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Regularity of flat free boundaries in two-phase problems for the p-Laplace operator

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Abstract

In this paper we continue the study in Lewis and Nyström (2010) [19], concerning the regularity of the free boundary in a general two-phase free boundary problem for the p-Laplace operator, by proving regularity of the free boundary assuming that the free boundary is close to a Lipschitz graph.

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1. Introduction

In [1–3] a theory for general two-phase free boundary problems for the Laplace operator was developed. In particular, in [1] Lipschitz free boundaries were shown to be $C^{1,\gamma}$-smooth for some $\gamma \in (0,1)$ and in [2] it was shown that free boundaries which are well approximated by Lipschitz graphs are in fact Lipschitz. Finally, in [3] the existence part of the theory was developed. In [19] we initiated our study of the corresponding problems for the p-Laplace operator by generalizing the results in [1] to the p-Laplace operator when $p \neq 2$, $1 < p < \infty$. In this paper we continue our study by establishing results similar to [2] for the p-Laplace operator when $p \neq 2$, $1 < p < \infty$. As in [19] we note that the generalization beyond the harmonic case, which corresponds to $p = 2$, is non-trivial due to the non-linear and degenerate character of the p-Laplace operator. In particular, our results and arguments rely heavily on the techniques developed in [14–20].

To properly state our results we need to introduce some notation. Points in Euclidean n-space $\mathbf{R}^n$ are denoted by $x = (x_1, \dots, x_n)$ or (x', x_n) where $x' = (x_1, \dots, x_{n-1}) \in \mathbf{R}^{n-1}$. Let $\bar{E}$, ∂E, $\operatorname{diam} E$ be the closure, boundary, and

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diameter of E. Let $\langle\cdot,\cdot\rangle$ denote the standard inner product on $\mathbf{R}^n$, $|x|=\langle x,x\rangle^{1/2}$, the Euclidean norm of x, and let dx be Lebesgue n-measure on $\mathbf{R}^n$. Given $x\in\mathbf{R}^n$, $r>0$ and $s>0$, put $B(x,r)=\{y\in\mathbf{R}^n\colon |x-y|<r\}$ and $Q_{r,s}(x)=\{y=(y',y_n)\colon |y'-x'|<r,\ |y_n-x_n|<s\}$. In case $r=s$, we write $Q_r(x)$ for $Q_{r,r}(x)$. Given $E,F\subset\mathbf{R}^n$, let $E+F$ denote the set $\{x+y\colon x\in E,\ y\in F\}$ and let $d(E,F)$ be the Euclidean distance from E to F. In case $E=\{y\}$, we write $d(y,F)$. Let

$$\breve{h}(E,F)=\max\Big\{\sup_{y\in E} d(y,F), \sup_{y\in F} d(y,E)\Big\}$$

be the Hausdorff distance from E to F.

If $O\subset\mathbf{R}^n$ is open and $1\leqslant q\leqslant\infty$, then by $W^{1,q}(O)$, we denote the space of equivalence classes of functions f with distributional gradient $\nabla f=(f_{x_1},\dots,f_{x_n})$, both of which are q-th power integrable on O. Let

$$\|f\|_{W^{1,q}(O)}=\|f\|_{L^q(O)}+\||\nabla f|\|_{L^q(O)}$$

be the norm in $W^{1,q}(O)$ where $\|\cdot\|_{L^q(O)}$ denotes the usual Lebesgue q-norm in O. Next let $C_0^\infty(O)$ be the set of infinitely differentiable functions with compact support in O and let $C(E)$, be the set of continuous functions on E.

Given $D\subset\mathbf{R}^n$ a bounded domain (i.e., a connected open set) and $1<p<\infty$, we say that u is p-harmonic in D provided $u\in W^{1,p}(O)$ whenever $\bar{O}\subset D$ and

$$\int |\nabla u|^{p-2}\langle\nabla u,\nabla\theta\rangle\,dx=0 \tag{1.1}$$

whenever $\theta\in C_0^\infty(D)$. Observe that if u is smooth enough and $\nabla u\neq 0$ in D, then

$$\nabla\cdot\big(|\nabla u|^{p-2}\nabla u\big)\equiv 0\quad\text{in } D \tag{1.2}$$

so u is a classical solution in D to the p-Laplace partial differential equation. Here, as in the sequel, $\nabla\cdot$ is the divergence operator. u is said to be p-subharmonic (p-superharmonic) in D provided $u\in W^{1,p}(O)$ whenever $\bar{O}\subset D$ and (1.1) holds with $=$ replaced by $\leqslant$ ($\geqslant$) whenever $\theta\geqslant 0$ in D. Let $u\in C(\bar{D})$ and put $D^+(u)=\{x\in D\colon u(x)>0\}$, $F(u)=\partial D^+(u)\cap D$. Let $D^-(u)$ be the interior of $\{x\in D\colon u(x)\leqslant 0\}$ and set $u^+=\max\{u,0\}$, $u^-=-\min\{u,0\}$. Assuming that $w\in F(u)$, and that $F(u)$ is smooth in a neighborhood of w, we let $\nu=\nu(w)$ denote the unit normal, to $F(u)$ at w, pointing into $D^+(u)$. Moreover, we let $u_\nu^+(w)$ and $u_\nu^-(w)$ denote the normal derivatives of u^+ and u^- at w in the direction of ν. Note that $u_\nu^+,-u_\nu^-\geqslant 0$. In this paper we consider weak solutions, defined and continuous in $\bar{D}$, to the following general two-phase free boundary problem,

$$\begin{aligned}
&\text{(i)}\quad \nabla\cdot\big(|\nabla u|^{p-2}\nabla u\big)=0\quad\text{in } D^+(u)\cup D^-(u),\\
&\text{(ii)}\quad u_\nu^+(w)=G\big(-u_\nu^-(w)\big)\quad\text{whenever } w\in F(u),\\
&\text{(iii)}\quad u=k\in C(\partial D)\quad\text{on } \partial D.
\end{aligned} \tag{1.3}$$

In (1.3)(ii) the function $G:[0,\infty)\to[0,\infty)$ defines the free boundary condition and the interface $F(u)$ is referred to as the free boundary. If we make no a priori classical regularity assumptions on the interface $F(u)$ then the free boundary condition in (1.3)(ii) must be interpreted in a weak sense and in particular a notion of weak solutions to the problem in (1.3) has to be introduced. Let $\langle\cdot,\cdot\rangle^+=\max\{\langle\cdot,\cdot\rangle,0\}$, $\langle\cdot,\cdot\rangle^-=-\min\{\langle\cdot,\cdot\rangle,0\}$. We will work with the following notion of weak solutions to the problem in (1.3).

Definition 1.4. Let $D\subset\mathbf{R}^n$ be a bounded domain, $u\in C(\bar{D})$ and $1<p<\infty$, be given. u is said to be a (weak) solution to the problem in (1.3) if u is p-harmonic in $D^+(u)\cup D^-(u)$, $u=k$, on ∂D and if the free boundary condition in (1.3)(ii) is satisfied in the following sense. Assume that $w\in F(u)$ and there exists a ball $B(\hat{w},\hat{\rho})$, $\hat{w}\in D^+(u)\cup D^-(u)$ with $w\in\partial B(\hat{w},\hat{\rho})$. If $\nu=(\hat{w}-w)/|\hat{w}-w|$, then the following holds, as $x\to w$, for some α, $\beta\in[0,\infty]$ with $\alpha=G(\beta)$,

$$\begin{aligned}
&\text{(i)}\quad \text{if } B(\hat{w},\hat{\rho})\subset D^+(u), \text{ then } u(x)=\alpha\langle x-w,\nu\rangle^+-\beta\langle x-w,\nu\rangle^-+o\big(|x-w|\big),\\
&\text{(ii)}\quad \text{if } B(\hat{w},\hat{\rho})\subset D^-(u), \text{ then } u(x)=\alpha\langle w-x,\nu\rangle^+-\beta\langle w-x,\nu\rangle^-+o\big(|x-w|\big).
\end{aligned}$$

Let $\theta(\nu,\tilde\nu)$ be the angle between $\nu,\tilde\nu\neq 0$ in $\mathbf{R}^n$. If $|\nu|=1$, $\theta_0\in(0,\pi/2]$, set

$$\begin{aligned}\Gamma(\nu,\theta_0)&:=\bigl\{\tilde\nu\colon\ |\tilde\nu|=1 \text{ and } \theta(\nu,\tilde\nu)<\theta_0\bigr\},\\ C(\nu,\theta_0)&:=\bigl\{t\tilde\nu\colon\ \tilde\nu\in\Gamma(\nu,\theta_0) \text{ and } 0<t<\infty\bigr\}.\end{aligned}\tag{1.5}$$

Given $\epsilon>0$ we say that u is ϵ-monotone in $O\subset\mathbf{R}^n$, with respect to the directions in $\Gamma(\nu,\theta_0)$ if

$$\sup_{B(x,\epsilon'\sin\theta_0)} u\bigl(y-\epsilon'\nu\bigr)\leqslant u(x)\tag{1.6}$$

whenever $\epsilon'\geqslant\epsilon$ and $x\in O$ with $B(x-\epsilon'\nu,\epsilon'\sin\theta_0)\subset O$. u is said to be monotone in $O\subset\mathbf{R}^n$ with respect to the directions in $\Gamma(\nu,\theta_0)$ if whenever $y\in B(x,r)\subset O$ and $\frac{y-x}{|y-x|}\in\Gamma(\nu,\theta_0)$, it is true that $u(y)\geqslant u(x)$. Note that if u is monotone in O and $B(x,r)\subset O$, then (1.6) holds whenever $0<\epsilon'\leqslant r/4$.

Recall that $f:E\to\mathbf{R}$ is Lipschitz on E provided there exists b, $0<b<\infty$, such that $|f(z)-f(w)|\leqslant b|z-w|$ whenever $z,w\in E$. The infimum of all b such that this inequality holds is called the Lipschitz norm of f on E, denoted by $\|f\|_{\mathrm{Lip}(E)}$. It is well known that if $E=\mathbf{R}^{n-1}$, then f is differentiable almost everywhere on $\mathbf{R}^{n-1}$ and $\|f\|_{\mathrm{Lip}(\mathbf{R}^{n-1})}=\||\nabla f|\|_{L^\infty(\mathbf{R}^{n-1})}$. In this paper we prove the following theorems.

Theorem 1. *Let $D\subset\mathbf{R}^n$ be a bounded domain, assume that $u\in C(\bar D)$ and that u is a weak solution in D, for some $1<p<\infty$, to the problem in* (1.3) *in the sense of Definition* 1.4. *Assume that the function $G\geqslant 0$ is strictly increasing and, for some $N>0$, that $s\to s^{-N}G(s)$ is decreasing on $(0,\infty)$. Assume $0\in F(u)$, $\bar Q_1(0)\subset D$, and that $\bar\theta\in(\pi/4,\pi/2)$. Then there exists $\bar\epsilon=\bar\epsilon(\bar\theta,p,n,N)>0$ such that if u is ϵ-monotone on $Q_1(0)$ with respect to the directions in the spherical cap, $\Gamma(e_n,\bar\theta)$, for some $\epsilon\in(0,\bar\epsilon)$, then u is monotone in $Q_{1/2}(0)$ with respect to the directions in $\Gamma(e_n,\bar\theta_1)$ where $\bar\theta_1>\pi/4$. In particular,*

$$\begin{aligned}D^+(u)\cap Q_{1/2}(0)&=\bigl\{\bigl(x',x_n\bigr)\in\mathbf{R}^n\colon\ x_n>\bar f\bigl(x'\bigr)\bigr\}\cap Q_{1/2}(0),\\ F(u)\cap Q_{1/2}(0)&=\bigl\{\bigl(x',x_n\bigr)\in\mathbf{R}^n\colon\ x_n=\bar f\bigl(x'\bigr)\bigr\}\cap Q_{1/2}(0),\end{aligned}$$

where $\bar f:\mathbf{R}^{n-1}\to\mathbf{R}$ is Lipschitz with $\|\bar f\|_{\mathrm{Lip}(\mathbf{R}^{n-1})}<1$.

Theorem 2. *Let $D\subset\mathbf{R}^n$ be a bounded domain, assume that $u\in C(\bar D)$ and that u is a solution in D, for some $1<p<\infty$, to the problem in* (1.3) *in the sense of Definition* 1.4. *Assume that $G\geqslant 0$ is strictly increasing, Lipschitz continuous with $\|G\|_{\mathrm{Lip}(\mathbf{R}^{n-1})}\leqslant C$, $G(0)>0$, and, for some $N>0$, that $s^{-N}G(s)$ is decreasing on $(0,\infty)$. If $0\in F(u)$ and $\bar Q_1(0)\subset D$, then there exist $\hat\epsilon>0$, $\hat\theta\in(\pi/4,\pi/2)$, both depending on p, n, C, N, $G(0)$, and $\max_{\bar Q_1(0)}u^-$, such that the following statement is valid. If $0<\epsilon\leqslant\hat\epsilon$, $\hat\theta\leqslant\theta\leqslant\pi/2$, and if u^+ is ϵ-monotone in $Q_1(0)$ with respect to the directions in $\Gamma(e_n,\theta)$, then u^+ is monotone in $Q_{1/2}(0)$ with respect to the directions in $\Gamma(e_n,\hat\theta_1)$ where $\hat\theta_1>\pi/4$.*

As a corollary to Theorem 2 we also prove the following.

Corollary 1. *Let $D\subset\mathbf{R}^n$ be a bounded domain, assume that $u\in C(\bar D)$ and that u is a solution in D, for some $1<p<\infty$, to the problem in* (1.3) *in the sense of Definition* 1.4. *Let G be as in the statement of Theorem* 2 *and assume that $0\in F(u)$ and $\bar Q_1(0)\subset D$. Assume that there exists $\eta\geqslant 1$ such that*

$$\eta^{-1}d\bigl(x,F(u)\bigr)\leqslant u(x)\leqslant \eta d\bigl(x,F(u)\bigr)\quad \text{whenever } x\in D^+(u)\cap\bar Q_1(0).$$

Then there exist $\hat\epsilon>0$, $\hat\theta\in(\pi/4,\pi/2)$, both depending on p, n, C, N, $G(0)$, $\max_{\bar Q_1(0)}u^-$ and η, such that the following statement is valid. If $0<\epsilon\leqslant\hat\epsilon$, $\hat\theta\leqslant\theta\leqslant\pi/2$, and if

$$\breve h\bigl(F(u)\cap\bar Q_1(0),\Lambda\cap\bar Q_1(0)\bigr)\leqslant\epsilon,$$

where $\Lambda=\{(x',\tilde f(x'))\colon\ x'\in\mathbf{R}^{n-1}\}$ and $\|\tilde f\|_{\mathrm{Lip}(\mathbf{R}^{n-1})}\leqslant\tan(\pi/2-\theta)$, then u^+ is monotone in $Q_{1/2}(0)$ with respect to the directions in $\Gamma(e_n,\hat\theta_1)$ where $\hat\theta_1>\pi/4$.

As mentioned earlier, Theorems 1, 2, and Corollary 1 are part of the program initiated in [19]. In particular, in [19] we proved the following theorem.

Theorem A. *Let $D \subset \mathbf{R}^n$ be a bounded domain, assume that $u \in C(\bar{D})$ and that u is a solution in D, for some $1 < p < \infty$, to the problem in* (1.3) *in the sense of Definition* 1.4. *Moreover, suppose that $G > 0$ is strictly increasing on* $[0, \infty)$ *and, for some $N > 0$, that $s^{-N} G(s)$ is decreasing on* $(0, \infty)$. *Assume that $0 \in F(u)$, $\bar{B}(0,2) \subset D$,* $\max_{B(0,2)} |u| = 1$ *and that,*

$$D^+(u) \cap B(0,2) = \Omega \cap B(0,2), \qquad F(u) \cap B(0,2) = \partial\Omega \cap B(0,2),$$
$$\Omega = \{ y = (y', y_n) \in \mathbf{R}^n \colon y_n > \psi(y') \},$$

in an appropriate coordinate system, where ψ is Lipschitz on $\mathbf{R}^{n-1}$ with $M = \|\psi\|_{\mathrm{Lip}(\mathbf{R}^{n-1})}$. Then there exists $\sigma = \sigma(p,n,M,N) \in (0,1)$ such that $\nabla\psi$ is Hölder continuous of order σ on $\{x' \colon (x', \psi(x')) \in B(0,1/8)\}$. The C^σ-Hölder norm of $\nabla\psi$ depends only on p, n, M, N.

Using Theorem A and invariance of the p-Laplacian under rotations, translations, and dilations, we deduce under the scenario of either Theorems 1, 2 or Corollary 1, that $F(u) \cap B(0,1/16)$ is of class $C^{1,\sigma}$. Furthermore, to indicate earlier work, for $p = 2$, Theorem A is given in [1] while Theorems 1, 2 and Corollary 1 can be found in [2]. The work in [1,2] was generalized in [22,23], to solutions of fully non-linear concave PDE of the form $F(D^2 u) = 0$, where F is homogeneous. Further analogues of the work in [1] were obtained for a class of nonisotropic operators in [7] and in [8] the concavity assumption in [22] was removed for viscosity solutions to fully non-linear PDE of the form, $F(D^2 u, Du) = 0$, where F is homogeneous in both arguments. Moreover, generalizations of the results in [1] were made in [9] to fully non-linear PDE of the form $F(D^2 u, x) = 0$ which have interior $C^{1,1}$-estimates. Generalizations of the work in [1], to linear divergence form PDE with variable Lipschitz continuous coefficients were obtained in [6,11]. Finally the work in [1,2] was generalized to viscosity solutions of certain linear nondivergence form elliptic PDE with drift term in [10]. However, Theorems A, 1, 2 and Corollary 1 are the first generalizations of [1,2] to divergence form operators of p-Laplacian type.

The rest of the paper is organized in the following way. In Section 2, which partly is of a preliminary nature, we collect a number of results from [14,15,20,19] concerning p-harmonic functions in Lipschitz domains. In Section 3 we construct appropriate p-subsolutions to the free boundary problem under consideration using results from [2,19, 22]. We also prove that if u is as in Theorem 1, then u satisfies

$$c^{-1} \frac{u(y)}{d(y, \partial\Omega)} \leqslant |\nabla u(y)| \leqslant c \frac{u(y)}{d(y, \partial\Omega)} \tag{1.7}$$

where $c = c(p,n)$, at points sufficiently far away from $\partial Q_1(0) \cup F(u)$. Finally, in Section 3 we use (1.7) to show that u is monotone in the directions $\Gamma(e_n, \bar{\theta})$ at points sufficiently far away from $\partial Q_1(0) \cup F(u)$. In Section 4 we then prove Theorem 1 leaving the proofs of Theorem 2 and Corollary 1 for Section 5. At the end of Section 5 we mention possible generalizations of Theorems 1, 2 and Corollary 1.

Concerning our proofs of Theorems 1, 2 and Corollary 1 we note that our argument combines the geometric approach developed in [1,2,23] with the analytic techniques for p-harmonic functions in Lipschitz domains, and in domains which are well approximated by Lipschitz graph domains in the Hausdorff distance sense, developed in [14–16,18,19]. In particular, based on the results in our previous papers, we are able to proceed along the lines of [2] and [23] to complete the proofs. The most tricky part of the argument, as compared to the harmonic case in [2], is to obtain a contradiction when the graph of the solution u and the graph of the carefully constructed subsolution v_t touch. In [19] we obtained a contradiction, and thus proved Theorem A, by using a boundary Harnack inequality – Hopf type maximum principle (see Theorem 2.9 below) as well as the fact that in [19] u satisfied the fundamental inequality (1.7) up to $F(u)$. In the proof of Theorem 1, we can no longer assume (1.7) up to $F(u)$. Instead we have to introduce several other comparison functions and prove these functions can be used to get the desired contradiction.

We emphasize that on the one hand this paper is not user friendly, as it relies heavily on previous rather technical work of the authors mentioned above. On the other hand, we state and give references for results which are used in this paper. In general our strategy in writing this paper has been to refer to previous work whenever possible, as well as, to provide details whenever our arguments differ from previous arguments. Thus as a minimum the interested reader should first be familiar with [2,19], and to have these papers at hand.

Finally the authors would like to thank the referee for some helpful comments.

2. Estimates for p-harmonic functions

We say that $\Omega \subset \mathbf{R}^n$ is a bounded Lipschitz domain if there exists a finite set of balls $\{B(x_i, r_i)\}$, with $x_i \in \partial\Omega$ and $r_i > 0$, such that $\{B(x_i, r_i)\}$ constitutes a covering of an open neighborhood of $\partial\Omega$ and such that, for each i,

$$\begin{aligned}\Omega \cap B(x_i, 4r_i) &= \{x = (x', x_n) \in \mathbf{R}^n:\ x_n > \phi_i(x')\} \cap B(x_i, 4r_i),\\ \partial\Omega \cap B(x_i, 4r_i) &= \{x = (x', x_n) \in \mathbf{R}^n:\ x_n = \phi_i(x')\} \cap B(x_i, 4r_i),\end{aligned} \tag{2.1}$$

in an appropriate coordinate system and for a Lipschitz function ϕ_i on $\mathbf{R}^{n-1}$. The Lipschitz constant of Ω is defined to be $M = \max_i \|\phi_i\|_{\mathrm{Lip}(\mathbf{R}^{n-1})}$. If $w \in \partial\Omega$, $0 < r < r_0$, we let $\Delta(w, r) = \partial\Omega \cap B(w, r)$ be the naturally defined surface ball and we let e_i, $1 \leqslant i \leqslant n$, denote the point in $\mathbf{R}^n$ with one in the i-th coordinate position and zeroes elsewhere. Moreover, throughout the paper c will denote, unless otherwise stated, a positive constant $\geqslant 1$, not necessarily the same at each occurrence, which only depends on p, n, and M. In general, $c(a_1, \dots, a_n)$ denotes a positive constant $\geqslant 1$, not necessarily the same at each occurrence, which depends on p, n, M and $a_1, \dots, a_n$. If $A \approx B$ then A/B is bounded from above and below by constants which, unless otherwise stated, depend on p, n and M at most. Moreover, we let $\max_{B(z,s)} \hat{u}$, $\min_{B(z,s)} \hat{u}$ be the essential supremum and infimum of $\hat{u}$ on $B(z, s)$ whenever $B(z, s) \subset \mathbf{R}^n$ and $\hat{u}$ is defined on $B(z, s)$.

We first state a number of basic lemmas in Lipschitz domains. As a general reference for proofs of the following lemmas we refer to [14]. Lemmas 2.2, 2.3, 2.5, are classical interior type estimates for non-linear partial differential equations in divergence form. Lemma 2.4 is well known for harmonic functions while Theorems 2.6, 2.7, are recent results of the authors. Their proofs use deformations of $\hat{u}$ into $\hat{v}$, similar to the one in (2.13), as well as uniform ellipticity estimates, similar to those in (2.14)–(2.17), and classical boundary Harnack inequalities for nondivergence uniformly elliptic partial differential equations.

Lemma 2.2. *Given p, $1 < p < \infty$, let $\hat{u}$ be a positive p-harmonic function in $B(w, 2r)$. Then*

$$\text{(i)} \quad \max_{B(w,r)} \hat{u} \leqslant c \min_{B(w,r)} \hat{u}.$$

Furthermore, there exists $\alpha = \alpha(p, n) \in (0, 1)$ such that if $x, y \in B(w, r)$, then

$$\text{(ii)} \quad \left|\hat{u}(x) - \hat{u}(y)\right| \leqslant c\left(\frac{|x-y|}{r}\right)^{\alpha} \max_{B(w,2r)} \hat{u}.$$

Lemma 2.3. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain and p given, $1 < p < \infty$. Let $w \in \partial\Omega$, $0 < r < r_0$, and suppose that $\hat{u}$ is a non-negative p-harmonic function in $\Omega \cap B(w, 2r)$. Assume also that $\hat{u}$ is continuous on $\bar{\Omega} \cap B(w, 2r)$ with $\hat{u} \equiv 0$ on $\Delta(w, 2r)$. There exist $c = c(p, n, M) \geqslant 1$ and $\alpha = \alpha(p, n, M) \in (0, 1)$ such that if $x, y \in \Omega \cap B(w, r)$, then*

$$\left|\hat{u}(x) - \hat{u}(y)\right| \leqslant c\left(\frac{|x-y|}{r}\right)^{\alpha} \max_{\Omega\cap B(w,2r)} \hat{u}.$$

Lemma 2.4. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain and p given, $1 < p < \infty$. Let $w \in \partial\Omega$, $0 < r < r_0$, and suppose that $\hat{u}$ is a non-negative p-harmonic function in $\Omega \cap B(w, 2r)$. Assume also that $\hat{u}$ is continuous in $\bar{\Omega} \cap B(w, 2r)$ with $\hat{u} \equiv 0$ on $\Delta(w, 2r)$. There exists $c = c(p, n, M)$, $1 \leqslant c < \infty$, such that if $\tilde{r} = r/c$, then*

$$\max_{\Omega\cap B(w,\tilde{r})} \hat{u} \leqslant c\hat{u}\big(a_{\tilde{r}}(w)\big),$$

where $a_{\tilde{r}}(w)$ is any point in $\Omega \cap B(w, \tilde{r})$ with $d(a_{\tilde{r}}(w), \partial\Omega) \geqslant \tilde{r}/c$.

Lemma 2.5. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain and p given, $1 < p < \infty$. Let $w \in \partial\Omega$, $0 < r < r_0$ and suppose that $\hat{u}$ is a non-negative p-harmonic function in $\Omega \cap B(w, 2r)$. Assume also that $\hat{u}$ is continuous in $\bar{\Omega} \cap B(w, 2r)$ with $\hat{u} \equiv 0$ on $\Delta(w, 2r)$. Extend $\hat{u}$ to $B(w, 2r)$ by defining $\hat{u} \equiv 0$ on $B(w, 2r) \setminus \Omega$. Then $\hat{u}$ has a representative in*

$W^{1,p}(B(w,2r))$ with Hölder continuous partial derivatives in $\Omega \cap B(w,2r)$. In particular, there exists $\sigma \in (0,1]$, depending only on p and n, such that if $x, y \in B(\tilde{w}, \tilde{r}/2)$, $B(\tilde{w}, 4\tilde{r}) \subset \Omega \cap B(w,2r)$, then

$$c^{-1}\left|\nabla\hat{u}(x) - \nabla\hat{u}(y)\right| \leqslant \left(|x-y|/\tilde{r}\right)^{\sigma} \max_{B(\tilde{w},\tilde{r})} |\nabla\hat{u}| \leqslant c\tilde{r}^{-1}\left(|x-y|/\tilde{r}\right)^{\sigma}\hat{u}(\tilde{w}).$$

Moreover, if for some $\beta \in (1,\infty)$,

$$\frac{\hat{u}(y)}{d(y,\partial\Omega)} \leqslant \beta\left|\nabla\hat{u}(y)\right| \quad \text{for all } y \in B(\tilde{w}, 2\tilde{r}),$$

then $\hat{u} \in C^{\infty}(B(\tilde{w}, 2\tilde{r}))$ and given a positive integer k there exists $\bar{c} \geqslant 1$, depending only on p, n, β, k, such that

$$\max_{B(\tilde{w},\frac{\tilde{r}}{2})} \left|D^k\hat{u}\right| \leqslant \bar{c}\frac{\hat{u}(\tilde{w})}{d(\tilde{w},\partial\Omega)^k} \quad \text{where } D^k\hat{u} \text{ denotes an arbitrary } k\text{-th order derivative of } \hat{u}.$$

Next we state two theorems proved in [14,15].

Theorem 2.6. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with constants M, r_0. Given p, $1 < p < \infty$, $w \in \partial\Omega$, $0 < r < r_0$, suppose that $\hat{u}$ and $\hat{v}$ are positive p-harmonic functions in $\Omega \cap B(w,2r)$. Assume also that $\hat{u}, \hat{v}$ are continuous in $\bar{\Omega} \cap B(w,2r)$ and that $\hat{u} \equiv 0 \equiv \hat{v}$ on $\Delta(w,2r)$. Under these assumptions there exists c_1, $1 \leqslant c_1 < \infty$, $\gamma > 0$, depending only on p, n, and M, such that if $x, y \in \Omega \cap B(w, r/c_1)$, then*

$$\left|\log\left(\frac{\hat{u}(y)}{\hat{v}(y)}\right) - \log\left(\frac{\hat{u}(x)}{\hat{v}(x)}\right)\right| \leqslant c_1\left(\frac{|x-y|}{r}\right)^{\gamma}.$$

Theorem 2.7. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with constants M, r_0. Let $w \in \partial\Omega$, $0 < r < r_0$, and suppose that (2.1) holds with x_i, r_i, ϕ_i, replaced by w, r, ϕ. Given p, $1 < p < \infty$, let $\hat{u}$ be a positive p-harmonic function in $\Omega \cap B(w,2r)$. Assume also that $\hat{u}$ is continuous in $\bar{\Omega} \cap B(w,2r)$ with $\hat{u} \equiv 0$ on $\Delta(w,2r)$. Then there exists $\theta_0 \in (0,\pi/2]$ and $1 \leqslant c_2 < \infty$, both depending only on p, n, M, such that*

$$c_2^{-1}\frac{\hat{u}(y)}{d(y,\partial\Omega)} \leqslant \left\langle\nabla\hat{u}(y), \xi\right\rangle \leqslant \left|\nabla\hat{u}(y)\right| \leqslant c_2\frac{\hat{u}(y)}{d(y,\partial\Omega)}$$

whenever $y \in \Omega \cap B(w, r/c_2)$ and $\xi \in \Gamma(e_n, \theta_0)$.

We note that in [14] we proved, under the assumptions stated in Theorem 2.6, that

$$\left|\log\left(\frac{\hat{u}(y)}{\hat{v}(y)}\right) - \log\left(\frac{\hat{u}(x)}{\hat{v}(x)}\right)\right| \leqslant c \quad \text{whenever } x, y \in \Omega \cap B(w, r/c). \tag{2.8}$$

Here $c = c(p,n,M)$. Using (2.8) we then obtained, essentially simultaneously, Theorems 2.6 and 2.7 in [15]. Furthermore, in [19] we also proved the following theorem.

Theorem 2.9. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with constants M, r_0. Let $w \in \partial\Omega$, $0 < r < r_0$, $1 < p < \infty$, and suppose that $\hat{u}$, $\hat{v}$ are positive p-harmonic functions in $\Omega \cap B(w,2r)$ with $\hat{v} \leqslant \hat{u}$. Assume also that $\hat{u}$, $\hat{v}$ are continuous in $\bar{\Omega} \cap B(w,2r)$ with $\hat{u} \equiv \hat{v} \equiv 0$ on $\Delta(w,2r)$. Then there exists c_3, $1 < c_3 < \infty$, such that*

$$\frac{\hat{u}(y) - \hat{v}(y)}{\hat{v}(y)} \leqslant c_3\frac{\hat{u}(x) - \hat{v}(x)}{\hat{v}(x)} \quad \text{whenever } x, y \in \Omega \cap B(w, r/c_3).$$

We note that Theorem 2.9 implies (2.8), as follows easily from the fact that the p-Laplacian is invariant under multiplication by a constant. Thus replacing $\hat{v}$ by $\delta\hat{v}$ in the above display, multiplying both sides by δ, and then letting $\delta \to 0$, we get (2.8). A slightly more involved argument (see Section 6 of [20]) also gives Theorem 2.6. Also for later use we observe from Theorem 2.6 that v can be replaced by u in the denominator of the display in Theorem 2.9.

Next we state Lemma 2.10 which gives a useful criteria for determining when a positive p-harmonic function satisfies the last inequality in Theorem 2.7 at a point. Note that Lemma 2.10 is similar to Lemmas 4.3 and 5.4 in [15], Lemma 3.1 in [18], and Lemma 3.18 in [20]. Hence we do not include a proof of this lemma here.

Lemma 2.10. *Let O be an open set, and suppose that $\hat u$, $\hat v$ are positive p-harmonic functions in O. Let $a \geqslant 1$, $y \in O$, $|\xi| = 1$, and assume that*

$$\frac{1}{a}\frac{\hat v(y)}{d(y,\partial O)} \leqslant \langle \nabla \hat v(y), \xi\rangle \leqslant \left|\nabla \hat v(y)\right| \leqslant a\frac{\hat v(y)}{d(y,\partial O)}.$$

Let $\tilde\epsilon^{-1} = (ca)^{(1+\sigma)/\sigma}$, where σ is as in Lemma 2.5 and $c = c(p,n)$. Then the following statement is true for $c = c(p,n)$ suitably large. If

$$(1-\tilde\epsilon)\tilde L \leqslant \frac{\hat v}{\hat u} \leqslant (1+\tilde\epsilon)\tilde L$$

in $B(y, \frac14 d(y,\partial O))$ for some $\tilde L$, $0 < \tilde L < \infty$, then

$$\frac{1}{ca}\frac{\hat u(y)}{d(y,\partial O)} \leqslant \langle \nabla \hat u(y), \xi\rangle \leqslant \left|\nabla \hat u(y)\right| \leqslant ca\frac{\hat u(y)}{d(y,\partial O)}.$$

To continue our basic estimates, we list some results for the difference of two p-harmonic functions. To this end, let $\hat u$, $\hat v$ be positive p-harmonic functions in an open set O, satisfying $1 \leqslant \hat u/\hat v \leqslant c_4$. Suppose also that $\hat v$ satisfies, for some $\hat\delta > 1$, the fundamental inequality

$$\hat\delta^{-1}\frac{\hat v(x)}{d(x,\partial O)} \leqslant \left|\nabla \hat v(x)\right| \leqslant \hat\delta\frac{\hat v(x)}{d(x,\partial O)}, \quad \text{whenever } x \in B(w,r), \tag{2.11}$$

where $\bar B(w,2r) \subset O$. Define

$$e(x) = \hat u(x) - \hat v(x) \quad \text{whenever } x \in B(w,r). \tag{2.12}$$

and put

$$u(x,\tau) = \tau\hat u(x) + (1-\tau)\hat v(x) \quad \text{whenever } x \in B(w,r), \text{ and } \tau \in [0,1]. \tag{2.13}$$

Clearly, $e(x) = u(x,1) - u(x,0)$. Using p-harmonicity of $\hat u$, $\hat v$ and that

$$|\xi|^{p-2}\xi - |\eta|^{p-2}\eta = \int_0^1 \frac{d\{|t\xi + (1-t)\eta|^{p-2}[t\xi + (1-t)\eta]\}}{dt}\,dt$$

whenever $\xi, \eta \in \mathbf{R}^n \setminus \{0\}$, it follows that e is a weak solution to

$$\hat L e := \sum_{i,j=1}^n \frac{\partial}{\partial y_i}\big(\hat b_{ij}(y) e_{y_j}(y)\big) = 0 \quad \text{whenever } y \in B(w,r) \tag{2.14}$$

where

$$\hat b_{ij}(y) = \int_0^1 b_{ij}(y,\tau)\,d\tau,$$

$$b_{ij}(y,\tau) = \left|\nabla u(y,\tau)\right|^{p-4}\big((p-2)u_{y_i}(y,\tau)u_{y_j}(y,\tau) + \delta_{ij}\left|\nabla u(y,\tau)\right|^2\big). \tag{2.15}$$

Here $i, j \in \{1,\dots,n\}$ and δ_{ij} is the Kronecker δ. If $y \in B(w,r)$, then from (2.14) we observe that $e = \hat u - \hat v$ is the solution in to a symmetric divergence form PDE with ellipticity constants at y estimated by

$$\min\{p-1,1\}|\xi|^2\hat\lambda(y) \leqslant \sum_{i,j=1}^n \hat b_{ij}(y)\xi_i\xi_j \leqslant \max\{p-1,1\}|\xi|^2\hat\lambda(y), \tag{2.16}$$

whenever $\xi \in \mathbf{R}^n$, and where

$$\hat{\lambda}(y) = \int_0^1 \big|\nabla u(y,\tau)\big|^{p-2}\, d\tau \approx \big(\big|\nabla \hat{u}(y)\big| + \big|\nabla \hat{v}(y)\big|\big)^{p-2} \approx \big(\hat{v}(y)/d(y,\partial O)\big)^{p-2}. \tag{2.17}$$

The right-hand side inequality in (2.17) was obtained by using Lemma 2.5 to estimate $|\nabla \hat{u}(\cdot)|$ in terms of $\hat{u}(\cdot)/d(\cdot,\partial O)$, the assumption that $\hat{u} \leqslant c_4 \hat{v}$, (2.11), and the fact that

$$1/2 \leqslant \frac{d(x,\partial O)}{d(z,\partial O)} \leqslant 2 \quad \text{for } x, z \in B(w,r)$$

since $\bar{B}(w,2r) \subset O$. In (2.17) the constants of proportionality depend only on p, n, $\hat{\delta}$, and c_4. In Sections 4 and 5 we will need the following interior Harnack inequality.

Lemma 2.18. *Let $\hat{u}$, $\hat{v}$, be positive p-harmonic functions in O with $1 \leqslant \hat{u}/\hat{v} \leqslant c_4$ and suppose that $\hat{v}$ satisfies* (2.11) *with $r = 2\tilde{r}$, $w = \tilde{w}$, where $\bar{B}(\tilde{w},4\tilde{r}) \subset O$. If $e = \hat{u} - \hat{v}$, then there exists a constant $c = c(p,n,\hat{\delta},c_4) > 1$ such that*

$$\max_{B(\tilde{w},\tilde{r})} e \leqslant c \min_{B(\tilde{w},\tilde{r})} e.$$

Proof. From (2.16), (2.17), and (2.11), we see that $\hat{L}$ is uniformly elliptic in $B(\tilde{w},2\tilde{r})$ with bounded measurable coefficients. Constants depend only on p, n, $\hat{\delta}$, c_4. The stated Harnack inequality now follows from classical arguments, see [21]. □

Finally in this section we prove a lemma concerning properties of a positive minimal p-harmonic function in a cone. More specifically, if $0 < \theta_0 < \pi$, recall the definition of the cone $C(e_n,\theta_0)$ in (1.5). We write $C(\theta_0)$ for $C(e_n,\theta_0)$. Given p, $1 < p < \infty$, we say that $\hat{u}$ is a minimal positive p-harmonic function in $C(\theta_0)$, relative to ∞, provided $\hat{u}$ is a positive p-harmonic function in $C(\theta_0)$ with continuous boundary value zero on $\partial C(\theta_0)$.

Lemma 2.19. *Given $\theta_0 \in (0,\pi]$, and $1 < p < \infty$, there exists a unique minimal positive p-harmonic function, $\hat{u} = \hat{u}(\cdot,\theta_0)$, in $C(\theta_0)$ with $\hat{u}(e_n) = 1$. Moreover, if $r = |x|$, $x_n = r\cos\theta$, $0 \leqslant \theta < \theta_0$, are spherical coordinates of x, then there exist $\psi \in C^\infty(\cos\theta_0, 1)$ and $\gamma > 0$ such that*

$$\hat{u}(x) = \hat{u}(r,\theta) = r^\gamma \psi(\cos\theta), \quad 0 \leqslant \theta < \theta_0.$$

Also, γ is a decreasing positive continuous function of $\theta_0 \in (0,\pi)$ with $\gamma(\pi/2) = 1$.

Proof. We note that in [12], homogeneous p-harmonic functions of the above form are constructed in cones. To begin the proof of Lemma 2.19, existence of a minimal positive p-harmonic function $\hat{u}$ relative to ∞ in $C(\theta_0)$ is easily shown. For example one can take $\hat{u}$ to be the limit of a subsequence of $(u_m)_1^\infty$ where u_m is a positive p-harmonic function in $C(\theta_0) \cap B(0,m)$ with continuous boundary value 0 on $\partial C(\theta_0) \cap B(0,m)$ and $u_m(e_n) = 1$. Existence of u_m, $m = 1, 2, \dots$, follows from a calculus of variations argument. Applying Lemmas 2.2–2.5 to u_m, $m = 1, 2, \dots$, and using Ascoli's theorem we get $\hat{u}$. To prove uniqueness of $\hat{u}$, let $\hat{v}$ be another minimal positive p-harmonic function in $C(\theta_0)$ with $\hat{v}(e_n) = 1$. Using Theorem 2.6 with $\Omega = C(\theta_0) \cap B(0,2r)$, $w = 0$, we get, upon letting $r \to \infty$, that $\hat{v} = \hat{u}$. Thus $\hat{u}$ is the unique minimal positive p-harmonic function in $C(\theta_0)$ with $\hat{u}(e_n) = 1$. To obtain the desired form for $\hat{u}$ we first note that uniqueness of $\hat{u}$ and invariance of the p-Laplace equation under rotations imply that $\hat{u}$ is symmetric about the x_n axis. Thus we write $\hat{u}(r,\theta)$ for $\hat{u}(x)$ when $x \in \bar{C}(\theta_0)$ and $r = |x|$, $x_n = r\cos\theta$, $0 \leqslant \theta \leqslant \theta_0$. Also since the p-Laplacian is invariant under dilations it follows from uniqueness of $\hat{u}$ that

$$\hat{u}(\lambda x) = \hat{u}(\lambda e_n)\hat{u}(x) \quad \text{whenever } \lambda > 0 \text{ and } x \in C(\theta_0). \tag{2.20}$$

Differentiating (2.20) with respect to λ and evaluating at $\lambda = 1$ we find that

$$r\hat{u}_r(r,\theta) = \langle x, \nabla \hat{u}(x)\rangle = \langle \nabla \hat{u}(e_n), e_n\rangle \hat{u}(r,\theta).$$

Dividing this equality by $r\hat{u}(r,\theta)$, integrating, and then exponentiating, we get $\hat{u}(r,\theta) = r^\gamma \psi(\cos\theta)$ where $\gamma = \langle e_n, \nabla \hat{u}(e_n)\rangle$. Continuity of γ once again follows from uniqueness of $\hat{u}(\cdot,\theta_0)$ and Lemmas 2.2–2.5. Also, $\gamma(\theta_0)$ is decreasing for $\theta_0 \in (0,\pi)$, as follows easily from comparing solutions in different cones and using the maximum principle for p-harmonic functions. Finally $\hat{u}(x) = x_n = r\cos\theta$ when $\theta_0 = \pi/2$, so $\gamma(\pi/2) = 1$. □

3. Preliminary reductions for Theorem 1

Recall that given a bounded domain D and $1 < p < \infty$, we say that u is p-subharmonic in D provided $u \in W^{1,p}(\Omega)$ whenever Ω is an open set with $\bar{\Omega} \subset D$ and

$$\int |\nabla u|^{p-2} \langle \nabla u, \nabla \theta \rangle \, dx \leqslant 0 \tag{3.1}$$

whenever $\theta \in C_0^\infty(D)$ and $\theta \geqslant 0$ on D. We say that u is p-superharmonic provided $-u$ is p-subharmonic. Moreover, we let $C^2(D)$ denote the set of functions which have continuous second partial derivatives in D. For $\phi \in C^2(D)$ we let $\nabla^2 \phi(x)$ denote the Hessian matrix of ϕ at $x \in D$.

Let $S(n)$ denote the set of all symmetric $n \times n$ matrices and let P be the Pucci type extremal operator (see [4]) defined, for $M \in S(n)$, as

$$P(M) = \inf_{A \in A_p} \sum_{i,j=1}^{n} a_{ij} M_{ij}. \tag{3.2}$$

Here A_p denotes the set of all symmetric $n \times n$ matrices $A = \{a_{ij}\}$ which satisfy

$$\min\{p-1, 1\}|\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \leqslant \max\{p-1, 1\}|\xi|^2 \quad \text{whenever } \xi \in \mathbf{R}^n. \tag{3.3}$$

For the proof of the following lemma we refer to [19, Lemma 3.5].

Lemma 3.4. *Let $\phi > 0$ be in $C^2(D)$, $\|\nabla \phi\|_{L^\infty(D)} \leqslant 1/2$, p fixed, $1 < p < \infty$, and suppose that*

$$\phi(x) P\big(\nabla^2 \phi(x)\big) \geqslant 50pn \big|\nabla \phi(x)\big|^2 \quad \text{whenever } x \in D.$$

Let u be continuous in an open set O containing the closure of $\bigcup_{x \in D} B(x, \phi(x))$ and define

$$v(x) = \max_{\bar{B}(x, \phi(x))} u$$

whenever $x \in D$. If u is p-harmonic in $O \setminus \{u = 0\}$, then v is continuous and p-subharmonic in $\{v \neq 0\} \cap D$.

Next we consider the asymptotic development, near $F(v)$, of the p-subharmonic function constructed in Lemma 3.4.

Lemma 3.5. *Let D, u, ϕ, O, and $v = v_\phi$ be as in the statement of Lemma 3.4 and let G be as in Theorem 1. Suppose also that (i), (ii) of Definition 1.4 hold for some α, β, whenever $w \in O \cap \partial\{u > 0\}$ and there exists $B(\hat{w}, \hat{\rho}) \subset O \setminus \partial\{u > 0\}$ with $w \in \partial B(\hat{w}, \hat{\rho})$. If $\tilde{w} \in F(v)$, then there exist $w^* \in D^+(v)$ and $\rho^* > 0$ such that $B(w^*, \rho^*) \subset D^+(v)$ and $\tilde{w} \in \partial B(w^*, \rho^*)$. Also, there exist $\tilde{\alpha}$, $\tilde{\beta} \in [0, \infty)$, such that the following hold, as $x \to \tilde{w}$, with $\tilde{\nu} = (w^* - \tilde{w})/|w^* - \tilde{w}|$,*

(a) $v(x) \geqslant \tilde{\alpha} \langle x - \tilde{w}, \tilde{\nu} \rangle^+ - \tilde{\beta} \langle x - \tilde{w}, \tilde{\nu} \rangle^- + o\big(|x - \tilde{w}|\big)$,

(b) $\dfrac{\tilde{\alpha}}{1 - |\nabla \phi(\tilde{w})|} \geqslant G\bigg(\dfrac{\tilde{\beta}}{1 + |\nabla \phi(\tilde{w})|}\bigg)$.

Proof. The proof of Lemma 3.5 for $p = 2$ can be found in Lemmas 10, 11 of [1]. The proof is based on a purely geometric argument using smoothness of ϕ, and the asymptotic expansion of u in balls tangent to $F(u)$. Hence it is also valid here. □

We will also use the following lemma.

Lemma 3.6. *Let D, ϕ, be as in Lemma* 3.4. *Assume $\epsilon > 0$ small, p fixed, $1 < p < \infty$, $\theta_0 \in (\pi/4, \pi/2)$ and let $\tilde{O}$ be an open set containing*

$$\Big\{y\colon\ |y-z| \leqslant 2\epsilon \text{ for some } z \text{ in the closure of } \bigcup_{x\in D} B\big(x,\phi(x)\big)\Big\}.$$

Assume that u is continuous, and ϵ-monotone in $\tilde{O}$ with respect to the directions in $\Gamma(e_n,\theta_0)$. Assume also that u is p-harmonic in $\tilde{O}\setminus\partial\{u>0\}$ and satisfies (*as in Lemma* 3.5) (i), (ii) *of Definition* 1.4 *at points of $\tilde{O}\cap\partial\{u>0\}$. Let G be as in Theorem* 1 *and define v relative to u, ϕ as in Lemma* 3.5. *If $0<\theta'\leqslant\theta_0$, $\frac{1}{2}\epsilon\sin\theta_0 < \phi(x) < \epsilon$, and*

$$\sin\theta' \leqslant \frac{1}{1+|\nabla\phi|(x)}\left(\sin\theta_0 - \frac{\epsilon}{2\phi(x)}(\cos\theta_0)^2 - |\nabla\phi|(x)\right)$$

for all $x\in D$, then v is monotone in D with respect to the directions in $\Gamma(e_n,\theta')$ and $F(v)\cap D$ is the graph of a Lipschitz function with constant M', $M'\leqslant\cot\theta'$.

Proof. A proof for $p=2$ is given in [2, Lemma 2]. The proof involves a purely geometric argument so can be repeated here. However for $p\neq 2$, $1<p<\infty$, certain issues should be clarified. Indeed, let $\hat{x}\in D$ and suppose $\hat{y}\in\partial B(\hat{x},\phi(\hat{x}))$ with $u(\hat{y}) = v(\hat{x})$. We consider several cases. If $\hat{y}\in\tilde{O}\setminus\partial\{x\colon u(x)>0\}$ and $\nabla u(\hat{y})\neq 0$, then u is p-harmonic, consequently infinitely differentiable in a neighborhood of $\hat{y}$, so we can argue as in Lemma 2 of [2] to get that v is increasing at $\hat{x}$ in the directions given by $\Gamma(e_n,\theta')$. If $\hat{y}\in\tilde{O}\cap\partial\{x\colon u(x)>0\}$ and $\tilde{\alpha}\neq 0$ in Lemma 3.5(a), we can once again use the geometric argument in [2] to conclude that v is increasing at $\hat{x}$ in the directions given by $\Gamma(e_n,\theta')\cap\{x\colon |x|=1\}$. Hence it remains to consider the cases when (a) $\hat{y}\in\tilde{O}\setminus\partial\{x\colon u(x)>0\}$, $\nabla u(\hat{y})=0$, and (b) $\hat{y}\in\tilde{O}\cap\partial\{x\colon u(x)>0\}$, $\tilde{\alpha}=0$. In case (a) it follows from the Hopf boundary maximum principle and the fact that $u(\hat{y}) = \max\{u(z)\colon z\in\bar{B}(\hat{x},\phi(\hat{x}))\}$ that $u\equiv u(\hat{y})$ in $\bar{B}(\hat{x},\phi(\hat{x}))$. In this case we note that

$$2\phi(\hat{x}) \geqslant \epsilon\sin\theta_0 \geqslant \epsilon(1-\sin\theta_0) + (\sqrt{2}-1)\epsilon.$$

Thus

$$B\big(\hat{x}+\big(\phi(\hat{x})-\epsilon\big)e_n, \epsilon\sin\theta_0\big)\cap B\big(\hat{x},\phi(\hat{x})\big)\neq\emptyset$$

and so it follows from the definition of ϵ-monotonicity, that $u\geqslant u(\hat{x})$ in an open neighborhood of $\hat{x}+\phi(\hat{x})e_n$. Since $v(x)\geqslant u(x+\phi(x)e_n)$ we conclude that $v\geqslant v(\hat{x})$ in an open neighborhood of $\hat{x}$. In case (b) it follows from Definition 1.4 applied to u, and the Hopf boundary maximum principle, that $u\equiv 0$ in $\bar{B}(\hat{x},\phi(\hat{x}))$. Hence, once again using ϵ-monotonicity we have that $v\geqslant 0 = v(\hat{x})$ in an open neighborhood of $\hat{x}$. Thus v is monotone in D with respect to the directions in $\Gamma(e_n,\theta')$. Lipschitzness of $F(v)\cap D$ follows from an easy geometric argument using monotonicity of v and the definition of $F(v)$. The proof of Lemma 3.6 is now complete. □

Finally, we will use the following set of functions $\{\phi_t\}$, $0\leqslant t\leqslant 1$, to construct appropriate p-subharmonic functions to be used in the proof of Theorems 1 and 2.

Lemma 3.7. *Let $\Lambda = \{(x',x_n)\in\mathbf{R}^n\colon\ x_n = \lambda(x')\}$ where $\lambda\colon\mathbf{R}^{n-1}\to\mathbf{R}$, $\lambda(0)=0$, and $\|\lambda\|_{\mathrm{Lip}(\mathbf{R}^{n-1})}\leqslant M<\infty$ for some $M\geqslant 1$. Given h, $0<h<10^{-3}$, let $\Lambda(h) = \{(x',x_n)\colon\ |x_n-\lambda(x')|<h\}\cap\bar{Q}_{2,8M}(0)$. If $\beta\in(0,1)$, then there exists a family of functions $\{\phi_t\}$, $0\leqslant t\leqslant 1$, C^2-regular in $\Lambda(h)$, and $c=c(p,n,M,\beta)\geqslant 1$, $h_0=h_0(p,n,M,\beta)>0$, such that the following holds. There exists $\mu=\mu(p,n)$, $0<\mu\leqslant 2$, such that for $x\in\Lambda(h)$, $t\in[0,1]$, and $h\in(0,h_0]$, we have*

(i) $1\leqslant\phi_t(x)\leqslant 1+\mu t$,

(ii) $\big|\nabla\phi_t(x)\big|\leqslant cth^{\beta-1}$,

(iii) $\phi_t(x)P\big(\nabla^2\phi_t(x)\big)\geqslant 50pn\big|\nabla\phi_t(x)\big|^2$,

(iv) $\phi_t(x)\equiv 1$ *whenever* $x\in\Lambda(h)\setminus Q_{1-2h^{1-\beta},4M}(0)$,

(v) $1+\mu t-\phi_t(x)\leqslant cth^{\beta}$ *whenever* $x\in\Lambda(h)\cap Q_{1-100h^{1-\beta},4M}(0)$.

Proof. We could prove Lemma 3.7 by arguing as in [10] however we prefer to use a covering argument and the construction in [19]. This construction in turn was based on a construction in [22] (the construction in [23] appears incorrect). To begin the argument let $\{a_{ij}\} \in A_p$ and let L be the operator $\sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial}{\partial x_j})$. Given $\hat{x} \in \mathbf{R}^n$, $\rho \in (0, 10^{-2})$, we claim there exists $0 \leqslant \hat{f} \in C^2(\mathbf{R}^n \setminus \bar{B}(\hat{x}, \rho/2))$ satisfying

$$L\hat{f} \geqslant 0 \quad \text{in } \mathbf{R}^n \setminus B(\hat{x}, \rho) \quad \text{and} \quad \hat{f} \equiv 0 \quad \text{in } \mathbf{R}^n \setminus B(\hat{x}, 10). \tag{3.8}$$

Also, for some $k_+ = k_+(\rho, p, n) \geqslant 1$, we have

$$\begin{aligned} &\text{(i)} \quad k_+^{-1} \leqslant \min(\hat{f}, L\hat{f}) \quad \text{on } B(\hat{x}, 6) \setminus \bar{B}(\hat{x}, \rho), \\ &\text{(ii)} \quad |\nabla \hat{f}| \leqslant L\hat{f} \leqslant k_+ \quad \text{in } \mathbf{R}^n \setminus B(\hat{x}, \rho). \end{aligned} \tag{3.9}$$

For example, let N be a non-negative integer and let

$$\tilde{f}(x) = |\hat{x} - x|^{-2N} \quad \text{whenever } x \in \mathbf{R}^n \setminus B(\hat{x}, \rho/2).$$

It is easily checked that $\tilde{f}$ satisfies (3.8), (3.9) on $B(x, 10) \setminus B(x, \rho)$, for $N = N(\rho, p, n) > 0$ large enough, except that $\tilde{f}$ does not have support in $B(\hat{x}, 10)$. To remedy this let $\hat{f} = [\max(\tilde{f} - 10^{-2N}, 0)]^4$. Then $\hat{f}$ is C^2 on $\mathbf{R}^n \setminus \bar{B}(\hat{x}, \rho/2)$ and $\hat{f} \equiv 0$ in $\mathbf{R}^n \setminus B(\hat{x}, 10)$. From the definition of L and (3.9) (ii) we find that

$$L\hat{f} \geqslant 4\tilde{f}^3 L\tilde{f} \geqslant 4\tilde{f}^3|\nabla\tilde{f}| = |\nabla\hat{f}| \quad \text{whenever } \rho \leqslant |x - \hat{x}| < 10.$$

Thus (3.8), (3.9) are valid.

Next we choose $\rho = (100M)^{-1}$ and use a well-known covering lemma to get $\{B(w_i, \rho h)\}$ with $\{B(w_i, \rho h/10)\}$ pairwise disjoint, and

$$\begin{aligned} &\text{(a)} \quad w_i \in \Lambda(3h) \setminus \Lambda(2h), \quad i = 1, 2, \ldots. \\ &\text{(b)} \quad \bigcup B(w_i, \rho h) \cap \Lambda(h) = \emptyset. \\ &\text{(c)} \quad \Lambda(h) \cap Q_{1-50h^{1-\beta},4M}(0) \subset \bigcup B(w_i, 6h). \\ &\text{(d)} \quad \big(\Lambda(h) \setminus Q_{1-2h^{1-\beta},4M}(0)\big) \cap \bigcup B(w_i, 10h) = \emptyset. \end{aligned} \tag{3.10}$$

Existence of $\{w_i\}$ satisfying (a), (c), (d) is easily seen. Note that (b) follows from (a) our choice of ρ, and the fact that λ has Lipschitz norm $\leqslant M$. Next given w_i we take $\hat{x} = w_i$ in the definition of $\hat{f}$ and set

$$\hat{f}_i(x) = \hat{f}\left(w_i + \frac{x - w_i}{h}\right), \quad x \in \mathbf{R}^n \setminus B(w_i, \rho h),\ i = 1, 2, \ldots.$$

Let

$$\psi(x) = h^\beta \sum \hat{f}_i(x), \quad \text{when } x \in \mathbf{R}^n \setminus \bigcup B(w_i, \rho h).$$

Observe from (3.8) that $\hat{f}_i$ has support in $\bar{B}(w_i, 10h)$. Using this fact, as well as the disjointness of $\{B(w_i, \rho h/10)\}$, we deduce that if $x \in \mathbf{R}^n \setminus \bigcup B(w_i, \rho h)$, then $\{i\colon\ \hat{f}_i(x) \neq 0\}$ has cardinality at most $\tilde{c}$ where $\tilde{c} = \tilde{c}(p, n, M) \geqslant 1$. Using these facts (3.8)–(3.10) we see for some $c_- = c_-(p, n, M, \beta) \geqslant 1$ that

$$\begin{aligned} &(\alpha) \quad L\psi(x) \geqslant c_-^{-1} h^{\beta-2} \quad \text{when } x \in \Lambda(h) \cap Q_{1-50h^{1-\beta},4M}(0), \\ &(\beta) \quad L\psi \geqslant h^{-1}|\nabla\psi| \quad \text{on } \Lambda(h), \\ &(\gamma) \quad \max\big(\psi, h|\nabla\psi|\big) \leqslant c_- h^\beta \quad \text{on } \Lambda(h), \\ &(\delta) \quad \psi \equiv 0 \quad \text{on } \Lambda(h) \setminus Q_{1-2h^{1-\beta},4M}(0). \end{aligned} \tag{3.11}$$

Let $\theta \in C_0^\infty(\mathbf{R}^{n-1})$ with $0 \leqslant \theta \leqslant 1$ and

$$\begin{aligned} &(\alpha) \quad \theta \equiv 1 \quad \text{on } \big\{x' \in \mathbf{R}^{n-1}\colon\ |x'| \leqslant 1 - 100h^{1-\beta}\big\}, \\ &(\beta) \quad \theta \equiv 0 \quad \text{on } \big\{x' \in \mathbf{R}^{n-1}\colon\ |x'| \geqslant 1 - 50h^{1-\beta}\big\}, \\ &(\gamma) \quad h^{1-\beta} \sum_{i,j=1}^{n-1} \left|\frac{\partial^2\theta}{\partial x_i \partial x_j}\right| + \sum_{i=1}^{n-1} \left|\frac{\partial\theta}{\partial x_i}\right| \leqslant c h^{\beta-1} \quad \text{on } \mathbf{R}^{n-1}, \end{aligned} \tag{3.12}$$

where $c = c(p, n, M, \beta)$. Finally put

$$\phi_t(x) = 1 + t\big[\theta(x') + \psi(x)\big], \quad \text{when } x = \big(x', x_n\big) \in \Lambda(h).$$

Then (i), (v) of Lemma 3.7 are easily deduced from (3.11)(γ) and (3.12)(α), (β). (ii) of Lemma 3.7 is implied by (3.11)(γ) and (3.12)(γ) while (iv) of this lemma is a consequence of (3.11)(δ), (3.12)(β). (iii) for $x \in \Lambda(h) \cap Q_{1-50h^{1-\beta}}(0)$ with P replaced by L follows from (3.11)(α), (3.12)(γ), and Lemma 3.7(ii). (iii) for $x \in \Lambda(h) \setminus Q_{1-50h^{1-\beta}}(0)$ with P replaced by L follows for $h_0 = h_0(p, n, M, \beta) > 0$ small enough from (3.11)(β), (γ) and (3.12)(β). Taking the infimum over all $\{a_{ij}\} \in A_p$, we get (iii) for P. The proof of Lemma 3.7 is now complete. □

Next we prove

Lemma 3.13. *Let u, D, G, $\bar\theta$, be as in Theorem* 1. *Assume $\hat x \in F(u)$ and $\bar Q_r(\hat x) \subset D$. Then there exist $\epsilon_* = \epsilon_*(p, n) > 0$, $c_* = c_*(p, n) \geqslant 1$, and $\theta^* = \theta^*(p, n) \in (0, \pi/2)$, such that if $0 < \epsilon \leqslant \epsilon_*$, then*

$$c_*^{-1}|u|(x)/d\big(x, F(u)\big) \leqslant \big\langle \nabla u(x), \xi\big\rangle \leqslant \big|\nabla u(x)\big| \leqslant c_*|u|(x)/d\big(x, F(u)\big)$$

whenever $r \geqslant c_*^3\epsilon$, $x \in Q_{r/c_*}(\hat x)$, $\xi \in \Gamma(e_n, \theta^*)$, and $c_*\epsilon \leqslant d(x, F(u))$.

Proof. To begin the proof of Lemma 3.13 assume $r \geqslant 10^{10}\epsilon$, and let $Q = Q_{r-100\epsilon, r-50\epsilon}(\hat x)$, $z \in F(u) \cap \bar Q$. We first show that

$$\begin{aligned}
&\text{(i)} \quad 0 = u(z) < u(w) \text{ whenever } w \in Q_r(\hat x) \cap B(z + \rho e_n, \rho \sin\bar\theta),\ \epsilon \leqslant \rho,\ \text{and } d\big(w, \partial Q_r(\hat x)\big) \geqslant 20\epsilon,\\
&\text{(ii)} \quad \text{Either } u(w) \equiv 0 \text{ or } u(w) < 0 \text{ for all } w \in Q_r(\hat x) \cap B(z - \rho e_n, \rho \sin\bar\theta)\\
&\qquad \text{with } \epsilon \leqslant \rho \text{ and } d\big(w, \partial Q_r(\hat x)\big) \geqslant 20\epsilon.
\end{aligned} \tag{3.14}$$

Indeed first assume $\epsilon \leqslant \rho \leqslant 20\epsilon$. Then replacing $<$ in (i), (ii) by $\leqslant$, we deduce easily that the amended (i), (ii), follow from ϵ-monotonicity of u in the directions given by $\Gamma(e_n, \bar\theta)$. Moreover if $u(w) = 0$ for some w as in (3.14) (i), then $u \leqslant 0$ in an open neighborhood of z which contradicts $z \in F(u)$. Also, if $u(w) = 0$ for some w as in (3.14) (ii), then $u \equiv 0$ in $Q_r(\hat x) \cap B(z - \rho e_n, \rho \sin\theta)$ follows from the maximum principle for p-harmonic functions. If $\rho > 20\epsilon$, then to prove (i), we can use convexity of $Q_r(\hat x)$ and choose successive points w_j, $1 \leqslant j \leqslant k$, on the ray from z to w with $w_1 = z$, $w_k = w$, and $\epsilon < |w_{j+1} - w_j| \leqslant 5\epsilon$, $1 \leqslant j \leqslant k - 1$. Using '$\epsilon$-monotonicity' once again it follows that $u(w_j) \leqslant u(w_{j+1})$ and thereupon from the case $\rho \leqslant 20\epsilon$ that $u(w) > u(z)$. Hence (i) is true. (ii) is proved similarly when $\rho > 20\epsilon$, we omit the details.

Let $C(e_n, \bar\theta)$ be the cone with axis parallel to e_n and of angle opening $\bar\theta$ (as in (1.5)). Put

$$\begin{aligned}
\Sigma(x) &= x + C(e_n, \bar\theta) = \big\{x + \zeta\colon\ \zeta \in C(e_n, \bar\theta)\big\},\\
\breve\Omega &= \bigcup_{x \in F(u) \cap \bar Q} \Sigma(x).
\end{aligned}$$

If $y' \in \mathbf{R}^{n-1}$ let $\tau(y') = \inf\{y_n\colon (y', y_n) \in \breve\Omega\}$. Then τ is Lipschitz with

$$\|\tau\|_{\mathrm{Lip}(\mathbf{R}^{n-1})} \leqslant \tan(\pi/2 - \bar\theta) < 1 \quad \text{and} \quad \big\{\big(y', \tau\big(y'\big)\big)\colon\ y' \in \mathbf{R}^{n-1}\big\} = \partial\breve\Omega. \tag{3.15}$$

We claim that

$$\check h\big(F(u) \cap \bar Q, \partial\breve\Omega \cap \bar Q\big) \leqslant \epsilon. \tag{3.16}$$

To prove claim (3.16) note from (3.14) with $z = \hat x$ that for given y' with $|y' - \hat x'| \leqslant r - 100\epsilon$, we have $E(y') = F(u) \cap \{(y', t)\colon |t - \hat x_n| \leqslant r - 50\epsilon\} \neq \emptyset$. If $y = (y', y_n) \in E(y')$, then from the definition of τ, (3.14), and (3.15) we see that $\tau(y') \leqslant y_n < r - 100\epsilon + \hat x_n$. Next from a compactness argument we find that $(y', \tau(y'))$ is in $\bar\Sigma(z)$ for some $z \in F(u) \cap \bar Q$. Now $y_n - \tau(y') \leqslant \epsilon$ since otherwise $y \in Q \cap \Sigma(z)$, and $y_n - z_n > \epsilon$ which violates (3.14)(i). Since $y \in E(y')$ is arbitrary we conclude the validity of (3.16) from this contradiction.

Let

$$\begin{aligned}
\Omega_1 &= \big\{y \in Q\colon\ y_n > \tau\big(y'\big) + 2\epsilon\big\},\\
\Omega_2 &= \big\{y \in Q\colon\ y_n > \tau\big(y'\big) - 2\epsilon\big\}.
\end{aligned}$$

Then from (3.15), (3.16) we get

$$\Omega_1 \subset D^+(u) \cap Q \subset \Omega_2. \tag{3.17}$$

Let u_1 be the p-harmonic function in Ω_1 which is continuous in $\bar{\Omega}_1$ with boundary values

$$\begin{aligned} &\text{(a)} \quad u_1 \equiv 0 \quad \text{on } \partial\Omega_1 \cap Q, \\ &\text{(b)} \quad u_1(y) = u(y) \quad \text{when } y \in \partial\Omega_1 \cap \partial Q \text{ and } y_n \geqslant \tau(y') + 3\epsilon, \\ &\text{(c)} \quad u_1(y) = \frac{(y_n - 2\epsilon - \tau(y'))}{\epsilon} u\bigl(y', \tau(y') + 3\epsilon\bigr) \quad \text{when } y \in \partial\Omega_1 \cap \partial Q \\ &\qquad \text{and } \tau\bigl(y'\bigr) + 2\epsilon \leqslant y_n < \tau\bigl(y'\bigr) + 3\epsilon. \end{aligned} \tag{3.18}$$

Let u_2 be the p-harmonic function in Ω_2 which is continuous in $\bar{\Omega}_2$ with boundary values $u_2 = u^+$ on $\partial\Omega_2$. From the maximum principle for p-harmonic functions and (3.17) we deduce that

$$u_1 \leqslant u \leqslant u_2 \quad \text{in } \Omega_1. \tag{3.19}$$

Let $u_3(x) = u_2(x', x_n - 4\epsilon)$ whenever $x \in \Omega_1$. Observe that $u_3 > 0$ is p-harmonic in Ω_1 with $u_3 \equiv 0$ on $\partial\Omega_1 \cap Q$. We claim that

$$u_3 \leqslant u_1. \tag{3.20}$$

The claim in (3.20) follows from the boundary maximum principle once we show that $u_3 \leqslant u_1$ on $\partial\Omega_1$. In fact, if $y \in \partial\Omega_1 \cap \partial Q$ and $y_n \geqslant \tau(y') + 3\epsilon$, then

$$u_3(y) = u^+\bigl(y', y_n - 4\epsilon\bigr) \leqslant u\bigl(y', y_n\bigr) = u_1\bigl(y', y_n\bigr).$$

If $y \in \partial\Omega_1 \cap \partial Q$ and $\tau(y') + 2\epsilon \leqslant y_n < \tau(y') + 3\epsilon$ or $y \in \partial\Omega_1 \cap Q$, then $u_3(y) = 0 \leqslant u_1(y)$ as we see from ϵ-monotonicity of u and the fact that $F(u)$ is contained in the closure of $\breve{\Omega}$. Hence (3.20) is valid. From (3.19), (3.20), we have

$$1 \leqslant u/u_1 \leqslant u_2/u_3. \tag{3.21}$$

Next we note from Lemma 2.2 that if $x \in \Omega_1 \cap Q_{r/2}(\hat{x})$ and $d(x, F(u)) \geqslant A\epsilon$, for some $A \geqslant 100$, then

$$0 \leqslant u_2(x) - u_3(x) = u_2\bigl(x', x_n\bigr) - u_2\bigl(x', x_n - 4\epsilon\bigr) \leqslant cA^{-\alpha} u_3(x) \tag{3.22}$$

where $c \geqslant 1$ and $\alpha > 0$ depend only on p, n. Putting (3.21), (3.22) together we have

$$1 \leqslant u/u_1 \leqslant 1 + cA^{-\alpha}. \tag{3.23}$$

Finally we observe from Theorem 2.7 that there exist $c = c(p,n) \geqslant 1$ and $\theta' = \theta'(p,n) \in (0, \pi/2)$ such that

$$c^{-1} u_1(x)/d(x, \partial\Omega_1) \leqslant \bigl\langle \nabla u_1(x), \xi \bigr\rangle \leqslant \bigl|\nabla u_1(x)\bigr| \leqslant c u_1(x)/d(x, \partial\Omega_1) \tag{3.24}$$

whenever $x \in \Omega_1 \cap Q_{r/c}(\hat{x})$ and $\xi \in \Gamma(e_n, \theta')$. From (3.23), (3.24), we see that Lemma 2.10 can be applied for $A = A(p,n)$ large enough with $\hat{v} = u_1$, $\hat{u} = u$. Doing this we get Lemma 3.13 when $u(x) > 0$.

Similarly, to prove Lemma 3.13 when $u(x) \leqslant 0$, let

$$\tilde{\Sigma}(w) = w + C(-e_n, \bar{\theta}),$$
$$\tilde{\Omega} = \bigcup_{w \in \bar{Q} \cap F(u)} \tilde{\Sigma}(w)$$

and for $y' \in \mathbf{R}^{n-1}$ let $\tilde{\tau}(y') = \sup\{y_n\colon (y', y_n) \in \tilde{\Omega}\}$. Then (3.15) holds with τ replaced by $\tilde{\tau}$. Also (3.16) is true with $\breve{\Omega}$ replaced by $\tilde{\Omega}$, as follows from an argument similar to the one used earlier for (3.16). Let

$$\tilde{\Omega}_1 = \bigl\{y \in Q\colon\ y_n < \tilde{\tau}\bigl(y'\bigr) + 2\epsilon\bigr\},$$
$$\tilde{\Omega}_2 = \bigl\{y \in Q\colon\ y_n < \tilde{\tau}\bigl(y'\bigr) - 2\epsilon\bigr\}.$$

Then from the new versions of (3.15), (3.16), we see that

$$\tilde{\Omega}_2 \subset Q \setminus \bar{D}^+(u) \subset \tilde{\Omega}_1.$$

From this relationship and (3.14) we see that either $u \equiv 0$ on $\tilde{\Omega}_2$ or $u(x) < 0$ whenever $x \in \tilde{\Omega}_2$. If $u \equiv 0$ on $\tilde{\Omega}_2$, then Lemma 3.13 is trivially true. Otherwise we can repeat the argument following (3.17) to (3.24) with u, Ω_1, Ω_2, replaced by $-u$, $\tilde{\Omega}_2$, $\tilde{\Omega}_1$, respectively. The proof of Lemma 3.13 is now complete. □

Finally in this section we prove the following lemma.

Lemma 3.25. *Let $u, D, G, \bar{\theta}$, be as in Theorem 1. Assume $\hat{x} \in F(u)$ and $\bar{Q}_r(\hat{x}) \subset D$. Then there exist $\breve{\epsilon} = \breve{\epsilon}(p,n) > 0$ and $\breve{c} = \breve{c}(p,n) \geqslant 1$, such that if $0 < \epsilon \leqslant \breve{\epsilon}$, then $\langle \nabla u(x), \xi \rangle \geqslant 0$ whenever $r \geqslant \breve{c}^3\epsilon$, $x \in Q_{r/\breve{c}}(\hat{x})$, $\xi \in \Gamma(e_n, \bar{\theta})$, and $\breve{c}\epsilon \leqslant d(x, F(u))$.*

Proof. Fix $\xi \in \Gamma(e_n, \bar{\theta})$, suppose $r \geqslant c_*^4\epsilon$, and $w \in Q_{r/c_*^2}(\hat{x})$ with $c_*^2\epsilon \leqslant d(w, F(u))$ where c_* is the constant in Lemma 3.13. Put $d(w, F(u)) = 5A\epsilon$ and note from Lemma 3.13, as well as Harnack's inequality for p-harmonic functions, that

$$(cA\epsilon)^{-1}u(w) \leqslant c\left|\nabla u(y)\right| \leqslant c^2 u_{y_n}(y) \leqslant c^3(A\epsilon)^{-1}u(w) \tag{3.26}$$

for some $c = c(p,n)$ whenever $y \in B(w, 4A\epsilon)$. If $\eta \in [1, A)$ is fixed and $\xi \in \Gamma(e_n, \bar{\theta})$, set

$$e_\eta(x) = u(x + \eta\epsilon\xi + w) - u(x + w) \quad \text{whenever } x \in B(0, 3A\epsilon).$$

From (3.26), (2.11)–(2.15), as well as Lemma 2.5, we see that

$$0 = \hat{L}e(y) = \sum_{i,j=1}^{n} \frac{\partial}{\partial y_i}\big(\hat{b}_{ij}(y)(e_\eta)_{y_j}(y)\big) = 0 \quad \text{whenever } y \in B(0, 3A\epsilon), \tag{3.27}$$

where, if $u(y,\tau) = \tau u(y + \eta\epsilon\xi + w) + (1-\tau)u(y + w)$, for $\tau \in [0,1]$ and $y \in B(0, 3A\epsilon)$, then

$$\hat{b}_{ij}(y) = \int_0^1 b_{ij}(y,\tau)\,d\tau, \quad \text{where}$$

$$b_{ij}(y,\tau) = \left|\nabla u(y,\tau)\right|^{p-4}\big((p-2)u_{y_i}(y,\tau)u_{y_j}(y,\tau) + \delta_{ij}\left|\nabla u(y,\tau)\right|^2\big). \tag{3.28}$$

Let $\tilde{b}_{ij} = (A\epsilon/u(w))^{p-2}\hat{b}_{ij}$. Then as in (2.16), (2.17), we see from (3.26) that $(\tilde{b}_{ij})$ are uniformly elliptic and bounded with constants depending only on p, n. Also from Lemma 2.5 and (3.26) we see, for a given positive integer k, that

$$\left|D^k\tilde{b}_{ij}(y)\right| \leqslant c(A\epsilon)^{-k} \quad \text{for } y \in B(0, 2A\epsilon),\ 1 \leqslant i, j \leqslant n, \tag{3.29}$$

where $c = c(p,n,k)$ and D^k denotes an arbitrary k-th order derivative. Next observe from 'ϵ-monotonicity' that $e_\eta \geqslant 0$ in $B(0, 3A\epsilon)$. To continue, using (3.26)–(3.29), the above observations, basic Schauder type estimates, and Harnack's inequality for uniformly elliptic PDE in divergence form we get, for some $c = c(p,n) \geqslant 1$ and $\eta \in [2,3]$, that

$$\left|\nabla e_\eta(0)\right| \leqslant ce_\eta(0)/(A\epsilon) \leqslant c^2e_1(0)/(A\epsilon). \tag{3.30}$$

Moreover,

$$c^{-1}e_1(0) \leqslant e_1(2\epsilon\xi) = u(3\epsilon\xi + w) - u(2\epsilon\xi + w) = \epsilon\int_2^3 \big\langle \nabla u(\eta\epsilon\xi + w), \xi\big\rangle\,d\eta. \tag{3.31}$$

(3.31) can be rewritten using that $\langle \nabla e_\eta(0), \xi\rangle = \langle \nabla u(\eta\epsilon\xi + w), \xi\rangle - \langle \nabla u(w), \xi\rangle$. Doing this we find that

$$\tilde{c}^{-1}e_1(0) \leqslant \epsilon\int_2^3 \big\langle \nabla e_\eta(0), \xi\big\rangle\,d\eta + \big\langle \nabla u(w), \xi\big\rangle, \tag{3.32}$$

where $\tilde{c} \geqslant 1$ depends only on p, n. Using (3.30) to make simple estimates in (3.32) we can conclude that

$$\bar{c}^{-1}e_1(0) \leqslant \frac{c^2}{A}e_1(0) + \langle \nabla u(w), \xi \rangle. \tag{3.33}$$

In particular, if $A = A(p,n)$ is large enough, and $0 < \epsilon \leqslant \check{\epsilon}$ small enough, then $\langle \nabla u(w), \xi \rangle \geqslant 0$ whenever $\xi \in \Gamma(e_n, \bar{\theta})$. This completes the proof of Lemma 3.25. □

4. Proof of Theorem 1

In this section we prove Theorem 1. To begin the proof, recall that u is p-harmonic, for some fixed p, $1 < p < \infty$, and that u is ϵ-monotone in $D \supset \bar{Q}_1(0)$, in the spherical cap of directions, $\Gamma(e_n, \bar{\theta})$, for some fixed $\bar{\theta} \in (\pi/4, \pi/2)$. Also u is a weak solution to the free boundary problem in (1.3), as defined in Definition 1.4. In view of Lemmas 3.13, 3.25, we may assume, without loss of generality, that for some constants $A \geqslant 1000$, $\theta' \in (0, \pi/2)$, $\epsilon_0 > 0$, depending only on p, n, that

$$A^{-1}|u|(x)/d\big(x, F(u)\big) \leqslant \big|\nabla u(x)\big| \leqslant A\big\langle \nabla u(x), \xi\big\rangle \leqslant A^2|u|(x)/d\big(x, F(u)\big) \tag{4.1}$$

whenever $x \in Q_1(0)$, $\xi \in \Gamma(e_n, \theta')$, and $d(x, F(u)) \geqslant A\epsilon$, $0 < \epsilon \leqslant \epsilon_0$. Also,

$$\big\langle \nabla u(x), \xi\big\rangle \geqslant 0 \quad \text{whenever } \xi \in \Gamma(e_n, \bar{\theta}) \text{ and } x \in Q_1(0) \quad \text{with } d\big(x, F(u)\big) \geqslant A\epsilon. \tag{4.2}$$

Indeed, otherwise we consider $u^*(x) = u(\hat{x} + x/c)$, $x \in Q_1(0)$, for fixed $\hat{x} \in F(u) \cap Q_{1/2}(0)$ and $c \geqslant 1$ large. Then u^* is p-harmonic in $Q_1(0) \setminus F(u^*)$ as we see from translation and dilation invariance of the p-Laplacian. Also for $c = c(p,n)$ large enough, u^* satisfies (4.1), (4.2) with u replaced by u^* thanks to Lemmas 3.13, 3.25 (provided ϵ_0, $1/A$ are sufficiently small). Finally u^* is a weak solution to the free boundary problem in (1.3), as stated in Definition 1.4, with G replaced by G^* where $G^*(s) = c^{-1}G(cs)$, $s \in [0, \infty)$. Proving Theorem 1 for u^* and translating back we get that $F(u) \cap Q_{1/(2c)}(\hat{x})$ is the graph of a Lipschitz function. Using this result and covering $\bar{Q}_{1/2}(0)$ by cylinders of the form $Q_{1/(2c)}(\hat{x})$, $\hat{x} \in F(u) \cap Q_{1/2}(0)$ we get Theorem 1. Hence throughout the proof of Theorem 1 we assume that (4.1), (4.2) hold. Let $O = O_A = \{x \in Q_1(0)\colon d(x, F(u)) > 2A\epsilon\}$ for A large and note that (4.2) implies, for $A = A(p,n)$ large enough, that

$$u \text{ is monotone in } O = O_A \text{ in the spherical cap of directions } \Gamma(e_n, \bar{\theta}). \tag{4.3}$$

Thus we may also assume that (4.3) holds. From (3.15), (3.16) with $Q = Q_{1-100\epsilon, 1-50\epsilon}(0)$, we see that there exists a Lipschitz function $\tau : \mathbf{R}^{n-1} \to \mathbf{R}$ with

$$\|\tau\|_{\mathrm{Lip}(\mathbf{R}^{n-1})} < 1 \quad \text{and} \quad \check{h}\big(F(u) \cap Q, \qquad \big\{\big(y', \tau(y')\big)\colon y' \in \mathbf{R}^{n-1}\big\} \cap Q\big) \leqslant \epsilon.$$

Convoluting τ with a suitable approximate identity on $\mathbf{R}^{n-1}$ we get $\tilde{\tau} \in C^\infty(\mathbf{R}^{n-1})$ with

$$\begin{aligned} &\check{h}\big(F(u) \cap Q, \qquad \big\{\big(y', \tilde{\tau}(y')\big)\colon y' \in \mathbf{R}^{n-1}\big\} \cap Q\big) \leqslant 2\epsilon \quad \text{while} \\ &\|\tilde{\tau}\|_{\mathrm{Lip}(\mathbf{R}^{n-1})} < 1 \quad \text{and} \quad \big\|D^k\tilde{\tau}\big\|_{L^\infty(\mathbf{R}^{n-1})} \leqslant (c\epsilon)^{1-k} \quad \text{for } k \geqslant 2. \end{aligned} \tag{4.4}$$

Here $c = c(p,n,k)$ and D^k denotes an arbitrary k-th order derivative of $\tilde{\tau}$. As in Lemma 3.7 set

$$\Lambda(h) = \big\{\big(x', x_n\big) \in \mathbf{R}^n\colon \big|x_n - \tilde{\tau}\big(x'\big)\big| < h\big\} \cap \bar{Q}_{2,8}(0).$$

Using Lemma 3.7 with $\lambda = \tilde{\tau}$, $M = 1$, $\beta = 1/2$, and $h = 100A\epsilon$, we get $\{\phi_t(x),\ x \in \Lambda(h),\ 0 \leqslant t \leqslant 1\}$ satisfying (i)–(v) of this lemma. Next let $\mu = \mu(p,n) > 0$ be the constant in Lemma 3.7(i) and let $\gamma \in [7/8, 1)$ be the smallest number such that

$$k(\gamma) = \frac{\gamma \sin\bar{\theta}}{\sin\bar{\theta} - 1 + \gamma} \leqslant 1 + \mu. \tag{4.5}$$

Since k is decreasing on $[7/8, 1]$ with $k(1) = 1$ it is easily seen that $1 < k(\gamma) = \min(k(7/8), 1 + \mu)$. Also using $\bar{\theta} \in (\pi/4, \pi/2)$, one deduces

$$1/2 < \sin\bar{\theta} + \gamma - 1 < 1. \tag{4.6}$$

Let $\epsilon' \in (\epsilon, 2\epsilon)$ and put $\sigma = \epsilon'(\sin\bar{\theta} + \gamma - 1)$. Observe from (4.6) that $\sigma \in (\epsilon'/2, \epsilon')$. Also set

$$v_t(x) = \max_{y \in B(x, \sigma\phi_t(x))} u\big(y - \gamma\epsilon' e_n\big) \quad \text{whenever } \bar{B}\big(x, \sigma\phi_t(x)\big) \subset Q_1(0),\ t \in [0,1].$$

From (i)–(iii) of Lemma 3.7 and Lemma 3.4 we deduce that v_t is p-subharmonic in $(\Lambda(h/2) \cap Q_{1-8\epsilon}(0)) \setminus F(v_t)$ for $0 \leqslant t \leqslant 1$. Also note from Lemma 3.7, with $\beta = 1/2$, $h = 100A\epsilon$, and the above observation, that

$$\epsilon'/2 \leqslant \sigma\phi_t \leqslant c\epsilon \quad \text{and} \quad \sigma|\nabla\phi_t| \leqslant c\epsilon^{1/2} \quad \text{on } \Lambda(h), \tag{4.7}$$

for some $c = c(p,n) \geqslant 1$. Using (4.7) we first see that

$$\begin{aligned} &\frac{1}{1+\sigma|\nabla\phi_t|(x)}\left(\sin\bar\theta - \frac{\epsilon'}{2\sigma\phi_t(x)}(\cos\bar\theta)^2 - \sigma|\nabla\phi_t|(x)\right) \\ &\quad\geqslant \frac{1}{1+c\epsilon^{1/2}}\bigl(\sin\bar\theta - (\cos\bar\theta)^2 - c\epsilon^{1/2}\bigr) > 0, \end{aligned} \tag{4.8}$$

for ϵ_0 sufficiently small, $0 < \epsilon \leqslant \epsilon_0$, whenever $x \in \Lambda(h) \cap Q_{2,8}(0)$. Hence, using Lemma 3.6 with ϵ replaced by ϵ', we deduce the existence of θ', $0 < \theta' \leqslant \bar\theta$, such that

$$\begin{aligned} &v_t \text{ is monotone in } \Lambda(h/2) \cap Q_{1-8\epsilon}(0) \text{ in the set of directions } \Gamma(e_n,\theta') \text{ while} \\ &F(v) \cap \Lambda(h/2) \cap Q_{1-8\epsilon}(0) \text{ is the graph of a Lipschitz function with norm} \leqslant c\cot\theta'. \end{aligned} \tag{4.9}$$

Let $\Omega = \Lambda(h/2) \cap Q_{1-h}(0)$ where once again $h = 100A\epsilon$. From Lemmas 3.4 and 3.7 we see that v_t is p-subharmonic in $\Omega \setminus F(v_t)$. Next we prove the following lemma.

Lemma 4.10. *If $\epsilon_0 > 0$ is small enough, then there exists $c' \geqslant 1$, depending only on $p, n, \bar\theta$, such that if*

$$1 + \mu\bar t = \frac{\gamma(\sin\bar\theta - c'\epsilon^{1/4})}{\sin\bar\theta - 1 + \gamma},$$

then $\bar t \in (0,1)$, and for $t \in [0,\bar t]$,

$$\begin{aligned} &(+) \qquad v_t \leqslant u \quad \text{on } \partial\Omega \quad \text{and} \quad v_t \leqslant \bigl(1-\epsilon^{1/4}\bigr)u \quad \text{on } \bar\Omega \setminus \bar\Lambda(h/16), \\ &(++) \quad u > 0 \quad \text{on } F(v_t) \cap \bar\Omega \setminus Q_{1-h^{1/2}}(0) = F(v_0) \cap \bar\Omega \setminus Q_{1-h^{1/2}}(0). \end{aligned}$$

Proof. From basic geometry and the definition of σ we note that

$$B\bigl(x - \gamma\epsilon' e_n, \epsilon'(\sin\bar\theta + \gamma - 1)\bigr) = B\bigl(x - \gamma\epsilon' e_n, \sigma\bigr) \subset B\bigl(x - \epsilon' e_n, \epsilon'\sin\bar\theta\bigr). \tag{4.11}$$

From (4.11), Lemma 3.7(i), (iv), and ϵ-monotonicity of u in the spherical cap of directions $\Gamma(e_n,\bar\theta)$ we have $v_0 \leqslant u$ in $\bar\Omega$ and $v_t \equiv v_0$ in $\bar\Omega \setminus Q_{1-h^{1/2}}(0)$. Also $u(x) > 0$ whenever $x \in F(v_0) \cap \bar\Omega$, since otherwise we could use $\epsilon' \in (\epsilon, 2\epsilon)$ and argue as in the proof of (3.14)(i) to get a contradiction. Using these facts it is easily seen that (++) of Lemma 4.10 is valid and $v_t \leqslant u$ on $\bar\Omega \setminus Q_{1-h^{1/2}}(0)$. To complete the proof of Lemma 4.10 suppose $x \in \bar\Omega \setminus \bar\Lambda(h/16)$. Then from (4.3) we see that

$$a = \max_{y \in \bar B(x, \gamma\epsilon'\sin\bar\theta)} u\bigl(y - \gamma\epsilon' e_n\bigr) \leqslant u(x). \tag{4.12}$$

From (4.12), Lemma 3.7(i), and our choice of $\bar t$, we deduce that

$$v_t(x) \leqslant v_{\bar t}(x) \leqslant \max_{y \in \bar B(x, \sigma(1+\mu\bar t))} u\bigl(y - \gamma\epsilon' e_n\bigr) = \max_{y \in \bar B(x, \gamma\epsilon'(\sin\bar\theta - c'\epsilon^{1/4}))} u\bigl(y - \gamma\epsilon' e_n\bigr) = b. \tag{4.13}$$

Finally from (4.12), (4.13), and (4.1) we get, for some $c = c(p,n)$, that

$$b \leqslant a - c'\epsilon^{5/4}\frac{u(x)}{c\epsilon} \leqslant \bigl(1-\epsilon^{1/4}\bigr)u(x) \quad \text{whenever } x \in \bar\Omega \setminus \bar\Lambda(h/16),$$

provided c' is large enough. The proof of Lemma 4.10 is now complete. □

To complete the proof of Theorem 1 we use, as in [1,2], a method of continuity type argument. In particular, let

$$\Theta = \bigl\{t:\ t \in [0,\bar t],\ v_t \leqslant u \text{ on } \Omega\bigr\}$$

where $\bar{t}$ is as stated in Lemma 4.10. We will prove that

$$\Theta = [0, \bar{t}]. \tag{4.14}$$

To proceed we first note that $0 \in \Theta$ as we pointed out after (4.11). Moreover, the continuity of u and v_t implies that Θ is closed. Thus to prove (4.14) it suffices to prove that Θ is relatively open. Note that if $t \in \Theta$, then $D^+(v_t) = \{x \in \Omega:\ v_t(x) > 0\} \subseteq D^+(u)$. Also from Lemma 4.10(++) we see, for $0 \leqslant t \leqslant \bar{t}$, that $F(v_t) \cap \Omega \setminus Q_{1-h^{1/2}}(0)$ lies strictly above $F(u) \cap \Omega \setminus Q_{1-h^{1/2}}(0)$ and hence the two sets have an empty intersection. Also $v_t \leqslant u$ on $\partial\Omega$ and $F(v_0) \cap F(u) = \emptyset$. Since v_t is p-subharmonic in $\Omega \setminus F(v_t)$ it follows that either (4.14) is true or there exists $t \in \Theta$ with

$$F(u) \cap F(v_t) \cap \Omega \cap \bar{Q}_{1-h^{1/2}}(0) \neq \emptyset. \tag{4.15}$$

To get a contradiction to (4.15) suppose $\tilde{w} \in F(u) \cap F(v_t) \cap \bar{Q}_{1-h^{1/2}}(0)$. From Lemma 3.5 we see that there exists $w^* \in D^+(v_t)$, and $\rho^* > 0$ such that $B(w^*, \rho^*) \subset D^+(v_t)$ with $\tilde{w} \in \partial B(w^*, \rho^*)$. Moreover if $\tilde{\nu} = (w^* - \tilde{w})/|w^* - \tilde{w}|$, then there exist $\bar{\alpha}, \bar{\beta} \in [0, \infty)$, such that

$$v_t(x) \geqslant \bar{\alpha}\langle x - \tilde{w}, \tilde{\nu}\rangle^+ - \bar{\beta}\langle x - \tilde{w}, \tilde{\nu}\rangle^- + o\big(|x - \tilde{w}|\big), \tag{4.16}$$

near $\tilde{w}$. Furthermore,

$$\frac{\bar{\alpha}}{1 - \sigma|\nabla\phi_t(\tilde{w})|} \geqslant G\bigg(\frac{\bar{\beta}}{1 + \sigma|\nabla\phi_t(\tilde{w})|}\bigg). \tag{4.17}$$

Since $D^+(v_t) \cap \Omega \subset D^+(u) \cap \Omega$, we see that $B(w^*, \rho^*)$ is also a tangent ball for $D^+(u)$. Using the fact that u is a weak solution to the free boundary problem in (1.3), as defined in Definition 1.4, we obtain

$$u(x) = \alpha\langle x - \tilde{w}, \tilde{\nu}\rangle^+ - \beta\langle x - \tilde{w}, \tilde{\nu}\rangle^- + o\big(|x - \tilde{w}|\big), \tag{4.18}$$

as $x \to \tilde{w}$, for some $\alpha, \beta \in [0, \infty)$ with $\alpha = G(\beta)$.

We claim that

$$0 \leqslant \bar{\alpha} \leqslant \alpha\big(1 - \epsilon^{1/4}/c\big) \tag{4.19}$$

for some $c = c(p, n, \bar{\theta}) \geqslant 1$. (4.16)–(4.19) easily lead to a contradiction. In fact, from (4.16), (4.18), and $t \in \Theta$ we see that $\bar{\alpha} \leqslant \alpha$ while $\beta \leqslant \bar{\beta}$. Using the assumptions on G in Theorem 1, (4.17), (4.19), and Lemma 3.7(ii) we find that if $\bar{\beta} \neq 0$, then

$$\begin{aligned} G(\beta) \leqslant G(\bar{\beta}) &\leqslant \bar{\beta}^N\bigg(\frac{\bar{\beta}}{1 + \sigma|\nabla\phi_t(\tilde{w})|}\bigg)^{-N} G\bigg(\frac{\bar{\beta}}{1 + \sigma|\nabla\phi_t(\tilde{w})|}\bigg) \\ &\leqslant \frac{(1 + \sigma|\nabla\phi_t(\tilde{w})|)^N}{1 - \sigma|\nabla\phi_t(\tilde{w})|}\bar{\alpha} \leqslant \frac{(1 + c\epsilon^{1/2})^N}{1 - c\epsilon^{1/2}}\bar{\alpha} < \alpha, \end{aligned} \tag{4.20}$$

provided ϵ_0 is small enough, thanks to (4.19). If $\bar{\beta} = 0$, we can omit the second inequality in (4.20) and still get $G(0) = G(\beta) < \alpha$. Since $\alpha = G(\beta)$, we have reached a contradiction in either case. Thus (4.14) follows from (4.19).

As for claim (4.19) we first observe from (4.9) that there exists $\lambda : \mathbf{R}^{n-1} \to \mathbf{R}$, with $\|\lambda\|_{\mathrm{Lip}(\mathbf{R}^{n-1})} \leqslant \tan(\pi/2 - \theta')$, such that

$$F(v_t) \cap \Omega = \big\{\big(x', x_n\big):\ x_n = \lambda\big(x'\big)\big\}. \tag{4.21}$$

Also, from the definition of v_t, and ϵ-monotonicity of u in the cap of directions $\Gamma(e_n, \bar{\theta})$, we deduce that

$$\check{h}\big(F(v_t) \cap \bar{\Omega}, F(u) \cap \bar{\Omega}\big) \leqslant 8\epsilon \quad \text{and hence} \quad F(v_t) \cap \bar{\Omega} \subset \Lambda(h/100). \tag{4.22}$$

Let $U = \Omega \cap D^+(v_t)$ and let f_1 be the p-harmonic function in U with continuous boundary values

(a) $f_1\big(x', \lambda\big(x'\big)\big) \equiv 0$ when $x = \big(x', x_n\big) \in F(v_t) \cap \partial U$.

(b) $f_1(x) \equiv v_t(x)$ for $x \in \partial U \cap \partial Q_{1-h}(0) \cap \big\{x = \big(x', x_n\big):\ x_n \leqslant \lambda\big(x'\big) + h/8\big\}$.

(c) $f_1(x) = \min\{\max(v_t(x), \alpha_1(x)), (1-\epsilon^{1/4})u(x)\}$ for $x = (x', x_n) \in \partial U \cap \partial Q_{1-h}(0)$ and $h/8 < x_n - \lambda(x') \leqslant h/4$, where

$$\alpha_1(x', x_n) = v_t(x', \lambda(x') + h/8) + \frac{8(x_n - \lambda(x') - h/8)}{h}\big[(1-\epsilon^{1/4})u(x', \lambda(x') + h/4) - v_t(x', \lambda(x') + h/8)\big].$$

(d) $f_1(x', x_n) = (1-\epsilon^{1/4})u(x', x_n)$ when $x \in \partial U$ and $x_n > \lambda(x') + h/4$. (4.23)

Next let f_2 be the p-harmonic function in U with continuous boundary values,

(a) $f_2(x', \lambda(x')) \equiv 0$ when $x = (x', x_n) \in F(v_t) \cap \partial U$.

(b) $f_2(x) \equiv v_t(x)$ for $x \in \partial U \cap \partial Q_{1-h}(0) \cap \{x = (x', x_n):\ x_n \leqslant \lambda(x') + h/8\}$.

(c) $f_2(x) = \min\{\max(v_t(x), \alpha_2(x)), u(x)\}$ for $x = (x', x_n) \in \partial U \cap \partial Q_{1-h}(0)$ and $h/8 < x_n - \lambda(x') \leqslant h/4$, where

$$\alpha_2(x', x_n) = v_t(x', \lambda(x') + h/8) + \frac{8(x_n - \lambda(x') - h/8)}{h}\big[u(x', \lambda(x') + h/4) - v_t(x', \lambda(x') + h/8)\big].$$

(d) $f_2(x', x_n) = u(x', x_n)$ when $x \in \partial U$ and $x_n > \lambda(x') + h/4$. (4.24)

From (4.22), (4.23) and (+) of Lemma 4.10 we see that $v_t \leqslant f_1 \leqslant f_2 \leqslant u$ on ∂U. Since v_t is p-subharmonic it follows from the boundary maximum principle for p-harmonic functions that

$$v_t \leqslant f_1 \leqslant f_2 \leqslant u \quad \text{in } U. \tag{4.25}$$

From Theorem 2.6 we see that

$$\chi = \lim_{\substack{x \to \tilde{w} \\ x \in U}} \frac{f_1(x)}{f_2(x)} \text{ exists.} \tag{4.26}$$

Using (4.25), (4.26), (4.16), and (4.18) we deduce that

$$\bar{\alpha} \leqslant \liminf_{t\to 0} t^{-1} v_t(\tilde{w} + t\tilde{\nu}) \leqslant \liminf_{t\to 0} t^{-1} f_1(\tilde{w} + t\tilde{\nu}) \leqslant \chi \liminf_{t\to 0} t^{-1} f_2(\tilde{w} + t\tilde{\nu}) \leqslant \chi\alpha. \tag{4.27}$$

From (4.27) we conclude that in order to prove (4.19), and thus complete the proof of (4.14), we only need to prove that

$$\chi \leqslant 1 - \epsilon^{1/4}/\breve{c} \tag{4.28}$$

for some $\breve{c} = \breve{c}(p, n, \bar{\theta})$. To prove (4.28) we note from Theorem 2.9, with $r = A\epsilon$, $\hat{u} = f_2$, $\hat{v} = f_1$, and the observation following that theorem, that we have

$$\frac{f_2(x) - f_1(x)}{f_2(x)} \geqslant c^{-1} \frac{f_2(\tilde{w} + A\epsilon e_n/c) - f_1(\tilde{w} + A\epsilon e_n/c)}{f_2(w + A\epsilon e_n/c)} =: \hat{C}$$

whenever $x \in B(\tilde{w}, A\epsilon e_n/c^2)$. Letting $x \to \tilde{w}$ in the last display it follows that $1 - \chi \geqslant \hat{C}$. Thus to get (4.28) it suffices to show that

$$\hat{C} \geqslant \tilde{c}^{-1}\epsilon^{1/4} \tag{4.29}$$

for some $\tilde{c}$ having the same dependence as $\breve{c}$ in (4.28). To prove (4.29) we would like to use the fact that $f_2 - f_1 = \epsilon^{1/4}u$ on $\partial U \cap \partial\Lambda(h/2)$ as well as an iterative argument using a Harnack inequality for $f_2 - f_1$. Unfortunately however we do not know if the left-hand inequality in (2.11) holds for either $\hat{v} = f_1$ or $\hat{v} = f_2$ in a Harnack chain of balls connecting points in U near $\partial U \cap \partial\Lambda(h/2)$ to $\tilde{w} + A\epsilon e_n/c$. Thus for some balls in our Harnack chain we are not able to control the ellipticity in the PDE satisfied by $f_2 - f_1$ (see (2.14)–(2.16)). To overcome this difficulty we introduce another p-harmonic function f which is continuous in $\bar{U}$ and satisfies

$$
\begin{aligned}
&\text{(a)}\quad f\big(x',\lambda(x')\big)\equiv 0 \quad \text{when } x=(x',x_n)\in F(v_t)\cap\partial U.\\
&\text{(b)}\quad f(x)\equiv 0 \quad \text{for } x\in\partial U\cap\partial Q_{1-h}(0)\cap\big\{x=(x',x_n)\colon x_n\leqslant\lambda(x')+h/8\big\}.\\
&\text{(c)}\quad f(x',x_n)=\min\bigg\{\big(1-\epsilon^{1/4}\big)u(x',x_n),\ \frac{8(x_n-\lambda(x')-h/8)}{h}\Big[\big(1-\epsilon^{1/4}\big)u\big(x',\lambda(x')+h/4\big)\Big]\bigg\}\\
&\qquad \text{when } x\in\partial U\cap\partial Q_{1-h}(0) \text{ and } h/8<x_n-\lambda(x')\leqslant h/4.\\
&\text{(d)}\quad f(x',x_n)=\big(1-\epsilon^{1/4}\big)u(x',x_n) \quad \text{when } x\in\partial U \text{ and } x_n>\lambda(x')+h/4.
\end{aligned}
\tag{4.30}
$$

Observe from (4.30) that

$$
0\leqslant f\leqslant \min\big\{\big(1-\epsilon^{1/4}\big)u,\ f_1\big\} \tag{4.31}
$$

on ∂U. Hence, by the maximum principle for p-harmonic functions (4.31) also holds in U. To prove (4.29), and thus finally get (4.19), we prove that

$$
\frac{f_2(\tilde w+A\epsilon e_n/c)-f(\tilde w+A\epsilon e_n/c)}{f_2(\tilde w+A\epsilon e_n/c)}\geqslant \epsilon^{1/4}/c \quad\text{and}\quad \frac{f_1(\tilde w+A\epsilon e_n/c)-f(\tilde w+A\epsilon e_n/c)}{f_2(\tilde w+A\epsilon e_n/c)}\leqslant\epsilon \tag{4.32}
$$

for some $c=c(p,n,\bar\theta)\geqslant 1$. To do this we first assert that

$$
\big|\nabla f(x)\big|\geqslant\frac{\partial f}{\partial x_n}(x)\geqslant c^{-1}f(x)/h \quad \text{whenever } x\in U \tag{4.33}
$$

for some $c=c(p,n)\geqslant 1$ is large enough. Indeed, for given $0<\delta<10^{-3}h$, let

$$
D_\delta f(x)=\frac{f(x+\delta e_n)-f(x)}{\delta}\quad\text{and}\quad U_\delta=\{x\in U\colon x+\delta e_n\in U\}.
$$

To prove (4.33) we start by comparing the values of $D_\delta f$ and f on ∂U_δ. Note from (4.31) that $D_\delta f\geqslant 0=f$ on $F(v_t)\cap\bar Q_{1-h}(0)$. We observe from (4.22) and $h=100A\epsilon$, that (4.1) holds at points $x\in\partial U\cap\partial Q_{1-h}(0)$ with $x_n\geqslant\lambda(x')+h/16$. Using this observation and (4.31) we see that if $x\in\partial U_\delta\cap\partial\Lambda(-\delta+h/2)$, then

$$
\begin{aligned}
D_\delta f(x)&=\frac{(1-\epsilon^{1/4})u(x+\delta e_n)-f(x)}{\delta}\geqslant\frac{(1-\epsilon^{1/4})[u(x+\delta e_n)-u(x)]}{\delta}\\
&\geqslant c^{-1}u(x)/h\geqslant c^{-1}f(x)/h.
\end{aligned}
$$

Moreover, if $x\in\partial U_\delta\cap\partial Q_{1-h}(0)$ and $\lambda(x')+h/8<x_n$, then we deduce from (4.1) and the definition of f that $x_n\to f(x',x_n)$ is an increasing Lipschitz function, hence absolutely continuous, and

$$
\frac{\partial f}{\partial x_n}\big(x',x_n\big)\geqslant c^{-1}u(x)/h\geqslant c^{-1}f(x)/h,
$$

almost everywhere with respect to one-dimensional Lebesgue measure. Integrating this inequality and using Harnack's inequality we deduce that

$$
D_\delta f(x)\geqslant c^{-1}f(x)/h \tag{4.34}
$$

whenever $x=(x',x_n)\in\partial U_\delta\cap\partial Q_{1-h}(0)$ and $x_n>\lambda(x')+h/8$. Finally, if $x\in\partial U_\delta\cap\partial Q_{1-h}(0)$, and $x_n\leqslant\lambda(x')+h/8$, then $f(x)=0$, $D_\delta f\geqslant 0$. We now conclude that (4.34) holds on ∂U_δ and thereupon, by the maximum principle for p-harmonic functions, that (4.34) holds in U_δ. Letting $\delta\to 0$ we obtain from (4.34) that assertion (4.33) is true.

To continue our proof of (4.32) recall that $\partial U\cap\partial\Lambda(h/2)=\{x\colon x_n=\tilde\tau(x')+h/2\}$ and that $\tilde\tau$ satisfies (4.4). Given $\hat x\in\partial U\cap\partial\Lambda(h/2)\cap Q_{1-2h}$ put $G=\{y\in B(0,1)\colon \hat x+(h/4)y\in U\cap B(\hat x,h/4)\}$ and $\Gamma=\{y\in B(0,1)\colon \hat x+(h/4)y\in\partial U\cap B(\hat x,h/4)\}$. If $f'\in\{f,f_1,f_2\}$, set $f''(y)=f'(\hat x+(h/4)y)$ and $u'(y)=u(\hat x+(h/4)y)$, $y\in G$. From (4.4) we see that $\Gamma\cap B(0,1)$ is C^2 with C^2-constants depending only on p,n. Also, from (4.1) and Lemma 2.5 we see for k a positive integer that u' has continuous k-th order derivatives in $\bar G$, with L^∞-norm bounded by $cu(\hat x)$ where c depends only on p,n,k. Using these facts we deduce that Theorem 1 in [13] can be applied to conclude that f'' has a Hölder continuous extension to $\bar G\cap B(0,1/2)$. In particular, $|\nabla f''|\leqslant cu(\hat x)$ in $G\cap B(0,1/2)$. Transferring this inequality to f' we conclude that

$$
\big|\nabla f'\big|\leqslant cu(\hat x)/h \quad \text{in } U\cap B(\hat x,h/8) \text{ whenever } f'\in\{f,f_1,f_2\}. \tag{4.35}
$$

We observe from the boundary values of f, (4.35), and the mean value theorem from elementary calculus that, for some $\tilde{c} = \tilde{c}(p, n)$,

$$u(y)/\tilde{c} \leqslant u(\hat{x}) \leqslant \tilde{c} f(y) \quad \text{whenever } y \in U \cap B(\hat{x}, h/\tilde{c}). \tag{4.36}$$

Let $\bar{x} \in F(v_t)$ with $\bar{x}' = \hat{x}'$. Then from Theorem 2.6, (4.25), and (4.31) we see, for some $c = c(p, n)$, that

$$1 \leqslant \frac{f_2}{f} \leqslant c \frac{u(\bar{x} + e_n h/c)}{f(\bar{x} + e_n h/c)} \quad \text{on } B(\bar{x}, h/c).$$

Also from Harnack's inequality and (4.36) we find that

$$\frac{u(\bar{x} + he_n/c)}{f(\bar{x} + he_n/c)} \leqslant c^+$$

where $c^+ = c^+(p, n)$. Combining the above inequalities and using arbitrariness of $\hat{x}$, it follows that

$$1 \leqslant \frac{f_2(x)}{f(x)} \leqslant c' \quad \text{whenever } x \in U \cap Q_{1-2h}(0). \tag{4.37}$$

Again $c' = c'(p, n)$. Similarly from (4.33), (4.35), (4.36), and Theorem 2.7 we deduce that

$$c^{-1} \frac{f(x)}{d(x, F(v_t))} \leqslant \left|\nabla f(x)\right| \leqslant c \frac{f(x)}{d(x, F(v_t))} \quad \text{whenever } x \in U \cap Q_{1-2h}(0). \tag{4.38}$$

Now

$$\left|\nabla f_2(x)\right| \leqslant c \frac{f_2(x)}{d(x, F(v_t))} \quad \text{whenever } x \in U \cap Q_{1-2h}(0) \tag{4.39}$$

as follows from (4.35) and Lemma 2.5. Let $e = f_2 - f$. From (4.37)–(4.39) and (2.12)–(2.17) with $\hat{u} = f_2$, $\hat{v} = f$ we see that e satisfies a locally uniformly elliptic divergence form PDE in U for which solutions satisfy a Harnack inequality as in Lemma 2.18. Moreover, this PDE is uniformly elliptic in $U \cap B(\hat{x}, h/8)$ whenever $\hat{x} \in \partial U \cap \partial \Lambda(h/2)$. Using results for such solutions similar to those in Lemma 2.3 (see [5]), and examining the boundary values of e, we deduce that $ce \geqslant \epsilon^{1/4} u(\hat{x})$ on $U \cap B(\hat{x}, h/8)$. Let $\hat{x} \in \partial U \cap \partial \Lambda(h/2)$, with $\hat{x}' = \tilde{w}'$, where $\tilde{w}$ is as in (4.32). Then from the above deduction, Harnack's inequality for e, (4.36), and (4.37), we get for A, c as in (4.32) that

$$e(\tilde{w} + A\epsilon e_n/c)/f_2(\tilde{w} + A\epsilon e_n/c) \geqslant c^{-1} \epsilon^{1/4} \tag{4.40}$$

which is the left-hand inequality in (4.32).

To prove the right-hand inequality in (4.32), let i be a positive integer and let M_i denote the maximum of $\bar{e} = f_1 - f$ in $\bar{U} \cap \bar{Q}_{1-ih}(0)$ for $1 \leqslant i \leqslant h^{-1/2}$. We next prove, for some $\eta = \eta(p, n)$, $0 < \eta < 1$, that

$$M_1 \leqslant cu(e_n/2) \quad \text{and} \quad M_{i+1} \leqslant \eta M_i \quad \text{whenever } 2 \leqslant i + 1 \leqslant h^{-1/2}. \tag{4.41}$$

The left-hand inequality in (4.41) follows from (4.25), (4.31), and ϵ-monotonicity of u in the directions $\Gamma(e_n, \bar{\theta})$. To prove the right-hand inequality in (4.41) we note from (4.25) that (4.37) holds with f_2 replaced by f_1. Also (4.39) is valid with f_2 replaced by f_1. Arguing as below (4.39) we see that $\bar{e}$ satisfies a locally uniformly elliptic PDE for which positive solutions satisfy a Harnack inequality as in Lemma 2.18. Moreover, if $\hat{x} \in \bar{Q}_{1-(i+1)h}(0) \cap \partial \Lambda(h/2)$, then this PDE is uniformly elliptic in $U \cap B(\hat{x}, h/8)$ and $\bar{e} \equiv 0$ on $\partial U \cap B(\hat{x}, h/8)$. We can now conclude, arguing as in [5], that $\bar{e}$ is Hölder continuous in a neighborhood of $\hat{x}$. In particular, there exists $c = c(p, n) \geqslant 1$ such that

$$\bar{e}(x) \leqslant M_i/2 \quad \text{whenever } x \in U \cap B(\hat{x}, h/c). \tag{4.42}$$

Let $\bar{x} \in F(v_t)$ with $\bar{x}' = \hat{x}'$. Then from Theorem 2.9 applied to $\bar{e}$, f_1 we have

$$c^{-1} \frac{\bar{e}(x)}{f_1(x)} \leqslant \frac{\bar{e}(\bar{x} + he_n/c)}{f_1(\bar{x} + he_n/c)} \leqslant \frac{cM_i}{f_1(\bar{x} + he_n/c)}$$

for some $c = c(p, n)$ and $x \in B(\bar{x}, h/c)$. From this display, and Lemmas 2.3 and 2.4 for f_1, we deduce, for some $c' = c'(p, n) \geqslant 1$, that

$$\bar{e}(x) \leqslant M_i/2 \quad \text{whenever } x \in U \cap B\big(\bar{x}, h/c'\big). \tag{4.43}$$

Let $E = M_i - \bar{e}$. Using (4.42), (4.43), and Harnack's inequality for E we conclude that $E \geqslant M_i/c$ on $U \cap \partial Q_{1-(i+1)h}(0)$ for some $c = c(p,n) \geqslant 1$. Thus $\bar{e} \leqslant (1-1/c)M_i$ holds on $U \cap \partial Q_{1-(i+1)h}(0)$. Since $\bar{e} \equiv 0$ on the rest of the boundary of $U \cap Q_{1-(i+1)h}(0)$ it follows once again from the boundary maximum principle for p-harmonic functions that the right-hand inequality in (4.41) is true. Finally we use (4.41) to prove the right-hand inequality in (4.32). Recall that $\tilde{w} \in \bar{U} \cap Q_{1-h^{1/2}}(0)$. Using this fact and iterating (4.41) we see for some $c^* = c^*(p,n) \geqslant 1$ that

$$\bar{e}(\tilde{w} + A\epsilon e_n/c) = (f_2 - f_1)(\tilde{w} + A\epsilon e_n/c) \leqslant \exp\bigl[-1/\bigl(c^* h^{1/2}\bigr)\bigr]u(e_n/2) \tag{4.44}$$

where c is the constant in (4.32). From Harnack's inequality we deduce, for some $\tilde{c} = \tilde{c}(p,n)$, that

$$u(e_n/2) \leqslant \epsilon^{-\tilde{c}} u(\tilde{w} + A\epsilon e_n/c). \tag{4.45}$$

Combining (4.44), (4.45) we get for ϵ_0 sufficiently small that

$$\bar{e}(\tilde{w} + A\epsilon e_n/c) \leqslant \epsilon^2 u(\tilde{w} + A\epsilon e_n/c) \leqslant \epsilon f_2(\tilde{w} + A\epsilon e_n/c),$$

where the last inequality follows from the display above (4.37). Thus the right-hand inequality in (4.32) is valid for sufficiently small $\epsilon > 0$. From earlier work we can now conclude first the validity of (4.32) and then that (4.19) is valid. Finally, we get (4.14) from (4.19) as we proved after that display.

Proof of Theorem 1. The rest of the proof of Theorem 1 follows as in [2, Section 7]. More specifically, from (4.14) we have $v_{\bar{t}} \leqslant u$ whenever $x \in U$. In view of the definition of $\bar{t}$, γ, and Lemma 3.7(v), we deduce the existence of θ^*, $c_* = c_*(p,n,\bar{\theta}) \geqslant 1$, such that $0 \leqslant \bar{\theta} - \theta^* \leqslant c_* \epsilon^{1/4}$ and

$$\max_{B(x,\gamma\epsilon'\sin\theta^*)} u\bigl(y - \gamma\epsilon' e_n\bigr) \leqslant u(x) \quad \text{whenever } x \in U \cap Q_{1-4h^{1/2}}(0) \text{ and } \epsilon' \in (\epsilon, 2\epsilon). \tag{4.46}$$

Clearly (4.46) and (4.2) imply, for $\epsilon_0 = \epsilon_0(p,n,\bar{\theta}) > 0$ sufficiently small, that u is $(\gamma\epsilon)$-monotone in $Q_1(0)$ in the directions $\Gamma(e_n, \theta^*)$. We can now proceed by an iterative argument to obtain Theorem 1. That is, we repeat the argument in Section 4 with ϵ replaced by $\gamma\epsilon$ and $Q_1(0)$ replaced by $Q_{1-8h^{1/2}}(0)$ to get that u is $(\gamma^2\epsilon)$-monotone, in a certain cap of directions in $Q_\rho(0)$ where $\rho = 1 - 8h^{1/2} - 8(\gamma h)^{1/2}$, etc. On the surface each iteration may yield constants which depend on the angle opening of the cap yielding the directions of monotonicity. However these constants can also be chosen to depend only on $\bar{\theta}$ as we could have chosen the constants in each iteration to depend only on $\bar{\theta}_1 = \frac{\bar{\theta}}{2} + \frac{\pi}{8}$ (since $\Gamma(e_n, \bar{\theta}_1) \subset \Gamma(e_n, \bar{\theta})$) provided we first choose ϵ_0 so small that for the new c_* above (4.46) we have

$$c_* \epsilon^{1/4} \sum_{m=0}^{\infty} \gamma^{m/4} < \frac{\bar{\theta}}{2} - \frac{\pi}{8}.$$

Since $\bar{\theta}_1 = \bar{\theta}_1(\bar{\theta})$ it follows that we can choose all constants to depend only on $\bar{\theta}$. Continuing the induction or iterative process we eventually conclude that u is η monotone in the cap $\Gamma(e_n, \bar{\theta}_1)$ in $Q_{1/2}(0)$ whenever $\eta > 0$. Clearly this conclusion implies that u is monotone in $Q_{1/2}(0)$. The proof of Theorem 1 is now complete. □

5. Proof of Theorem 2 and Corollary 1

To begin the proof of Theorem 2 we remark that much of the proof of Theorem 1 remains valid (with modest changes) under the weaker assumption that u^+ is ϵ-monotone in D. More specifically Lemma 3.13 remains valid under the additional assumption that $u(x) > 0$. In fact, arguing as in (3.14)–(3.16) we get $\tau : \mathbf{R}^{n-1} \to \mathbf{R}$ such that if $\breve{\Omega} = \{(x', x_n):\ x_n > \tau(x')\}$, then

$$\begin{aligned} &(+) \qquad \breve{h}\bigl(F(u) \cap \bar{Q}, \partial\breve{\Omega} \cap \bar{Q}\bigr) \leqslant \epsilon, \\ &(++) \qquad \|\tau\|_{\mathrm{Lip}(\mathbf{R}^{n-1})} \leqslant \tan(\pi/2 - \hat{\theta}) \ll 1, \end{aligned} \tag{5.1}$$

where $Q = Q_{1-100\epsilon, 1-50\epsilon}(0)$. Using this fact and repeating the argument from (3.17) to (3.24) we get Lemma 3.13 when $u(x) > 0$. Also, Lemma 3.25 holds (under the assumptions of Theorem 2) with $\bar{\theta}$ replaced by $\hat{\theta}$ when $u(x) > 0$. Using the amended form of these lemmas we can now assume, as in Section 4 (see the remark after (4.2)), that for some $A \geqslant 1000$, $\theta' \in (0, \pi/2)$, $\hat{\epsilon} > 0$ that

$$A^{-1} u(x)/d\bigl(x, F(u)\bigr) \leqslant \bigl|\nabla u(x)\bigr| \leqslant A\bigl\langle \nabla u(x), \xi\bigr\rangle \leqslant A^2 u(x)/d\bigl(x, F(u)\bigr) \tag{5.2}$$

whenever $x \in D^+(u) \cap Q_1(0)$, $\xi \in \Gamma(e_n, \theta')$, and $d(x, F(u)) \geqslant A\epsilon$, $0 < \epsilon \leqslant \hat{\epsilon}$. Furthermore,

$$\langle \nabla u(x), \xi \rangle \geqslant 0 \quad \text{whenever } \xi \in \Gamma(e_n, \hat{\theta}), \text{ and } x \in D^+(u) \cap Q_1(0) \quad \text{with } d\big(x, F(u)\big) \geqslant A\epsilon. \tag{5.3}$$

We can now repeat, essentially verbatim, the argument leading to (4.19). Unfortunately however, in this case, we cannot use (4.19) to obtain the contradiction in (4.20). In fact we only have $v_t^+ \leqslant u^+$ so we do not know that $\beta \leqslant \bar{\beta}$. To overcome this obstacle we follow closely the proof scheme in [2,23]. Indeed, if $M = \max_{\bar{Q}_1(0)} u^-$, then we first prove the following.

Lemma 5.4. *Under the assumptions in Theorem* 2, *there exist* $\tilde{\epsilon} > 0$, $\tilde{\theta} \in (3\pi/8, \pi/2)$, $a > 0$, $c \geqslant 1$, *all depending only on* p, n, *such that if* $0 < \epsilon \leqslant \tilde{\epsilon}$, $\tilde{\theta} \leqslant \theta < \pi/2$, *and* $u^-(-e_n/2) \geqslant M\epsilon^{1/2}$, *then* u *is* ϵ^a*-monotone in* $Q_{3/4}(0) \cap \{y\colon y_n \geqslant -1/c\}$ *in the cap of directions* $\Gamma(e_n, 5\pi/16)$.

Proof. Let τ be as in (5.1) and for $z = (z', \tau(z') + 2\epsilon) \in D^+(u) \cap Q_{7/8}(0)$ let $K(z)$ be the set of all points in $Q_1(0)$ that are not in the closure of the cone $z + C(e_n, \tilde{\theta})$. Observe from (5.1) that $D^-(u) \cap Q_1(0) \subset K(z)$. Let h be the p-harmonic function in $K(z)$ with continuous boundary values u^- on $\partial K(z)$. If $y = z - t\epsilon e_n \in D^-(u) \cap Q_{7/8}(0)$, where $-3 \leqslant t \leqslant 3$, then from Lemma 2.19 and Theorem 2.6 we see, for some $c = c(p, n)$, $b = b(p, n, \tilde{\theta}) \geqslant 1$, that

$$u^-(y) \leqslant cM\epsilon^b \quad \text{where } b \to 1 \text{ as } \tilde{\theta} \to \pi/2. \tag{5.5}$$

Let $\Omega' = \{w \in Q_{7/8}(0)\colon w_n < \tau(w') - 2\epsilon\}$. Then from (5.1) we find that

$$\Omega' \subset D^-(u) \cap Q_{7/8}(0). \tag{5.6}$$

Let $\hat{u}$ be the p-harmonic function in Ω' which is continuous in $\bar{\Omega}'$ with boundary values

$$\begin{aligned} &\text{(a)} \quad \hat{u} \equiv 0 \quad \text{on } \partial\Omega' \cap Q_{7/8}(0). \\ &\text{(b)} \quad \hat{u}(y) = u^-(y) \quad \text{when } y \in \partial\Omega' \cap \partial Q_{7/8}(0) \text{ and } y_n \leqslant \tau\big(y'\big) - 3\epsilon. \\ &\text{(c)} \quad \hat{u}(y) = \min\left\{u^-(y), \frac{(-y_n + \tau(y') - 2\epsilon)}{\epsilon} u^-\big(y', \tau(y') - 3\epsilon\big)\right\} \quad \text{when } y \in \partial\Omega' \cap \partial Q_{7/8}(0) \\ &\qquad \text{and } \tau\big(y'\big) - 3\epsilon < y_n \leqslant \tau\big(y'\big) - 2\epsilon. \end{aligned} \tag{5.7}$$

From the maximum principle for p-harmonic functions and (5.5) we deduce that

$$\hat{u} \leqslant u^- \leqslant \hat{u} + cM\epsilon^b \quad \text{in } \Omega'. \tag{5.8}$$

Next let $\sigma \in \Gamma(e_n, 5\pi/16)$, $w = x - s\sigma$, where $\epsilon^a \leqslant s \leqslant c^{-1}$, $0 < \epsilon \leqslant \tilde{\epsilon}$, and set $\phi(x) = u(x) - u(w)$ whenever $x \in \bar{Q}_{3/4}(0)$. We show for c large enough that there exists $a = a(p, n) > 0$ with $\phi(x) \geqslant 0$ whenever $x \in Q_{3/4}(0)$ with $x_n \geqslant -c^{-1}$, provided $\tilde{\theta} = \tilde{\theta}(p, n)$ is near enough $\pi/2$ and $\tilde{\epsilon} = \tilde{\epsilon}(p, n)$ is small enough. From ϵ-monotonicity of u^+ it is easily seen that we only need to consider the case when x, w are in $D^-(u) \cap Q_{3/4}(0)$. From (5.1), (5.5), we see that if $-2\epsilon + \tau(x') \leqslant x_n$, then

$$\phi(x) \geqslant -u(w) - c'M\epsilon^b \geqslant \hat{u}(w) - c'M\epsilon^b. \tag{5.9}$$

Using Lemma 2.19 applied to cones within Ω', and arguing as in the proof of (5.5), we deduce, for small $\epsilon > 0$, that there exists $d = d(p, n) > 1$ with

$$\hat{u}(w) \geqslant \epsilon^{ad} u^-(-e_n/2) \quad \text{where } d \to 1 \text{ as } \tilde{\theta} \to \pi/2. \tag{5.10}$$

Combining (5.9), (5.10), and using the hypotheses in Lemma 5.4 we find that

$$\phi(x) \geqslant \big(\epsilon^{ad+1/2} - c'\epsilon^b\big)M > 0 \tag{5.11}$$

for small $\epsilon > 0$ provided $ad + 1/2 < b$. If $x_n < -2\epsilon + \tau(x')$ then $x, w \in \Omega'$ and we find from (5.8) that

$$\phi \geqslant \hat{u}(w) - \hat{u}(x) - cM\epsilon^b. \tag{5.12}$$

We note that Theorem 2.7 is valid for the current $\hat{u}$ with $\theta_0 = 5\pi/16$. From this note we deduce that if $x_n \geqslant -c^{-1}$, then $\hat{u}$ is increasing on the line segment from x to w. Let y be the point on this line segment with $|w - y| = \frac{1}{2}\epsilon^a$. Then from Theorem 2.7, the mean value theorem from calculus, Harnack's inequality, and the same estimate as in (5.10), we find that

$$\hat{u}(w) - \hat{u}(x) \geqslant \hat{u}(w) - \hat{u}(y) \geqslant \tilde{c}^{-1}\epsilon^a \hat{u}(w)/d\big(w, \partial\Omega'\big) \geqslant \epsilon^{ad} u^-(-e_n/2). \tag{5.13}$$

Using (5.13) in (5.12) we see that (5.11) is valid.

From arbitrariness of x, σ, s, in (5.11), we now conclude Lemma 5.4. □

From Lemma 5.4 we see that if $u^-(-e_n/2) \geqslant \epsilon^{1/2}M$, then u is ϵ^a-monotone in $Q_{3/4} \cap \{w\colon\ y_n \geqslant -1/c\}$. Hence we can essentially repeat the proof of Theorem 1 with ϵ replaced by ϵ^a and $Q_1(0)$ by $Q_{3/4}(0) \cap \{w\colon\ y_n \geqslant -1/c\}$ to prove Theorem 2. Thus throughout the rest of the proof of Theorem 2 we assume that

$$u^-(-e_n/2) \leqslant \epsilon^{1/2}M. \tag{5.14}$$

From (5.14) and Harnack's inequality applied to u^- we see, for $\hat{\epsilon}$ sufficiently small, that there exists $\kappa = \kappa(p,n)$, $0 < \kappa < 1/100$, such that

$$u^-(x) \leqslant \epsilon^{7/16}M \quad \text{when } x = \big(x', x_n\big) \in Q_{1-\epsilon^\kappa}(0) \text{ and } x_n \leqslant \tau\big(x'\big) - \epsilon^\kappa. \tag{5.15}$$

Next suppose that $w \in F(u) \cap Q_{1-2\epsilon^{\kappa/2}}(0)$ and that there exists a ball $B(\hat{w}, \rho)$, $\hat{w} \in D^+(u)$ with $w \in \partial B(\hat{w}, \rho)$ and $\epsilon/100 \leqslant \rho \leqslant 100\epsilon$. From Definition 1.4 we obtain for $\nu = (\hat{w} - w)/|\hat{w} - w|$, and some $\alpha, \beta \in [0, \infty]$ with $\alpha = G(\beta)$, that

$$u(x) = \alpha\langle x - w, \nu\rangle^+ - \beta\langle x - w, \nu\rangle^- + o\big(|x - w|\big)$$

as $x \to w$. To proceed we prove the following lemma.

Lemma 5.16. *With the above notation and under the assumptions in Theorem* 2, (5.14)*, there exist $\theta_+ = \theta_+(p,n) \in (5\pi/8, \pi/2)$ and $\epsilon_+ = \epsilon_+(p,n,M) > 0$, such that if $\theta_+ \leqslant \theta < \pi/2$, $0 < \epsilon \leqslant \epsilon_+$, then $\beta \leqslant \epsilon^{3/8}$.*

Proof. Let ψ be the p-harmonic function in $B(\hat{w}, 4\rho) \setminus B(\hat{w}, \rho)$ with continuous boundary values 1 on $\partial B(\hat{w}, 4\rho)$ and 0 on $\partial B(\hat{w}, \rho)$. We note that $\psi(x) = a_1|x - \hat{w}|^{(p-n)/(p-1)} + a_2$ for properly chosen a_1, a_2 when $p \neq n$, and $\psi(x) = a_1 \log|x - \hat{w}| + a_2$ for $p = n$. From the maximum principle for p-harmonic functions it follows that

$$t^{-1}u^-(w - t\nu) \leqslant t^{-1}\psi(w - t\nu) \max_{B(\hat{w},4\rho)} u^-.$$

Letting $t \to 0$ we get, for some $c = c(p,n)$, that

$$\beta \leqslant c\epsilon^{-1} \max_{B(\hat{w},4\rho)} u^-. \tag{5.17}$$

Let $0 \leqslant H \leqslant 1$ be p-harmonic in $G = B(w, 2\epsilon^\kappa) \setminus \{x\colon\ x_n < \tau(x') + 2\epsilon\}$ with continuous boundary values and with $H \equiv 1$ on $\partial G \cap \partial B(w, 2\epsilon^\kappa)$ while $H \equiv 0$ on $\partial G \cap B(w, \epsilon^\kappa)$. We claim, for $\epsilon_+ > 0$ sufficiently small, that

$$\max_{B(\hat{w},4\rho)} u^- \leqslant c\epsilon^{7/16} M H(w - 4\epsilon e_n) \tag{5.18}$$

whenever $0 < \epsilon \leqslant \epsilon_+$, where $c = c(p,n)$. Once (5.18) is proved we get Lemma 5.16 from the following argument. Using Lemma 2.19 for H, as in the proof of (5.5), we have,

$$H(w - 4\epsilon e_n) \leqslant c\epsilon^{b(1-\kappa)} \quad \text{where } b \to 1 \text{ as } \theta_+ \to \pi/2. \tag{5.19}$$

Combining (5.17)–(5.19) we get $\beta \leqslant \epsilon^{3/8}$ by first choosing θ_+ near enough $\pi/2$, so that $b(1-\kappa) > 15/16$, and then $\epsilon_+ > 0$ small enough (depending on p, n, M).

To prove (5.18) we let

$$\Omega^* = Q_{1-\epsilon^\kappa}(0) \cap \big\{x = \big(x', x_n\big)\colon\ \tau\big(x'\big) - 2\epsilon^\kappa < x_n < \tau\big(x'\big) + 2\epsilon\big\}.$$

Let F, $0 \leqslant F \leqslant M$, be the p-harmonic function in Ω^* with continuous boundary values, $F \equiv 0$ on $\partial\Omega^* \cap Q_{1-\epsilon^\kappa}(0)$ and $F \equiv u^-$ on $\partial\Omega^* \cap \partial Q_{1-\epsilon^\kappa}(0) \cap \{x\colon\ x_n \geqslant \tau(x') - \epsilon^\kappa\}$. Existence of F follows easily from (5.1). Put $u^* = (-u - \epsilon^{7/16}M)^+$ and note from (5.15), as well as the definition of F, that u^* is p-subharmonic in Ω^* with $u^* \leqslant F$ on $\partial\Omega^*$. Thus by the boundary maximum principle for these functions, $u^* \leqslant F$ in Ω^*. Using this fact, $w \in Q_{1-2\epsilon^{\kappa/2}}(0)$, as well as Lemmas 2.2–2.4 for F, we can now argue as in the proofs of (4.41), (4.44), to obtain, for some $c = c(p,n) \geqslant 1$, that

$$u^*(x) \leqslant F(x) \leqslant M \exp\bigl[-1/\bigl(c\epsilon^{\kappa/2}\bigr)\bigr] \quad \text{whenever } x \in Q_{1-\epsilon^{\kappa/2}}(0).$$

From this inequality, the maximum principle for p-harmonic functions, and Lemma 2.4 applied to H we first conclude (5.18) and then Lemma 5.16. □

Next we prove the following lemma.

Lemma 5.20. *Under the assumptions of Theorem* 2 *and* (5.14) *there exist* $\gamma \in [7/8, 1)$ *and* $c \geqslant 1$*, both depending only on* p, n*, such that* u^+ *is* $(\gamma\epsilon)$*-monotone in* $Q_{1-c\epsilon^{\kappa/2}}(0)$ *in the cap of directions* $\Gamma(e_n, \hat{\theta} - c\epsilon^{1/4})$.

Proof. Armed with Lemma 5.16 we are now in a position to prove this lemma by following closely the proof of Theorem 1 in Section 4. Let τ be as in (5.1), let $h = 100A\epsilon$, and let $\Lambda(h)$ be as defined in Section 4 relative to τ. Let ϕ_t, $t \in [0,1]$, be the family of functions defined in Lemma 3.7 with $\beta = 1 - \kappa/2$. As pointed out at the beginning of Section 5, (4.1)–(4.4) remain valid with $\bar{\theta}$ replaced by $\hat{\theta}$ and for $x \in D^+(u)$. We also define γ as in (4.5) to be the smallest number in $[7/8, 1)$ such that

$$\frac{\gamma \sin\hat{\theta}}{\sin\hat{\theta} - 1 + \gamma} \leqslant 1 + \mu$$

where μ is as in Lemma 3.7. Put $\sigma = \epsilon'(\sin\hat{\theta} + \gamma - 1)$ whenever $\epsilon' \in (\epsilon, 2\epsilon)$. From Lemma 3.7 we see that

$$\epsilon'/2 \leqslant \sigma\phi_t \leqslant c\epsilon \quad \text{and} \quad \sigma|\nabla\phi_t| \leqslant c\epsilon^{1-\kappa/2} \quad \text{on } \Lambda(h) \cap Q_{2,8}(0), \tag{5.21}$$

some $c = c(p,n) \geqslant 1$. Next set

$$v_t(x) = \max_{y \in B(x, \sigma\phi_t(x))} u^+\bigl(y - \gamma\epsilon' e_n\bigr) \quad \text{whenever } \bar{B}\bigl(x, \sigma\phi_t(x)\bigr) \subset Q_1(0),\ t \in [0,1].$$

From (i)–(iii) of Lemmas 3.7 and 3.4 we deduce that v_t is p-subharmonic in $\Lambda(h/2) \cap Q_{1-8\epsilon}(0) \setminus F(v_t)$ for $0 \leqslant t \leqslant 1$. Using (5.21) we can also argue as in (4.7) and (4.8) to get monotonicity of v_t in a cap of directions (see (4.9)). Let $\Omega = \Lambda(h/2) \cap Q_{1-h}(0)$. Then as in Lemma 4.10 we define $\bar{t}$ by

$$1 + \mu\bar{t} = \frac{\gamma(\sin\hat{\theta} - c'\epsilon^{1/4})}{\sin\hat{\theta} - 1 + \gamma}$$

and we observe, for $\hat{\epsilon} > 0$ sufficiently small, that $\bar{t} \in (0,1)$, and, for $t \in [0, \bar{t}]$,

$$\begin{aligned} &(*) \quad v_t \leqslant u^+ \quad \text{on } \partial\Omega \quad \text{and} \quad v_t \leqslant \bigl(1 - \epsilon^{1/4}\bigr)u^+ \quad \text{on } \bar{\Omega} \setminus \bar{\Lambda}(h/16), \\ &(**) \quad u > 0 \quad \text{on } F(v_t) \setminus Q_{1-h^{\kappa/2}}(0) = F(v_0) \setminus Q_{1-h^{\kappa/2}}(0). \end{aligned} \tag{5.22}$$

(5.22) follows from the argument after Lemma 4.10 (see (4.11)–(4.15)).

Next let

$$\Theta = \bigl\{t\colon\ t \in [0, \bar{t}],\ v_t \leqslant u \text{ on } D^+(u) \cap \Omega\bigr\}.$$

Once again we use a contradiction argument to prove that $\Theta = [0, \bar{t}]$. If not, then repeating the argument after (4.14) we see that there exists, for some $t \in [0, \bar{t})$, $\tilde{w} \in F(u) \cap F(v_t) \cap \bar{Q}_{1-h^{\kappa/2}}(0)$ and $w^* \in D^+(v_t)$, $\rho^* > 0$, such that $B(w^*, \rho^*) \subset D^+(v_t)$, $\tilde{w} \in \partial B(w^*, \rho^*)$. Moreover if $\tilde{\nu} = (w^* - \tilde{w})/|w^* - \tilde{w}|$, then there exist, $\bar{\alpha}, \bar{\beta}, \in [0, \infty)$, such that

$$v_t(x) \geqslant \bar{\alpha}\langle x - \tilde{w}, \tilde{\nu}\rangle^+ - \bar{\beta}\langle x - \tilde{w}, \tilde{\nu}\rangle^- + o\bigl(|x - \tilde{w}|\bigr),$$

near $\tilde{w}$. Here,

$$\frac{\bar{\alpha}}{1 - \sigma|\nabla\phi_t(\tilde{w})|} \geqslant G\biggl(\frac{\bar{\beta}}{1 + \sigma|\nabla\phi_t(\tilde{w})|}\biggr).$$

Also,

$$u(x)=\alpha\langle x-\tilde{w},\tilde{\nu}\rangle^{+}-\beta\langle x-\tilde{w},\tilde{\nu}\rangle^{-}+o\big(|x-\tilde{w}|\big),$$

as $x\to\tilde{w}$, for some $\alpha,\beta\in[0,\infty)$ with $\alpha=G(\beta)$.

As in (4.19) we claim that

$$0\leqslant\bar{\alpha}\leqslant\alpha\big(1-\epsilon^{1/4}/c\big),\tag{5.23}$$

for some $c=c(p,n)\geqslant 1$. To obtain a contradiction from (5.23), we first note that $\epsilon/100\leqslant\rho^{*}\leqslant 100\epsilon$, as shown in [23, p. 1511]. Thus the hypotheses of Lemma 5.16 are satisfied so that $\beta\leqslant\epsilon^{3/8}$. Using this note and the assumptions on G in Theorem 2 it follows that

$$\begin{aligned}\alpha=G(\beta)&\leqslant G\big(\epsilon^{3/8}\big)\leqslant C\epsilon^{3/8}+G(0)\leqslant C\epsilon^{3/8}+G(\bar{\beta})\\&\leqslant C\epsilon^{3/8}+\bar{\beta}^{N}\left(\frac{\bar{\beta}}{1+\sigma|\nabla\phi_t(\tilde{w})|}\right)^{-N}G\left(\frac{\bar{\beta}}{1+\sigma|\nabla\phi_t(\tilde{w})|}\right)\\&\leqslant C\epsilon^{3/8}+\frac{(1+\sigma|\nabla\phi_t(\tilde{w})|)^{N}}{1-\sigma|\nabla\phi_t(\tilde{w})|}\bar{\alpha}\leqslant C\epsilon^{3/8}+\frac{(1+c\epsilon^{1-\kappa/2})^{N}}{1-c\epsilon^{1-\kappa/2}}\bar{\alpha}<\alpha,\end{aligned}\tag{5.24}$$

thanks to (5.23), provided $\hat{\epsilon}$ is small enough (depending on $p,n,M,G(0)$). Here we have used the fact that $\alpha\geqslant G(0)>0$ and that $\kappa<1/100$. From this contradiction we first get that $\Theta=[0,\bar{t}]$ and then Lemma 5.20 as in the discussion after (4.46). The proof of (5.23) is exactly the same as the proof of (4.19). Therefore, we omit the details. □

Proof of Theorem 2. As mentioned earlier, Theorem 2 is true if (5.14) is false. If (5.14) is true, we can apply Lemma 5.20 to get that u^{+} is $(\gamma\epsilon)$-monotone in $Q_{1-c\epsilon^{\kappa/2}}(0)$ in the cap of directions $\Gamma(e_n,\bar{\theta}-c\epsilon^{1/4})$. If now (5.14) is false with ϵ replaced by $\gamma\epsilon$, we get Theorem 2 from Lemma 5.4 and the argument in Theorem 1. Otherwise we repeat the argument leading to Lemma 5.20 in order to get that u^{+} is $(\gamma^{2}\epsilon)$-monotone in the directions $\Gamma(e_n,\hat{\theta}-c\epsilon^{1/4}-c(\gamma\epsilon)^{1/4})$. Continuing in this manner, we obtain Theorem 2. □

Proof of Corollary 1. To avoid confusion we write $\tilde{\epsilon},\tilde{\theta}$ for $\hat{\epsilon},\hat{\theta}$ in Theorem 2. To prove Corollary 1 we show, that u^{+} is $(c\epsilon)$-monotone in $Q_{3/4}(0)\cap\{y\colon y_n\leqslant 1/c\}$ for some $c=c(p,n,\eta,\tilde{\theta})$ provided $\hat{\theta}=\tilde{\theta}/2+\pi/4$ and $0<\hat{\epsilon}\ll\tilde{\epsilon}$. The proof is essentially the same as in [23], thanks to Theorem 2.7. For the readers convenience we include the details. Let $\tilde{f},\hat{\theta},\hat{\epsilon}$ be as in Corollary 1 and suppose that $\tilde{\theta}$ is near enough $\pi/2$ so that if $\Omega_{+}=\{x\colon x_n>\tilde{f}(x')+2\epsilon\}\cap Q_{3/4}(0)$, then $\Omega_{+}\subset D^{+}(u)\cap Q_{3/4}(0)$. Let v be the p-harmonic function in Ω_{+} with continuous boundary values,

(a) $v\equiv 0$ on $\partial\Omega_{+}\cap Q_{3/4}(0)$.

(b) $v(y)=u^{+}(y)$ when $y\in\partial\Omega_{+}\cap\partial Q_{3/4}(0)$ and $y_n\geqslant\tilde{f}\big(y'\big)+3\epsilon$.

(c) $v(y)=\min\left\{u^{+}(y),\dfrac{(y_n-\tilde{f}(y')-2\epsilon)}{\epsilon}u^{+}\big(y',\tau\big(y'\big)+3\epsilon\big)\right\}$ when $y\in\partial\Omega_{+}\cap\partial Q_{3/4}(0)$ and $\tilde{f}\big(y'\big)+2\epsilon<y_n\leqslant\tilde{f}\big(y'\big)+3\epsilon$.

From the maximum principle for p-harmonic functions and the assumptions on u^{+} in Corollary 1 we deduce that

$$v\leqslant u^{+}\leqslant v+8\eta^{-1}\epsilon\tag{5.25}$$

provided $\tilde{\theta}$ is near enough $\pi/2$. From Theorem 2.7, (5.25), and our choice of $\hat{\theta}$, we deduce for all $\xi\in\Gamma(e_n,\tilde{\theta})$ and some $\tilde{c}=\tilde{c}(p,n,\tilde{\theta},\eta)$, $c_{+}=c_{+}(p,n)\geqslant 1$ that

$$\tilde{c}\big\langle\nabla v(y),\xi\big\rangle\geqslant 1\quad\text{in }\big\{y\in\Omega_{+}\colon\tilde{c}\epsilon\leqslant d(y,\partial\Omega_{+})\leqslant 100c_{+}^{-1}\big\}=K.\tag{5.26}$$

Let $c^{*}\gg\tilde{c}$ and for given $\xi\in\Gamma(e_n,\tilde{\theta})$, consider $e(x)=u^{+}(x)-u^{+}(x-s\xi)$, when $x\in Q_{5/8}(0)$ and $c^{*}\epsilon\leqslant s\leqslant 2c_{+}^{-1}$. If $u^{+}(x-s\xi)=0$, then trivially $e(x)\geqslant 0$. Also, if $d(x-s\xi,\partial\Omega_{+})\leqslant 4\tilde{c}\epsilon$ we can suppose c^{*} large enough so that $e(x)>0$ as we see from the assumptions on u in Corollary 1 and a geometric argument using Lipschitzness of $\tilde{f}$. Otherwise if $x,x-s\xi\in K$, we can use (5.25), (5.26) to conclude that

$$e(x)\geqslant v(x)-v\big(x-c^{*}\epsilon\xi\big)-8\epsilon\eta^{-1}\geqslant\tilde{c}^{-1}c^{*}\epsilon-8\epsilon\eta^{-1}>0\tag{5.27}$$

provided $c^* = c^*(p, n, \tilde{\theta}, \eta)$ is large enough. It follows that u^+ is $(c^*\epsilon)$-monotone in $F(u) \cap Q_{5/8}(0) \cap \{y\colon\ y_n \leqslant c_+^{-1}\}$. We can now repeat the argument in Theorem 2 with $Q_1(0)$ replaced by $Q_{5/8}(0) \cap \{y\colon\ y_n \leqslant c_+^{-1}\}$ or essentially just apply Theorem 2 in order to conclude Corollary 1. □

Closing remarks. Theorem 2 remains valid when G in Theorem 2 is allowed to depend Lipschitz continuously on x, ν uniformly on bounded subsets of u_ν and causes no new problems. Also, one can state a version of Theorem 1 in [19], and Theorems 1 and 2 in the present paper, when u is p-harmonic in $D^+(u)$ and q-harmonic in $D^-(u)$ where $1 < p, q < \infty$. These more general theorems also appear likely to be true with minor changes in the proofs of the corresponding theorems.

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