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On the large-distance asymptotics of steady state solutions of the Navier–Stokes equations in 3D exterior domains

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Abstract

We identify the leading term describing the behavior at large distances of the steady state solutions of the Navier–Stokes equations in 3D exterior domains with vanishing velocity at the spatial infinity.

1. Introduction

We consider the 3D steady-state Navier–Stokes equations in exterior domains and study the behavior of the solutions "near infinity". The equations are

$$\left.\begin{aligned} &-\Delta u + u\nabla u + \nabla p = 0,\\ &\operatorname{div} u = 0 \end{aligned}\right\} \quad \text{in } \mathbb{R}^3 \setminus \bar{B}_{R_0}, \tag{1.1}$$

where B_{R_0} denotes the ball of radius R_0 centered at the origin. Our main assumption about the solutions will be the decay condition

$$\left|u(x)\right| \leqslant \frac{C_*}{R_0 + |x|} \quad \text{in } \mathbb{R}^3 \setminus \bar{B}_{R_0} \tag{1.2}$$

for sufficiently small C_*. The specific boundary conditions at ∂B_{R_0} will play no role in our results.

We note that if u solves the equations in (1.1) in $\mathbb{R}^3 \setminus B_R$, then (1.2) is satisfied for some $R_0 > 0$ if

$$\limsup_{x\to\infty} |x|\left|u(x)\right| < \frac{C_*}{2}. \tag{1.3}$$

Naively one might think that the behavior near infinity of the above solutions should be given by the linearized equation. An immediate well-known objection[2] to that is that we expect decay $|\nabla^k u| = O(|x|^{-k-1})$ and

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1 Supported in part by NSF Grant DMS-0457061.
2 The objection was probably raised already by the classics in the 19th century.

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$|\nabla^k p| = O(|x|^{-k-2})$ as $x \to \infty$, and therefore the non-linear term in the equation should have the same order of magnitude as the linear terms, making the accuracy of the linearization questionable. This heuristics is made rigorous in [5], where it is proved that the leading order term describing the behavior of the solutions cannot be given by the linearized equation.

In this paper we identify explicitly the leading order behavior of the above solutions near infinity. We show that it is given by the explicit solutions calculated by L.D. Landau in 1943 see [8,9]. In this paper the formulae are presented in Section 3, (3.1). Landau's calculations were revisited and certain extensions were obtained in [4,15]. The Landau solutions were recently characterized in [13] as the only solutions of the steady Navier–Stokes equation in $\mathbb{R}^3 \setminus \{0\}$ which are smooth and (-1)-homogeneous in $\mathbb{R}^3 \setminus \{0\}$. The Landau solutions in $\mathbb{R}^3 \setminus \{0\}$ can be parametrized by vectors $b \in \mathbb{R}^3$ in the following way: For each $b \in \mathbb{R}^3$ there exists a unique (-1)-homogeneous solution U^b of the steady Navier–Stokes equations together with an associated pressure P^b which is (-2)-homogeneous, such that U^b, P^b are smooth in $\mathbb{R}^3 \setminus \{0\}$, U^b is weakly div-free across the origin and satisfies

$$-\Delta U^b + \operatorname{div}\big(U^b \otimes U^b\big) + \nabla P^b = b\delta(x) \tag{1.4}$$

in the sense of distributions. Here $\delta(x)$ denotes the Dirac function and we use the standard notation $u \otimes v$ for the tensor field $u_i v_j$ defined by the tensor product of the vector fields $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$. We also use the standard notation $\operatorname{div} T$ for the vector field $\frac{\partial}{\partial x_j} T_{ij}$. (The uniqueness of U^b, P^b is much easier to prove if we add the requirement that U^b be axi-symmetric with respect to the axis passing through the origin in the direction of the vector b, but as was shown in [13], this additional symmetry assumption is not necessary.) As noticed by Landau, the solutions can be calculated explicitly in terms of elementary functions, see formulae (3.1), (3.4) and (3.5). (It was observed in [13] that the Landau solutions are in a natural one-to-one correspondence with the group of conformal transformations of the two-dimensional sphere, and Landau's formulae can be also derived from this observation by using some standard geometry.)

If u, p is a solution of (1.1), we will denote by

$$T_{ij} = T_{ij}(u,p) = p\delta_{ij} + u_i u_j - \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1.5}$$

the momentum flux density tensor in the fluid. Our main result is the following:

Theorem 1. *For each $\alpha \in (1,2)$ there exists $\varepsilon = \varepsilon(\alpha) > 0$ such that the following statement holds true: Let u, p be a solution of (1.1) in $\mathbb{R}^3 \setminus \bar{B}_{R_0}$ satisfying (1.2) or (1.3) with $C_* \leqslant \varepsilon$. Let $b = (b_1, b_2, b_3)$ be defined by*

$$b_i = \int_{\partial B_{R_1}} T_{ij}(u,p) n_j(x) \tag{1.6}$$

for some $R_1 > R_0$. (Note that the integral is independent of R_1.) Let U^b be the Landau solution corresponding to the vector b. Then

$$u(x) = U^b(x) + O\big(|x|^{-\alpha}\big) \quad \text{as } |x| \to \infty \tag{1.7}$$

and, for a suitable constant p_0,

$$p(x) - p_0 = P^b(x) + O\big(|x|^{-\alpha-1}\big) \quad \text{as } |x| \to \infty. \tag{1.8}$$

Remark 1. Standard estimates for the linear Stokes system (such as estimate (2.5) in the next section), together with the scaling symmetry $u(x) \to \lambda u(\lambda x)$ of Navier–Stokes can be used to show that any solution of (1.1) satisfying (1.7) will also satisfy

$$\nabla^k u(x) = \nabla^k U^b(x) + O\big(|x|^{-k-\alpha}\big) \quad \text{as } |x| \to \infty,\ \text{for } k = 1, 2, \dots, \tag{1.9}$$

and

$$\nabla^k p = \nabla^k P^b + O\big(|x|^{-k-1-\alpha}\big) \quad \text{as } |x| \to \infty,\ \text{for } k = 1, 2, \dots. \tag{1.10}$$

See for example [14] for an argument of this type.

As suggested by a referee, it may be worth pointing out that Theorem 1 together with Remark 1 imply that if $|u(x)| = o(\frac{1}{|x|})$, then $u = O(\frac{1}{|x|^{2-\varepsilon}})$ for each $\varepsilon > 0$.

The existence of expansions similar to (1.7) with U^b replaced by a less specific term, namely a (-1)-homogeneous function, was studied in [11]. The main result of that paper is, roughly speaking, that under smallness conditions similar to (1.2), the solutions of (1.1) are "asymptotically (-1)-homogeneous". As is shown in [13], the leading term of the asymptotical expansion at ∞ of any solution of (1.1) which is asymptotically (-1)-homogeneous must be given by a Landau solution. (This result remains true even for large data, since the proof is not based on perturbative arguments.) Therefore the results in [11] together with the results in [13] imply a version of Theorem 1. Our proof in this paper is much simpler than the proof one could get by combining [11,13]. Also, if one tried to evaluate explicitly the values of the constants in the smallness conditions, the constants coming from the proof here would probably be more favorable.

The proof of Theorem 1 is given in the following sections. The main idea of the proof is as follows. First, it is clear that (1.3) implies that (1.2) is satisfied for some $R_0 > 0$, so we can work with (1.2) in what follows. Assume now for simplicity that u satisfies the "no outflow to infinity condition"

$$\int_{\partial B_{R_1}} u(x) \cdot n(x) = 0 \tag{1.11}$$

for some $R_1 > R_0$, where $n(x)$ denotes the unit normal to ∂B_{R_1}. (Since u is div-free, the last integral does not depend on R_1.) We extend the fields u, p to fields defined in all $\mathbb{R}^3$ and satisfying the inhomogeneous equation

$$\left.\begin{array}{l} -\Delta u + u\nabla u + \nabla p = f, \\ \operatorname{div} u = 0 \end{array}\right\} \quad \text{in } \mathbb{R}^3. \tag{1.12}$$

The extension needs to be done in a way which enables one to control smooth norms of the extended function by the corresponding norms of the original function. We have $b = \int_{\mathbb{R}^3} f$. We then search for solutions of (1.12) in the form $u = \tilde{U}^b + v$, where $\tilde{U}^b$ is a suitable regularization of the Landau solution U^b. The equation for v is solved (for small data) by a standard perturbation analysis in the space of continuous functions with decay $O(|x|^{-\alpha})$ as $|x| \to \infty$. For this argument one needs both an existence result (for v) and a uniqueness result (for u), and therefore it looks unlikely that our method could be used in a large data situation.

The general situation when the outflow $\int_{\partial B_{R_1}} u \cdot n$ does not vanish can be handled by a standard method of writing $u = a + v$ where v has no outflow, and a is a suitable multiple of the canonical outflow field $\frac{x}{|x|^3}$. See for example [6, Section 2.2], [7, Chapter IX], or [11, Remark 3.2]. Roughly speaking, the part of the flow which produces a non-zero outflow has decay $O(|x|^{-2})$, and therefore it does not influence the main term in (1.7) at large distances. See Section 4 for details.

One can see easily by looking at the Landau solutions that the best possible decay rate α for which the result might still be true is $\alpha = 2$. We conjecture that the result indeed remains true for $\alpha = 2$. However, as the example in Remark 2 shows, one would probably need to go beyond the elementary perturbation theory used in this paper to prove that.

The problem of steady-state solutions of the Navier–Stokes equations in exterior 3D domains has a long history going back to Leray's paper [10]. Leray proved the existence of solutions with finite energy. Such solutions are easily seen to be smooth since the steady state equation is subcritical with respect to the energy estimate. However, the precise behavior of these solutions as $|x| \to \infty$ is a more subtle problem. This problem shares some features with the (super-critical) regularity problem for the time-dependent equation, since there seems to be some vague duality between regularity (or short-distance behavior) of super-critical problems and asymptotics at large times/distances (or long distance behavior) of sub-critical problems. In particular, in both cases it seems to be important to obtain some local control of the energy flux which is stronger than what one can immediately get from the known conservation laws. In this paper we will not address these difficult issues, which arise for large data, and we will only treat the small data situation, which can be handled by a simple perturbation theory, and is independent of the energy methods. For the steady state exterior problem this approach was pioneered by Finn, see e.g. [6,7]. We note that the 3D exterior problem with non-zero velocity at ∞ has been more or less fully solved, even for large data, see [2,7], since the non-zero velocity at infinity sufficiently regularizes the flow. In the 2D situation many problems remain open even in the case of non-zero velocity at infinity, see [1,7].

How reasonable are our assumptions? We note that Finn proved in [6] the existence of solutions satisfying our assumptions for quite general boundary-value problems in exterior domains under smallness assumptions on the data. See also [7] for extensions of these results (still under smallness assumptions). It is quite conceivable that Theorem 1 remains true even without assuming that the constant C_* in (1.2) is small. The proof of such a result would however require to go beyond the perturbation theory and the standard energy methods.

The following example, taken from [12], shows that the question of relaxing the decay condition (1.2) to a slower decay might be quite subtle. Consider the equation

$$-\Delta u + (1-a)u\nabla u + \frac{a}{2}\nabla|u|^2 + \frac{1}{2}u \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3, \tag{1.13}$$

for vector fields u in $\mathbb{R}^3$. The number $a \in (0,1)$ is a parameter. (For $a = \frac{1}{2}$ the non-linear term in (1.13) can be written as $\operatorname{div} Q(u,u)$ for a suitable quadratic expression Q.) The equation has the same energy estimate as the Navier–Stokes equations. It turns our that (1.13) has a nontrivial global smooth solution $\bar{u}$ satisfying $|\bar{u}(x)| \sim |x|^{-2/3}$ as $x \to \infty$. Since it appears that the various perturbation and energy methods used for the steady Navier–Stokes should also work for (1.13), at least in the case $a = \frac{1}{2}$, the properties of $\bar{u}$ indicate some limitations to these methods. On the other hand, we should remark that steady Navier–Stokes does have some special properties which are probably not shared by (1.13). For example, for steady Navier–Stokes the quantity $\frac{1}{2}|u|^2 + p$ satisfies a maximum principle, see e.g. [1].

The paper is organized as follows: In Section 2 we explain the material necessary for extending the solutions to $\mathbb{R}^3$ in a controlled manner. In Section 3 we recall the necessary facts about Landau's solutions. Finally, in Section 4 we explain the perturbation argument.

2. Preliminaries

We consider the solutions of the steady Navier–Stokes equation

$$\begin{aligned} &-\Delta u + u\nabla u + \nabla p = 0, \\ &\operatorname{div} u = 0 \end{aligned} \tag{2.1}$$

which are defined "in the neighborhood of infinity", i.e. in the region $\mathbb{R}^3 \setminus \bar{B}_{R_0}$, where B_{R_0} denotes the ball of radius R_0 centered at the origin. In this section we will be interested in the solutions which satisfy

$$\left|u(x)\right| \leqslant \frac{C_*}{R_0 + |x|} \quad \text{in } \mathbb{R}^3 \setminus \bar{B}_{R_0}, \tag{2.2}$$

and the "no outflow to infinity" condition

$$\int_{\partial B_{R_1}} u \cdot n = 0 \tag{2.3}$$

for some $R_1 > R_0$. (We note that the integral in (2.3) is independent of R_1, since $\operatorname{div} u = 0$.)

The constant C_* above will play a special role and we distinguish it from the "generic constants" which will be denoted by c. If c depends on a parameter X and we want to emphasize this dependence, we will write $c(X)$ instead of c. The value of c can change from line to line.

By a solution of (2.1) we mean a smooth function vector field u in $\mathbb{R}^3 \setminus \bar{B}_{R_0}$ which satisfies (2.1) for a suitable p. (The pressure p will be considered only as a "secondary" variable: Instead of saying "the solution (u,p)", we can just say "the solution u", with the understanding that (2.1) is satisfied for a suitable p.) Various other notions of solutions are used in the literature (e.g. weak solutions), but under the assumption (2.2) they all coincide are equivalent to the one defined above.

One reason that the above way of thinking of p only as an auxiliary variable works quite well is that the linear steady-state Stokes system

$$-\Delta u + \nabla p = \operatorname{div} f,$$
$$\operatorname{div} u = 0 \tag{2.4}$$

satisfies local elliptic estimates of the form

$$\|\nabla u\|_{X(B_{x_0,R})} + \big\|p - (p)_{B_{x_0,R}}\big\|_{X(B_{x_0,R})} \leqslant c(R,X)\|f\|_{X(B_{x_0,2R})} + \tilde{c}(R,X)\|u\|_{L^1(B_{x_0,2R})}, \tag{2.5}$$

where $B_{x_0,R}$ denotes the ball of radius R centered at x_0, $(p)_{B_{x_0,R}}$ is the average of p over the ball $B_{x_0,R}$ and X can be any space in which classical elliptic estimates work, such as an L^p-space with $p \in (1,\infty)$ or a Hölder space. The main point of estimate (2.5) is that there is no p on the right-hand side. See for example [14] for details.

The linear estimate (2.5) combined with the standard bootstrapping and scaling arguments (using the scaling symmetry $u(x) \to \lambda u(\lambda x)$) imply that solutions of (2.1) satisfying estimate (2.2) with $C_* \leqslant M$ also satisfy

$$\big|\nabla^k u(x)\big| \leqslant c(k,M)\frac{C_*}{(R_0+|x|)^{k+1}} \quad \text{in } \mathbb{R}^3 \setminus B_{2R_0}, \text{ for } k = 1,2,\ldots. \tag{2.6}$$

We now relate the solutions of (2.1) in $\mathbb{R}^3 \setminus \bar{B}_{R_0}$ to the solutions of the equation in $\mathbb{R}^3$ with non-trivial right-hand side:

$$\left.\begin{aligned} -\Delta u + u\nabla u + \nabla p &= f, \\ \operatorname{div} u &= 0 \end{aligned}\right\} \quad \text{in } \mathbb{R}^3. \tag{2.7}$$

Let u be a solution of (2.1) in $\mathbb{R}^3 \setminus \bar{B}_{R_0}$ satisfying (2.2) with $C_* \leqslant M$ and let p be the associated pressure, defined up to a constant. Using (2.6) we see that we can in fact choose a "normalized" p so that, for $C_* \leqslant M$, we have

$$\big|\nabla^k p\big| \leqslant c(k,M)\frac{C_*}{(R_0+|x|)^{k+2}} \quad \text{in } \mathbb{R}^3 \setminus B_{2R_0}, \text{ for } k = 0,1,2,\ldots. \tag{2.8}$$

We can now extend u, p from $\mathbb{R}^3 \setminus B_{3R_0}$ to $\tilde{u}$, $\tilde{p}$ defined in $\mathbb{R}^3$ such that $\operatorname{div}\tilde{u} = 0$ in $\mathbb{R}^3$ and

$$\big|\nabla^k \tilde{u}(x)\big| \leqslant c(k,M)\frac{C_*}{(R_0+|x|)^{k+1}} \quad \text{in } \mathbb{R}^3, \text{ for } k = 0,1,2,\ldots, \tag{2.9}$$

together with

$$\big|\nabla^k \tilde{p}(x)\big| \leqslant c(k,M)\frac{C_*}{(R_0+|x|)^{k+2}} \quad \text{in } \mathbb{R}^3, \text{ for } k = 0,1,2,\ldots. \tag{2.10}$$

The construction of the extension $p \to \tilde{p}$ is standard. To be able to construct the extension $u \to \tilde{u}$, we of course need condition (2.3). With (2.3) satisfied, the existence of a smooth div-free extension $\tilde{u}$ (not necessarily satisfying (2.9)) is also classical. The construction of a div-free extension satisfying (2.9) can be carried out in many ways. One can proceed for example as follows: Let $\eta : [0,\infty) \to [0,1]$ be a smooth function such that $\eta(r) = 0$ for $r \leqslant 2$ and $\eta(r) = 1$ for $r \geqslant \frac{5}{2}$, and let $\eta_{R_0}(r) = \eta(\frac{r}{R_0})$. Now set $\tilde{u} = \eta_{R_0}u + v$, where v is a suitable solution of the equation $\operatorname{div} v = -u\nabla\eta_{R_0} = g$ which is compactly supported in B_{3R_0}. The equation $\operatorname{div} v = g$ has of course many compactly supported solutions, but it is possible to construct a solution operator $S : g \to v = Sg$ which has the required regularity properties. (Such an operator is sometimes called a Bogovskii operator.) See for example [3] or [7, Chapter III.3] for details.[3]

We have

$$\left.\begin{aligned} -\Delta \tilde{u} + \tilde{u}\nabla \tilde{u} + \nabla \tilde{p} &= f, \\ \operatorname{div} \tilde{u} &= 0 \end{aligned}\right\} \quad \text{in } \mathbb{R}^3, \tag{2.11}$$

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[3] There are many ways to construct S. For example, one can follow Bogovskii and define S by $Sg(x) = \int_{B_{3R_0}} K(x,y)g(y)\,dy$, where the kernel $K(x,y)$ is given by $K(x,y) = \frac{x-y}{|x-y|^n}m(x,y)$, with $m(x,y) = \int_{|x-y|}^{\infty} \omega_{R_0}(y + t\frac{x-y}{|x-y|})t^{n-1}\,dt$ and $\omega_{R_0}(x) = {R_0}^{-n}\omega(x/R_0)$ for a suitable smooth function ω compactly supported in B_3 and satisfying $\int_{B_3}\omega = 1$. This operator gains one derivative in the usual elliptic regularity spaces, and also has the right scaling. This is enough to obtain the required estimates for a solution v of $\operatorname{div} v = g = -u\nabla\eta_{R_0}$. (Note that if we re-scale the problem to $R_0 = 1$, we do not need that v "gain" one full derivative over g in the sup-norms, and therefore the sup-norm in the required estimate presents no problem.)

where the right-hand side f is supported in $\bar{B}_{3R_0}$ and satisfies

$$\left|\nabla^k f(x)\right| \leqslant c(k, M)\frac{C_*}{(R_0+|x|)^{k+3}} \quad \text{in } \mathbb{R}^3,\ \text{for } k=0,1,2,\ldots. \tag{2.12}$$

Dropping the tildes and changing R_0, if necessary, we see that the study of solutions of Navier–Stokes defined in the neighborhood of infinity and satisfying the "no outflow" condition (2.3) and the growth condition (2.2) can be reduced to the study of the solutions u of the inhomogeneous equation (2.7) with f supported in B_{R_0} and satisfying (2.12), and u satisfying (2.2) globally, with C_* replaced by cC_*. The "no outflow condition" (2.3) is easily seen to be necessary for this reduction to be possible. It is also easy to check by integrating the first equation of (2.11) over B_{R_1} that the vector b in (1.6) can be obtained from f as $b=\int_{\mathbb{R}^3} f$.

3. The Landau solutions

The Landau solutions are smooth (-1)-homogeneous solutions of the steady-state Navier–Stokes equations defined in $\mathbb{R}^3\setminus\{0\}$. Under the additional assumption of axial symmetry, these were first calculated by L.D. Landau in 1943, see [8,9]. In [13] it was proved that we do not get any new solutions if the assumption of axial symmetry is dropped. To write down the explicit formulae, we will use the standard polar coordinates r,θ,φ defined by

$$\begin{aligned} x_1 &= r\sin\theta\cos\varphi,\\ x_2 &= r\sin\theta\sin\varphi,\\ x_3 &= r\cos\theta. \end{aligned}$$

The explicit formulae in polar coordinates for the Landau solution U and the corresponding pressure P are as follows

$$\begin{aligned} U_r &= \frac{2}{r}\left[\frac{A^2-1}{(A-\cos\theta)^2}-1\right],\\ U_\theta &= -\frac{2\sin\theta}{r(A-\cos\theta)},\\ U_\varphi &= 0,\\ P &= -\frac{4(A\cos\theta-1)}{r^2(A-\cos\theta)^2}. \end{aligned} \tag{3.1}$$

In the above formulae, A is a parameter satisfying $A>1$. The velocity field U can also be expressed in terms of the stream function

$$\psi = \frac{2r\sin^2\theta}{A-\cos\theta} \tag{3.2}$$

as

$$\begin{aligned} U_r &= \frac{1}{r\sin\theta}\frac{\partial\psi}{r\partial\theta},\\ U_\theta &= -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r},\\ U_\varphi &= 0. \end{aligned} \tag{3.3}$$

The integral curves of the velocity field U are given the equations $\psi=\text{const.}$ and $\varphi=\text{const.}$

Clearly U, $U\otimes U$ and P are locally integrable, and a direct calculation (see e.g. [9]) gives

$$-\Delta U+\operatorname{div}(U\otimes U)+\nabla P=\beta(A)e_3\delta(x), \tag{3.4}$$

where e_3 is the unit vector in the positive x_3-direction and

$$\beta(A)=16\pi\left(A+\frac{1}{2}A^2\log\frac{A-1}{A+1}+\frac{4A}{3(A^2-1)}\right). \tag{3.5}$$

It is not hard to check that the function $\beta(A)$ is monotonically decreasing in $(1,\infty)$ and maps this interval onto $(0,\infty)$. In particular, β has an inverse function $\gamma:(0,\infty)\to(1,\infty)$.

It is instructive to compare the formula (3.2) with the corresponding formula for the linear Stokes system. Namely, the solution of

$$\left.\begin{aligned} &-\Delta U_{\mathrm{lin}}+\nabla P_{\mathrm{lin}}=e_3\delta(x),\\ &\operatorname{div}U_{\mathrm{lin}}=0\end{aligned}\right\}\quad \text{in } \mathbb{R}^3, \tag{3.6}$$

satisfying $U_{\mathrm{lin}}(x)\to 0$ as $x\to\infty$ is given by the stream function

$$\psi_{\mathrm{lin}}=\frac{1}{8\pi}r\sin^2\theta. \tag{3.7}$$

We can get another useful comparison if we express the solution of the problem

$$\left.\begin{aligned} &-\Delta u_\varepsilon+\varepsilon\operatorname{div}(u_\varepsilon\otimes u_\varepsilon)+\nabla p_\varepsilon=e_3\delta(x),\\ &\operatorname{div}u_\varepsilon=0\end{aligned}\right\}\quad \text{in } \mathbb{R}^3, \tag{3.8}$$

with the condition $u_\varepsilon(x)\to 0$ as $x\to\infty$ in terms of the Landau solutions. The formula (for $\varepsilon>0$) is

$$u_\varepsilon=\frac{1}{\varepsilon}U\bigg|_{A=\gamma(\varepsilon)} \tag{3.9}$$

and u_ε is given by the stream function

$$\psi_\varepsilon=\frac{r\sin^2\theta}{\varepsilon\gamma(\varepsilon)-\varepsilon\cos\theta}. \tag{3.10}$$

We will now regularize the Landau solutions near the origin in the following way. Let $r_0>0$. Consider a smooth function $\rho:[0,\infty)\to[0,\infty)$ such that $\rho(r)=0$ for $r\leqslant r_0$, $\rho(r)=r$ for $r\geqslant 2r_0$ and the k-th derivative $\rho^{(k)}(r)$ is bounded by $c(k)r^{1-k}$, and define

$$\tilde\psi=\tilde\psi_{A,r_0}=\frac{2\rho(r)\sin^2\theta}{A-\cos\theta}. \tag{3.11}$$

With the help of $\tilde\psi$ we now define the regularized velocity field $\tilde U=\tilde U_{A,r_0}$ by the formulae (3.3), with ψ replaced by $\tilde\psi$. We also define

$$\tilde P=\tilde P_{A,r_0}=-\frac{4\rho(r)(A\cos\theta-1)}{r^3(A-\cos\theta)^2}. \tag{3.12}$$

It is easy to check that for $A\geqslant A_0>1$ we have

$$\left|\nabla^k\tilde U\right|\leqslant\frac{c(k,A_0)}{A(r_0+|x|)^{k+1}}\quad \text{in } \mathbb{R}^3,\ \text{for } k=0,1,\ldots \tag{3.13}$$

and

$$\left|\nabla^k\tilde P\right|\leqslant\frac{c(k,A_0)}{A(r_0+|x|)^{k+2}}\quad \text{in } \mathbb{R}^3,\ \text{for } k=0,1,\ldots. \tag{3.14}$$

So far we have mostly considered the Landau solutions which are axi-symmetric with respect to the x_3 axis. However, it is clear from the above that for each non-zero vector $b\in\mathbb{R}^3$ there exist a unique Landau solution U^b and the associated pressure P^b which are axi-symmetric with respect to the axis $\mathbb{R}\cdot b$ and satisfy

$$\begin{aligned} &-\Delta U^b+\operatorname{div}\bigl(U^b\otimes U^b\bigr)+\nabla P^b=b\delta(x),\\ &\operatorname{div}U^b=0.\end{aligned} \tag{3.15}$$

We also set $U^0=0$. For each U^b, P^b the above construction of the regularized solutions gives the regularized fields $\tilde U^b=\tilde U^b_{r_0}$ and $\tilde P^b=\tilde P^b_{r_0}$ which, for $|b|\leqslant M$ will satisfy the estimates

$$\left|\nabla^k \tilde{U}^b\right| \leqslant c(k, M)\frac{|b|}{(r_0 + |x|)^{k+1}} \quad \text{in } \mathbb{R}^3, \text{ for } k = 0, 1, \ldots \tag{3.16}$$

and

$$\left|\nabla^k \tilde{P}^b\right| \leqslant c(k, M)\frac{|b|}{(r_0 + |x|)^{k+2}} \quad \text{in } \mathbb{R}^3, \text{ for } k = 0, 1, \ldots. \tag{3.17}$$

4. Perturbation analysis

Let f be a sufficiently regular compactly supported vector field in $\mathbb{R}^3$. Let $b = \int_{\mathbb{R}^3} f$. Let $r_0 = 1$ and let $\tilde{U} = \tilde{U}^b = \tilde{U}^b_{r_0}$ and $\tilde{P} = \tilde{P}^b = \tilde{P}^b_{r_0}$ be the regularizations of the Landau solutions U^b, P^b corresponding to the vector b constructed in the previous section. We will seek solutions of the steady Navier–Stokes equation

$$\left.\begin{aligned} &-\Delta u + u\nabla u + \nabla p = f, \\ &\operatorname{div} u = 0 \end{aligned}\right\} \quad \text{in } \mathbb{R}^3 \tag{4.1}$$

in the form $u = \tilde{U} + v$. We set

$$\tilde{F} = -\Delta\tilde{U} + \tilde{U}\nabla\tilde{U} + \nabla\tilde{P}. \tag{4.2}$$

It is easy to check by integrating (4.2) over B_{R_1} that $\int_{\mathbb{R}^3} \tilde{F} = b = \int_{\mathbb{R}^3} f$. The equation for v becomes

$$\left.\begin{aligned} &-\Delta v + \tilde{U}\nabla v + v\nabla\tilde{U} + v\nabla v + \nabla q = f - \tilde{F}, \\ &\operatorname{div} v = 0 \end{aligned}\right\} \quad \text{in } \mathbb{R}^3. \tag{4.3}$$

Let us choose a fixed $\alpha \in (1, 2)$. We will prove that under some smallness assumptions Eq. (4.3) has a unique solution v with decay $O(|x|^{-\alpha})$ as $|x| \to \infty$. An important point is that, by our construction, $\int_{\mathbb{R}^3}(f - \tilde{F}) = 0$. Using the scaling symmetry, we see that we can assume $r_0 = 1$ without loss of generality.

Let $G = G_{ij}$ be the Green tensor of the linear Stokes operator. We note that the vector field G_{i3} is given by the stream function (3.7). Another explicit formula for G is

$$G_{ij}(x) = \frac{1}{8\pi}\left(-\delta_{ij}\Delta + \frac{\partial^2}{\partial x_i \partial x_j}\right)|x|. \tag{4.4}$$

For our purposes here we will only need the following obvious estimate

$$\left|\nabla G(x)\right| \leqslant \frac{c}{|x|^2}. \tag{4.5}$$

The required solutions of (4.3) will be found for small data by a standard perturbation argument. Let X_α be the space of all continuous div-free vector fields u in $\mathbb{R}^3$ satisfying $u(x) = O(|x|^{-\alpha})$ as $x \to \infty$. A natural norm in X_α is given for example by

$$[u]_\alpha = \sup_x \left(1 + |x|\right)^\alpha \left|u(x)\right|. \tag{4.6}$$

Our perturbation analysis is based on the following elementary estimates:

Lemma 1. *Using the notation above, let $b = \int_{\mathbb{R}^3} f$ be the vector used in the construction of $\tilde{U}$. Then for $|b| \leqslant M$ we have*

$$\left[G * \operatorname{div}(\tilde{U} \otimes v + v \otimes \tilde{U})\right]_\alpha \leqslant c(\alpha, M)|b|[v]_\alpha, \tag{4.7}$$

and

$$\left[G * \operatorname{div}(v \otimes w)\right]_\alpha \leqslant c(\alpha)[v]_\alpha[w]_\alpha. \tag{4.8}$$

Proof. The proof these estimates is standard. We move the derivatives to G, use the definition of the norm, after which the only remaining task is to estimate the integral

$$I(x)=\int_{\mathbb{R}^3}\frac{dy}{|x-y|^2(1+|y|)^{\alpha+\beta}},\tag{4.9}$$

where $\beta\in\{1,\alpha\}$. It is enough to consider only the case $\beta=1$. Clearly $I(x)$ is bounded for $|x|\leqslant 1$. To estimate $I(x)$ when $|x|$ is large, let us write $x=te$ with $|e|=1$ and make the substitution $y=tz$ in (4.9) (with $\beta=1$). We obtain

$$I(te)=t^{-\alpha}\int_{\mathbb{R}^3}\frac{dz}{|e-z|^2(t^{-1}+|z|)^{\alpha+1}}\leqslant t^{-\alpha}\int_{\mathbb{R}^3}\frac{dz}{|e-z|^2|z|^{\alpha+1}}.\tag{4.10}$$

Since we assume $\alpha\in(1,2)$, the last integral is bounded, and we see that

$$t^\alpha I(te)\leqslant c(\alpha).\tag{4.11}$$

Combining this estimate with the estimate of $I(x)$ for $|x|\leqslant 1$ we see that

$$\bigl(1+|x|\bigr)^\alpha I(x)\leqslant c(\alpha)\quad\text{for all } x\in\mathbb{R}^3.\tag{4.12}$$

This completes the proof of estimates (4.7) and (4.8). □

We have shown that the linear operator

$$T_{\tilde U}:v\to G*\operatorname{div}(\tilde U\otimes v+v\otimes\tilde U)$$

is continuous from X_α to X_α, and its norm is bounded by $c(\alpha,M)|b|$. Also, we have shown that the bi-linear operator

$$B:(v,w)\to G*\operatorname{div}(v\otimes w)$$

is continuous from $X_\alpha\times X_\alpha\to X_\alpha$, with the bound

$$\bigl[B(v,w)\bigr]_\alpha\leqslant c(\alpha)[v]_\alpha[w]_\alpha.$$

We let $V=G*(f-\tilde F)$ and re-write Eq. (4.3) as

$$v+T_{\tilde U}(v)+B(v,v)=V.\tag{4.13}$$

Since $\int_{\mathbb{R}^3}(f-\tilde F)=0$, we have $V=O(|x|^{-2})$ as $|x|\to\infty$. Standard perturbation arguments (such as the Implicit Function Theorem) now imply that Eq. (4.13) has a solution v when V is sufficiently small in X_α. (A simple sufficient condition for that is that, in addition to $\int_{\mathbb{R}^3}(f-\tilde F)=0$ and the restriction on the support on $f-\tilde F$, the field $f-\tilde F$ be small in $L^{\frac{3}{2}+\delta}$ with some $\delta>0$.) Moreover, the solution is unique in some small ball in X_α (centered at the origin). These statements can be made more quantitative if we use the special form of the perturbation (namely that it is quadratic in v). For example, one can use the following folklore lemma:

Lemma 2. *Let X be a Banach space. Let $T:X\to X$ be linear with $\|Tx\|\leqslant\varepsilon\|x\|$ for all $x\in X$, and let $B:X\times X\to X$ be bilinear with $\|B(x_1,x_2)\|\leqslant c\|x_1\|\|x_2\|$ for all $x_1,x_2\in X$. Let $y\in X$ with $\|y\|<\frac{(1-\varepsilon)^2}{4c}$. Let $0<\xi_1<\xi_2$ be the two roots of the equation $\xi=\|y\|+\varepsilon\xi+c\xi^2$, i.e. $\xi_{1,2}=\frac{(1-\varepsilon)\mp\sqrt{(1-\varepsilon)^2-4c\|y\|}}{2c}$. Then the equation*

$$x+Tx+B(x,x)=y\tag{4.14}$$

has a solution $\bar x$ satisfying $\|\bar x\|\leqslant\xi_1$. Moreover, the solution $\bar x$ is unique in the open ball $\{x\in X,\ \|x\|<\xi_2\}$.

Proof. The proof is standard and we include it for the convenience of the reader. Consider the map $F(x)=y-Tx-B(x,x)$. We have $\|F(x)\|\leqslant\|y\|+\varepsilon\|x\|+c\|x\|^2$ which shows that for $\xi_1<\|x\|<\xi_2$ we have $\|F(x)\|<\|x\|$ and that, for any $\delta>0$, the iterates $F(x), F^2(x)=F(F(x)),\dots,F^k(x),\dots$ enter the ball of radius $\xi_1+\delta$ after finitely many steps. At the same time, we have $\|F(x_1)-F(x_2)\|\leqslant\varepsilon\|x_1-x_2\|+c\|x_1-x_2\|(\|x_1\|+\|x_2\|)$ which shows that F is a contraction of any closed ball of radius $\xi\in[\xi_1,\frac{\xi_1+\xi_2}{2})$. □

Proof of Theorem 1. Let us first assume that the "no outflow to infinity" condition (1.11) is satisfied. In this case the statement of Theorem 1 is a direct consequence of the construction of the extensions in Section 2 and Lemmata 1 and 2. Note that we not only need existence and uniqueness for v in (4.3), but we also need uniqueness (with smallness assumptions) for u in (4.1). The uniqueness of u in our situation is well known (see e.g. [6] or [7]), and can also be easily proved from Lemma 2 and (an obvious modification of) Lemma 1.

The situation when we have some outflow to infinity can be handled by a standard method of using the canonical outflow field $\frac{x}{|x|^3}$, see for example [6, Section 2.2], [7, Chapter IX], or [11, Remark 3.2]. Assume $R_0 = 1$ without loss of generality. Let a be the multiple of the vector field $\frac{x}{|x|^3}$ which has the same outflow as u. Note that a satisfies the Navier–Stokes equation (1.1) in $\mathbb{R}^3 \setminus \{0\}$ with the associated pressure field $\pi_a = -\frac{1}{2}|a|^2$. Let us write $u = a + w$ and $p = \pi_a + p_w$. The field w satisfies the no outflow condition, and we can extend it to a div-free field $\tilde{w}$ with the control similar to (2.9). We can also regularize a and π_a in B_1 (while not changing them outside B_1) so that estimates similar to (2.9) and (2.10) are satisfied. Let us denote be $\tilde{a}$ and $\tilde{\pi}_a$ these regularized functions. Finally, we extend p_w to $\tilde{p}_w$ with control similar to (2.10). Let $\tilde{u} = \tilde{a} + \tilde{w}$ and $\tilde{p} = \tilde{\pi}_a + \tilde{p}_w$. (Note that $\tilde{u}$ is not div-free in B_1.) Let $\tilde{f} = \operatorname{div} \tilde{T}$, where $\tilde{T} = \tilde{T}(\tilde{u}, \tilde{p})$ is given by (1.5) with u, p replaced by $\tilde{u}, \tilde{p}$. We note that the vector b given by (1.6) can also be expressed as $b = \int_{\mathbb{R}^3} \tilde{f}$. We will now search a div-free vector field z and a function p_z satisfying $\operatorname{div} \tilde{T}(\tilde{a} + z, \tilde{\pi}_a + p_z) = \tilde{f}$. We seek z, p_z in the form $z = \tilde{U}^b + v$, $p_z = \tilde{P}^b + q$, where $\tilde{U}^b, \tilde{P}^b$ are the regularizations of the Landau solutions U^b, P^b constructed in Section 3. It is now easy to check that the perturbation theory of Section 4 gives the required solution. □

Remark 2. The borderline space X_α in which a more sophisticated perturbation analysis might possibly work is the space X_2. (This corresponds to the naturally expected decay $O(|x|^{-2})$ for v.) However, a perturbation analysis in X_2 cannot be based only on the decay properties of $\tilde{U}$ (as was the case with our simpler analysis for $\alpha < 2$). To see this, let $\varepsilon \in (0, 1)$ and consider the equation

$$-\Delta u + \frac{\varepsilon(n-\varepsilon)}{1-\varepsilon} \operatorname{div}\left(\frac{x}{|x|^2} u\right) = f(x) \quad \text{in } \mathbb{R}^n. \tag{4.15}$$

We can think of this equation as an analogue of the linearization of (4.3) at $v = 0$, see also Eq. (4.16) below. A direct calculation shows that when $f = 0$ the function $x_1|x|^{-n+\varepsilon}$ is a solution of (4.15) away from the origin. Let η be a smooth function in $\mathbb{R}^n$ which vanishes in the unit ball and is equal to 1 outside of the ball of radius 2. An easy calculation shows that the function $u = \eta x_1 |x|^{-n+\varepsilon}$ satisfies (4.15) with $\int_{\mathbb{R}^3} f = 0$. Moreover, one can change the coefficients of the equation in the unit ball so that they become smooth. For $n = 3$ and $0 < \varepsilon < 1$, the decay of these solution for $x \to \infty$ is slower than $O(|x|^{-2})$.

To get results in the space X_2, one would probably have to prove optimal decay estimates for the linear equation

$$\left.\begin{array}{l} -\Delta v + \tilde{U}\nabla v + v\nabla\tilde{U} + \nabla q = f - \tilde{F}, \\ \operatorname{div} v = 0 \end{array}\right\} \quad \text{in } \mathbb{R}^3 \tag{4.16}$$

by a non-perturbative approach, and then treat the quadratic term in (4.3) perturbatively. The above example shows that to get the optimal decay $O(|x|^{-2})$, one would need to use more information about $\tilde{U}$ than just its decay properties at ∞. We conjecture that for large $|x|$ the perturbation v from the Landau solution U^b indeed has the decay $v(x) = O(|x|^{-2})$, at least for small data.

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