Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN 28 (2011) 395–411

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

# Higher differentiability of minimizers of convex variational integrals

Menita Carozza [a], Jan Kristensen [b], Antonia Passarelli di Napoli [c,*]

[a] *Università del Sannio, Piazza Arechi II – Palazzo De Simone, 82100 Benevento, Italy*
[b] *Mathematical Institute, University of Oxford, 24-29 St. Giles', Oxford OX1 3LB, England, United Kingdom*
[c] *Università di Napoli 'Federico', Dipartimento di Mat. e Appl. 'R. Caccioppoli', Via Cintia, 80126 Napoli, Italy*

Received 22 September 2010; received in revised form 19 January 2011; accepted 12 February 2011

Available online 18 February 2011

**Abstract**

In this paper we consider integral functionals of the form

$$\mathfrak{F}(v,\Omega)=\int\limits_{\Omega} F\big(Dv(x)\big)\,\mathrm{d}x$$

with convex integrand satisfying $(p,q)$ growth conditions. We prove local higher differentiability results for bounded minimizers of the functional $\mathfrak{F}$ under dimension-free conditions on the gap between the growth and the coercivity exponents.

As a novel feature, the main results are achieved through uniform higher differentiability estimates for solutions to a class of auxiliary problems, constructed adding singular higher order perturbations to the integrand.


*MSC:* 49N15; 49N60; 49N99

## 1. Introduction and statement of results

We prove higher differentiability results for minimizers of the convex variational integrals

$$\mathfrak{F}(v,O)=\int\limits_{O} F\big(Dv(x)\big)\,\mathrm{d}x \tag{1.1}$$

with integrands $F$ satisfying $(p,q)$ growth conditions. The functionals $\mathfrak{F}$ are defined for Sobolev maps $v\in \mathrm{W}^{1,p}(\Omega,\mathbb{R}^N)$ and open subsets $O$ of a fixed bounded and open subset $\Omega$ of $\mathbb{R}^n$. Our main concern is the multi-dimensional vectorial case $n,N\geqslant 2$, but some of our results remain new also in the scalar case $N=1$.

The integral in (1.1) is understood in the usual sense of Lebesgue integration. In fact, as is well known, for autonomous convex integrands $F$ there is no issue as to how one should define the functional $\mathfrak{F}(v,O)$ for non-smooth maps $v$. All reasonable definitions lead to the same result, (1.1) as a Lebesgue integral. The situation, as regards to both existence and regularity of minimizers, is different and more delicate for non-autonomous or non-convex

* Corresponding author.
*E-mail addresses:* carozza@unisannio.it (M. Carozza), kristens@maths.ox.ac.uk (J. Kristensen), antpassa@unina.it (A. Passarelli di Napoli).

0294-1449/$ – see front matter 
doi:10.1016/j.anihpc.2011.02.005

integrals. We refer the interested reader to [1,11,13,24–26,28,36,38,39] and the references therein. Our results here are correspondingly stronger, and we obtain higher differentiability results for a priori bounded minimizers under dimension-free conditions on the gap, namely $q - p < 1$ without any control on the second derivative $F''$ of the integrand, and $q - p < 2$ with control on the second derivative. A special feature of our approach is that we establish the higher differentiability results through uniform estimates for a family of auxiliary problems defined through singular perturbation. In this connection a nonlinear Gagliardo–Nirenberg inequality is established and used in connection with the a priori boundedness assumption on the minimizer. We hereby avoid the usual difference quotient method, and the technicalities its use entails in the $(p, q)$ growth setup.

In order to state the results precisely we shall briefly introduce and discuss our hypotheses.

It is convenient to express the convexity and growth conditions for the integrands in terms of two auxiliary functions defined for all $\xi \in \mathbb{R}^{N\times n}$ as

$$\langle \xi \rangle = \langle \xi \rangle_\mu := \left(\mu^2 + |\xi|^2\right)^{\frac{1}{2}} \tag{1.2}$$

and

$$V(\xi) = V_{p,\mu}(\xi) := \left(\mu^2 + |\xi|^2\right)^{\frac{p-2}{4}} \xi, \tag{1.3}$$

where $\mu \geqslant 0$ and $p > 1$ are parameters. Clearly, $\langle \xi \rangle_\mu^p$ is the $p$-Dirichlet integrand for $\mu = 0$, whereas it is the reference (non-degenerate) convex functional of $p$-growth for $\mu > 0$. Indeed, as is well known, regardless of the dimensions $n$, $N$, the minimizers for the corresponding variational integral $\int_\Omega \langle Dv \rangle_\mu^p$ are $C^\infty$ smooth in the case $\mu > 0$ and $C^{1,\alpha}$ smooth for $\mu = 0$, where $\alpha = \alpha(n, N, p) < 1$ $(p \neq 2)$, and in both cases these qualitative regularity results can be expressed very efficiently in terms of estimates between various norms of the minimizers.

Let $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ be an integrand satisfying for exponents $q \geqslant p \geqslant 2$ and positive constants $L, \ell > 0$ and $\mu \geqslant 0$, the following growth and convexity hypotheses:

$$0 \leqslant F(\xi) \leqslant L\langle \xi \rangle_\mu^q, \tag{H1}$$

$$\xi \mapsto F(\xi) - \ell\langle \xi \rangle_\mu^p \quad \text{is convex ,} \tag{H2}$$

for all $\xi \in \mathbb{R}^{N\times n}$. Assumption (H2) is a uniform strong $p$-convexity condition for the function $F$, and is similar to the one considered in [16]. In fact, when $F$ is $C^2$ then (H2) is equivalent to the following standard strong $p$-ellipticity condition

$$F''(\xi)[\eta, \eta] \geqslant c\langle \xi \rangle_\mu^{p-2} |\eta|^2 \tag{H3}$$

for all $\xi, \eta \in \mathbb{R}^{N\times n}$, where $c$ is a positive constant of form $c = c(p)\ell$. It is well known that for convex $C^1$ integrands, the growth condition (H1) implies a Lipschitz condition:

$$\left|F'(\xi)\right| \leqslant c\langle \xi \rangle_\mu^{q-1} \tag{1.4}$$

for all $\xi \in \mathbb{R}^{N\times n}$, where we can use $c = 2^q L$. In the sequel we shall mostly suppress the dependence on parameters $\mu$, $p$ in the notation writing simply $\langle \xi \rangle$ and $V(\xi)$ where the precise values of $\mu$, $p$ will be clear from the context.

Let us give the definition of local minimizer, where we emphasize the importance of the local integrability requirement on $F(Du)$:

**Definition 1.** A mapping $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is a local $F$-minimizer if $F(Du) \in L^1_{\mathrm{loc}}(\Omega)$ and

$$\int_O F(Du)\,dx \leqslant \int_O F(Dv)\,dx$$

for any $O \Subset \Omega$ and any $v \in W^{1,p}_u(O, \mathbb{R}^N)$.

The notation $O \Subset \Omega$ signifies as usual that $O$ is an open set whose closure, $\overline{O}$, is compact and contained in $\Omega$. Furthermore, $W^{1,p}_u(O, \mathbb{R}^N)$ denotes the Dirichlet class of Sobolev maps $v$ such that $v - u|_O \in W^{1,p}_0(O, \mathbb{R}^N)$, where the latter is defined as the closure of the space of smooth compactly supported test maps, $\mathrm{C}_c^\infty(O,\mathbb{R}^N)$, in $\mathrm{W}^{1,p}(O,\mathbb{R}^N)$. We shall adhere to standard notation for functions and function spaces, by and large following [40].

The assumptions (H1), (H2) (or (H3)) clearly entail a $(p,q)$ growth condition, namely that there exists a constant $c=c(L/\ell,p,q,\mu)>0$ such that

$$\frac{1}{c}|\xi|^p-c\leqslant F(\xi)\leqslant c\big(|\xi|^q+1\big) \tag{1.5}$$

for all $\xi\in\mathbb{R}^{N\times n}$.

The study of the regularity of minimizers of such functionals started with the celebrated papers by Marcellini (see in particular [30–32]) and has since attracted much attention. From very early on it has been clear that no regularity can be expected if the coercivity and growth exponents, denoted $p$ and $q$, respectively, are too far apart (see [29,20]). On the other hand, many regularity results are available if the ratio $q/p$ is bounded above by a suitable constant depending on the dimension $n$, and converging to 1 when $n$ tends to infinity (incl. [2–7,13,14,17,22,33,34]).

We recall from [37] that even under hypotheses (H1) and (H3) with $p=q=2$ vectorial $F$-minimizers might be locally unbounded when $n\geqslant 5$, while they must be locally bounded when $n\leqslant 4$ by classical results (see [9,21,19]). However, under additional structure conditions on the integrand and under assumptions (H1) and (H3) with $p\leqslant q$ it is possible to establish weak maximum principles that imply boundedness of minimizers provided the boundary conditions are bounded (see [27]). For such bounded minimizers, it has been shown in [10], for integrands $F$ of special structure and with a growth condition on the second derivative $F''$, that the dimension-free condition $q<p+1$ implies higher integrability for the gradient. We also recall that in [13], under the hypothesis $p>n$, higher integrability was established without additional structure conditions on $F$ provided $q<p+2$ and the second derivative $F''$ has $q-2$ growth. More recently, some refinements have been obtained in [8] by imposing various additional structure conditions on the integrand, including so-called splitting type conditions, Uhlenbeck-structure and growth conditions on $F''$ as in [10,13].

In all the above quoted papers, regularity of the minimizers is deduced by higher differentiability of the gradient which is obtained by means of difference quotient methods. Let us also remark that in the context of non-standard growth, higher integrability has been established by an argument involving a Gehring-type result only under very severe structure conditions on the integrand, see [18]. Zhikov obtained Gehring-type higher integrability results under $p(x)$-growth conditions for the integrand.

The first result in this paper is a higher differentiability result for the gradient of a locally bounded $F$-minimizer. We do not require any structure assumptions for the integrand, merely that the growth exponent $q$ satisfies $q<p+1$, independently of $n$. More precisely we have the following

**Theorem 2.** *Let $F:\mathbb{R}^{N\times n}\to\mathbb{R}$ be an integrand satisfying the assumptions* (H1), (H2) *with $q<p+1$ and $\mu\in[0,1]$, where $2\leqslant p<n$. If $u\in\mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)\cap\mathrm{L}^\infty_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ is a local $F$-minimizer, then*

$$V_{p,\mu}(Du)\in\mathrm{W}^{1,2}_{\mathrm{loc}}\big(\Omega,\mathbb{R}^{N\times n}\big)$$

*where we recall that*

$$V_{p,\mu}(Du):=\big(\mu^2+|Du|^2\big)^{\frac{p-2}{4}}Du.$$

*Furthermore, there exists a constant $c=c(n,N,L/\ell,p,q)$ such that whenever $B_{2R}\subset\Omega$ we have the Caccioppoli inequality*

$$\int\limits_{B_R}\big|D\big[V_{p,\mu}(Du)\big]\big|^2\leqslant\frac{c}{R^2}\Bigg[|B_{2R}|+\Bigg(\bigg(\frac{\|u\|_{\mathrm{L}^\infty}}{R}\bigg)^{\frac{2}{p+1-q}}+1\Bigg)\int\limits_{B_{2R}}\langle Du\rangle_\mu^p\Bigg]. \tag{1.6}$$

*For $p\geqslant n$ the conclusion persists for minimizers $u\in\mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ under the weaker assumption $q<p+\frac{p}{n}$ provided the term $\|u\|_{\mathrm{L}^\infty}/R$ in* (1.6) *is replaced by $\|Du\|_{\mathrm{L}^p(B_{2R})}$.*

We remark that in the case $p\geqslant n$ our result is not new. In [14] (see also [15]) it is shown that for integrands satisfying (H1), (H2) with $q<p+\frac{p}{n}$ the corresponding minimizers enjoy the above higher differentiability property.

While aspects of our proof in this case are similar to that of [14] our approach via higher order singular perturbations appears less involved.

When the integrand $F$ is $C^2$, satisfies (H3), and we strengthen the growth condition (H1) to

$$\|F''(\xi)\| \leqslant \Lambda\big(\mu^2+|\xi|^2\big)^{\frac{q-2}{2}}, \tag{H4}$$

we can improve Theorem 2 whose conclusion then still holds under the weaker assumption $q < p+2$:

**Theorem 3.** *Let $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ be a $C^2$ integrand satisfying the conditions* (H1), (H3) *and* (H4) *with $q < p+2$ and $\mu \in [0,1]$, where $2 \leqslant p < n$. If $u \in W^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^N) \cap L^\infty_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ is a local $F$-minimizer, then*

$$V_{p,\mu}(Du) \in W^{1,2}_{\mathrm{loc}}\big(\Omega,\mathbb{R}^{N\times n}\big)$$

*and* (1.6) *still holds.*

*For $p \geqslant n$ the conclusion persists for minimizers $u \in W^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ under the weaker assumption $q < p+\frac{2p}{n}$.*

We remark that our result was known already in the case $q < p+\frac{2p}{n}$, see [13,12].

Our proof relies on an approximation procedure. This approach has features in common with [14], and also [13,15,16], but differs in essence at one point: whereas the auxiliary problems in [14] are obtained by suitably approximating the integrand $F$ by smooth integrands satisfying standard $q$-growth and $q$-convexity conditions, ours are obtained by adding singular higher order perturbations. Next, we establish uniform higher differentiability estimates for the solutions of the auxiliary problems, mainly through the use of a nonlinear Gagliardo–Nirenberg inequality that is probably known to specialists, but as we could not find a reference in the literature we have included its proof in Appendix A. It seems that, avoiding the use of the difference quotient method has the advantage of making the ensuing calculations easier, as the solutions are smooth, and our calculations here follow at some points [16]. Finally, we remark that because we do not impose any special structure conditions on the integrand we also have to add an additional term in the auxiliary problems that allows us to exploit the a priori boundedness assumption on minimizers.

We remark that a byproduct of our proofs is that, under the assumptions of Theorem 2, the higher integrability exponent of the gradient of a minimizer is at the least $p+2$. That it is dimension-free is due to the fact that we assume the minimizer is bounded when $p < n$. Dimension-free higher integrability exponents have been established in [23] for the *second gradient* of the minimizer, but under much more severe growth conditions (*viz.* (H3), (H4) with $p = q = 2$).

**Remark 4.** It is not difficult to check that the higher differentiability results of Theorems 2 and 3 can be extended to minimizers of more general autonomous convex integrals of the form $\int_\Omega F(v, Dv)$. For instance, one possible generalization of Theorem 2 would apply to minimizers of such integrals with integrands $F : \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ that instead of hypotheses (H1), (H2) satisfy

$$0 \leqslant F(y,\xi) \leqslant L\big(\mu^2+|y|^2+|\xi|^2\big)^{\frac{q}{2}} \tag{H1$'$}$$

and

$$(y,\xi) \mapsto F(y,\xi) - \ell\big(\mu^2+|\xi|^2\big)^{\frac{p}{2}} \quad \text{is convex.} \tag{H2$'$}$$

Note that we only require convexity in the $y$-variable, and not its strong variant. Under the conditions $q < p+1$ and $u \in L^\infty_{\mathrm{loc}}$ or $p \geqslant n$ and $q < p+\frac{p}{n}$, we have again that $V_{p,\mu}(Du) \in W^{1,2}_{\mathrm{loc}}$. We leave further detailed statements to the interested reader. A brief sketch of the proof is given at the end of Section 3. Finally, we remark that without convexity in the $y$-variables the situation is very different, and in particular is complicated by the appearance of the Lavrentiev phenomenon. Using different methods it is however still possible to obtain a degree of higher differentiability when $p = q$, see [24–26].

**Remark 5.** Moreover, with slight modifications, the higher differentiability results of Theorem 2 and Remark 4 still hold for minimizers of general convex integrals of the form $\int_\Omega F(x, v, Dv)\,dx$, provided if, for instance, we assume (H1$'$), (H2$'$) hold uniformly in $x$ and in addition that

$$(y,\xi) \mapsto F(x,y,\xi) \in C^1\big(\mathbb{R}^N \times \mathbb{R}^{N\times n}\big)$$

and

$$\big|F'_{(y,\xi)}(x_1,y,\xi) - F'_{(y,\xi)}(x_2,y,\xi)\big| \leqslant L|x_1-x_2|\big(\mu^2+|y|^2+|\xi|^2\big)^{\frac{q-1}{2}}.$$

It is worth pointing out that, under the Lipschitz dependence on the $x$-variable of the first derivative of $F$, the bound $q < p\frac{n+1}{n}$ that we find in Theorem 2 in case $p \geqslant n$, cannot be improved, by virtue of a counterexample given in [15]. On the other hand, the bound $q < p+1$ (which is clearly a better one when $p < n$) is obtained in case of a priori bounded minimizers.

The plan of the paper is the following. We have collected standard preliminary material in Section 2, which at the same time serves as our reference for notation. The proofs of the higher differentiability results stated in Theorems 2 and 3 are presented in Sections 3 and 4, respectively. Finally, Appendix A contains the proof of a nonlinear Gagliardo–Nirenberg inequality.

## 2. Preliminaries

For matrices $\xi, \eta \in \mathbb{R}^{N\times n}$ we write $\langle \xi, \eta\rangle := \operatorname{trace}(\xi^T\eta)$ for the usual inner product of $\xi$ and $\eta$, and $|\xi| := \langle \xi, \xi\rangle^{\frac{1}{2}}$ for the corresponding Euclidean norm. When $a \in \mathbb{R}^N$ and $b \in \mathbb{R}^n$ we write $a \otimes b \in \mathbb{R}^{N\times n}$ for the tensor product defined as the matrix that has the element $a_r b_s$ in its $r$-th row and $s$-th column. Observe that $|a\otimes b| = |a||b|$, where $|a|$, $|b|$ denote the usual Euclidean norms of $a$ in $\mathbb{R}^N$, $b$ in $\mathbb{R}^n$, respectively.

When $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ is sufficiently differentiable we write

$$F'(\xi)[\eta] := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} F(\xi+t\eta) \quad \text{and} \quad F''(\xi)[\eta,\eta] := \frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} F(\xi+t\eta)$$

for $\xi, \eta \in \mathbb{R}^{N\times n}$. Hereby we think of $F'(\xi)$ both as an $N\times n$ matrix and as the corresponding linear form on $\mathbb{R}^{N\times n}$, though $|F'(\xi)|$ will always denote the Euclidean norm of the matrix $F'(\xi)$. The second derivative, $F''(\xi)$, is a symmetric real bilinear form on $\mathbb{R}^{N\times n}$. We express growth conditions for the second derivative of the integrand in terms of the operator norm on bilinear forms:

$$\big\|F''(\xi)\big\| := \sup_{|\eta|\leqslant 1,\, |\zeta|\leqslant 1} F''(\xi)[\eta,\zeta].$$

**Lemma 6.** *Let $\Phi : [\frac{R}{2}, R] \to \mathbb{R}$ be a bounded nonnegative function on the interval $[\frac{R}{2}, R]$ where $R > 0$. Assume that for all $\frac{R}{2} \leqslant r < s \leqslant R$ we have*

$$\Phi(r) \leqslant \vartheta\Phi(s) + A + \frac{B}{(s-r)^2} + \frac{C}{(s-r)^\alpha} + \frac{D}{(s-r)^\beta}$$

*where $\vartheta \in (0,1)$, $A, B, C, D \geqslant 0$ and $0 < \alpha < \beta$ are constants. Then there exists a constant $c = c(\vartheta, \beta)$ such that*

$$\Phi\left(\frac{R}{2}\right) \leqslant c\left(A + \frac{B}{R^2} + \frac{C}{R^\alpha} + \frac{D}{R^\beta}\right).$$

See for instance [21, pp. 191–192], for a proof that can easily be adapted to cover the above statement too.

For the proof of Theorem 2 (and Remark 4) we require the following approximation result that is well known to specialists, and closely related to the approximation results established and used in [14] and [16].

**Lemma 7.** *Let $\mu \in [0,1]$, $0 < \ell \leqslant L$ and $2 \leqslant p \leqslant q \leqslant p+1$. Assume that $F : \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies*

$$0 \leqslant F(y,\xi) \leqslant L\big(\mu^2+|y|^2+|\xi|^2\big)^{\frac{q}{2}} \tag{2.1}$$

*and*

$$(y,\xi) \mapsto F(y,\xi) - \ell\big(\mu^2+|\xi|^2\big)^{\frac{p}{2}} \quad \textit{is convex.} \tag{2.2}$$

*Then there exist $F_j : \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ that are $C^\infty$ smooth, and satisfy for a constant $c = c(p)\ell > 0$:*

$$0 \leqslant F_j(y,\xi) \leqslant L\big(\mu^2 + |y|^2 + |\xi|^2\big)^{\frac{q}{2}} + \frac{1}{j}\big(1 + |y|^2 + |\xi|^2\big)^{\frac{p}{2}}, \tag{2.3}$$

$$F_j''(y,\xi)\big[(z,\eta),(z,\eta)\big] \geqslant c\big(\mu^2 + |\xi|^2\big)^{\frac{p-2}{2}}|\eta|^2 \tag{2.4}$$

*and*

$$F(y,\xi) \leqslant F_j(y,\xi) \leqslant F(y,\xi) + \frac{1}{j}\big(1 + |y|^2 + |\xi|^2\big)^{\frac{p}{2}} \tag{2.5}$$

*for all $(y,\xi)$, $(z,\eta) \in \mathbb{R}^N \times \mathbb{R}^{N\times n}$ and $j \in \mathbb{N}$.*

**Proof.** First note that (2.1) and convexity transpire to give local Lipschitz continuity with

$$\big|F'_{(y,\xi)}(y,\xi)\big| \leqslant 2^q L\big(\mu^2 + |y|^2 + |\xi|^2\big)^{\frac{q-1}{2}} \tag{2.6}$$

for almost all $(y,\xi) \in \mathbb{R}^N \times \mathbb{R}^{N\times n}$. Next we use the following radially symmetric and smooth convolution kernel

$$\Phi(y,\xi) := \begin{cases} c\exp(\frac{1}{|y|^2+|\xi|^2-1}) & \text{for } |y|^2 + |\xi|^2 < 1, \\ 0 & \text{for } |y|^2 + |\xi|^2 \geqslant 1, \end{cases}$$

where the constant $c = c(n,N) > 0$ is chosen so $\int_{\mathbb{R}^N \times \mathbb{R}^{N\times n}} \Phi = 1$.

Put for each $\varepsilon > 0$, $\Phi_\varepsilon(y,\xi) = \varepsilon^{-N(n+1)}\Phi(\varepsilon^{-1}y, \varepsilon^{-1}\xi)$. It is clear that the convolution $F_\varepsilon := \Phi_\varepsilon * F$ is $C^\infty$ smooth and convex, and owing to the convexity of $F$ and (2.6) we get in a routine manner

$$F(y,\xi) \leqslant F_\varepsilon(y,\xi) \leqslant F(y,\xi) + c\big(\mu^2 + \varepsilon^2 + |y|^2 + |\xi|^2\big)^{\frac{q-1}{2}}\varepsilon.$$

Now for $\varepsilon \in (0,1]$ we estimate using that $0 \leqslant \mu \leqslant 1$, $q \leqslant p+1$:

$$\big(\mu^2 + \varepsilon^2 + |y|^2 + |\xi|^2\big)^{\frac{q-1}{2}} \leqslant c\big(1 + |y|^2 + |\xi|^2\big)^{\frac{p}{2}},$$

hence

$$\begin{aligned} F(y,\xi) \leqslant F_\varepsilon(y,\xi) &\leqslant F(y,\xi) + c\big(1 + |y|^2 + |\xi|^2\big)^{\frac{p}{2}}\varepsilon \\ &\leqslant L\big(\mu^2 + \varepsilon^2 + |y|^2 + |\xi|^2\big)^{\frac{q}{2}} + c\big(1 + |y|^2 + |\xi|^2\big)^{\frac{p}{2}}\varepsilon. \end{aligned}$$

By virtue of (2.2) we find some constant $c = c(p)\ell > 0$ such that (recall $p \geqslant 2$):

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} F_\varepsilon(y + tz, \xi + t\eta) \geqslant c\big(\mu^2 + |\xi|^2\big)^{\frac{p-2}{2}}|\eta|^2.$$

The conclusion follows if we take $\varepsilon = 1/cj$ and $F_j := F_\varepsilon$. □

## 3. Proof of Theorem 2

Throughout this section $u \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ designates a local $F$-minimizer. For the sake of simplicity, we shall give the proof in case the integrand $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ is $C^2$ and satisfies the hypotheses (H1) and (H3), with $q < p+1$ when $p < n$ and $q < p + \frac{p}{n}$ when $p \geqslant n$. The general case can be treated by a combination of the arguments below and the approximation result of Lemma 7. More precisely, disregarding the $y$-dependence, it yields $C^\infty$ integrands $F_j = F_j(\xi)$ that satisfy (2.3), (2.4) and (2.5) uniformly in $j$. Apart from the additional term $(1 + |\xi|^2)^{\frac{p}{2}}/j$ on the right-hand side of (2.3), $F_j$ satisfy (H1), (H3) uniformly in $j$ and converges to $F$ according to (2.5). As the additional term appearing in (2.3) (and the term controlling the rate of approximation in (2.5)) is of $p$-th order it is under control. Hence in the argument below we then simply replace $F$ by $F_j$ for suitably chosen $j = j(\varepsilon)$. We skip the details of this as it brings nothing new (except for additional bookkeeping of terms that are under control). Instead we shall as mentioned impose the conditions (H1), (H3) on $F$ directly, and turn to the remaining parts of the proof. We start with the case $p < n$, and assume that $u \in \mathrm{L}^{\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$.

Our aim is to show that $V(Du) \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega)$, where we recall the definition of the auxiliary functions as

$$V(\xi) := \langle \xi \rangle^{\frac{p-2}{2}} \xi, \qquad \langle \xi \rangle := \sqrt{\mu^2 + |\xi|^2}.$$

For later reference we note that for a $\mathrm{C}^2$ map $w$ a routine calculation yields

$$\bigl|D\bigl[V(Dw)\bigr]\bigr|^2 \leqslant \left(\frac{p-2}{2} + 1\right)^2 \langle Dw \rangle^{p-2} \bigl|D^2 w\bigr|^2. \tag{3.1}$$

Fix a subdomain with a smooth boundary $\Omega' \Subset \Omega$. Take a subdomain $\Omega' \Subset \Omega'' \Subset \Omega$ and for $u \in \mathrm{L}^{\infty}_{\mathrm{loc}}(\Omega)$, choose numbers $a > \|u\|_{\mathrm{L}^{\infty}(\Omega'')}$ and $k \in \mathbb{N}$, the latter so large that we have the continuous embedding $\mathrm{W}^{k,2}(\Omega') \hookrightarrow \mathrm{C}^2(\overline{\Omega'})$ and, for technical reasons that will become clear later, also

$$2k > p + 2. \tag{3.2}$$

For a smooth kernel $\phi \in \mathrm{C}^{\infty}_c(B_1(0))$ with $\phi \geqslant 0$ and $\int_{B_1(0)} \phi = 1$, we consider the corresponding family of mollifiers $(\phi_\varepsilon)_{\varepsilon>0}$ and put $\tilde{u}_\varepsilon := \phi_\varepsilon * u$ on $\Omega'$ for each positive $\varepsilon < \operatorname{dist}(\Omega', \partial\Omega)$. We record that

$$\begin{cases} \tilde{u}_\varepsilon \to u \quad \text{as } \varepsilon \searrow 0 \text{ strongly in } \mathrm{W}^{1,p}(\Omega'), \\ \|\tilde{u}_\varepsilon\|_{\mathrm{L}^\infty(\Omega')} \leqslant \|u\|_{\mathrm{L}^\infty(\Omega'')} < a \quad \text{for } 0 < \varepsilon < \operatorname{dist}(\Omega', \partial\Omega''), \end{cases} \tag{3.3}$$

and that for a suitable function $\tilde{\varepsilon} = \tilde{\varepsilon}(\varepsilon)$ with $\tilde{\varepsilon} \searrow 0$ as $\varepsilon \searrow 0$, also

$$\tilde{\varepsilon} \int_{\Omega'} \bigl|D^k \tilde{u}_\varepsilon\bigr|^2 \to 0 \quad \text{as } \varepsilon \searrow 0. \tag{3.4}$$

For large $m \in \mathbb{N}$ satisfying at the least

$$m > \max\left\{2, \frac{2q-p}{2(p+1-q)}\right\} \tag{3.5}$$

and small $\varepsilon > 0$ we let $u_\varepsilon \in \mathrm{W}^{k,2}(\Omega') \cap \mathrm{W}^{1,p}_{\tilde{u}_\varepsilon}(\Omega')$ denote a minimizer to the functional

$$v \mapsto \int_{\Omega'} \left(F(Dv) + \bigl(|v|^2 - a^2\bigr)_+^m + \frac{\tilde{\varepsilon}}{2}\bigl|D^k v\bigr|^2\right)$$

on the Sobolev class $\mathrm{W}^{k,2}(\Omega') \cap \mathrm{W}^{1,p}_{\tilde{u}_\varepsilon}(\Omega')$. Here $t_+$ for a real number $t$ denotes its nonnegative part, that is, $t_+ := \max\{t, 0\}$. The existence of $u_\varepsilon$ is easily established by the direct method. Our next goal is the following:

**Lemma 8.** *For each $\varphi \in \mathrm{W}^{k,2}(\Omega') \cap \mathrm{W}^{1,p}_0(\Omega')$,*

$$0 = \int_{\Omega'} \Bigl(\bigl\langle F'(Du_\varepsilon), D\varphi\bigr\rangle + 2m\bigl(|u_\varepsilon|^2 - a^2\bigr)_+^{m-1} u_\varepsilon \cdot \varphi + \tilde{\varepsilon}\bigl\langle D^k u_\varepsilon, D^k \varphi\bigr\rangle\Bigr). \tag{3.6}$$

*Furthermore, $u_\varepsilon \in \mathrm{W}^{2k,2}_{\mathrm{loc}}(\Omega')$.*

**Proof.** The minimality of $u_\varepsilon$ yields the weak form of the Euler–Lagrange system (3.6) by straightforward means (regardless of the growth of $F$ because our choice of $k$ in particular means that $Du_\varepsilon, D\varphi \in \mathrm{L}^\infty(\Omega')$). The additional regularity of $u_\varepsilon$ then follows from standard elliptic regularity theory if we notice that (3.6) can be rewritten as

$$\Delta^k u_\varepsilon = \frac{(-1)^k}{\tilde{\varepsilon}}\bigl(\operatorname{div} F'(Du_\varepsilon) - 2m\bigl(|u_\varepsilon|^2 - a^2\bigr)_+^{m-1} u_\varepsilon\bigr) \tag{3.7}$$

where the composition of $k$ Laplacians on the left-hand side acts row-wise, and as usual is understood in the distributional sense on $\Omega'$. Since by our choice of $k$, $u_\varepsilon \in \mathrm{C}^2(\overline{\Omega'})$, the right-hand side of (3.7) belongs in particular to $\mathrm{L}^2(\Omega')$ from which we deduce $u_\varepsilon \in \mathrm{W}^{2k,2}_{\mathrm{loc}}(\Omega')$. $\square$

**Lemma 9.** *As $\varepsilon \searrow 0$, $u_\varepsilon \to u$ strongly in* $\mathrm{W}^{1,p}(\Omega')$,

$$\int_{\Omega'} \left( \left(|u_\varepsilon|^2 - a^2\right)_+^m + \frac{\tilde{\varepsilon}}{2}\left|D^k u_\varepsilon\right|^2 \right) \to 0 \quad \text{and} \quad \int_{\Omega'} F(Du_\varepsilon) \to \int_{\Omega'} F(Du).$$

**Proof.** By minimality,

$$\int_{\Omega'} \left( F(Du_\varepsilon) + \left(|u_\varepsilon|^2 - a^2\right)_+^m + \frac{\tilde{\varepsilon}}{2}\left|D^k u_\varepsilon\right|^2 \right) \leqslant \int_{\Omega'} \left( F(D\tilde{u}_\varepsilon) + \left(|\tilde{u}_\varepsilon|^2 - a^2\right)_+^m + \frac{\tilde{\varepsilon}}{2}\left|D^k \tilde{u}_\varepsilon\right|^2 \right) \tag{3.8}$$

for all $0 < \varepsilon \leqslant \varepsilon_0$ for some $\varepsilon_0 > 0$. Now since $F$ is convex and $F(Du) \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ we get by use of Jensen's inequality and standard properties of mollifiers that

$$\int_{\Omega'} F(D\tilde{u}_\varepsilon) \leqslant \int_{\Omega'} \phi_\varepsilon * F(Du) \to \int_{\Omega'} F(Du)$$

as $\varepsilon \searrow 0$. Fatou's lemma then allows us to conclude that

$$\int_{\Omega'} F(D\tilde{u}_\varepsilon) \to \int_{\Omega'} F(Du) \quad \text{as } \varepsilon \searrow 0.$$

In view of (3.3), (3.4), estimate (3.8) implies

$$\limsup_{\varepsilon \searrow 0} \int_{\Omega'} \left( F(Du_\varepsilon) + \left(|u_\varepsilon|^2 - a^2\right)_+^m + \frac{\tilde{\varepsilon}}{2}\left|D^k u_\varepsilon\right|^2 \right) \leqslant \int_{\Omega'} F(Du). \tag{3.9}$$

By virtue of the convexity condition (H3) we can find positive constants $c_1, c_2$ such that $F(\xi) \geqslant c_1|\xi|^p - c_2$ for all $\xi$. The family $(Du_\varepsilon)$ is therefore in particular bounded in $\mathrm{L}^p(\Omega')$ and since $u_\varepsilon = \tilde{u}_\varepsilon$ in the sense of trace on $\partial\Omega'$ a standard lower semicontinuity result together with the minimality of $u$ allow us to conclude that

$$\liminf_{\varepsilon \searrow 0} \int_{\Omega'} F(Du_\varepsilon) \geqslant \int_{\Omega'} F(Du).$$

By virtue of (3.9) this implies that

$$\int_{\Omega'} \left( \left(|u_\varepsilon|^2 - a^2\right)_+^m + \frac{\tilde{\varepsilon}}{2}\left|D^k u_\varepsilon\right|^2 \right) \to 0 \tag{3.10}$$

and

$$\int_{\Omega'} F(Du_\varepsilon) \to \int_{\Omega'} F(Du)$$

as $\varepsilon \searrow 0$. In order to conclude the proof, it is sufficient to note that standard calculations, by virtue of (H3), imply that

$$\int_{\Omega'} \left(\mu^2 + |D\tilde{u}_\varepsilon|^2 + |Du_\varepsilon|^2\right)^{\frac{p-2}{2}} |D\tilde{u}_\varepsilon - Du_\varepsilon|^2 \leqslant c \int_{\Omega'} \left( F(D\tilde{u}_\varepsilon) - F(Du_\varepsilon) - \left\langle F'(Du_\varepsilon), D\tilde{u}_\varepsilon - Du_\varepsilon \right\rangle \right).$$

Here we have by Lemma 8,

$$\int_{\Omega'} \left\langle F'(Du_\varepsilon), D\tilde{u}_\varepsilon - Du_\varepsilon \right\rangle = -2m \int_{\Omega'} \left(|u_\varepsilon|^2 - a^2\right)_+^{m-1} u_\varepsilon \cdot (\tilde{u}_\varepsilon - u_\varepsilon) - \tilde{\varepsilon} \int_{\Omega'} \left\langle D^k u_\varepsilon, D^k \tilde{u}_\varepsilon - D^k u_\varepsilon \right\rangle.$$

Thanks to (3.10) we have that

$$\limsup_{\varepsilon \searrow 0} \int_{\Omega'} |u_\varepsilon|^{2m} \leqslant \left|\Omega'\right| a^{2m}. \tag{3.11}$$

Hence using the last formula together with Hölder's inequality, (3.4) and (3.10), it follows that

$$\int_{\Omega'} \left(\mu^2 + |D\tilde{u}_\varepsilon|^2 + |Du_\varepsilon|^2\right)^{\frac{p-2}{2}} |D\tilde{u}_\varepsilon - Du_\varepsilon|^2 \to 0 \quad \text{as } \varepsilon \searrow 0.$$

Because $p \geqslant 2$ we conclude by well-known means that $u_\varepsilon \to u$ strongly in $\mathrm{W}^{1,p}(\Omega', \mathbb{R}^N)$. $\square$

We are now ready to embark on the core of the proof of Theorem 2, where also the growth hypothesis (H1) will be used.

**Proof of Theorem 2.** Fix $B_{2R} = B_{2R}(x_0) \subset \Omega'$, radii $R \leqslant r < s \leqslant 2R \leqslant 2$ and a smooth cut-off function $\rho$ satisfying $1_{B_r} \leqslant \rho \leqslant 1_{B_s}$ and $|D^i \rho| \leqslant (\frac{2}{s-r})^i$ for each $i \in \mathbb{N}$. Our choice of test map and the ensuing computation is inspired by [16] and [14]. According to Lemma 8 we can test the Euler–Lagrange system with $\varphi = \rho^{2k} D_j^2 u_\varepsilon$ for each direction $1 \leqslant j \leqslant n$:

$$\begin{aligned} 0 &= \int_{\Omega'} \left\langle F'(Du_\varepsilon), D_j^2 Du_\varepsilon \right\rangle \rho^{2k} + \int_{\Omega'} \left\langle F'(Du_\varepsilon), D_j^2 u_\varepsilon \otimes D\left(\rho^{2k}\right) \right\rangle \\ &\quad + \int_{\Omega'} 2m\left(|u_\varepsilon|^2 - a^2\right)_+^{m-1} u_\varepsilon \cdot D_j^2 u_\varepsilon \rho^{2k} + \tilde{\varepsilon} \int_{\Omega'} \left\langle D^k u_\varepsilon, D^k\left(D_j^2 u_\varepsilon \rho^{2k}\right)\right\rangle \\ &=: I + II + III + IV. \end{aligned} \tag{3.12}$$

Integration by parts yields

$$\begin{aligned} I &= -\int_{\Omega'} \left(\rho^{2k}\left\langle D_j\left(F'(Du_\varepsilon)\right), D_j Du_\varepsilon\right\rangle + 2k\rho^{2k-1} D_j\rho \left\langle F'(Du_\varepsilon), D_j Du_\varepsilon\right\rangle\right) \\ &= -\int_{\Omega'} \left(\rho^{2k} F''(Du_\varepsilon)[D_j Du_\varepsilon, D_j Du_\varepsilon] + 2k\left\langle \rho^{k-1} D_j \rho F'(Du_\varepsilon), \rho^k D_j Du_\varepsilon\right\rangle\right) \\ &\leqslant -\int_{\Omega'} \rho^{2k} \langle Du_\varepsilon\rangle^{p-2} |D_j Du_\varepsilon|^2 \\ &\quad + \int_{\Omega'} \left(4k^2 \rho^{2(k-1)} |D_j\rho|^2 \frac{|F'(Du_\varepsilon)|^2}{\langle Du_\varepsilon\rangle^{p-2}} + \frac{1}{4}\rho^{2k}\langle Du_\varepsilon\rangle^{p-2} |D_j Du_\varepsilon|^2\right), \end{aligned}$$

where we used (H3) and Young's inequality. Hence invoking (1.4) we arrive at the estimate

$$I \leqslant -\frac{3}{4} \int_{\Omega'} \rho^{2k} \langle Du_\varepsilon\rangle^{p-2} |D_j Du_\varepsilon|^2 + c \int_{\Omega'} k^2 \rho^{2(k-1)} |D_j \rho|^2 \langle Du_\varepsilon\rangle^{2q-p}. \tag{3.13}$$

By virtue of (1.4) and Cauchy–Schwarz' inequality,

$$\begin{aligned} II &\leqslant c \int_{\Omega'} \langle Du_\varepsilon\rangle^{q-1} \rho^{2k-1} |D\rho| \left|D_j^2 u_\varepsilon\right| \\ &\leqslant c \int_{\Omega'} \langle Du_\varepsilon\rangle^{2q-p} \rho^{2(k-1)} |D\rho|^2 + \frac{1}{4} \int_{\Omega'} \rho^{2k} \langle Du_\varepsilon\rangle^{p-2} |D_j Du_\varepsilon|^2, \end{aligned} \tag{3.14}$$

where $c$ is independent of $m$, $\varepsilon$. Integration by parts gives

$$\begin{aligned}
III &= -2m\int_{\Omega'}\rho^{2k}D_j\big[\big(|u_\varepsilon|^2-a^2\big)_+^{m-1}u_\varepsilon\big]D_ju_\varepsilon - 2m\int_{\Omega'}2k\big(|u_\varepsilon|^2-a^2\big)_+^{m-1}u_\varepsilon\rho^{k-1}D_j\rho\cdot\rho^kD_ju_\varepsilon\\
&\leqslant -2m\int_{\Omega'}\rho^{2k}\big(2(m-1)\big(|u_\varepsilon|^2-a^2\big)_+^{m-2}(u_\varepsilon\cdot D_ju_\varepsilon)^2+\big(|u_\varepsilon|^2-a^2\big)_+^{m-1}|D_ju_\varepsilon|^2\big)\\
&\quad+\int_{\Omega'}\big(|u_\varepsilon|^2-a^2\big)_+^{m-1}\left(\frac{1}{2}|D_ju_\varepsilon|^2\rho^{2k}+32m^2k^2|u_\varepsilon|^2\rho^{2(k-1)}|D_j\rho|^2\right)\\
&\leqslant -\frac{1}{2}\int_{\Omega'}\rho^{2k}\big(|u_\varepsilon|^2-a^2\big)_+^{m-1}|D_ju_\varepsilon|^2+32m^2k^2\int_{\Omega'}\big(|u_\varepsilon|^2-a^2\big)_+^{m-1}|u_\varepsilon|^2|D_j\rho|^2\rho^{2(k-1)}.
\end{aligned}\tag{3.15}$$

For the estimation of $IV$ we rewrite

$$IV=\tilde\varepsilon\int_{\Omega'}\big\langle D^ku_\varepsilon, D_jD^k\big(\rho^{2k}D_ju_\varepsilon\big)-D^k\big(D_j\big(\rho^{2k}\big)D_ju_\varepsilon\big)\big\rangle$$

and integrating the first term by parts,

$$\begin{aligned}
IV &= -\tilde\varepsilon\int_{\Omega'}\left(\big\langle D_jD^ku_\varepsilon, D^k\big(\rho^{2k}D_ju_\varepsilon\big)\big\rangle-\tilde\varepsilon\int_{\Omega'}\big\langle D^ku_\varepsilon, D^k\big(D_j\big(\rho^{2k}\big)D_ju_\varepsilon\big)\big\rangle\right)\\
&=: IV_1+IV_2.
\end{aligned}$$

We estimate these terms by use of Cauchy–Schwarz' inequality, Leibniz' product formula and the assumed bounds on $D^i\rho$ (simplifying also by use of $s-r\leqslant 1$):

$$\begin{aligned}
IV_1 &\leqslant -\tilde\varepsilon\int_{\Omega'}\rho^{2k}\big|D_jD^ku_\varepsilon\big|^2+\frac{c_k\tilde\varepsilon}{(s-r)^k}\int_{\Omega'}\rho^k\big|D_jD^ku_\varepsilon\big|\sum_{i=0}^{k-1}\big|D^iD_ju_\varepsilon\big|\\
&\leqslant -\frac{2\tilde\varepsilon}{3}\int_{\Omega'}\rho^{2k}\big|D_jD^ku_\varepsilon\big|^2+\frac{c_k\tilde\varepsilon}{(s-r)^{2k}}\int_{B_{2R}}\left(\sum_{i=0}^{k-1}\big|D^iD_ju_\varepsilon\big|\right)^2\\
&\leqslant -\frac{2\tilde\varepsilon}{3}\int_{\Omega'}\rho^{2k}\big|D_jD^ku_\varepsilon\big|^2+\frac{c_k\tilde\varepsilon}{(s-r)^{2k}}\int_{B_{2R}}\sum_{i=0}^{k-1}\big|D^iD_ju_\varepsilon\big|^2
\end{aligned}$$

for a (new) constant $c_k$. Likewise,

$$IV_2\leqslant\frac{\tilde\varepsilon}{3}\int_{\Omega'}\rho^{2k}\big|D_jD^ku_\varepsilon\big|^2+\frac{c_k\tilde\varepsilon}{(s-r)^{2k+2}}\int_{B_{2R}}\left(\sum_{i=0}^{k-1}\big|D^iD_ju_\varepsilon\big|^2+\big|D^ku_\varepsilon\big|^2\right),$$

where we remark that the increased power of the factor $(s-r)$ is due to the presence of an additional $D_j$-derivative on $\rho^{2k}$ in $IV_2$. Collecting the above bounds and adjusting the constant $c_k$ we arrive at

$$IV\leqslant -\frac{\tilde\varepsilon}{3}\int_{\Omega'}\rho^{2k}\big|D_jD^ku_\varepsilon\big|^2+\frac{c_k\tilde\varepsilon}{(s-r)^{2k+2}}\int_{B_{2R}}\left(\sum_{i=0}^{k-1}\big|D^iD_ju_\varepsilon\big|^2+\big|D^ku_\varepsilon\big|^2\right).\tag{3.16}$$

Inserting the bounds (3.13), (3.14), (3.15), (3.16) in (3.12) and using the properties of $\rho$ we get for each $1\leqslant j\leqslant n$:

$$\frac{1}{2}\int_{\Omega'}\rho^{2k}\langle Du_\varepsilon\rangle^{p-2}|D_jDu_\varepsilon|^2+\frac{1}{2}\int_{\Omega'}\rho^{2k}\big(|u_\varepsilon|^2-a^2\big)_+^{m-1}|D_ju_\varepsilon|^2+\frac{\tilde\varepsilon}{3}\int_{\Omega'}\rho^{2k}\big|D_jD^ku_\varepsilon\big|^2$$

$$\leqslant \frac{c}{(s-r)^2} \int_{B_s \setminus B_r} \langle Du_\varepsilon \rangle^{2q-p} + \frac{cm^2}{(s-r)^2} \int_{B_{2R}} \left(|u_\varepsilon|^2 - a^2\right)_+^{m-1} |u_\varepsilon|^2$$
$$+ \frac{c\tilde{\varepsilon}}{(s-r)^{2k+2}} \int_{B_{2R}} \left( \sum_{i=0}^{k-1} \left|D_j D^i u_\varepsilon\right|^2 + \left|D^k u_\varepsilon\right|^2 \right).$$

Adding up these inequalities over $j \in \{1, \dots, n\}$ and adjusting the constants we arrive at

$$\int_{\Omega'} \rho^{2k} \langle Du_\varepsilon \rangle^{p-2} \left|D^2 u_\varepsilon\right|^2 + \int_{\Omega'} \rho^{2k} \left(|u_\varepsilon|^2 - a^2\right)_+^{m-1} |Du_\varepsilon|^2 + \frac{2\tilde{\varepsilon}}{3} \int_{\Omega'} \rho^{2k} \left|D^{k+1} u_\varepsilon\right|^2$$
$$\leqslant \frac{c}{(s-r)^2} \int_{B_s \setminus B_r} \langle Du_\varepsilon \rangle^{2q-p} + \frac{A(\varepsilon)}{(s-r)^2} + \frac{B(\varepsilon)}{(s-r)^{2k+2}},$$

where $c$ is a constant depending only on $n$, $N$, $q$, $k$, $L$ and where $A(\varepsilon)$, $B(\varepsilon)$ are independent of $r$, $s$ and where by virtue of Lemma 9, through the Gagliardo–Nirenberg interpolation inequality, for each fixed $m$

$$A(\varepsilon) \to 0, \qquad B(\varepsilon) \to 0 \quad \text{as } \varepsilon \searrow 0.$$

Omitting the second and third terms on the left-hand side, the above inequality simplifies to

$$\int_{\Omega'} \rho^{2k} \langle Du_\varepsilon \rangle^{p-2} \left|D^2 u_\varepsilon\right|^2 \leqslant \frac{c}{(s-r)^2} \int_{B_s \setminus B_r} \langle Du_\varepsilon \rangle^{2q-p} + \frac{A(\varepsilon)}{(s-r)^2} + \frac{B(\varepsilon)}{(s-r)^{2k+2}}. \tag{3.17}$$

Next, we invoke the following Gagliardo–Nirenberg-type inequality that we state as a lemma. The elementary proof is deferred to Appendix A.

**Lemma 10.** *For $\psi \in \mathrm{C}_c^1(\Omega')$ with $\psi \geqslant 0$ and $\mathrm{C}^2$ maps $v : \Omega' \to \mathbb{R}^N$ we have*

$$\int_{\Omega'} \psi^{\frac{m}{m+1}(p+2)} |Dv|^{\frac{m}{m+1}(p+2)} \leqslant (p+2)^2 \left( \int_{\Omega'} \psi^{\frac{m}{m+1}(p+2)} |v|^{2m} \right)^{\frac{1}{m+1}}$$
$$\times \left[ \left( \int_{\Omega'} \psi^{\frac{m}{m+1}(p+2)} |D\psi|^2 |Dv|^p \right)^{\frac{m}{m+1}} + n \left( \int_{\Omega'} \psi^{\frac{m}{m+1}(p+2)} |Dv|^{p-2} \left|D^2 v\right|^2 \right)^{\frac{m}{m+1}} \right]$$

*where $p \in (1, \infty)$ and $m > 1$.*

In view of (3.2) the function $\psi := \rho^{2k \frac{m+1}{m} \frac{1}{p+2}}$ is of class $\mathrm{C}_c^1(\Omega')$ and so is an admissible choice together with $v = u_\varepsilon$ in Lemma 10, which then yields (using $p \geqslant 2$ and properties of $\rho$)

$$\int_{B_r} |Du_\varepsilon|^{\frac{m}{m+1}(p+2)} \leqslant c \left( \int_{B_{2R}} |u_\varepsilon|^{2m} \right)^{\frac{1}{m+1}} \left( \int_{\Omega'} \rho^{2k} \langle Du_\varepsilon \rangle^{p-2} \left|D^2 u_\varepsilon\right|^2 \right)^{\frac{m}{m+1}}$$
$$+ c \left( \int_{B_{2R}} |u_\varepsilon|^{2m} \right)^{\frac{1}{m+1}} \left( \int_{B_s \setminus B_r} \frac{|Du_\varepsilon|^p}{(s-r)^2} \right)^{\frac{m}{m+1}}$$

for a constant $c$ that only depends on $n$, $p$. In order to simplify notation we recall that by virtue of Lemma 9 (see in particular (3.11)) we have for each $\tilde{a} > a$ that

$$\left( \int_{\Omega'} |u_\varepsilon|^{2m} \right)^{\frac{1}{m+1}} < \tilde{a}^2$$

provided that $\varepsilon \leqslant \varepsilon_0(\tilde{a})$ and $m \geqslant m_0(\tilde{a})$, which we assume in the following. Hence

$$\int_{B_r} |Du_\varepsilon|^{\frac{m}{m+1}(p+2)} \leqslant c\tilde{a}^2 \biggl(\int_{\Omega'} \rho^{2k} \langle Du_\varepsilon\rangle^{p-2} \bigl|D^2 u_\varepsilon\bigr|^2\biggr)^{\frac{m}{m+1}} + \frac{c\tilde{a}^2}{(s-r)^2} \biggl(\int_{B_{2R}} |Du_\varepsilon|^p\biggr)^{\frac{m}{m+1}} \tag{3.18}$$

where we simplified with $1/(s-r)^{\frac{2m}{m+1}} \leqslant 1/(s-r)^2$. Combining (3.17) and (3.18) we get (simplifying with $m/(m+1) < 1$ and $(\alpha+\beta)^{\frac{m}{m+1}} \leqslant \alpha^{\frac{m}{m+1}} + \beta^{\frac{m}{m+1}}$),

$$\begin{aligned}\int_{B_r} |Du_\varepsilon|^{\frac{m}{m+1}(p+2)} &\leqslant \frac{c\tilde{a}^2}{(s-r)^2} \biggl(\int_{B_{2R}} |Du_\varepsilon|^p\biggr)^{\frac{m}{m+1}} \\ &\quad + \frac{c\tilde{a}^2}{(s-r)^2} \biggl[\biggl(\int_{B_s\setminus B_r} \langle Du_\varepsilon\rangle^{2q-p}\biggr)^{\frac{m}{m+1}} + A_1(\varepsilon) + \frac{B_1(\varepsilon)}{(s-r)^{2k}}\biggr],\end{aligned} \tag{3.19}$$

where $A_1(\varepsilon) := A(\varepsilon)^{\frac{m}{m+1}}$ and $B_1(\varepsilon) := B(\varepsilon)^{\frac{m}{m+1}}$. In view of (3.5) we can write

$$\frac{1}{2q-p} = \frac{\theta}{\frac{m}{m+1}(p+2)} + \frac{1-\theta}{p},$$

where, since $q < p+1$,

$$\theta = \frac{2m(p+2)}{2m-p} \times \frac{q-p}{2q-p} \in (0,1).$$

Hence by Hölder's inequality

$$\biggl(\int_{B_s\setminus B_r} \langle Du_\varepsilon\rangle^{2q-p}\biggr)^{\frac{m}{m+1}} \leqslant \biggl(\int_{B_{2R}} \langle Du_\varepsilon\rangle^p\biggr)^{\frac{2q-p}{p}\frac{m}{m+1}(1-\theta)} \biggl(\int_{B_s\setminus B_r} \langle Du_\varepsilon\rangle^{\frac{m}{m+1}(p+2)}\biggr)^{\frac{2q-p}{p+2}\theta}.$$

Observe that by virtue of (3.5)

$$\frac{2q-p}{p+2}\theta = (q-p)\frac{2m}{2m-p} \in (0,1).$$

Applying Young's inequality with the exponents

$$d = \frac{2m-p}{2m} \times \frac{1}{q-p} > 1 \quad \text{and} \quad d' = \frac{2m-p}{2m(p+1-q)-p}$$

we find (abbreviating $\chi_m = \frac{m}{m+1}\frac{2m(p+1-q)-2q+p}{2m(p+1-q)-p}$)

$$\frac{c\tilde{a}^2}{(s-r)^2} \biggl(\int_{B_s\setminus B_r} \langle Du_\varepsilon\rangle^{2q-p}\biggr)^{\frac{m}{m+1}} \leqslant \frac{c^{d'}\tilde{a}^{2d'}}{(s-r)^{2d'}} \biggl(\int_{B_{2R}} \langle Du_\varepsilon\rangle^p\biggr)^{\chi_m} + \int_{B_s\setminus B_r} \langle Du_\varepsilon\rangle^{\frac{m}{m+1}(p+2)}.$$

Insert this bound in (3.19) and use $\langle Du_\varepsilon\rangle^{\frac{m}{m+1}(p+2)} \leqslant c(|Du_\varepsilon|^{\frac{m}{m+1}(p+2)} + \mu^{\frac{m}{m+1}(p+2)})$ to get

$$\begin{aligned}\int_{B_r} \langle Du_\varepsilon\rangle^{\frac{m}{m+1}(p+2)} &\leqslant \int_{B_s\setminus B_r} \langle Du_\varepsilon\rangle^{\frac{m}{m+1}(p+2)} + c\mu^{\frac{m}{m+1}(p+2)}|B_{2R}| + \frac{c^{d'}\tilde{a}^{2d'}}{(s-r)^{2d'}} \biggl(\int_{B_{2R}} \langle Du_\varepsilon\rangle^p\biggr)^{\chi_m} \\ &\quad + \frac{c\tilde{a}^2}{(s-r)^2} \biggl[\biggl(\int_{B_{2R}} \langle Du_\varepsilon\rangle^p\biggr)^{\frac{m}{m+1}} + A_1(\varepsilon) + \frac{B_1(\varepsilon)}{(s-r)^{2k}}\biggr].\end{aligned}$$

As this estimate is valid for all radii $R \leqslant r < s \leqslant 2R$ we can apply the hole-filling method of Widman: add $\int_{B_r} \langle Du_\varepsilon \rangle^{\frac{m}{m+1}(p+2)}$ to both sides of the inequality, normalize and note that we are in the situation of Lemma 6 with $\vartheta = 1/2$ (and $A$, $B$, $C$, $D$ given by the obvious corresponding terms). Thus

$$\int_{B_R} \langle Du_\varepsilon \rangle^{\frac{m}{m+1}(p+2)} \leqslant c\mu^{\frac{m}{m+1}(p+2)}|B_{2R}| + \frac{c^{d'}\tilde{a}^{2d'}}{R^{2d'}}\biggl(\int_{B_{2R}} \langle Du_\varepsilon \rangle^{p}\biggr)^{\chi_m} + \frac{c\tilde{a}^2}{R^2}\biggl[\biggl(\int_{B_{2R}} \langle Du_\varepsilon \rangle^{p}\biggr)^{\frac{m}{m+1}} + A_1(\varepsilon) + \frac{B_1(\varepsilon)}{R^{2k}}\biggr], \tag{3.20}$$

for some constant $c = c(n, N, p, q, L, k)$. In particular it follows that $(Du_\varepsilon)$ is bounded in $\mathrm{L}^{2q-p}(B_R, \mathbb{R}^{N\times n})$ and so, by the arbitrariness of the ball $B_{2R}(x_0) \subset \Omega'$ and a simple covering argument, we conclude that $(Du_\varepsilon)$ is bounded in $\mathrm{L}^{2q-p}_{\mathrm{loc}}(\Omega', \mathbb{R}^{N\times n})$. In view of (3.1) and (3.17) it then also follows that $(V(Du_\varepsilon))$ is bounded in $\mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega', \mathbb{R}^{N\times n})$. Therefore by passing to the limits (first $\varepsilon \searrow 0$, then $m \nearrow \infty$ and then $\tilde{a} \searrow \|u\|_{\mathrm{L}^\infty}$) we infer that

$$\int_{B_R} \langle Du \rangle^{p+2} \leqslant c\mu^{\frac{m}{m+1}(p+2)}|B_{2R}| + \biggl(\biggl(\frac{c\|u\|_{\mathrm{L}^\infty}}{R}\biggr)^{\frac{2}{p+1-q}} + 1\biggr)\int_{B_{2R}} \langle Du \rangle^{p} \tag{3.21}$$

and hence

$$\int_{B_R} \bigl|D\bigl[V(Du)\bigr]\bigr|^2 \leqslant \frac{c}{R^2}\biggl[\mu^{\frac{m}{m+1}(p+2)}|B_{2R}| + \biggl(\biggl(\frac{\|u\|_{\mathrm{L}^\infty}}{R}\biggr)^{\frac{2}{p+1-q}} + 1\biggr)\int_{B_{2R}} \langle Du \rangle^{p}\biggr]$$

for a constant $c = c(n, N, p, q, L, k)$. This concludes the proof for the case $p < n$.

Finally we shall indicate the necessary modifications when $p \geqslant n$ and $q < p + \frac{p}{n}$. In this situation we do not assume that the minimizer $u$ is bounded (it is of course bounded when $p > n$), and the resulting proof is a bit easier.

In the above notation we let $u_\varepsilon \in \mathrm{W}^{k,2}(\Omega') \cap \mathrm{W}^{1,p}_{\tilde{u}_\varepsilon}(\Omega')$ denote a minimizer of the functional

$$v \mapsto \int_{\Omega'} \biggl(F(Dv) + \frac{\tilde{\varepsilon}}{2}\bigl|D^k v\bigr|^2\biggr)$$

on the Sobolev class $\mathrm{W}^{k,2}(\Omega') \cap \mathrm{W}^{1,p}_{\tilde{u}_\varepsilon}(\Omega')$.

With the obvious modifications both Lemmas 8 and 9 remain true, and so we can proceed along the lines of the previous proof. Hereby we arrive at an estimate of the form (3.17). The proof is now completed using (3.1), Sobolev embedding theorem and Hölder's inequality exactly as in [14] (see (4.6), (4.7) there and proceed as in the subsequent proof on pp. 260–262). □

**Sketch of the proof of Remark 4.** The proof is very similar to that of Theorem 2 and so we confine ourselves to a very brief sketch, leaving details to the interested reader. First we remark that the general case is treated by use of the approximation result in Lemma 7, and that we shall confine our comments to the case where $F = F(y, \xi)$ is $C^\infty$ smooth and satisfies (H1′) and

$$F''(y, \xi)\bigl[(z, \eta), (z, \eta)\bigr] \geqslant c\bigl(\mu^2 + |\xi|^2\bigr)^{\frac{p-2}{2}}|\eta|^2. \tag{H3′}$$

We now simply mimic the proof of Theorem 2. The Euler–Lagrange equation (3.6) reads: for each $\varphi \in \mathrm{W}^{k,2}(\Omega') \cap \mathrm{W}^{1,p}_0(\Omega')$,

$$0 = \int_{\Omega'} \bigl(\bigl\langle F'(u_\varepsilon, Du_\varepsilon), (\varphi, D\varphi)\bigr\rangle + 2m\bigl(|u_\varepsilon|^2 - a^2\bigr)_+^{m-1} u_\varepsilon \cdot \varphi + \tilde{\varepsilon}\bigl\langle D^k u_\varepsilon, D^k\varphi\bigr\rangle\bigr). \tag{3.22}$$

Because of the joint convexity of $F(y, \xi)$ the conclusions of Lemmas 8, 9 remain the same. The only complication comes in the core part of the proof, which is slightly more cumbersome due to the additional terms resulting from the $y$-dependence. However, as an interested reader can readily check, these terms are well-behaved lower order terms that are easily estimated and controlled using a Poincaré inequality and that $(u_\varepsilon)$ is bounded in $\mathrm{W}^{1,p}$. □

## 4. Proof of Theorem 3

We only treat the points where the proof differs from the one of Theorem 2.

**Proof of Theorem 3.** Also here we start with the case $p < n$, where it is assumed that $q < p + 2$ and $u \in \mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$. Our starting point is the identity (3.12), where, thanks to assumption (H4), we can estimate integrals $I$ and $II$ in a more efficient way. The estimates for $III$ and $IV$ remain as (3.15) and (3.16). More precisely, we proceed as follows

$$\begin{aligned} I + II &= -\int_{\Omega'} \rho^{2k}\big\langle D_j\big(F'(Du_\varepsilon)\big), D_j Du_\varepsilon\big\rangle - 2k\int_{\Omega'} \rho^{2k-1} D_j\rho\big\langle F'(Du_\varepsilon), D_j Du_\varepsilon\big\rangle \\ &\quad - 2k\int_{\Omega'} \rho^{2k-1}\big\langle F'(Du_\varepsilon), D_j^2 u_\varepsilon \otimes D_j\rho\big\rangle \\ &= -\int_{\Omega'} \rho^{2k} F''(Du_\varepsilon)[D_j Du_\varepsilon, D_j Du_\varepsilon] - 2k\int_{\Omega'} \big\langle \rho^{2k-1} D_j\rho F'(Du_\varepsilon), D_j Du_\varepsilon\big\rangle \\ &\quad - 2k\int_{\Omega'} \rho^{2k-1}\big\langle F'(Du_\varepsilon), D_j^2 u_\varepsilon \otimes D_j\rho\big\rangle \\ &=: J + JJ + JJJ. \end{aligned} \tag{4.1}$$

In order to estimate $JJ$, we integrate by parts whereby

$$\begin{aligned} JJ &= 2k\int_{\Omega'} \big\langle D_j\big(\rho^{2k-1} D_j\rho F'(Du_\varepsilon)\big), Du_\varepsilon\big\rangle \\ &= 2k\int_{\Omega'} \rho^{2k-1} D_j\rho F''(Du_\varepsilon)[D_j Du_\varepsilon, Du_\varepsilon] + 2k\int_{\Omega'} \big\langle D_j\big(\rho^{2k-1} D_j\rho\big) F'(Du_\varepsilon), Du_\varepsilon\big\rangle. \end{aligned}$$

In view of (H3) the bilinear form $(\xi, \eta) \mapsto F''(Du_\varepsilon(x))[\xi, \eta]$ defines for each $x$ an inner product on $\mathbb{R}^{N\times n}$, and so we have in particular by use of Cauchy–Schwarz' inequality

$$F''(Du_\varepsilon)[D_j Du_\varepsilon, Du_\varepsilon] \leqslant F''(Du_\varepsilon)[D_j Du_\varepsilon, D_j Du_\varepsilon]^{\frac{1}{2}} F''(Du_\varepsilon)[Du_\varepsilon, Du_\varepsilon]^{\frac{1}{2}}$$

on $\Omega'$. Hence invoking Young's inequality we arrive at

$$\begin{aligned} JJ &\leqslant \frac{1}{4}\int_{\Omega'} \rho^{2k} F''(Du_\varepsilon)[D_j Du_\varepsilon, D_j Du_\varepsilon] \\ &\quad + 16k^2\int_{\Omega'} \rho^{2k-2}|D\rho|^2\big|F''(Du_\varepsilon)\big||Du_\varepsilon|^2 + 4k^2\int_{\Omega'} \rho^{2k-2}\big|D^2\rho\big|\big|F'(Du_\varepsilon)\big||Du_\varepsilon| \\ &\leqslant \frac{1}{4}\int_{\Omega'} \rho^{2k} F''(Du_\varepsilon)[D_j Du_\varepsilon, D_j Du_\varepsilon] + \frac{c}{(s-r)^2}\int_{\Omega'} |Du_\varepsilon|^q \end{aligned}$$

where, in the last line, we used (1.4) and (H4). The integral $JJJ$ can be estimated in exactly the same way, and then (4.1) becomes

$$\begin{aligned} I + II &\leqslant -\int_{\Omega'} \rho^{2k} F''(Du_\varepsilon)[D_j Du_\varepsilon, D_j Du_\varepsilon] + \frac{1}{2}\int_{\Omega'} \rho^{2k} F''(Du_\varepsilon)[D_j Du_\varepsilon, D_j Du_\varepsilon] + \frac{c}{(s-r)^2}\int_{\Omega'} |Du_\varepsilon|^q \\ &\leqslant -\frac{1}{2}\int_{\Omega'} \rho^{2k}\langle Du_\varepsilon\rangle^{p-2}|D_j Du_\varepsilon|^2 + \frac{c}{(s-r)^2}\int_{\Omega'} |Du_\varepsilon|^q \end{aligned} \tag{4.2}$$

where we again used the ellipticity condition (H3). Inserting (4.2), (3.15) and (3.16) in (3.12), summing on $j$ we obtain

$$\int_{\Omega'} \rho^{2k} \langle Du_\varepsilon \rangle^{p-2} \left| D^2 u_\varepsilon \right|^2 \leqslant \frac{c}{(s-r)^2} \int_{B_s \setminus B_r} \langle Du_\varepsilon \rangle^q + \frac{A(\varepsilon)}{(s-r)^2} + \frac{B(\varepsilon)}{(s-r)^{2k+2}}, \tag{4.3}$$

where $c$ is a constant depending on $n$, $N$, $q$, $k$, $L$, $\Lambda$. Here $A(\varepsilon)$, $B(\varepsilon)$ are independent of $r$, $s$ and, by virtue of Lemma 9, for each fixed $m$

$$A(\varepsilon) \to 0, \qquad B(\varepsilon) \to 0 \quad \text{as } \varepsilon \searrow 0.$$

Combining (4.3) and Lemma 10 we get

$$\begin{aligned} \int_{B_r} |Du_\varepsilon|^{\frac{m}{m+1}(p+2)} &\leqslant \frac{c(a+1)^2}{(s-r)^2} \left( \int_{B_{2R}} |Du_\varepsilon|^p \right)^{\frac{m}{m+1}} \\ &\quad + \frac{c(a+1)^2}{(s-r)^2} \left[ \left( \int_{B_s \setminus B_r} \langle Du_\varepsilon \rangle^q \right)^{\frac{m}{m+1}} + A_1(\varepsilon) + \frac{B_1(\varepsilon)}{(s-r)^{2k}} \right]. \end{aligned} \tag{4.4}$$

Because $q < p+2$ we have for $m$ large enough that $p < q < \frac{m}{m+1}(p+2)$, and hence we may write

$$\frac{1}{q} = \frac{\theta}{\frac{m}{m+1}(p+2)} + \frac{1-\theta}{p},$$

where

$$\theta = \frac{m(p+2)}{2m-p} \times \frac{q-p}{q} \in (0,1).$$

Finally, by Hölder's inequality

$$\left( \int_{B_s \setminus B_r} \langle Du_\varepsilon \rangle^q \right)^{\frac{m}{m+1}} \leqslant \left( \int_{B_{2R}} \langle Du_\varepsilon \rangle^p \right)^{\frac{q}{p}\frac{m}{m+1}(1-\theta)} \left( \int_{B_s \setminus B_r} \langle Du_\varepsilon \rangle^{\frac{m}{m+1}(p+2)} \right)^{\frac{q}{p+2}\theta},$$

and we observe that since $m > p/(p+2-q)$ we have

$$\frac{q}{p+2}\theta = (q-p)\frac{m}{2m-p} \in (0,1).$$

In order to conclude the proof it suffices to use Young's inequality and the hole filling method exactly as done in the proof of Theorem 2. This completes the proof for the case $p < n$.

Finally in the remaining case $p \geqslant n$ where we suppose $q < p + \frac{2p}{n}$ (and no a priori boundedness condition on the minimizer) the proof is completed using (3.1), the Sobolev embedding theorem and Hölder's inequality along the lines of [14] as in the proof of Theorem 2. □

## Acknowledgements

Parts of the research were done while M.C. and A.P.d.N. were visiting the Oxford Centre for Nonlinear Partial Differential Equations (EPSRC Science and Innovation Award EP/E035027/1), and while J.K. was visiting Dept. Maths. 'R. Caccioppoli' in Napoli. We wish to thank both institutions for financial support and hospitality. J.K. was also partially supported by ERC grant 207573 'Vectorial Problems'.

## Appendix A

In order to conclude the proofs from the previous sections it only remains to prove Lemma 10. We note that it's an elementary variant of the more sophisticated interpolation inequality [35, Theorem 3.1]; whereas the latter is proved by use of the $H^1$–BMO duality we simply use Hölder's inequality.

**Proof of Lemma 10.** It suffices to prove the result for a $C^2$ function $v : \Omega' \to \mathbb{R}$. Let $t > 0$ and denote $2t + 2 = \frac{m}{m+1}(p+2)$. As in [35] we integrate by parts to get

$$\int_{\Omega'} \psi^{2t+2}|Dv|^{2t+2} = -\int_{\Omega'} v \cdot \operatorname{div} w,$$

where $w = \psi^{2t+2}|Dv|^{2t}Dv$. Here

$$\operatorname{div} w = (2t+2)\psi^{2t+1}|Dv|^{2t}D\psi \cdot Dv + 2t\psi^{2t+2}|Dv|^{2t-2}\sum_{j=1}^{n}(Dv \cdot D_jDv)D_jv + \psi^{2t+2}|Dv|^{2t}\Delta v.$$

We may therefore estimate as follows

$$\int_{\Omega'} \psi^{2t+2}|Dv|^{2t+2} \leqslant 2(t+1)\int_{\Omega'} \psi^{2t+1}|D\psi||Dv|^{2t+1}|v| + 2(t+1)\sum_{j=1}^{n}\int_{\Omega'} \psi^{2t+2}|v||Dv|^{2t}|D_jDv|.$$

Considering $|v||Dv|^{2t}|D_jDv| = |v||Dv|^{2t-\frac{p-2}{2}} \times |Dv|^{\frac{p-2}{2}}|D_jDv|$ and using Hölder's inequality twice (with weight $\psi^{2t+2}$) we get

$$\begin{aligned}
&\int_{\Omega'} \psi^{2t+2}|v||Dv|^{2t}|D_jDv| \\
&\quad\leqslant \left(\int_{\Omega'} \psi^{2t+2}|v|^2|Dv|^{4t-p+2}\right)^{\frac{1}{2}}\left(\int_{\Omega'} \psi^{2t+2}|Dv|^{p-2}|D_jDv|^2\right)^{\frac{1}{2}} \\
&\quad\leqslant \left(\int_{\Omega'} \psi^{2t+2}|v|^{2m}\right)^{\frac{1}{2m}}\left(\int_{\Omega'} \psi^{2t+2}|Dv|^{2t+2}\right)^{\frac{m-1}{2m}}\left(\int_{\Omega'} \psi^{2t+2}|Dv|^{p-2}|D_jDv|^2\right)^{\frac{1}{2}},
\end{aligned}$$

where we used that $2t+2 = (4t-p+2)\frac{m}{m-1}$. Likewise Hölder's inequality yields

$$\int_{\Omega'} \psi^{2t+2}|v||Dv|^{2t+1}|D\psi| \leqslant \left(\int_{\Omega'} \psi^{2t+2}|v|^{2m}\right)^{\frac{1}{2m}}\left(\int_{\Omega'} \psi^{2t+2}|Dv|^{2t+2}\right)^{\frac{m-1}{2m}}\left(\int_{\Omega'} \psi^{2t+2}|Dv|^{p}|D\psi|^2\right)^{\frac{1}{2}},$$

and collecting the bounds a routine estimation allows us to conclude. □

## References


[1] E. Acerbi, G. Bouchitté, I. Fonseca, Relaxation of convex functionals: The gap problem, Ann. I. H. Poincaré – AN 20 (3) (2003) 359–390.

[2] E. Acerbi, N. Fusco, Partial regularity under anisotropic $(p,q)$ growth conditions, J. Differential Equations 107 (1994) 46–67.

[3] E. Acerbi, G. Mingione, Regularity results for a class of functionals with non-standard growth, Arch. Ration. Mech. Anal. 156 (2001) 121–140.

[4] M. Bildhauer, Convex Variational Problems. Linear, Nearly Linear and Anisotropic Growth Conditions, Lecture Notes in Math., vol. 1818, Springer-Verlag, Berlin, 2003.

[5] M. Bildhauer, M. Fuchs, Partial regularity for variational integrals with $(s,\mu,q)$-growth, Calc. Var. Partial Differ. Eq. 13 (2001) 537–560.

[6] M. Bildhauer, M. Fuchs, Partial regularity for a class of anisotropic variational integrals with convex hull property, Asymptot. Anal. 32 (3–4) (2002) 293–315.

[7] M. Bildhauer, M. Fuchs, Higher integrability of the gradient for vectorial minimizers of decomposable variational integrals, Manuscripta Math. 123 (2007) 269–283.

[8] M. Bildhauer, M. Fuchs, Variational integrals of splitting-type: higher integrability under general growth conditions, Ann. Mat. Pura Appl. (4) 188 (3) (2009) 467–496.
[9] S. Campanato, Elliptic systems with nonlinearity $q$ greater or equal to two. Regularity of the solution of the Dirichlet problem, Ann. Mat. Pura Appl. (4) 147 (1987) 117–150.
[10] H.J. Choe, Interior behaviour of minimizers for certain functionals with nonstandard growth, Nonlinear Anal. Theory Methods Appl. 19 (10) (1992) 933–945.
[11] F. Duzaar, J. Kristensen, G. Mingione, The existence of regular boundary points for non-linear elliptic systems, J. Reine Angew. Math. 602 (2007) 17–58.
[12] L. Esposito, F. Leonetti, G. Mingione, Regularity for minimizers of functionals with $p - q$ growth, NoDEA Nonlinear Differ. Eq. Appl. 6 (1999) 133–148.
[13] L. Esposito, F. Leonetti, G. Mingione, Higher integrability for minimizers of integral functionals with $(p, q)$ growth, J. Differential Equations 157 (1999) 414–438.
[14] L. Esposito, F. Leonetti, G. Mingione, Regularity results for minimizers of irregular integrals with $(p, q)$ growth, Forum Mathematicum 14 (2002) 245–272.
[15] L. Esposito, F. Leonetti, G. Mingione, Sharp regularity for functionals with $(p, q)$ growth, J. Differential Equations 204 (1) (2004) 5–55.
[16] I. Fonseca, N. Fusco, Regularity results for anisotropic image segmentation models, Ann. Sc. Norm. Super. Pisa 24 (1997) 463–499.
[17] N. Fusco, C. Sbordone, Local boundedness of minimizers in a limit case, Manuscripta Math. 69 (1990) 19–25.
[18] N. Fusco, C. Sbordone, Higher integrability of the gradient of minimizers of functionals with nonstandard growth conditions, Comm. Pure Appl. Math. 43 (1990) 673–683.
[19] M. Giaquinta, Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems, Annals of Math. Studies, vol. 105, Princeton Univ. Press, 1983.
[20] M. Giaquinta, Growth conditions and regularity, a counterexample, Manuscripta Math. 59 (1987) 245–248.
[21] E. Giusti, Direct Methods in the Calculus of Variations, World Scientific, 2003.
[22] M.C. Hong, Some remarks on the minimizers of variational integrals with nonstandard growth conditions, Boll. Un. Mat. Ital. A (7) 6 (1992) 91–101.
[23] J. Kristensen, C. Melcher, Regularity in oscillatory nonlinear elliptic systems, Math. Z. 260 (2008) 813–847.
[24] J. Kristensen, G. Mingione, The singular set of $\omega$-minima, Arch. Ration. Mech. Anal. 177 (2005) 93–114.
[25] J. Kristensen, G. Mingione, The singular set of minima of integral functionals, Arch. Ration. Mech. Anal. 180 (2006) 331–398.
[26] J. Kristensen, G. Mingione, Boundary regularity in variational problems, Arch. Ration. Mech. Anal. 198 (2010) 369–455.
[27] F. Leonetti, Maximum principles for vector-valued minimizers of some integral functionals, Boll. Un. Mat. Ital. A (6) 5 (1991) 51–56.
[28] P. Marcellini, On the definition and the lower semicontinuity of certain quasiconvex integrals, Ann. I. H. Poincaré – AN 3 (1986) 391–409.
[29] P. Marcellini, Un example de solution discontinue d'un probéme variationel dans le cas scalaire, preprint, Ist. U. Dini, Firenze, 1987–1988.
[30] P. Marcellini, Regularity of minimizers of integrals of the calculus of variations with non-standard growth conditions, Arch. Ration. Mech. Anal. 105 (1989) 267–284.
[31] P. Marcellini, Regularity and existence of solutions of elliptic equations with $(p, q)$ growth conditions, J. Differential Equations 90 (1991) 1–30.
[32] P. Marcellini, Everywhere regularity for a class of elliptic systems without growth conditions, Ann. Sc. Norm. Super. Pisa Cl. Sci. 23 (1996) 1–25.
[33] P. Marcellini, G. Papi, Nonlinear elliptic systems with general growth, J. Differential Equations 221 (2006) 412–443.
[34] A. Passarelli di Napoli, F. Siepe, A regularity result for a class of anisotropic systems, Rend. Ist. Mat di Trieste (1997) 13–31.
[35] T. Riviére, P. Strzelecki, A sharp nonlinear Gagliardo–Nirenberg-type estimate and applications to the regularity of elliptic systems, Comm. Part. Diff. Eq. 30 (2005) 589–604.
[36] T. Schmidt, Regularity of minimizers of $W^{1,p}$-quasiconvex variational integrals with $(p, q)$-growth, Calc. Var. 32 (2008) 1–24.
[37] V. Šverák, X. Yan, Non-Lipschitz minimizers of smooth uniformly convex variational integrals, Proc. Natl. Acad. Sci. USA 99 (2002) 15269–15276.
[38] V.V. Zhikov, On Lavrentiev's phenomenon, Russian J. Math. Phys. 3 (1995) 249–269.
[39] V.V. Zhikov, On some variational problems, Russian J. Math. Phys. 5 (1997) 105–116.
[40] W.P. Ziemer, Weakly Differentiable Functions, Grad. Texts in Math., vol. 120, Springer-Verlag, 1989.