  $c_4$  locally in dyadic regions, and then sum up the results using an appropriate partition of unity. We consider several cases:

(a)  $\lambda \ll \alpha \leq \mu$ . Then  $\xi_1, \xi_3 \sim \mu$  and  $|\xi_0 - \xi_2| \sim \alpha$ . Then the extension of  $b_4$  is defined using the formula

$$b_4(\xi_0, \xi_1, \xi_2, \xi_3) = \frac{d(\xi_0) + d(\xi_1) - d(\xi_2) - d(\xi_3)}{2(\xi_0 - \xi_2)(\xi_0 - \xi_3)}.$$

Its size and regularity properties are straightforward since  $|\xi_0 - \xi_2| \approx \alpha$  and  $|\xi_0 - \xi_3| \approx \mu$ . By (75) we obtain

$$c_4 = \frac{d(\xi_0) + d(\xi_1) - d(\xi_2) - d(\xi_3) - b_4(\xi_0^2 + \xi_1^2 - \xi_2^2 - \xi_3^2)}{\xi_0 + \xi_1 - \xi_2 - \xi_3} = b_4(\xi_0 - \xi_1 - \xi_2 - \xi_3).$$

The bounds for  $c_4$  are also obvious.

(b)  $\lambda \approx \alpha \ll \mu$ . Then  $|\xi_0|, |\xi_2| \sim \alpha$  and  $|\xi_1|, |\xi_3| \sim \mu$ . We define the extension of  $b_4$  using the formula

$$b_4(\xi_0, \xi_1, \xi_2, \xi_3) = \frac{d(\xi_0) - d(\xi_2)}{2(\xi_0 - \xi_2)(\xi_0 - \xi_3)} + \frac{d(\xi_1) - d(\xi_3)}{2(\xi_3 - \xi_1)(\xi_3 - \xi_0)}$$

and, as above,

$$c_4 = \frac{d(\xi_0) - d(\xi_2)}{2(\xi_0 - \xi_2)(\xi_0 - \xi_3)}(\xi_0 - \xi_1 - \xi_2 - \xi_3) + \frac{d(\xi_1) - d(\xi_3)}{2(\xi_3 - \xi_1)(\xi_3 - \xi_0)}(\xi_3 - \xi_0 - \xi_1 - \xi_2).$$

Again the estimates are immediate.

(c)  $\lambda \approx \alpha \approx \mu$ . We define the extension of  $b_4$  by

$$\begin{aligned} b_4(\xi_0, \xi_1, \xi_2, \xi_3) &= \frac{d(\xi_1) - d(\xi_3)}{2(\xi_3 - \xi_0)(\xi_3 - \xi_1)} + \frac{d(\xi_0) - d(\xi_1 + (\xi_0 - \xi_3))}{2(\xi_3 - \xi_0)(\xi_3 - \xi_1)} \\ &= \frac{q(\xi_1, \xi_3)}{2(\xi_0 - \xi_3)} + \frac{q(\xi_1 + (\xi_0 - \xi_3), \xi_1 + (\xi_0 - \xi_3))}{2(\xi_1 - \xi_3)} \end{aligned}$$

where  $q$  is the smooth function

$$q(\xi, \eta) = \frac{d(\xi) - d(\eta)}{\xi - \eta}.$$

Then

$$c_4 = b_4(\xi_3 - \xi_0 - \xi_1 - \xi_2)$$

and the estimates follow immediately.  $\square$

Using the above lemma, the contribution of  $E_1$  to the energy is easy to control:

**Lemma 6.4.** Assume that  $a \in S_A$ . Then

$$|E_1(u)| \lesssim E_0(u) \|u\|_{H_A^{-1/2}}^2. \tag{76}$$

**Proof.** The proof is easier than the proof of the more essential result below. Nevertheless it introduces some useful techniques. We expand the quadrilinear expression in the dyadic frequency components. Then for  $\lambda \leq \alpha \leq \mu$  we consider the expression

$$\left| \int b_4(\xi_0, \xi_1, \xi_2, \xi_0 + \xi_1 - \xi_2) \hat{u}_\lambda \hat{u}_\mu \overline{\hat{u}_\alpha} \hat{u}_\mu d\xi_0 d\xi_1 d\xi_2 \right| \tag{77}$$

where the ranges of the  $\xi_j$ 's are as in (73). In this range we can express  $b_4$  in the form

$$b_4(\xi_0, \xi_1, \xi_2, \xi_3) = a(\lambda)\alpha^{-1}\mu^{-1}\eta(\xi_0/\lambda, \xi_1/\mu, \xi_2/\alpha, \xi_3/\mu), \tag{78}$$

where  $\eta$  is compactly supported and smooth with bounds independent of  $\lambda, \alpha$  and  $\mu$ .

Due to (74) we can expand  $\eta$  into a rapidly convergent Fourier series. Since complex exponentials are products of complex exponentials in the coordinates, and since multiplication by a complex exponential of the Fourier transform corresponds to a translation in  $x$  space we can separate variables and reduce the problem to the case when  $b_4$  is simply replaced by  $a(\lambda)\lambda^{-1}\alpha^{-1}$ . Then using Bernstein to bound the low frequency factors in  $L^\infty$  we obtain for the expression in (77)

$$(77) = a(\lambda)\alpha^{-1}\mu^{-1} \int u_\lambda u_\mu \overline{u_\alpha} u_\mu dx \lesssim a(\lambda)\lambda^{\frac{1}{2}}\alpha^{-\frac{1}{2}}\mu^{-1} \|u_\lambda\|_{L^2} \|u_\alpha\|_{L^2} \|u_\mu\|_{L^2} \|u_\mu\|_{L^2}.$$

We estimate the high frequencies in  $H_A^{-1/2}$  and sum with respect to  $\lambda, \alpha$  and  $\mu$ .  $\square$

The more difficult result we need to prove is

**Lemma 6.5.** *Assume that  $a \in S_A$  is as above. Then we have*

$$\left| \int_0^1 R_6(u) dx dt \right| \lesssim \|u\|_{X^a \cap X_{le}^a}^2 \|u\|_{X_A^{-1/4} \cap X_{A,le}^{-1/4}}^4. \tag{79}$$

**Proof.** We consider a full dyadic decomposition of all factors and express the above integral in the Fourier space as a sum of terms of the form

$$K = \int_0^1 \int_{P_6} b_4(\xi_1, \xi_2, \xi_3, \xi_0) \hat{u}_{\lambda_1}(\xi_1) \overline{\hat{u}_{\lambda_2}(\xi_2)} \hat{u}_{\lambda_3}(\xi_3) P_{\lambda_0}(\overline{\hat{u}_{\lambda_4}(\xi_4)} \hat{u}_{\lambda_5}(\xi_5) \overline{\hat{u}_{\lambda_6}(\xi_6)}) d\xi dt$$

where

$$P_6 = \{\xi_1 + \xi_3 + \xi_5 = \xi_2 + \xi_4 + \xi_6\}, \quad \xi_0 = \xi_1 - \xi_2 + \xi_3.$$

For each of the dyadic factors  $u_{\lambda_j}$  we will only use the  $U_{\Delta}^2$  norm, which is controlled as in (23).

As in the previous lemma, since  $b_4$  is smooth in each variable on the corresponding dyadic scale we can expand it into a rapidly convergent Fourier series. This allows us to separate variables and reduce the problem to the case when  $b_4$  has separated variables,

$$b_4(\xi_1, \xi_2, \xi_3, \xi_0) = a(\lambda)\alpha^{-1}\mu^{-1}\chi^1(\xi_1)\chi^2(\xi_2)\chi^3(\xi_3)\chi^0(\xi_0)$$

where  $\chi_i$ 's are unit size bumps which are smooth on the respective dyadic scales and  $\{\lambda_1, \lambda_2, \lambda_3, \lambda_0\} = \{\lambda, \alpha, \mu, \mu\}$ . By definition  $\chi^i(D)$  are bounded in the  $X_{\lambda}$  spaces, therefore we can discard  $\chi^1, \chi^2$  and  $\chi^3$  and incorporate  $\chi^0$  into  $P_{\lambda_0}$ .

Similarly to above we may expand the Fourier multiplier  $\eta$  for  $P_{\lambda_0}$  into a Fourier integral. We obtain for a Schwartz function  $\rho$

$$\eta(\xi_0) = \int \rho(y) e^{i\xi_0 y / \lambda_0} dy.$$

Since in the domain of integration for  $K$  we have  $\xi_0 = \xi_1 - \xi_2 + \xi_3$ , we can separate the exponential into three factors which can be harmlessly absorbed into  $u_{\lambda_1}, u_{\lambda_2}$  and  $u_{\lambda_3}$ . Thus we may as above simply drop  $P_{\lambda_0}$  whenever we wish to do so. We discard  $P_{\lambda_0}$  if  $\lambda_0$  is large. On the other hand, if it is smaller than  $\lambda_4, \lambda_5$  and  $\lambda_6$  then we keep it to get better estimates. The disadvantage in that case is that  $P_0$  prevents us from using bilinear  $L^2$  estimates for factors located across  $P_{\lambda_0}$ . To summarize, we have reduced the problem to the case when  $K$  has the form

$$K = a(\lambda)\alpha^{-1}\mu^{-1} \int_0^1 \int_{\mathbb{R}} u_{\lambda_1} \overline{u_{\lambda_2}} u_{\lambda_3} P_{\lambda_0}(\overline{u_{\lambda_4}} u_{\lambda_5} \overline{u_{\lambda_6}}) dx dt$$

where we have the additional freedom to discard  $P_{\lambda_0}$  as needed. The placement of the complex conjugates is irrelevant here therefore we may always assume without any restriction in generality that

$$\lambda_1 \leq \lambda_2 \leq \lambda_3, \quad \lambda_4 \leq \lambda_5 \leq \lambda_6.$$

It is also convenient to reorganize the indices in an increasing fashion

$$\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\} = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6\}, \quad \mu_5 \sim \mu_6.$$

We also recall that  $\lambda, \alpha$  and  $\mu$  are given by the increasing rearrangement

$$\{\lambda_0, \lambda_1, \lambda_2, \lambda_3\} = \{\lambda, \alpha, \mu, \mu\}.$$

The  $X_A^{-1/4}$  norms involve space and time localizations. We will disregard those at first and consider the simpler question of estimating the integral



$$L = \int_{\mathbb{R}^2} u_{\lambda_1} \overline{u_{\lambda_2}} u_{\lambda_3} P_{\lambda_0}(\overline{u_{\lambda_4}} u_{\lambda_5} \overline{u_{\lambda_6}}) dx dt$$

in terms of the  $U_{\Delta}^2$  norm of each factor,

$$|L| \lesssim C_L \prod_{j=1}^6 \|u_{\lambda_j}\|_{U_{\Delta}^2}, \quad C_L = C_L(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6). \tag{80}$$

In all cases  $C$  will have a polynomial dependence on the  $\lambda$ 's and a zero order homogeneity. Before we set to the task of estimating  $C$  in all the cases, we consider the simpler question of the transition from the estimate for  $L$  to the estimate for  $K$ . Precisely, we claim that (80) implies that

$$|K| \lesssim C_K \prod_{j=1}^2 \|u_{\mu_j}\|_{X^a} \prod_{j=3}^6 \|u_{\mu_j}\|_{X_{\Lambda}^{-\frac{1}{4}} \cap X_{\Lambda,le}^{-\frac{1}{4}}}, \quad C_K = \frac{a(\lambda)}{\sqrt{a(\mu_1)a(\mu_2)} \alpha \mu} (\mu_3 \mu_4 \mu_5 \mu_6)^{\frac{1}{4}} (\mu_1 \mu_2)^{\frac{1}{2}} C_L. \tag{81}$$

Compared to  $C_L$ , the constant  $C_K$  contains the additional trivial frequency factors coming from the Sobolev regularity, plus the more interesting factor  $(\mu_1 \mu_2)^{\frac{1}{2}}$  coming from the time interval summation. For  $C_K$  we want to have  $C_K \leq 1$ , plus some additional off-diagonal decay to allow for the summation with respect to all  $\lambda_j$ 's. We remark that since  $s = -\frac{1}{4}$ ,  $C_K$  has homogeneity zero if  $a$  is homogeneous, therefore we do not have room for any losses.

We now prove that (80) implies (81). For convenience we simply omit the prefactor  $a(\lambda)/(\alpha \mu)$  in  $K$ , which plays no role here. We decompose each factor  $u_{\lambda_j}$  in space on the unit scale and in time according to the  $\delta t_{\lambda_j} = \lambda_j^{-1}$  scale, while preserving the frequency localization:

$$u_{\lambda_j} = \sum_{|I_j|=\lambda_j^{-1}} \sum_{k_j \in \mathbb{Z}} u_{\lambda_j, k_j}^{I_j}, \quad u_{\lambda_j, k_j}^{I_j} = \chi_{I_j} \tilde{P}_{\lambda_j}(\chi_{k_j} u_{\lambda_j}).$$

Then  $K$  is decomposed into

$$K = \sum_{I_j \text{ nested}} \sum_{|I_j|=\lambda_j^{-1}} \sum_{k_j \in \mathbb{Z}} K((I_j)_{j=1, \dots, 6}, (k_j)_{j=1, \dots, 6}),$$

where

$$K((I_j), (k_j)) = \int_0^1 \int_{\mathbb{R}} u_{\lambda_1, k_1}^{I_1} \overline{u_{\lambda_2, k_2}^{I_2}} u_{\lambda_3, k_3}^{I_3} P_{\lambda_0}(\overline{u_{\lambda_4, k_4}^{I_4}} u_{\lambda_5, k_5}^{I_5} \overline{u_{\lambda_6, k_6}^{I_6}}) dx dt.$$

For these components we claim that we have

$$|K((I_j), (k_j))| \lesssim (1 + \max |k_i - k_j|)^{-N} C_L \prod_{j=1}^6 \|\chi_{k_j} u_{\lambda_j}\|_{U^2[I_j]}. \tag{82}$$

If  $\max |k_i - k_j| \lesssim 1$  then this follows directly from (80). Otherwise we further decompose  $K((I_j), (k_j))$  as

$$K((I_j), (k_j)) = \sum_k \int_0^1 \int_{\mathbb{R}} \chi_k u_{\lambda_1, k_1}^{I_1} \overline{u_{\lambda_2, k_2}^{I_2}} u_{\lambda_3, k_3}^{I_3} P_{\lambda_0}(\overline{u_{\lambda_4, k_4}^{I_4}} u_{\lambda_5, k_5}^{I_5} \overline{u_{\lambda_6, k_6}^{I_6}}) dx dt.$$

For each  $k \in \mathbb{Z}$  we can find some  $j$  so that  $|k - k_j| \gtrsim \max\{k_i - k_j\}$ . To keep the notations simple let us take  $j = 1$ . Then we apply the bound (59) to  $\chi_k u_{\lambda_1, k_1}^{I_1}$ ; this shows that

$$\|P_{\mu} \chi_k u_{\lambda_1, k_1}^{I_1}\|_{U_{\Delta}^2} \lesssim \lambda_1^{-N} \mu^{-N} (\max\{k_i - k_j\} + |k - k_1|)^{-N} \|\chi_{k_1} u_{\lambda_1}\|_{U_{\Delta}^2[I_1]}.$$

Hence (82) follows by applying (80) to each of the terms in the above sum after a summation with respect to  $\mu$  and  $k$ .

We obtain the bound for  $K$  by summing (82) over the nested intervals  $I_j$  and  $k_j$ . We switch the frequencies to the  $\mu_j$  notation. Using the fact that  $\mu_5 = \mu_6$  and therefore  $I_5 = I_6$ , by Cauchy–Schwarz we obtain

$$\begin{aligned} |K| &\lesssim \sum_{\{k_i\}} (1 + \max |k_i - k_m|)^{-N} \prod_{j=1}^4 \max_{I_j} \|\chi_{k_j} u_{\mu_j}\|_{U_{\Delta}^2[I_j]} \prod_{j=5}^6 \left( \sum_{I_j} \|\chi_{k_j} u_{\mu_j}\|_{U_{\Delta}^2[I_j]}^2 \right)^{\frac{1}{2}} \\ &\lesssim \prod_{j=1}^2 \left( \sum_k \max_{I_j} \|\chi_k u_{\mu_j}\|_{U_{\Delta}^2[I_j]}^2 \right)^{\frac{1}{2}} \prod_{j=3}^4 \sup_k \max_{I_j} \|\chi_k u_{\mu_j}\|_{U_{\Delta}^2[I_j]} \prod_{j=5}^6 \sup_k \left( \sum_{I_j} \|\chi_k u_{\mu_j}\|_{U_{\Delta}^2[I_j]}^2 \right)^{\frac{1}{2}} \\ &\lesssim \prod_{j=1}^2 \left( \sum_{I_j} \sum_k \|\chi_k u_{\mu_j}\|_{U_{\Delta}^2[I_j]}^2 \right)^{\frac{1}{2}} \prod_{j=3}^6 \sup_k \left( \sum_{I_j} \|\chi_k u_{\mu_j}\|_{U_{\Delta}^2[I_j]}^2 \right)^{\frac{1}{2}} \\ &= (a(\mu_1)a(\mu_2))^{-\frac{1}{2}} (\mu_3\mu_4\mu_5\mu_6)^{\frac{1}{4}} (\mu_1\mu_2)^{\frac{1}{2}} \prod_{j=1}^2 \left( \sum_{I_j} \|u_{\mu_j}\|_{X^a}^2 \right)^{\frac{1}{2}} \prod_{j=3}^6 \|u_{\mu_j}\|_{X_{le}^{-\frac{1}{4}}}. \end{aligned}$$

Thus (81) is proved. It remains to estimate the constant  $C_L$  in (80). We need to distinguish two cases:

**Case A.**  $\lambda_0 \geq \mu_2$ . In this case we must have  $\lambda \geq \mu_1$ ,  $\alpha \geq \mu_2$  and  $\mu \geq \mu_3$ . We claim the following bound

$$C_L \lesssim (\mu_1\mu_3)^{\frac{1}{2}} \mu_6^{-1}. \tag{83}$$

This is not optimal in many cases, but it suffices for our purposes. In particular by (81) it implies that

$$C_K \lesssim \frac{(\mu_1\mu_3)^{\frac{1}{2}} a(\mu_1)}{\mu_6 \mu_2\mu_3} (\mu_1\mu_2)^{\frac{1}{2}} (a(\mu_1)a(\mu_2))^{-\frac{1}{2}} \mu_3^{\frac{1}{4}} \mu_4^{\frac{1}{4}} \mu_6^{\frac{1}{2}} \leq \frac{\mu_1 \sqrt{a(\mu_1)} \mu_2^{\frac{1}{4}}}{\mu_2 \sqrt{a(\mu_2)} \mu_6^{\frac{1}{4}}} \leq 1.$$

Note that we have rapid decay off the “diagonal”  $\lambda_0 = \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6$ , which suffices for the dyadic summation.

To prove (83) we drop  $P_{\lambda_0}$  and we consider three subcases:

**Case A1.**  $\mu_3 \ll \mu_6$ . Then we can use two bilinear  $L^2$  and two Bernstein to obtain the stronger bound

$$C_L \lesssim (\mu_1\mu_2)^{\frac{1}{2}} \mu_6^{-1}.$$

**Case A2.**  $\mu_1 \ll \mu_3 \sim \mu_6$ . Then we can only use a single bilinear  $L^2$  bound, one Bernstein and three  $L^6$  bounds to get

$$C_L \lesssim \mu_1^{\frac{1}{2}} \mu_6^{-\frac{1}{2}}$$

which still implies (83).

**Case A3.**  $\mu_1 \sim \mu_6$ . Then we simply use six  $L^6$  bounds to show that  $C_K \lesssim 1$ .

**Case B.**  $\lambda_0 \ll \mu_2$ . In this case we claim that the following bound holds:

$$C_L \lesssim \lambda_0 \mu_4^{-\frac{1}{2}} \mu_6^{-\frac{1}{2}}. \tag{84}$$

To see that this suffices we consider two cases. If  $\lambda_0 \ll \mu_1$  then we have  $\lambda = \lambda_0$ ,  $\alpha \geq \mu_1$  and  $\mu \geq \mu_3$ . Then by (81) we obtain

$$C_K \lesssim \frac{\lambda_0 a(\lambda_0)}{\mu_4^{\frac{1}{2}} \mu_6^{\frac{1}{2}} \mu_1 \mu_3} (\mu_1\mu_2)^{\frac{1}{2}} (a(\mu_1)a(\mu_2))^{-\frac{1}{2}} \mu_3^{\frac{1}{4}} \mu_4^{\frac{1}{4}} \mu_6^{\frac{1}{2}} \leq \frac{\lambda_0 a(\lambda_0) \sqrt{\mu_1 a(\mu_1)} \mu_2^{\frac{1}{4}}}{\mu_1 a(\mu_1) \sqrt{\mu_2 a(\mu_2)} \mu_4^{\frac{1}{4}}} \leq 1.$$

On the other hand if  $\mu_1 \leq \lambda_0 \ll \mu_2$  then we have  $\lambda \geq \mu_1$ ,  $\alpha \geq \lambda_0$  and  $\mu \geq \mu_3$ . Then by (81) we obtain

$$C_K \lesssim \frac{\lambda_0 a(\mu_1)}{\mu_4^{\frac{1}{2}} \mu_6^{\frac{1}{2}} \lambda_0 \mu_3} (\mu_1\mu_2)^{\frac{1}{2}} (a(\mu_1)a(\mu_2))^{-\frac{1}{2}} \mu_3^{\frac{1}{4}} \mu_4^{\frac{1}{4}} \mu_6^{\frac{1}{2}} \leq \frac{\sqrt{\mu_1 a(\mu_1)} \mu_2^{\frac{1}{4}}}{\sqrt{\mu_2 a(\mu_2)} \mu_4^{\frac{1}{4}}} \leq 1.$$

In both cases we have decay off the expanded diagonal  $\lambda_0 = \mu_1 = \mu_2 = \mu_3 = \mu_4, \mu_5 = \mu_6$ , which still suffices for the dyadic summation.

It remains to prove the bound (84). If  $\lambda_0 \ll \mu_2$  then we must have  $\lambda_0 \ll \lambda_2 = \lambda_3$  and  $\lambda_0 \ll \lambda_5 = \lambda_6$ . By symmetry we can assume that  $\lambda_6 = \mu_6$ . Then we have two cases to consider:

**Case B1.**  $\lambda_3 \gtrsim \mu_4$ . Applying twice (58) we have

$$\|P_{\lambda_0}(u_{\lambda_1} \bar{u}_{\lambda_2} u_{\lambda_3})\|_{L^2} \lesssim \lambda_0^{\frac{1}{2}} \lambda_3^{-\frac{1}{2}} \prod_{j=1^3} \|u_{\lambda_j}\|_{U_{\Delta}^2}, \quad \|P_{\lambda_0}(u_{\lambda_4} \bar{u}_{\lambda_5} u_{\lambda_6})\|_{L^2} \lesssim \lambda_0^{\frac{1}{2}} \lambda_6^{-\frac{1}{2}} \prod_{j=4^6} \|u_{\lambda_j}\|_{U_{\Delta}^2}$$

which imply (84).

**Case B2.**  $\lambda_3 \ll \mu_4$ . Then the frequencies must be ordered as follows:

$$\lambda_1 \lesssim \lambda_2 = \lambda_3 \ll \lambda_4 \lesssim \lambda_5 = \lambda_6.$$

The key observation here is that, regardless of the presence of  $P_{\lambda_0}$ , the multilinear interaction in  $L$  is nonresonant, i.e. at least one of the factors must have high modulation  $\gtrsim \lambda_4 \lambda_6$ . This is similar to the proof of (53). For the following argument it does not matter which is the high factor modulation factor. To fix the notations we assume this is the  $\lambda_4$  factor; this is actually the worst case. Then we write

$$\begin{aligned} |L| &\lesssim \lambda_0 \|u_{\lambda_1} u_{\lambda_2} u_{\lambda_3}\|_{L^4 L^1} \|Q_{>\lambda_4 \lambda_6} u_{\lambda_4} u_{\lambda_5} u_{\lambda_6}\|_{L^{\frac{4}{3}} L^1} \\ &\lesssim \lambda_0 \|u_{\lambda_1}\|_{L^\infty L^2} \|Q_{>\lambda_4 \lambda_6} u_{\lambda_4}\|_{L^2} \prod_{j=2,3,5,6} \|u_j\|_{L^8 L^4} \\ &\lesssim \lambda_0 (\lambda_4 \lambda_6)^{-\frac{1}{2}} \prod_{j=1}^6 \|u_j\|_{U_{\Delta}^2}. \end{aligned}$$

This gives (84) in this case, and concludes the proof of the lemma.  $\square$

### 7. Local energy decay

In this section we consider the weighted local energy decay estimates for (1). Our main goal is to prove Proposition 7.5, which is the local energy counterpart of Proposition 6.2 in the previous section. Proposition 7.5, together with Proposition 6.2, will be used in the last section to derive the second part of Proposition 1.6, namely the bound (19). To keep the argument as simple as possible, in this section we only consider the extreme case  $s = -\frac{1}{4}$ . The benefit of doing this is that at  $s = \frac{1}{4}$  we can work with the same unit spatial scale for all frequencies.

Let  $\phi$  be an odd smooth function whose derivative has the form  $\phi' = \psi^2$  where  $\psi$  is positive, with rapidly decaying and with Fourier transform supported in  $[-1, 1]$ . Let  $a$  be as in the previous section. We define an odd monotone smooth function  $\tilde{a} \in S_{\Lambda}(a)$  by

$$\tilde{a} = \begin{cases} a(\xi) & \text{if } \xi > \Lambda, \\ -a(\xi) & \text{if } \xi < -\Lambda, \\ \Lambda^{-1} \xi a(\xi) & \text{if } |\xi| < \Lambda/2 \end{cases}$$

and consider the indefinite quadratic form

$$\tilde{E}_0(u) = \frac{1}{2} \int (\phi \tilde{a}(D) + \tilde{a}(D) \phi) u \bar{u} \, dx.$$

A small modification of the calculation of the previous section gives

$$\frac{d}{dt} \tilde{E}_0(u) = \tilde{R}_2(u) \pm \tilde{R}_4(u)$$

where

$$\begin{aligned} \tilde{R}_2(u) &= i \langle (\phi \tilde{a}(D) + \tilde{a}(D) \phi) u_{xx}, u \rangle - i \langle (\phi \tilde{a}(D) + \tilde{a}(D) \phi) u, u_{xx} \rangle \\ &= \langle (\phi' \tilde{a}(D) + \tilde{a}(D) \phi') Du, u \rangle + \langle (\phi' \tilde{a}(D) + \tilde{a}(D) \phi') u, Du \rangle \end{aligned}$$

and

$$\tilde{R}_4(u) = 2\Re\{i(\tilde{a}(D)\phi + \phi\tilde{a}(D))u, |u|^2u\}.$$

The term  $\tilde{R}_2$ , which was zero in the computation of the previous section, has a positive principal symbol and will be used to measure the local energy.

We now turn our attention to the quadrilinear form  $\tilde{R}_4$ . In the Fourier space we represent this term in the form

$$\begin{aligned} \tilde{R}_4(u) &= \int_{\mathbb{R}} \phi(x) e^{ix\xi} \int_{P_\xi} i(\tilde{a}(\xi_0 - \xi) + \tilde{a}(\xi_0)) \hat{u}(\xi_0) \hat{u}(\xi_1) \overline{\hat{u}(\xi_2) \hat{u}(\xi_3)} d\xi_i d\xi dx \\ &= \int_{\mathbb{R}} \phi(x) e^{ix\xi} \int_{P_\xi} i\tilde{a}_4(\xi_0, \xi_1, \xi_2, \xi_3) \hat{u}(\xi_0) \hat{u}(\xi_1) \overline{\hat{u}(\xi_2) \hat{u}(\xi_3)} d\xi_i d\xi dx \end{aligned}$$

where

$$P_\xi = \{\xi_0 + \xi_1 - \xi_2 - \xi_3 = \xi\}$$

and the symbol  $\tilde{a}_4$  is obtained by symmetrizing the symbol  $\tilde{a}(\xi_0 - \xi) + \tilde{a}(\xi_0)$ . We can view this as a function of  $\xi_0$  with a smooth dependence on the parameter  $\xi$ , which is invariant with respect to the symmetrization. Here we only need uniformity with respect to  $\xi$  in a compact set  $[-1, 1]$ . Hence we can apply Proposition 6.3, keeping  $\xi$  as a parameter, in order to represent the symbol  $\tilde{a}_4$  in the form

$$\tilde{a}_4 = \tilde{b}_4(\xi_0^2 + \xi_1^2 - \xi_2^2 - \xi_3^2) + \tilde{c}_4(\xi_0 + \xi_1 - \xi_2 - \xi_3) = \tilde{b}_4(\xi_0^2 + \xi_1^2 - \xi_2^2 - \xi_3^2) + \tilde{c}_4\xi$$

where  $\tilde{b}_4$  and  $\tilde{c}_4$  are viewed as functions of  $\xi_0, \xi_1, \xi_2, \xi_3$  and  $\xi$  which are smooth in  $\xi$  in a compact set and are smooth on dyadic scales  $|\xi_j| \sim \lambda_j$  and have size

$$\tilde{b}_4 \sim a(\lambda)\alpha^{-1}\mu^{-1}, \quad \tilde{c}_4 \sim a(\lambda)\alpha^{-1}, \quad \{\lambda_0, \lambda_1, \lambda_2, \lambda_3\} = \{\lambda, \alpha, \mu, \mu\}, \quad \lambda \leq \alpha \leq \mu.$$

This leads to a decomposition

$$\tilde{R}_4(u) = \tilde{B}_4(u) + \tilde{C}_4(u).$$

The  $\tilde{C}_4$  term is better behaved, as one can see in the following integration by parts:

$$\begin{aligned} \tilde{C}_4(u) &= \int \phi(x) e^{ix\xi} \xi \int_{P_\xi} i\tilde{c}_4(\xi_0, \xi_1, \xi_2, \xi_3, \xi) u(\xi_1) u(\xi_2) \overline{u(\xi_3) u(\xi_4)} d\xi_1 d\xi_2 d\xi_3 d\xi dx \\ &= - \int \phi'(x) e^{ix\xi} \int_{P_\xi} \tilde{c}_4(\xi_0, \xi_1, \xi_2, \xi_3, \xi) u(\xi_0) u(\xi_1) \overline{u(\xi_2) u(\xi_3)} d\xi_1 d\xi_2 d\xi_3 dx d\xi dx. \end{aligned}$$

We will estimate  $\tilde{C}_4$  directly. For the  $\tilde{B}_4$  term, on the other hand, we introduce an energy correction

$$\tilde{E}_1(u) = \int \phi(x) e^{ix\xi} \int_{P_\xi} \tilde{b}_4(\xi_0, \xi_1, \xi_2, \xi_3, \xi) u(\xi_0) u(\xi_1) \overline{u(\xi_2) u(\xi_3)} d\xi_1 d\xi_2 d\xi_3 d\xi dx.$$

Then

$$\frac{d}{dt} \tilde{E}_1(u) = \tilde{B}_4(u) \pm \tilde{R}_6(u)$$

where  $\tilde{R}_6(u)$  is given by

$$R_6(u) = 2\Re \int \phi(x) e^{ix\xi} \int_{P_\xi} i b_4(\xi_0, \xi_1, \xi_2, \xi_3) \widehat{|u|^2 u}(\xi_0) \hat{u}(\xi_1) \overline{\hat{u}(\xi_2) \hat{u}(\xi_3)} d\xi_j d\xi dx.$$

With all the notations above, our full energy relation reads

$$\frac{d}{dt}(\tilde{E}_0(u) \mp \tilde{E}_1(u)) = \tilde{R}_2(u) \pm \tilde{C}_4(u) \mp \tilde{R}_6(u)$$

where the choice of the signs depends on the focusing or defocusing character of the problem, and plays no role in our analysis. Our goal is to use this relation to estimate the time integral of  $\tilde{R}_2(u)$ , which in turn controls the local energy. Integrating between 0 and 1 we obtain

$$\int_0^1 \tilde{R}_2(u) dt = (\tilde{E}_0(u) \mp \tilde{E}_1(u))\Big|_0^1 \mp \int_0^1 \tilde{C}_4(u) dt \pm \int_0^1 \tilde{R}_6(u) dt. \tag{85}$$

Following the steps in the previous section we bound the terms on the right. We begin with  $\tilde{E}_0$  and  $\tilde{E}_1$ :

**Lemma 7.1.** *Let  $a \in S_\Lambda$  and  $\tilde{a}, \phi$  be as above. Then at fixed time we have*

$$|\tilde{E}_0(u)| \lesssim E_0(u) \tag{86}$$

respectively

$$|\tilde{E}_1(u)| \lesssim E_0(u) \|u\|_{H_\Lambda^{-1/2}}^2. \tag{87}$$

**Proof.** The proof repeats the proof of Lemma 6.4. One begins with a Littlewood–Paley decomposition. Separating variables the symbols  $\tilde{a}$  and  $\tilde{b}_4$  can be replaced by their sizes for each dyadic piece. Once this is done, we observe that, since  $\phi$  is bounded and has a compactly supported Fourier transform, it can be harmlessly included in either factor and discarded. The proof is concluded as in Lemma 6.4.  $\square$

We continue with the bound for  $\tilde{C}_4$ :

**Lemma 7.2.** *Let  $a \in S_\Lambda$  and  $\tilde{a}, \phi$  be as above. Then*

$$\left| \int_0^1 \tilde{C}_4(u) dt \right| \lesssim \|u\|_{X^a}^2 \|u\|_{X_\Lambda^{-1/4} \cap X_{\Lambda,le}^{-1/4}}^2. \tag{88}$$

This result does not have a counterpart in the previous section, and requires a complete proof. In order to keep the argument fluid we postpone the proof for the end of the section. Finally we have the bound for  $\tilde{R}_6$ :

**Lemma 7.3.** *Let  $a \in S_\Lambda$  and  $\tilde{a}, \phi$  be as above. Then*

$$\left| \int_0^1 \tilde{R}_6(u) dt \right| \lesssim \|u\|_{X^a}^2 \|u\|_{X_{\Lambda,le}^{-1/4}}^4. \tag{89}$$

**Proof.** The proof repeats the proof of Lemma 6.5. Since  $\hat{\phi}$  has compact support, it does not affect any of the dyadic frequency localizations. Since  $\phi$  is bounded, it does not affect any of the  $L^p$  bounds. Finally, since  $\phi$  is time independent, it does not affect any of the nonresonance considerations in Case B2.  $\square$

Finally, we turn our attention to  $\tilde{R}_2$ . Since  $\phi' = \psi^2$ , we can rewrite  $\tilde{R}_2(u)$  in the form

$$\tilde{R}_2(u) = \|\psi(D\tilde{a}(D))^{\frac{1}{2}}u\|_{L^2}^2 + \langle r^{2,w}(x, D)u, u \rangle$$

where, by a slight abuse of notation,  $\phi'(D\tilde{a}(D))^{\frac{1}{2}}$  stands for the smooth odd square root, and the operator  $r^{2,w}$  accounts for the lower order terms, with its symbol  $r^2$  satisfying

$$|\partial_x^a \partial_\xi^b r^2(x, \xi)| \lesssim \langle x \rangle^{-N} \langle \Lambda^2 + \xi^2 \rangle^{-b/2} a(\xi)$$

and hence it has a negligible effect

$$|\langle r^{2,w}(x, D)u, u \rangle| \lesssim E_0(u). \tag{90}$$

Thus we obtain

**Lemma 7.4.** *The quadratic form  $\tilde{R}_2$  satisfies the bound*

$$\|\psi(D\tilde{a}(D))^{\frac{1}{2}}u\|_{L^2}^2 \leq \tilde{R}_2(u) + cE_0(u). \tag{91}$$

Taking into account the last four lemmas, we have proved that

$$\int_0^1 \|\psi(D\tilde{a}(D))^{\frac{1}{2}}u\|_{L^2}^2 dt \lesssim \sup_t \|u(t)\|_{H^a}^2 (1 + \|u(t)\|_{H_A^{-\frac{1}{2}}}^2) + \|u\|_{X^a}^2 (\|u\|_{X_A^{-1/4} \cap X_{A,le}^{-1/4}}^2 + \|u\|_{X_{A,le}^{-1/4}}^4).$$

The right hand side is translation invariant but the left hand side is not. Hence we can replace  $\psi$  by  $\psi(\cdot + x_0)$  and take the supremum over  $x_0$ . But some straightforward computations show that

$$\sup_j \sum_{\lambda \geq \Lambda} \lambda^{-1} a(\lambda) \|\chi_j \partial_x u_\lambda\|_{L^2([0,1] \times \mathbb{R})}^2 \lesssim \sup_{x_0} \|\psi(\cdot + x_0)(D\tilde{a}(D))^{\frac{1}{2}}u\|_{L^2([0,1] \times \mathbb{R})}^2.$$

Hence we have proved the main result of this section:

**Proposition 7.5.** *Let  $a \in S_\Lambda$ . Then all  $L^2$  solutions  $u$  to (1) satisfy the following bound in the time interval  $[0, 1]$ :*

$$\sup_j \sum_{\lambda \geq \Lambda} \lambda^{-1} a(\lambda) \|\chi_j \partial_x u_\lambda\|_{L^2}^2 \lesssim \sup_t \|u(t)\|_{H^a}^2 (1 + \|u(t)\|_{H_A^{-\frac{1}{2}}}^2) + \|u\|_{X^a}^2 (\|u\|_{X_A^{-\frac{1}{4}} \cap X_{A,le}^{-\frac{1}{4}}}^2 + \|u\|_{X_{A,le}^{-\frac{1}{4}}}^4). \tag{92}$$

**Proof of Lemma 7.2.** We recall that

$$\tilde{C}_4(u) = - \int \phi'(x) e^{ix\xi} \int_{P_\xi} \tilde{c}_4(\xi_0, \xi_1, \xi_2, \xi_3) u(\xi_0) u(\xi_1) \overline{u(\xi_2) u(\xi_3)} d\xi_j d\xi dx.$$

We use a Littlewood–Paley decomposition for all factors, denoting the corresponding frequencies by  $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ . Since  $\hat{\phi}$  has compact support, we can organize the four frequencies as usual  $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \{\lambda, \alpha, \mu, \mu\}$  with  $\lambda \leq \alpha \leq \mu$ . Within each dyadic term the symbol  $c_4$  has size  $a(\lambda)\alpha^{-1}$ . Hence we can separate variables and reduce the problem to estimating the expressions

$$c_{\lambda\alpha\mu} = a(\lambda)\alpha^{-1} \int_0^1 \int_{\mathbb{R}} \phi'(x) u_\lambda \bar{u}_\alpha u_\mu \bar{u}_\mu dx dt.$$

The position of the complex conjugates is of minor importance; above we chose the most interesting case. The argument applies to the other case without major changes. We split time into time intervals  $I$  of size  $\mu^{-1}$ ; the interval summation is accomplished due to the fact that we use the local energy norms.

**Case 1:** If  $\lambda \ll \mu$  then we expand each factor with respect to the  $\chi_j$  partition of unity on the unit spatial scale and use two bilinear  $L^2$  bounds plus Hölder’s inequality to obtain

$$\begin{aligned} |c_{\lambda\alpha\mu}| &\lesssim a(\lambda)\alpha^{-1}\mu^{-1} \sup_j \sum_I \|\chi_I \chi_j u_\lambda\|_{U^2} \|\chi_I \chi_j u_\alpha\|_{U^2} \|\chi_I \chi_j u_\mu\|_{U^2} \|\chi_I \chi_j u_\mu\|_{U^2} \\ &\lesssim a(\lambda)^{\frac{1}{2}} a(\alpha)^{-\frac{1}{2}} \alpha^{-1} \|u_\lambda\|_{X^a} \|u_\alpha\|_{X^a} \|u_\mu\|_{X_{A,le}^{-\frac{1}{2}}} \|u_\mu\|_{X_{A,le}^{-\frac{1}{2}}} \end{aligned}$$

with an easy summation in  $\alpha, \lambda$  and  $\mu$ .

**Case 2:** If  $\Lambda \ll \lambda \sim \mu$  then we use three  $L^6$  bounds, one energy and one Hölder inequality in time. We obtain a constant

$$a(\mu)\mu^{-\frac{3}{2}}$$

which is again more than we need for the summation.

**Case 3:** If  $\Lambda \sim \lambda \sim \mu$  then we have an additional difficulty, because we can no longer use the full strength of the local energy decay. In this case we can assume that  $\tilde{c}_4$  is constant (and nonzero) so there is no help from there. In the defocusing case this term comes with the right sign, as in the classical Morawetz estimate. However, in the focusing case we need to bound it, and there is a potential obstruction which is due to the existence of solitons. Indeed, consider the soliton

$$u = Q_\sigma e^{-i\sigma^2 t}, \quad \sigma = \Lambda^{\frac{1}{2}}$$

where the scale for  $\sigma$  is chosen so that this soliton has the largest mass among the zero speed solitons of  $H_\lambda^{-\frac{1}{4}}$  size less than one.

Suppose  $Q e^{it}$  is the normalized soliton. Then by rescaling we produce frequency  $\sigma$  solitons

$$Q_\sigma e^{i\sigma^2 t}, \quad Q_\sigma = \sigma Q(\sigma x)$$

with mass  $\sigma^{\frac{1}{2}}$ . Such a soliton has norm less than 1 in  $H_\Lambda^s$  provided that

$$\lambda \leq \Lambda^{2s}.$$

Now measure the same soliton in our space  $X_{\Lambda,le}^s$ . We loose  $\Lambda^{-2s}$  in the time interval summation. On the other hand we gain  $\sigma/\Lambda$  because of the  $\partial_x$  operator in the definition of  $X_{\Lambda,le}^s$ . Thus for all  $\sigma$  as above we must have

$$\Lambda^{-2s} \frac{\sigma}{\Lambda} \lesssim 1.$$

This gives exactly the threshold  $s = -\frac{1}{4}$  which corresponds to  $\sigma = \Lambda^{\frac{1}{2}}$ . Hence not only our full result (i.e. with large  $\Lambda$ ) is false for  $s < -\frac{1}{4}$ , but also the  $\partial_x$  operator in the definition of  $X_{\Lambda,le}^s$  cannot be relaxed at all if  $s = -\frac{1}{4}$ .

Then the low frequency part of the integral in the lemma has the form

$$\frac{a(\Lambda)}{\Lambda} \int_0^1 \int |Q_\sigma|^4 dx dt \sim \frac{a(\Lambda)}{\Lambda} \sigma^3 = a(\Lambda) \Lambda^{\frac{1}{2}}$$

which is a tight bound. This shows that  $s = -\frac{1}{4}$  is the actual threshold for this lemma, and also that in proving the lemma in this case we need to be careful about the concentration scale associated to the above soliton.

We consider a further dyadic decomposition

$$u_\Lambda = \sum_{\sqrt{\Lambda} \leq \lambda \leq \Lambda} u_\lambda$$

where  $u_{\sqrt{\Lambda}}$  also contains all the lower frequencies. Then  $c_{\Lambda\Lambda\Lambda}$  is decomposed into

$$c_{\Lambda\Lambda\Lambda} = \sum c_{\lambda\alpha\mu}, \quad c_{\lambda\alpha\mu} = a(\Lambda) \Lambda^{-1} \int_0^1 \int_{\mathbb{R}} \psi^2 u_\lambda \bar{u}_\alpha u_\mu \bar{u}_\mu dx dt$$

where  $\lambda \leq \alpha \leq \mu$  are in the range  $[\sqrt{\Lambda}, \Lambda]$ . This is a slight abuse of notation, since  $\lambda, \alpha$  and  $\mu$  here are in a different range from the one previously considered. We look at several cases:

**Case 3a:** If  $\lambda \ll \mu$  then we can use two bilinear  $L^2$  estimates to get

$$|c_{\lambda\alpha\mu}| \lesssim a(\Lambda) \Lambda^{-1} \mu^{-1} \sum_{|I|=\Lambda^{-1}} \|u_\lambda\|_{U_\Delta^2[I]} \|u_\alpha\|_{U_\Delta^2[I]} \|\psi u_\mu\|_{U_\Delta^2[I]} \|\psi u_\mu\|_{U_\Delta^2[I]}.$$

For the first two factors we use the  $X^a$  norm to get a uniform bound with respect to  $I$ . For the second two we use the local energy norm to gain  $l^2$  summability with respect to  $I$ , but there is a price to pay, namely a  $\Lambda/\mu$  factor for each due to the  $\partial_x$  operator in the definition of  $X_{\Lambda,le}^s$ . Using Cauchy–Schwarz with respect to  $I$  for the last two factors to obtain

$$\begin{aligned} |c_{\lambda\alpha\mu}| &\lesssim a(\Lambda)\Lambda^{-1}\mu^{-1}\frac{\Lambda^{\frac{1}{2}}}{a(\Lambda)}\left(\frac{\Lambda}{\mu}\right)^2\|u_\lambda\|_{X^a}\|u_\alpha\|_{X^a}\|u_\mu\|_{X_{\Lambda,le}^{-\frac{1}{4}}}\|u_\mu\|_{X_{\Lambda,le}^{-\frac{1}{4}}} \\ &= \frac{\Lambda^{\frac{3}{2}}}{\mu^3}\|u_\alpha\|_{X^a}\|u_\mu\|_{X_{\Lambda,le}^{-\frac{1}{4}}}\|u_\mu\|_{X_{\Lambda,le}^{-\frac{1}{4}}} \end{aligned}$$

where the factor  $\Lambda^{\frac{1}{2}}/a(\Lambda)$  is due to the  $L^2$  normalizations of the four factors. Since  $\mu \geq \Lambda^{\frac{1}{2}}$  the above coefficient is less than 1 and we have an easy summation with respect to  $\lambda, \alpha$  and  $\mu$ .

**Case 3b:** If  $\lambda \sim \mu$  then we cannot use bilinear  $L^2$  bounds. To understand this difficulty we consider first the extreme case:

**Case 3b(i):**  $\lambda \sim \mu \sim \Lambda^{\frac{1}{2}}$ . Then we neglect the local energy norms and simply use the energy for each factor combined with two Bernstein inequalities and Holder in time. We obtain

$$|c_{\mu\mu\mu}| \lesssim a(\Lambda)\Lambda^{-1}\mu\|u_\mu\|_{L^\infty L^2}^4 \lesssim \frac{\mu}{\Lambda^{\frac{1}{2}}}\|u_\mu\|_{X^a}^2\|u_\mu\|_{X_{\Lambda}^{\frac{1}{4}}}^2$$

which is favorable exactly when  $\mu = \Lambda^{\frac{1}{2}}$ . We continue with the last case:

**Case 3b(ii):**  $\lambda \sim \mu \gg \Lambda^{\frac{1}{2}}$ . We begin with the frequencies  $\xi_i$  for the four factors. Due to the compact frequency support of  $\psi$ , these are restricted to a unit neighborhood of the set  $P_0 = \{\xi_0 + \xi_1 - \xi_2 - \xi_3 = 0\}$ . We consider the dyadic scale  $100\sigma \sim 100 + \max |\xi_i - \xi_j|$ . The idea is now to produce a decomposition of  $\tilde{C}_4$  with respect to  $\sigma$ . To achieve that we begin with a corresponding decomposition of  $\mathbb{R}^4$ . For each dyadic  $\sigma \geq 1$  we consider the family  $\mathcal{Q}_\sigma$  of dyadic cubes  $Q$  with side-length  $\sigma$ , indexed by their position  $(k_0, k_1, k_2, k_3)$ . Then we consider a Whitney type partition of  $\mathbb{R}^4$  with respect to the distance to the diagonal  $\{\xi_0 = \xi_1 = \xi_2 = \xi_3\}$

$$\mathcal{Q}^1 = \bigcup_{\sigma \geq 1} \mathcal{Q}_\sigma^1, \quad \mathcal{Q}_\sigma^1 = \mathcal{Q}^1 \cap \mathcal{Q}_\sigma$$

where for  $\sigma > 1$  the cubes in  $\mathcal{Q}_\sigma^1$  are at distance  $\sim 100\sigma$  from the diagonal, while the cubes in  $\mathcal{Q}_1^1$  are within distance  $\lesssim 100$  of the diagonal. To this partition of  $\mathbb{R}^4$  we associate a corresponding partition of unit

$$1 = \sum_{Q \in \mathcal{Q}_1} \chi_Q(\xi_0, \xi_1, \xi_2, \xi_3)$$

where  $\chi_Q$  is smooth on the  $\sigma$  scale for  $Q \in \mathcal{Q}_\sigma$ . This is possible since each two neighboring cubes in  $\mathcal{Q}^1$  have comparable size. The functions  $\chi_Q$  do not have separated variables, but we can separate variables as before, and, by a slight abuse of notation, assume that  $\chi_Q$  has the form

$$\chi_Q = \chi_\sigma^{k_0}(\xi_0)\chi_\sigma^{k_1}(\xi_1)\chi_\sigma^{k_2}(\xi_2)\chi_\sigma^{k_3}(\xi_3)$$

for  $Q \in \mathcal{Q}_\sigma^1$  at position  $(k_0, k_1, k_2, k_3)$ . Thus we obtain a simultaneous quadrilinear decomposition

$$c_{\mu\mu\mu} = \sum_{1 \leq \sigma \leq \mu} c_{\mu\mu\mu}[\sigma] := a(\Lambda)\Lambda^{-1} \sum_{\sigma \geq 1} \sum_{(k_i) \in \mathcal{K}(\sigma)} \int_0^1 \int_{\mathbb{R}} \psi^2 P_\sigma^{k_0} u_\mu P_\sigma^{k_1} u_\mu \overline{P_\sigma^{k_2} u_\mu} \overline{P_\sigma^{k_3} u_\mu} dx dt$$

where the positions  $k_i$  are restricted to a range  $\mathcal{K}(\sigma)$  with the property that

$$\begin{aligned} \mathcal{K}(\sigma) &\subset \{|k_i - k_j| \lesssim 1000, \max |k_i - k_j| \geq 100, |k_0 + k_1 - k_2 - k_3| \leq 10\}, \quad \sigma > 1, \\ \mathcal{K}(\sigma) &\subset \{|k_i - k_j| \lesssim 1000, |k_0 + k_1 - k_2 - k_3| \leq 10\}, \quad \sigma = 1. \end{aligned}$$

For each integral we can apply either the argument in Case 3a or the argument in Case 3b(i). In both cases the summation with respect to  $k_j$  is diagonal and causes no losses by the Cauchy–Schwarz inequality.



For the Case 3a argument we split the quadrilinear form in two bilinear products with  $\sigma$  frequency separation to obtain

$$|c_{\mu\mu\mu}[\sigma]| \lesssim \frac{\Lambda^{\frac{3}{2}}}{\sigma\mu^2} \|u_\mu\|_{X^a}^2 \|u_\mu\|_{X^{\frac{1}{4}}_{\Lambda,le}}^2.$$

For the Case 3b argument we use Bernstein in  $\sigma$  frequency intervals to obtain

$$|c_{\mu\mu\mu}[\sigma]| \lesssim \frac{\sigma}{\Lambda^{\frac{1}{2}}} \|u_\mu\|_{X^a}^2 \|u_\mu\|_{X^{\frac{1}{4}}_{\Lambda,le}}^2.$$

Combining the two we have

$$|c_{\mu\mu\mu}[\sigma]| \lesssim \min\left\{ \frac{\Lambda^{\frac{3}{2}}}{\sigma\mu^2}, \frac{\sigma}{\Lambda^{\frac{1}{2}}} \right\} \|u_\mu\|_{X^a}^2 \|u_\mu\|_{X^{\frac{1}{4}}_{\Lambda,le}}^2.$$

We remark that the two factors balance when  $\sigma = \Lambda/\mu \leq \Lambda^{\frac{1}{2}}$ . In the second case the frequency separation is not needed, therefore the most natural decomposition would be obtained by restricting  $\sigma$  to the range  $\Lambda/\mu \leq \sigma \leq \Lambda$ .

Since  $\mu \geq \sigma$  and  $\mu \geq \Lambda^{\frac{1}{2}}$  it is now easy to check that the above coefficient is at most 1, with decay off the diagonal  $\sigma = \mu = \Lambda^{\frac{1}{2}}$ . The proof of the lemma is concluded.  $\square$

### 8. Conclusion

The final step in the paper is to use Propositions 6.2, 7.5 in order to conclude the proof of Proposition 1.6. For  $a \in S_\Lambda$  we can combine the results in Propositions 6.2, 7.5 to obtain

$$\begin{aligned} & \sup_t \sum_{\lambda \geq \Lambda} a(\lambda) \|u_\lambda(t)\|_{L^2}^2 + \sup_j \sum_{\lambda \geq \Lambda} a(\lambda) \lambda^{-1} \|\chi_j^\lambda \partial_x u_\lambda(t)\|_{L^2}^2 \\ & \lesssim \|u_0\|_{H^a}^2 + \|u\|_{L^\infty H^a}^2 \|u\|_{L^\infty H_\Lambda^{-\frac{1}{2}}}^2 + \|u\|_{X^a}^2 \left( \|u\|_{X_\Lambda^{-\frac{1}{4}} \cap X_{\Lambda,le}^{-\frac{1}{4}}}^2 + \|u\|_{X_{\Lambda,le}^{-\frac{1}{4}}}^4 \right). \end{aligned} \tag{93}$$

For  $\mu \geq \Lambda$  we apply this inequality for the symbols  $a_\mu \in S_\Lambda$  given by

$$a_\mu(\xi) = \mu^{-\frac{1}{2}} (1 + \xi^2/\mu^2)^{-\frac{1}{4}-\varepsilon} \tag{94}$$

with small  $\varepsilon$ . Restricting the left hand side in the above inequality to  $\lambda = \mu$  we have

$$\begin{aligned} & \mu^{-\frac{1}{2}} \|u_\mu\|_{L^\infty L^2}^2 + \sup_j \mu^{-\frac{3}{2}} \|\chi_j^\mu \partial_x u_\mu(t)\|_{L^2}^2 \\ & \lesssim \|u_0\|_{H^{a_\mu}}^2 + \|u\|_{L^\infty H^{a_\mu}}^2 \|u\|_{L^\infty H_\Lambda^{-\frac{1}{2}}}^2 + \|u\|_{X^{a_\mu}}^2 \left( \|u\|_{X_\Lambda^{-\frac{1}{4}} \cap X_{\Lambda,le}^{-\frac{1}{4}}}^2 + \|u\|_{X_{\Lambda,le}^{-\frac{1}{4}}}^4 \right). \end{aligned}$$

Finally, we sum up with respect to dyadic  $\mu \geq \Lambda$  to obtain

$$\|u\|_{l^2 L^\infty H_\Lambda^{-\frac{1}{4}}}^2 + \|u\|_{l^2 l^\infty L^2 H_\Lambda^{\frac{1}{4}}}^2 \lesssim \|u_0\|_{H_\Lambda^{-\frac{1}{4}}}^2 + \|u\|_{l^2 L^\infty H_\Lambda^{-\frac{1}{4}}}^2 \|u\|_{L^\infty H_\Lambda^{-\frac{1}{2}}}^2 + \|u\|_{X_\Lambda^{-\frac{1}{4}}}^2 \left( \|u\|_{X_\Lambda^{-\frac{1}{4}} \cap X_{\Lambda,le}^{-\frac{1}{4}}}^2 + \|u\|_{X_{\Lambda,le}^{-\frac{1}{4}}}^4 \right).$$

This implies both (18) and (19) and completes the proof of Proposition 1.6 for  $s = -\frac{1}{4}$ .

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