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On asymptotic stability in energy space of ground states of NLS in 2D

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Abstract

We transpose work by K. Yajima and by T. Mizumachi to prove dispersive and smoothing estimates for dispersive solutions of the linearization at a ground state of a Nonlinear Schrödinger equation (NLS) in 2D. As an application we extend to dimension 2D a result on asymptotic stability of ground states of NLS proved in the literature for all dimensions different from 2.
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Résumé

On utilise les travaux de K. Yajima et T. Mizumachi pour prouver des estimations dispersives et régularisantes des solutions de l'équation linéarisée aux états fondamentaux de NLS in 2D. On applique ces résultats pour obtenir des extensions en dimension 2D de la stabilité asymptotique prouvée en littérature pour toutes les dimensions différentes de 2.
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1. Introduction

We consider even solutions of a NLS

$$iu_t + \Delta u + \beta(|u|^2)u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^2, \qquad u(0,x) = u_0(x). \tag{1.1}$$

We assume:

(H1) $\beta(0) = 0$, $\beta \in C^\infty(\mathbb{R},\mathbb{R})$;
(H2) there exists a $p_0 \in (1,\infty)$ such that for every $k = 0, 1$,

$$\left|\frac{d^k}{dv^k}\beta(v^2)\right| \lesssim |v|^{p_0-k-1} \quad \text{if } |v| \geqslant 1;$$

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(H3) there exists an open interval $\mathcal{O}$ such that $\Delta u - \omega u + \beta(u^2)u = 0$ admits a C^1-family of ground states $\phi_\omega(x)$ for $\omega \in \mathcal{O}$;

(H4) $\frac{d}{d\omega}\|\phi_\omega\|^2_{L^2(\mathbb{R})} > 0$ for $\omega \in \mathcal{O}$;

(H5) Let $L_+ = -\Delta + \omega - \beta(\phi_\omega^2) - 2\beta'(\phi_\omega^2)\phi_\omega^2$ be the operator whose domain is $H^2_{\mathrm{rad}}(\mathbb{R}^2)$. We assume that L_+ has exactly one negative eigenvalue and that it has no (radial) kernel.
By [27] the $\omega \to \phi_\omega \in H^1(\mathbb{R}^2)$ is C^2 and by [38,13,14] (H4)–(H5) yields orbital stability of the ground state $e^{i\omega t}\phi_\omega(x)$. Here we investigate asymptotic stability. We need some additional hypotheses.

(H6) For any $x \in \mathbb{R}$, $u_0(x) = u_0(-x)$. That is, the initial data u_0 of (1.1) are even.
Consider the Pauli matrices σ_j and the linearization H_ω given by:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix};$$

$$H_\omega = \sigma_3\bigl[-\Delta + \omega - \beta\bigl(\phi_\omega^2\bigr) - \beta'\bigl(\phi_\omega^2\bigr)\phi_\omega^2\bigr] + i\beta'\bigl(\phi_\omega^2\bigr)\phi_\omega^2\sigma_2. \tag{1.2}$$

Then we assume:

(H7) Let H_ω be the linearized operator around $e^{it\omega}\phi_\omega$, see (1.2). H_ω has a positive simple eigenvalue $\lambda(\omega)$ for $\omega \in \mathcal{O}$ whose corresponding eigenfunctions are even functions. There exists an $N \in \mathbb{N}$ such that $N\lambda(\omega) < \omega < (N+1)\lambda(\omega)$.

(H8) The Fermi Golden Rule (FGR) holds (see Hypothesis 4.2 in Section 4).

(H9) The point spectrum of H_ω consists of 0 and $\pm\lambda(\omega)$. The points $\pm\omega$ are not resonances.

Then we prove:

Theorem 1.1. *Let $\omega_0 \in \mathcal{O}$ and $\phi_{\omega_0}(x)$ be a ground state in a family of ground states ϕ_ω. Let $u(t,x)$ be a solution to* (1.1). *Assume* (H1)–(H9). *In particular assume the* (*FGR*) *in Hypothesis* 4.2. *Then, there exist an $\epsilon_0 > 0$ and a $C > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ and for any u_0 with $\|u_0 - e^{i\gamma_0}\phi_{\omega_0}\|_{H^1} < \epsilon$, there exist $\omega_+ \in \mathcal{O}$, $\theta \in C^1(\mathbb{R};\mathbb{R})$, $\|h_\infty\|_{H^1} \leqslant C\epsilon$ and $|\omega_+ - \omega_0| \leqslant C\epsilon^2$ such that*

$$\lim_{t\to+\infty} \bigl\|u(t,\cdot) - e^{i\theta(t)}\phi_{\omega_+} - e^{it\Delta}h_\infty\bigr\|_{H^1} = 0.$$

Theorem 1.1 is the two dimensional version of Theorem 1.1 [10]. The one dimensional version is in [7]. We recall that results of the sort discussed here were pioneered by Soffer and Weinstein [29], see also [24], followed by Buslaev and Perelman [3,4], about 15 years ago. In this decade these early works were followed by a number of results [5,8, 9,15,21–23,25,29–31,33–36]. It was heuristically understood that the rate of the leaking of energy from the so called "internal modes" into radiation, is small and decreasing when N increases, producing technical difficulties in the closure of the nonlinear estimates. For this reason prior to Gang Zhou and Sigal [12], the literature treated only the case when $N = 1$ in (H6). [12] sheds light for $N > 1$. The results in [12] deal with all spatial dimensions different from 2 under the so called Fermi Golden Rule (FGR) hypothesis. [10,7] strengthen [12] by considering initial data in H^1, by showing that the (FGR) hypothesis is a consequence of what looks generic condition, Hypothesis 4.2 below, if (H8) is assumed. [10] treats also the case when there are many eigenvalues and Hypothesis 4.2 is replaced by a more stringent hypothesis which is a natural generalization of the (FGR) hypothesis in [12]. The same result with many eigenvalues case can be proved also here and in [7], but we skip for simplicity the proof. We recall that Mizumachi [21], resp. [22], extends to dimension 1, resp. 2, the results in [15] valid for small solitons obtained by bifurcation from ground states of a linear equation, while [20] extends in 2D the result in [30]. [7] transposes [21] to the case of large solitons, with the generalizations contained in [10]. Here we consider the case of dimension 2. Thanks to the work by [22], it is quite clear how to transpose to dimension 2 the higher dimensional arguments in [10]. The nonlinear arguments in [10] are not sensitive to the dimension except for the lack in 2D of the endpoint Strichartz estimate. Mizumachi [22] shows how to replace it with an appropriate smoothing estimate of Kato type. The estimate and its proof are suggested by [22]. In order to complete the proof of Theorem 1.1 we need some dispersive estimates on the linearization H_ω which in spatial dimension 2 are not yet proved in the literature. The main technical task of this paper is the transposition to H_ω of the proof of L^p boundedness of wave operators of Schrödinger operators in dimension 2 due to Yajima [40]. We use the following notation. We set $H_0(\omega) = \sigma_3(-\Delta + \omega)$; given normed spaces

X and Y we denote by $B(X,Y)$ the space of operators from X to Y and given $L \in B(X,Y)$ we denote by $\|L\|_{X,Y}$ or by $\|L\|_{B(X,Y)}$ its norm. We prove:

Proposition 1.2. *Assume the hypotheses of Theorem* 1.1. *The following limits are well defined isomorphism, inverse of each other*:

$$Wu = \lim_{t\to+\infty} e^{itH_\omega} e^{-itH_0(\omega)} u \quad \textit{for any } u \in L^2,$$
$$Zu = \lim_{t\to+\infty} e^{itH_0(\omega)} e^{-itH_\omega} \quad \textit{for any } u \in L^2_c(H_\omega) \text{ (defined in Section 2).}$$

For any $p \in (1,\infty)$ and any k the restrictions of W and Z to $L^2 \cap W^{k,p}$ extend into operators such that for $C(\omega) < \infty$ semicontinuous in ω

$$\|W\|_{W^{k,p}(\mathbb{R}^2), W^{k,p}_c(H_\omega)} + \|Z\|_{W^{k,p}_c(H_\omega), W^{k,p}(\mathbb{R}^2)} < C(\omega)$$

with $W^{k,p}_c(H_\omega)$ the closure in $W^{k,p}(\mathbb{R}^2)$ of $W^{k,p}(\mathbb{R}^2) \cap L^2_c(H_\omega)$.

We will set $L^{2,s}$ and $H^{m,s}$

$$\|u\|_{L^{2,s}} = \|\langle x\rangle^s u\|_{L^2(\mathbb{R}^2)} \quad \text{and} \quad \|u\|_{H^{m,s}} = \|\langle x\rangle^s u\|_{H^m(\mathbb{R}^2)},$$

where $m \in \mathbb{N}$, $s \in \mathbb{R}$ and $\langle x\rangle = (1+|x|^2)^{1/2}$. For $f(x)$ and $g(x)$ column vectors, their inner product is $\langle f,g\rangle = \int_{\mathbb{R}^2} {}^t f(x)\cdot \overline{g(x)}\,dx$. The adjoint H^* is defined by $\langle Hf,g\rangle = \langle f,H^*g\rangle$. Given an operator H, its resolvent is $R_H(z) = (H-z)^{-1}$. We will write $R_0(z) = (-\Delta - z)^{-1}$. We write $\|g(t,x)\|_{L^p_t L^q_x} = \|\|g(t,x)\|_{L^q_x}\|_{L^p_t}$ and $\|g(t,x)\|_{L^p_t L^{2,s}_x} = \|\|g(t,x)\|_{L^{2,s}_x}\|_{L^p_t}$.

2. Linearization, modulation and set up

We will use the following classical result, [38,13,14], see also [7]:

Theorem 2.1. *Suppose that $e^{i\omega t}\phi_\omega(x)$ satisfies* (H4). *Then $\exists \epsilon > 0$ and a $A_0(\omega) > 0$ such that for any $\|u(0,x) - \phi_\omega\|_{H^1} < \epsilon$ we have for the corresponding solution* $\inf\{\|u(t,x) - e^{i\gamma}\phi_\omega(x-x_0)\|_{H^1(x\in\mathbb{R}^2)}\colon \gamma \in \mathbb{R}$ *and* $x_0 \in \mathbb{R}^2\} < A_0(\omega)\epsilon$.

We can write the ansatz $u(t,x) = e^{i\Theta(t)}(\phi_{\omega(t)}(x) + r(t,x))$, $\Theta(t) = \int_0^t \omega(s)\,ds + \gamma(t)$. Inserting the ansatz into the equation we get

$$\begin{aligned} ir_t = {} & -\Delta r + \omega(t) r - \beta\big(\phi^2_{\omega(t)}\big) r - \beta'\big(\phi^2_{\omega(t)}\big)\phi^2_{\omega(t)} r - \beta'\big(\phi^2_{\omega(t)}\big)\phi^2_{\omega(t)}\bar r + \dot\gamma(t)\phi_{\omega(t)} \\ & - i\dot\omega(t)\partial_\omega \phi_{\omega(t)} + \dot\gamma(t) r + \mathrm{O}\big(r^2\big). \end{aligned}$$

We set ${}^tR = (r,\bar r)$, ${}^t\Phi = (\phi_\omega,\phi_\omega)$ and we rewrite the above equation as

$$iR_t = H_\omega R + \sigma_3\dot\gamma R + \sigma_3\dot\gamma\Phi - i\dot\omega\partial_\omega\Phi + \mathrm{O}\big(R^2\big). \tag{2.1}$$

Set $H_0(\omega) = \sigma_3(-\Delta + \omega)$ and $V(\omega) = H_\omega - H_0(\omega)$. The essential spectrum is

$$\sigma_e = \sigma_e(H_\omega) = \sigma_e\big(H_0(\omega)\big) = (-\infty,-\omega] \cup [\omega,+\infty),$$

0 is an isolated eigenvalue. Given an operator L we set $N_g(L) = \bigcup_{j\geqslant 1}\ker(L^j)$. [37] implies that, if $\{\cdot\}$ means span, $N_g(H^*_\omega) = \{\Phi, \sigma_3\partial_\omega\Phi\}$. $\lambda(\omega)$ has corresponding real eigenvector $\xi(\omega)$, which can be normalized so that $\langle \xi, \sigma_3\xi\rangle = 1$. $\sigma_1\xi(\omega)$ generates $\ker(H_\omega + \lambda(\omega))$. The function $(\omega,x) \in \mathcal{O}\times\mathbb{R} \to \xi(\omega,x)$ is C^2; $|\xi(\omega,x)| < ce^{-a|x|}$ for fixed $c > 0$ and $a > 0$ if $\omega \in K \subset \mathcal{O}$, K compact. $\xi(\omega,x)$ is even in x since by assumption we are restricting ourselves in the category of such functions. We have the H_ω invariant Jordan block decomposition

$$L^2 = N_g(H_\omega) \oplus \Big(\bigoplus_{j,\pm} \ker\big(H_\omega \mp \lambda(\omega)\big)\Big) \oplus L^2_c(H_\omega) = N_g(H_\omega) \oplus N_g^\perp\big(H^*_\omega\big)$$

where we set $L_c^2(H_\omega) = \{N_g(H_\omega^*) \oplus \bigoplus_\pm \ker(H_\omega^* \mp \lambda(\omega))\}^\perp$. We can impose

$$R(t) = (z\xi + \bar{z}\sigma_1\xi) + f(t) \in \left[\sum_\pm \ker\big(H_{\omega(t)} \mp \lambda\big(\omega(t)\big)\big)\right] \oplus L_c^2(H_{\omega(t)}). \tag{2.2}$$

The following claim admits an elementary proof which we skip:

Lemma 2.2. *There is a Taylor expansion at $R = 0$ of the nonlinearity $\mathrm{O}(R^2)$ in (2.1) with $R_{m,n}(\omega, x)$ and $A_{m,n}(\omega, x)$ real vectors and matrices rapidly decreasing in x:*

$$\mathrm{O}\big(R^2\big) = \sum_{2\leqslant m+n\leqslant 2N+1} R_{m,n}(\omega) z^m \bar{z}^n + \sum_{1\leqslant m+n\leqslant N} z^m \bar{z}^n A_{m,n}(\omega) f + \mathrm{O}\big(f^2 + |z|^{2N+2}\big).$$

In terms of the frame in (2.2) and the expansion in Lemma 2.2, (2.1) becomes

$$\begin{aligned} i f_t &= (H_{\omega(t)} + \sigma_3\dot{\gamma}) f + \sigma_3\dot{\gamma}\Phi(\omega) - i\dot{\omega}\partial_\omega\Phi(t) + \big(z\lambda(\omega) - i\dot{z}\big)\xi(\omega) \\ &\quad - \big(\bar{z}\lambda(\omega) + i\dot{\bar{z}}\big)\sigma_1\xi(\omega) + \sigma_3\dot{\gamma}(z\xi + \bar{z}\sigma_1\xi) - i\dot{\omega}(z\partial_\omega\xi + \bar{z}\sigma_1\partial_\omega\xi) \\ &\quad + \sum_{2\leqslant m+n\leqslant 2N+1} z^m\bar{z}^n R_{m,n}(\omega) + \sum_{1\leqslant m+n\leqslant N} z^m\bar{z}^n A_{m,n}(\omega) f + \mathrm{O}\big(f^2\big) + \mathrm{O}_{\mathrm{loc}}\big(\big|z^{2N+2}\big|\big) \end{aligned} \tag{2.3}$$

where by $\mathrm{O}_{\mathrm{loc}}$ we mean that the there is a factor $\chi(x)$ rapidly decaying to 0 as $|x| \to \infty$. By taking inner product of the equation with generators of $N_g(H_\omega^*)$ and $\ker(H_\omega^* - \lambda)$ we obtain modulation and discrete modes equations:

$$\begin{aligned} i\dot{\omega}\frac{d\|\phi_\omega\|_2^2}{d\omega} &= \Bigg\langle \sigma_3\dot{\gamma}(z\xi + \bar{z}\sigma_1\xi) - i\dot{\omega}(z\partial_\omega\xi + \bar{z}\sigma_1\partial_\omega\xi) + \sum_{m+n=2}^{2N+1} z^m\bar{z}^n R_{m,n}(\omega) \\ &\qquad + \Bigg(\sigma_3\dot{\gamma} + i\dot{\omega}\partial_\omega P_c + \sum_{m+n=1}^{N} z^m\bar{z}^n A_{m,n}(\omega)\Bigg) f + \mathrm{O}\big(f^2\big) + \mathrm{O}_{\mathrm{loc}}\big(\big|z^{2N+2}\big|\big), \Phi\Bigg\rangle, \\ \dot{\gamma}\frac{d\|\phi_\omega\|_2^2}{d\omega} &= \langle \text{same as above }, \sigma_3\partial_\omega\Phi\rangle, \\ i\dot{z} - \lambda(\omega)z &= \langle \text{same as above }, \sigma_3\xi\rangle. \end{aligned} \tag{2.4}$$

3. Spacetime estimates for H_ω

We need analogues of Lemmas 2.1–2.3 and Corollary 2.1 in [22]. We call admissible all pairs (p, q) with $1/p = 1/2 - 1/q$ and $2 \leqslant q < \infty$. We set $(p', q') = (p/(p-1), q/(q-1)))$. In the lemmas below we assume that the H_ω of the form (1.2) for which hypotheses (H3)–(H5), (H7) and (H9) hold.

Lemma 3.1 *(Strichartz estimate). There exists a positive number $C = C(\omega)$ upper semicontinuous in ω such that for any $k \in [0, 2]$:*

(a) *for any $f \in L_c^2(\omega)$ and any admissible all pairs (p, q),*

$$\big\|e^{-itH_\omega} f\big\|_{L_t^p W_x^{k,q}} \leqslant C\|f\|_{H^k};$$

(b) *for any $g(t, x) \in S(\mathbb{R}^2)$ and any couple of admissible pairs (p_1, q_1) (p_2, q_2) we have*

$$\left\| \int_0^t e^{-i(t-s)H_\omega} P_c(\omega) g(s, \cdot)\, ds \right\|_{L_t^{p_1} W_x^{k,q_1}} \leqslant C\|g\|_{L_t^{p_2'} W_x^{k,q_2'}}.$$

Lemma 3.1 follows immediately from Proposition 1.2 since W and Z intertwine $e^{-itH_\omega} P_c(H_\omega)$ and e^{-itH_0}.

Lemma 3.2. *Let $s>1$. $\exists C=C(\omega)$ upper semicontinuous in ω such that*:

(a) *for any $f \in S(\mathbb{R}^2)$*,
$$\left\| e^{-itH_\omega} P_c(\omega) f \right\|_{L_t^2 L_x^{2,-s}} \leqslant C \|f\|_{L^2};$$
(b) *for any $g(t,x) \in S(\mathbb{R}^2)$*
$$\left\| \int_{\mathbb{R}} e^{itH_\omega} P_c(\omega) g(t,\cdot)\, dt \right\|_{L_x^2} \leqslant C \|g\|_{L_t^2 L_x^{2,s}}.$$

Notice that (b) follows from (a) by duality.

Lemma 3.3. *Let $s>1$. $\exists C=C(\omega)$ as above such that $\forall g(t,x) \in S(\mathbb{R}^2)$ and $t \in \mathbb{R}$*:
$$\left\| \int_0^t e^{-i(t-s)H_\omega} P_c(\omega) g(s,\cdot)\, ds \right\|_{L_t^2 L_x^{2,-s}} \leqslant C \|g\|_{L_t^2 L_x^{2,s}}.$$

As a corollary from Christ and Kiselev [6], Lemmas 3.2 and 3.3 imply:

Lemma 3.4. *Let (p,q) be an admissible pair and let $s>1$. $\exists C=C(\omega)$ as above such that $\forall g(t,x) \in S(\mathbb{R}^2)$ and $t \in \mathbb{R}$*:
$$\left\| \int_0^t e^{-i(t-s)H_\omega} P(\omega) g(s,\cdot)\, ds \right\|_{L_t^p L_x^q} \leqslant C \|g\|_{L_t^2 L_x^{2,s}}.$$

Lemma 3.5. *Consider the diagonal matrices $E_+ = \mathrm{diag}(1,0)$, $E_- = \mathrm{diag}(0,1)$. Set $P_\pm(\omega) = Z(\omega) E_\pm W(\omega)$ with $Z(\omega)$ and $W(\omega)$ the wave operators associated to H_ω. Then we have for $u \in L_c^2(H_\omega)$*
$$P_+(\omega)u = \lim_{\epsilon\to 0^+} \frac{1}{2\pi i} \lim_{M\to+\infty} \int_\omega^M \left[R_{H_\omega}(\lambda+i\epsilon) - R_{H_\omega}(\lambda-i\epsilon) \right] u\, d\lambda,$$
$$P_-(\omega)u = \lim_{\epsilon\to 0^+} \frac{1}{2\pi i} \lim_{M\to+\infty} \int_{-M}^{-\omega} \left[R_{H_\omega}(\lambda+i\epsilon) - R_{H_\omega}(\lambda-i\epsilon) \right] u\, d\lambda \tag{1}$$
and for any s_1 and s_2 and for $C=C(s_1,s_2,\omega)$ upper semicontinuous in ω, we have
$$\left\| \left(P_+(\omega) - P_-(\omega) - P_c(\omega)\sigma_3 \right) f \right\|_{L^{2,s_1}} \leqslant C \|f\|_{L^{2,s_2}}. \tag{2}$$

Proof. Formulas (1) hold with $P_\pm(\omega)$ replaced by $E_\pm$ and H_ω replaced by H_0 and for any $u \in L^2(\mathbb{R}^2)$. Applying $W(\omega)$ we get (1) for H_ω. Estimate (2) follows by the proof of inequality (3) in Lemma 5.12 [7] which is valid for all dimensions. □

4. Proof of Theorem 1.1

We restate Theorem 1.1 in a more precise form:

Theorem 4.1. *Under the assumptions of Theorem 1.1 we can express*
$$u(t,x) = e^{i\Theta(t)} \left(\phi_{\omega(t)}(x) + \sum_{j=1}^{2N} p_j(z,\bar z) A_j\big(x,\omega(t)\big) + h(t,x) \right)$$

with $p_j(z,\bar z) = \mathrm{O}(z)$ *near 0, with* $\lim_{t\to+\infty}\omega(t)$ *convergent, with* $|A_j(x,\omega(t))| \leqslant Ce^{-a|x|}$ *for fixed* $C>0$ *and* $a>0$, $\lim_{t\to+\infty} z(t) = 0$, *and for fixed* $C>0$

$$\|z(t)\|_{L_t^{2N+2}}^{N+1} + \|h(t,x)\|_{L_t^\infty H_x^1 \cap L_t^3 W_x^{1,6}} < C\epsilon. \tag{1}$$

Furthermore, there exists $h_\infty \in H^1(\mathbb{R},\mathbb{C})$ *such that*

$$\lim_{t\to\infty} \left\| e^{i\int_0^t \omega(s)\,ds + i\gamma(t)} h(t) - e^{it\Delta} h_\infty \right\|_{H^1} = 0. \tag{2}$$

The proof of Theorem 4.1 consists in a normal forms expansion and in the closure of some nonlinear estimates. The normal forms expansion is exactly the same of [10,7], in turn adaptations of [12].

4.1. Normal form expansion

We repeat [10]. We pick $k = 1, 2, \dots, N$ and set $f = f_k$ for $k = 1$. The other f_k are defined below. In the ODE's there will be error terms of the form

$$E_{\mathrm{ODE}}(k) = \mathrm{O}\big(|z|^{2N+2}\big) + \mathrm{O}\big(z^{N+1} f_k\big) + \mathrm{O}\big(f_k^2\big) + \mathrm{O}\big(\beta\big(|f_k|^2\big) f_k\big).$$

In the PDE's there will be error terms of the form

$$E_{\mathrm{PDE}}(k) = \mathrm{O}_{\mathrm{loc}}\big(|z|^{N+2}\big) + \mathrm{O}_{\mathrm{loc}}(z f_k) + \mathrm{O}_{\mathrm{loc}}\big(f_k^2\big) + \mathrm{O}\Big(\beta\Big(|f_k|^2\Big) f_k\Big).$$

In the right-hand sides of Eqs. (2.3)–(2.4) we substitute $\dot\gamma$ and $\dot\omega$ using the modulation equations. We repeat the procedure a sufficient number of times until we can write for $k=1$ and $f_1 = f$

$$\begin{aligned}
&i\dot\omega \frac{d\|\phi_\omega\|_2^2}{d\omega} = \left\langle \sum_{m+n=2}^{2N+1} z^m \bar z^n \Lambda_{m,n}^{(k)}(\omega) + \sum_{m+n=1}^{N} z^m \bar z^n A_{m,n}^{(k)}(\omega) f_k + E_{\mathrm{ODE}}(k), \Phi(\omega) \right\rangle,\\
&i\dot z - \lambda z = \big\langle \text{same as above }, \sigma_3 \xi(\omega)\big\rangle,\\
&i\partial_t f_k = (H_\omega + \sigma_3\dot\gamma) f_k + E_{\mathrm{PDE}}(k) + \sum_{k+1\leqslant m+n\leqslant N+1} z^m \bar z^n R_{m,n}^{(k)}(\omega),
\end{aligned}$$

with $A_{m,n}^{(k)}$, $R_{m,n}^{(k)}$ and $\Lambda_{m,n}^{(k)}(\omega,x)$ real exponentially decreasing to 0 for $|x|\to\infty$ and continuous in (ω,x). Exploiting $|(m-n)\lambda(\omega)| < \omega$ for $m+n\leqslant N$, $m\geqslant 0$, $n\geqslant 0$, we define inductively f_k with $k\leqslant N$ by

$$f_{k-1} = -\sum_{m+n=k} z^m \bar z^n R_{H_\omega}\big((m-n)\lambda(\omega)\big) R_{m,n}^{(k-1)}(\omega) + f_k.$$

Notice that if $R_{m,n}^{(k-1)}(\omega,x)$ is real exponentially decreasing to 0 for $|x|\to\infty$, the same is true for $R_{H_\omega}((m-n)\lambda(\omega))R_{m,n}^{(k-1)}(\omega)$ by $|(m-n)\lambda(\omega)| < \omega$. By induction f_k solves the above equation with the above notifications. Now we manipulate the equation for f_N. We fix $\omega_1 = \omega(0)$. We write

$$\begin{aligned}
&i\partial_t P_c(\omega_1) f_N - \big\{H_{\omega_1} + (\dot\gamma + \omega - \omega_1)\big(P_+(\omega_1) - P_-(\omega_1)\big)\big\} P_c(\omega_1) f_N\\
&\quad = +P_c(\omega_1)\widetilde{E}_{\mathrm{PDE}}(N) + \sum_{m+n=N+1} z^m \bar z^n P_c(\omega_1) R_{m,n}^{(N)}(\omega_1)
\end{aligned} \tag{4.1}$$

where we split $P_c(\omega_1) = P_+(\omega_1) + P_-(\omega_1)$ with $P_\pm(\omega_1)$, see Lemma 3.5, where $P_+(\omega_1)$ are the projections in $\sigma_c(H_{\omega_1}) \cap \{\lambda\colon \pm\lambda \geqslant \omega_1\}$ and with

$$\begin{aligned}
&\widetilde{E}_{\mathrm{PDE}}(N) = E_{\mathrm{PDE}}(N) + \sum_{m+n=N+1} z^m \bar z^n \big(R_{m,n}^{(N)}(\omega) - R_{m,n}^{(N)}(\omega_1)\big) + \varphi(t,x) f_N,\\
&\varphi(t,x) := (\dot\gamma + \omega - \omega_1)\big(P_c(\omega_1)\sigma_3 - \big(P_+(\omega_1) - P_-(\omega_1)\big)\big) f_N + \big(V(\omega) - V(\omega_1)\big) f_N\\
&\qquad + (\dot\gamma + \omega - \omega_1)\big(P_c(\omega) - P_c(\omega_1)\big)\sigma_3 f_N.
\end{aligned} \tag{4.2}$$

By Lemma 3.5 for $C_N(\omega_1)$ upper semicontinuous in ω_0, $\forall N$ we have

$$\big\| \langle x\rangle^N \big(P_+(\omega_1) - P_-(\omega_1) - P_c(\omega_1)\sigma_3\big) f \big\|_{L^2_x} \leqslant C_N(\omega_1) \big\| \langle x\rangle^{-N} f \big\|_{L^2_x}. \tag{4.3}$$

The term $\varphi(t,x)$ in (4.2) can be treated as a small cutoff function. We write

$$f_N = -\sum_{m+n=N+1} z^m \bar z^n R_{H_{\omega_1}}\big((m-n)\lambda(\omega_1) + i0\big) P_c(\omega_1) R^{(N)}_{m,n}(\omega_1) + f_{N+1}. \tag{4.4}$$

Then

$$\begin{aligned} i\partial_t P_c(\omega_1) f_{N+1} &= \big(H_{\omega_1} + (\dot\gamma + \omega - \omega_1)\big(P_+(\omega_1) - P_-(\omega_1)\big)\big) P_c(\omega_1) f_{N+1} \\ &\quad + \sum_{\pm} \mathrm{O}\big(\epsilon |z|^{N+1}\big) R_{H_{\omega_1}}\big(\pm(N+1)\lambda(\omega_1) + i0\big) R_\pm(\omega_1) + P_c(\omega_1) \widehat{E}_{\mathrm{PDE}}(N) \end{aligned} \tag{4.5}$$

with $R_+ = R^{(N)}_{N+1,0}$ and $R_- = R^{(N)}_{0,N+1}$ and $\widehat{E}_{\mathrm{PDE}}(N) = \widetilde{E}_{\mathrm{PDE}}(N) + \mathrm{O}_{\mathrm{loc}}(\epsilon z^{N+1})$, where we have used that $(\omega - \omega_1) = \mathrm{O}(\epsilon)$ by Theorem 2.1. Notice that $R_{H_{\omega_0}}(\pm(N+1)\lambda(\omega_0) + i0) R_\pm(\omega_0) \in L^\infty$ do not decay spatially. In the ODE's with $k = N$, by the standard theory of normal forms and following the idea in Proposition 4.1 [5], see [10] for details, it is possible to introduce new unknowns

$$\begin{aligned} \tilde\omega &= \omega + q(\omega, z, \bar z) + \sum_{1\leqslant m+n\leqslant N} z^m \bar z^n \big\langle f_N, \alpha_{mn}(\omega)\big\rangle, \\ \tilde z &= z + p(\omega, z, \bar z) + \sum_{1\leqslant m+n\leqslant N} z^m \bar z^n \big\langle f_N, \beta_{mn}(\omega)\big\rangle, \end{aligned} \tag{4.6}$$

with $p(\omega, z, \bar z) = \sum p_{m,n}(\omega) z^m \bar z^n$ and $q(z, \bar z) = \sum q_{m,n}(\omega) z^m \bar z^n$ polynomials in $(z, \bar z)$ with real coefficients and $\mathrm{O}(|z|^2)$ near 0, such that we get

$$\begin{aligned} i\dot{\tilde\omega} &= \big\langle E_{\mathrm{PDE}}(N), \Phi\big\rangle, \\ i\dot{\tilde z} - \lambda(\omega)\tilde z &= \sum_{1\leqslant m\leqslant N} a_m(\omega) |\tilde z^m|^2 \tilde z + \big\langle E_{\mathrm{ODE}}(N), \sigma_3 \xi\big\rangle + \bar{\tilde z}^N \big\langle A^{(N)}_{0,N}(\omega) f_N, \sigma_3\xi\big\rangle \end{aligned} \tag{4.7}$$

with $a_m(\omega)$ real. Next step is to substitute f_N using (4.4). After eliminating by a new change of variables $\tilde z = \hat z + p(\omega, \hat z, \bar{\hat z})$ the resonant terms, with $p(\omega, \hat z, \bar{\hat z}) = \sum \hat p_{m,n}(\omega) z^m \bar z^n$ a polynomial in $(z, \bar z)$ with real coefficients $\mathrm{O}(|z|^2)$ near 0, we get

$$\begin{aligned} i\dot{\hat\omega} &= \big\langle E_{\mathrm{PDE}}(N), \Phi\big\rangle, \\ i\dot{\hat z} - \lambda(\omega)\hat z &= \sum_{1\leqslant m\leqslant N} \hat a_m(\omega) |\tilde z^m|^2 \hat z + \big\langle E_{\mathrm{ODE}}(N), \sigma_3\xi\big\rangle \\ &\quad - |\hat z^N|^2 \hat z \big\langle \hat A^{(N)}_{0,N}(\omega) R_{H_{\omega_0}}\big((N+1)\lambda(\omega_1) + i0\big) P_c(\omega_0) R^{(N)}_{N+1,0}(\omega_1), \sigma_3\xi\big\rangle \\ &\quad + \bar{\hat z}^N \big\langle \hat A^{(N)}_{0,N}(\omega) f_{N+1}, \sigma_3\xi\big\rangle \end{aligned} \tag{4.8}$$

with $\hat a_m$, $\hat A^{(N)}_{0,N}$ and $R^{(N)}_{N+1,0}$ real. By $\frac{1}{x-i0} = PV\frac{1}{x} + i\pi\delta_0(x)$ and by an elementary use of the wave operators, we can denote by $\Gamma(\omega, \omega_1)$ the quantity

$$\begin{aligned} \Gamma(\omega, \omega_1) &= \Im\big(\big\langle \hat A^{(N)}_{0,N}(\omega) R_{H_{\omega_1}}\big((N+1)\lambda(\omega_1) + i0\big) P_c(\omega_1) R^{(N)}_{N+1,0}(\omega_1)\sigma_3\xi(\omega)\big\rangle\big) \\ &= \pi\big\langle \hat A^{(N)}_{0,N}(\omega)\delta\big(H_{\omega_1} - (N+1)\lambda(\omega_1)\big) P_c(\omega_1) R^{(N)}_{N+1,0}(\omega_1)\sigma_3\xi(\omega)\big\rangle. \end{aligned}$$

Now we assume the following:

Hypothesis 4.2. There is a fixed constant $\Gamma > 0$ such that $|\Gamma(\omega, \omega)| > \Gamma$.

By continuity and by Hypothesis 4.2 we can assume $|\Gamma(\omega, \omega_1)| > \Gamma/2$. Then we write

$$\frac{d}{dt}\frac{|\hat z|^2}{2} = -\Gamma(\omega, \omega_1)|z|^{2N+2} + \Im\big(\big\langle \hat A^{(N)}_{0,N}(\omega) f_{N+1}, \sigma_3\xi(\omega)\big\rangle \bar{\hat z}^{N+1}\big) + \Im\big(\big\langle E_{\mathrm{ODE}}(N), \sigma_3\xi(\omega)\big\rangle \bar{\hat z}\big). \tag{4.9}$$

4.2. Nonlinear estimates

By an elementary continuation argument, the following a priori estimates imply inequality (1) in Theorem 4.1, so to prove (1) we focus on:

Lemma 4.3. *There are fixed constants C_0 and C_1 and $\epsilon_0 > 0$ such that for any $0 < \epsilon \leqslant \epsilon_0$ if we have*

$$\|\hat{z}\|^{N+1}_{L_t^{2N+2}} \leqslant 2C_0\epsilon \quad \text{and} \quad \|f_N\|_{L_t^\infty H_x^1 \cap L_t^3 W_x^{1,6} \cap L_t^{\frac{2p_0}{p_0-1}} W_x^{1,2p_0} \cap L_t^2 H^{1,-s}} \leqslant 2C_1\epsilon \tag{4.10}$$

then we obtain the improved inequalities

$$\|f_N\|_{L_t^\infty H_x^1 \cap L_t^3 W_x^{1,6} \cap L_t^{\frac{2p_0}{p_0-1}} W_x^{1,2p_0} \cap L_t^2 H^{1,-s}} \leqslant C_1\epsilon, \tag{4.11}$$

$$\|\hat{z}\|^{N+1}_{L_t^{2N+2}} \leqslant C_0\epsilon. \tag{4.12}$$

Proof. Set $\ell(t) := \gamma + \omega - \omega_1$. First of all, we have:

Lemma 4.4. *Let $g(0,x) \in H_x^1 \cap L_c^2(\omega_1)$ and let $\omega(t)$ be a continuous function. Consider $ig_t = \{H_{\omega_1} + \ell(t)(P_+(\omega_0) - P_-(\omega_0))\}g + P_c(\omega_1)F$. Then for a fixed $C = C(\omega_1, s)$ upper semicontinuous in ω_1 and $s > 1$ we have*

$$\|g\|_{L_t^\infty H_x^1 \cap L_t^3 W_x^{1,6} \cap L_t^{\frac{2p_0}{p_0-1}} W_x^{1,2p_0}} \leqslant C\big(\big\|g(0,x)\big\|_{H^1} + \|F\|_{L_t^1 H_x^1 + L_t^2 H_x^{1,s}}\big).$$

Lemma 4.4 follows easily from Lemmas 3.1–3.4 and

$$P_\pm(\omega_1)g(t) = e^{-itH_{\omega_1}} e^{-i\int_0^t \ell(\tau)\,d\tau} P_\pm(\omega_1)g(0) - i\int_0^t e^{-i(t-s)H_{\omega_1}} e^{\pm i\int_s^t \ell(\tau)\,d\tau} P_\pm(\omega_1)F(s)\,ds.$$

Lemma 4.5. *Consider Eq.* (4.1) *for f_N and assume* (4.10). *Then we can split $\widetilde{E}_{\mathrm{PDE}}(N) = X + \mathrm{O}(f_N^3) + \mathrm{O}(f_N^{p_0})$ such that $\|X\|_{L_t^2 H_x^{1,M}} \lesssim \epsilon^2$ for any fixed M and $\|\mathrm{O}(f_N^3) + \mathrm{O}(f_N^{p_0})\|_{L_t^1 H_x^1} \lesssim \epsilon^3$.*

Proof of Lemma 4.5. In the error terms for $k = N$ at the beginning of Section 4.1 we can write

$$\widetilde{E}_{\mathrm{PDE}}(N) = \mathrm{O}(\epsilon)\psi(x)f_N + \mathrm{O}_{\mathrm{loc}}\big(|z|^{N+2}\big) + \mathrm{O}_{\mathrm{loc}}(zf_N) + \mathrm{O}_{\mathrm{loc}}\big(f_N^2\big) + \mathrm{O}\big(f_N^3\big) + \mathrm{O}\big(f_N^{p_0}\big)$$

with $\psi(x)$ a rapidly decreasing function, p_0 the exponent in (H2) and with $\mathrm{O}(f_N^{p_0})$ relevant only for $p_0 > 3$. Denoting X the sum of all terms except the last one, setting $f = f_N$, by (4.10) we have:

(1) $\|\mathrm{O}(\epsilon)\psi(x)f\|_{L_t^2 H_x^{1,M}} \lesssim \epsilon\|f\|_{L_t^2 H_x^{1,-M}} \lesssim \epsilon^2$;
(2) $\|\mathrm{O}_{\mathrm{loc}}(zf)\|_{L_t^2 H_x^{1,M}} \lesssim \|z\|_\infty \|f\|_{L_t^2 H_x^{1,-M}} \lesssim \epsilon^2$;
(3) $\|\mathrm{O}_{\mathrm{loc}}(f^2)\|_{L_t^2 H_x^{1,M}} \lesssim \|f\|^2_{L_t^2 H_x^{1,-M}} \lesssim \epsilon^2$.

This yields $\|\langle x\rangle^M X\|_{H_x^1 L_t^2} \lesssim \epsilon^2$. To bound the remaining term observe:

(4) $\||f|^2 f\|_{L_t^1 H_x^1} \lesssim \|\|f\|_{W_x^{1,6}} \|f\|^2_{L_x^6}\|_{L_t^1} \leqslant \|f\|^3_{L_t^3 W_x^{1,6}} \lesssim \epsilon^3$;
(5) $\|\mathrm{O}(f^{p_0})\|_{L_t^1 H_x^1} \lesssim \|\|f\|_{W_x^{1,2p_0}} \|f\|^{p_0-1}_{L_x^{2p_0}}\|_{L_t^1} \leqslant \|f\|_{L_t^{\frac{2p_0}{p_0-1}} W_x^{1,2p_0}} \|f\|^{p_0-1}_{L_t^{2p_0\frac{p_0-1}{p_0+1}} W_x^{1,2p_0}} \lesssim \epsilon^{p_0}$, where in the last step we use $\|f\|_{L_t^{2p_0\frac{p_0-1}{p_0+1}} W_x^{1,2p_0}} \lesssim \|f\|^\alpha_{L_t^{\frac{2p_0}{p_0-1}} L_x^{2p_0}} \|f\|^{1-\alpha}_{L_t^\infty H_x^1}$ for some $0 < \alpha < 1$ by $p_0 > 3$, interpolation and Sobolev embedding. $\square$

Proof of (4.11). Recall that f_N satisfies Eq. (4.1) whose right-hand side is $P_c(\omega_1)\widetilde{E}_{\mathrm{PDE}}(N)+\mathrm{O}_{\mathrm{loc}}(z^{N+1})$. In addition to Lemma 4.5 we have the estimate $\|\mathrm{O}_{\mathrm{loc}}(z^{N+1})\|_{L_t^2H_x^{1,M}}\lesssim\|z\|^{N+1}_{L_t^{2N+1}}\lesssim 2C_0\epsilon$. So by Lemmas 3.1–3.4, for some fixed c_2 we get schematically

$$\|f_N\|_{L_t^\infty H_x^1\cap L_t^3W_x^{1,6}\cap L_t^{\frac{2p_0}{p_0-1}}W_x^{1,2p_0}}\leqslant 2c_2C_0\epsilon+\epsilon+\mathrm{O}\big(\epsilon^2\big)$$

where ϵ comes from initial data, $\mathrm{O}(\epsilon^2)$ from all the nonlinear terms save for the $R^{(N)}_{m,n}(\omega_0)z^m\bar{z}^n$ terms which contribute the $2c_2C_0\epsilon$. Let now $f_N=g+h$ with

$$\begin{aligned}
&ig_t=\big\{H_{\omega_1}+\ell(t)\big(P_+(\omega_1)-P_-(\omega_1)\big)\big\}g+X+\mathrm{O}_{\mathrm{loc}}\big(z^{N+1}\big),\quad g(0)=f_N(0),\\
&ih_t=\big\{H_{\omega_1}+\ell(t)\big(P_+(\omega_1)-P_-(\omega_1)\big)\big\}h+\mathrm{O}\big(f_N^3\big)+\mathrm{O}\big(f_N^{p_0}\big),\quad h(0)=0
\end{aligned}$$

in the notation of Lemma 4.5. Then, by Lemmas 3.2 and 3.3 and by the estimates in Lemma 4.5 we get $\|g\|_{L_t^2H_x^{1,-s}}\lesssim 2C_0\epsilon+\mathrm{O}(\epsilon^2)+c_0\epsilon$ for a fixed c_0. Finally,

$$\int_0^\infty\big\|e^{-i(t-s)H_{\omega_1}}e^{\pm i\int_s^t\ell(\tau)\,d\tau}\big(\mathrm{O}\big(f_N^3\big)+\mathrm{O}\big(f_N^{p_0}\big)\big)(s)\big\|_{L_t^2H^{1,-s}}\lesssim\int_0^\infty\big\|\big(\mathrm{O}\big(f_N^3\big)+\mathrm{O}\big(f_N^{p_0}\big)\big)(s)\big\|_{H^1}\lesssim\epsilon^3.$$

So if we set $C_1\approx 2C_0+c_0+1$ we obtain (4.11). We need to bound C_0.

Proof of (4.12). We first need:

Lemma 4.6. *We can decompose $f_{N+1}=h_1+h_2+h_3+h_4$ with for a fixed large $M>0$:*

(1) $\|h_1\|_{L_t^2L_x^{2,M}}\leqslant\mathrm{O}(\epsilon^2)$;
(2) $\|h_2\|_{L_t^2L_x^{2,M}}\leqslant\mathrm{O}(\epsilon^2)$;
(3) $\|h_3\|_{L_t^2L_x^{2,M}}\leqslant\mathrm{O}(\epsilon^2)$;
(4) $\|h_4\|_{L_t^2L_x^{2,M}}\leqslant c(\omega_1)\epsilon$ *for a fixed $c(\omega_1)$ upper semicontinuous in ω_1.*

Proof of Lemma 4.6. We set

$$\begin{aligned}
&i\partial_t h_1=\big(H_{\omega_1}+\ell(t)(P_+-P_-)\big)h_1,\\
&h_1(0)=\sum_{m+n=N+1}R_{H_{\omega_1}}\big((m-n)\lambda(\omega_1)+i0\big)R^{(N)}_{m,n}(\omega_1)z^m(0)\bar{z}^n(0).
\end{aligned}$$

We get $\|h_1\|_{L_t^2L_x^{2,-M}}\leqslant c(\omega_1)|z(0)|^2\sum\|R^{(N)}_{m,n}(\omega_1)\|_{L_x^{2,M}}=\mathrm{O}(\epsilon^2)$ by the following lemma:

Lemma 4.7. *There is a fixed s_0 such that for $s>s_0$,*

$$\begin{aligned}
&\big\|e^{-iH_\omega t}R_{H_\omega}(\Lambda+i0)P_c(\omega)\varphi\big\|_{L_t^2L_x^{2,-s}}<C_s(\Lambda,\omega)\big\|\varphi(x)\big\|_{L_x^{2,s}},\\
&\bigg\|\int_0^t e^{-iH_\omega(t-\tau)}R_{H_\omega}(\Lambda+i0)P_c(\omega)g(\tau)\,d\tau\bigg\|_{L_t^2L_x^{2,-s}}<C_s(\Lambda,\omega)\big\|g(t,x)\big\|_{L_t^2L_x^{2,s}}
\end{aligned}\tag{4.13}$$

with $C_s(\Lambda,\omega)$ upper semicontinuous in ω and in $\Lambda>\omega$.

Let us assume Lemma 4.7 for the moment, for the proof see Section 9. We set $h_2(0)=0$ and

$$\begin{aligned}
i\partial_t h_2={}&\big(H_{\omega_1}+\ell(t)(P_+-P_-)\big)h_2+\mathrm{O}\big(\epsilon z^{N+1}\big)R_{H_{\omega_1}}\big((N+1)\lambda(\omega_1)+i0\big)R^{(N)}_{N+1,0}(\omega_0)\\
&+\mathrm{O}\big(\epsilon z^{N+1}\big)R_{H_{\omega_1}}\big(-(N+1)\lambda(\omega_1)+i0\big)R^{(N)}_{0,N+1}(\omega_1).
\end{aligned}$$

Then we have $h_2 = h_{21} + h_{22}$ with $h_{2j} = \sum_{\pm} h_{2j\pm}$ with

$$h_{21\pm}(t) = \int_0^t e^{-iH_{\omega_1}(t-s)} e^{\pm i\int_s^t \ell(\tau)\,d\tau} P_\pm \mathrm{O}\big(\epsilon z^{N+1}\big) R_{H_{\omega_1}}\big((N+1)\lambda(\omega_1) + i0\big) R^{(N)}_{N+1,0}(\omega_1)\,ds$$

and $h_{22\pm}$ defined similarly but with $R_{H_{\omega_0}}(-(N+1)\lambda(\omega_1) + i0)R^{(N)}_{0,N+1}$. Now by (4.13) we get

$$\big\|h_{2j\pm}(t)\big\|_{L^2_t L^{2,-M}_x} \leqslant C\epsilon \|z\|^{N+1}_{L^{2N+2}_t}$$

and so $\|h_2(t)\|_{L^2_t L^{2,-M}_x} = \mathrm{O}(\epsilon^2)$. Let $h_3(0) = 0$ and

$$i\partial_t P_c(\omega_1)h_3 = \big(H_{\omega_1} + \ell(t)\big(P_+(\omega_1) - P_-(\omega_1)\big)\big)P_c(\omega_1)h_3 + P_c(\omega_1)\widetilde{E}_{\mathrm{PDE}}(N).$$

Then by the argument in the proof of (4.11) we get claim (3). Finally let $h_4(0) = f_N(0)$ and

$$i\partial_t P_c(\omega_1)h_4 = \big(H_{\omega_1} + \ell(t)\big(P_+(\omega_1) - P_-(\omega_1)\big)\big)P_c(\omega_1)h_4.$$

Then by Lemma 3.2 $\|\langle x\rangle^{-M} h_4\|_{L^2_{tx}} \lesssim \|f_N(0)\|_{L^2_x} \leqslant c(\omega_1)\epsilon$ we get (4). □

Continuation of proof of Lemma 4.3. We integrate (4.9) in time. Then by Theorem 2.1 and by Lemma 4.4 we get, for A_0 an upper bound of the constants $A_0(\omega)$ of Theorem 2.1,

$$\|\hat{z}\|^{2N+2}_{L^{2N+2}_t} \leqslant A_0\epsilon^2 + \epsilon\|\hat{z}\|^{N+1}_{L^{2N+2}_t} + \mathrm{o}\big(\epsilon^2\big).$$

Then we can pick $C_0 = (A_0 + 1)$ and this proves that (4.10) implies (4.12). Furthermore $\hat{z}(t) \to 0$ by $\frac{d}{dt}\hat{z}(t) = \mathrm{O}(\epsilon)$. □

As in [10,7] in the above argument we did not use the sign of $\Gamma(\omega, \omega_0)$. With the same argument in [10,7] one can prove

Corollary 4.8. *If Hypothesis* 4.2 *holds, then* $\Gamma(\omega, \omega) > \Gamma$.

The proof that, for ${}^t f_N(t) = (h(t), \bar{h}(t))$, $h(t)$ is asymptotically free for $t \to \infty$, is similar to the analogous one in [10] and we skip it.

5. Limiting absorption principle and L^2 theory for H_ω

In Sections 5–7 we prove Proposition 1.2. We start emphasizing two consequences of hypothesis (H9), in particular (b) clarifies the absence of resonance at $\pm\omega$:

(a) H_ω has no eigenvalues in $[\omega, +\infty) \cup (-\infty, -\omega]$;
(b) if $g \in W^{2,\infty}(\mathbb{R}^2, \mathbb{C}^2)$ satisfies $H_\omega g = \omega g$ or $H_\omega g = -\omega g$ then $g = 0$.

Because of the fact that H_ω is not a symmetric operator, we need some preparatory work to show that in fact H_ω is diagonalizable in the continuous spectrum. This work is done in Section 5 which ends with a formula for the wave operator W which is the basis to develop in Sections 6 and 7 a transposition of the work of Yajima [40].

We first need a preliminary on Schrödinger operators. We will denote by $q(x)$ a real valued function with: $q(x) \geqslant 0$ with $q(x) > 0$ at some points; $q(x) \in C_0^\infty(\mathbb{R}^2)$. We set $h_q = -\Delta + q(x)$. Then we have:

Lemma 5.1. *Let* $\mathbb{C}_+ = \{z \in \mathbb{C}\colon \Im z > 0\}$. *Suppose* $q(x) = 0$ *for* $r \geqslant r_0 > 0$. *Then we have the following facts.*

(1) *There exists* $s_0 > 0$ *and* $C_0 > 0$ *such that for* $s \geqslant s_0$, $R_{h_q}(z)$ *extends into a function* $z \to R^+_{h_q}(z)$ *which is in* $(L^\infty \cap C^0)(\overline{\mathbb{C}_+}, B(L^{2,s}, L^{2,-s}))$.

(2) *For any $n_0 \in \mathbb{N}$ there exists $s_0 > 0$ such that for any $a_0 > 0$ there is a choice of $C > 0$ such that for $n \leqslant n_0$*

$$\left\| \frac{d^n}{dz^n} R^+_{h_q}(z) : L^{2,s}(\mathbb{R}^2) \to L^{2,-s}(\mathbb{R}^2) \right\| \leqslant C_0 \langle z \rangle^{-\frac{1}{2}(1+n)}\ \forall z \in \overline{\mathbb{C}_+} \cap \{z\colon\ |z| \geqslant a_0\}.$$

(3) *The same argument can be repeated for $\mathbb{C}_- = \{z \in \mathbb{C}\colon\ \Im z < 0\}$ and $R^-_{h_q}(z)$.*

Claim (2) follows from [1] and [16] and claim (3) follows along the lines of the previous two claims. In view of (2), it is enough to prove (1) for $z \approx 0$. For $\zeta = re^{i\theta}$ with $\theta \in (-\pi, \pi)$ let $\sqrt{\zeta} = \sqrt{r}e^{i\theta/2}$. With this convention for $z \notin [0, \infty)$ for $R_0(z) = (-\Delta - z)^{-1}$ we have

$$R_0(z) = \frac{1}{2\pi} K_0\big(\sqrt{-z}\,|x|\big)* = \frac{i}{4} H^+_0\big(i\sqrt{-z}\,|x|\big)* = -\frac{i}{4} H^-_0\big(-i\sqrt{-z}\,|x|\big)*$$

for the Macdonald function K_0 and the Hankel functions $H^\pm_0$. We set $G_0 = -\frac{1}{2\pi}\log|x|*$, $P_0 f = \int_{\mathbb{R}^2} f\,dx$. We have for $M(z) = (1 + \sqrt{q}R_0(z)\sqrt{q}\,)$ the identity

$$R_{h_q}(z) = R_0(z) - R_0(z)\sqrt{q}M^{-1}(z)\sqrt{q}R_0(z). \tag{4}$$

From the expansion at 0 in $\mathbb{C}_+$ of H^+_0 and by the argument in Lemma 5 [26] we have in $B(L^{2,s}, L^{2,-s})$, for s sufficiently large,

$$R_0(z) = c(z)P_0 - G_0 + \mathrm{O}\big(-z\log\sqrt{-z}\big), \quad c(z) = \frac{i}{4} - \frac{\gamma}{2\pi} - \frac{1}{2\pi}\log\big(\sqrt{-z}/2\big). \tag{5}$$

Consider the projections in $L^2(\mathbb{R}^2)$, $P = \sqrt{q}\langle\cdot, \sqrt{q}\,\rangle/\|q\|_{L^1}$ and $Q = 1 - P$. Let $T = 1 + \sqrt{q}G_0\sqrt{q}$. Then QTQ is invertible in $QL^2(\mathbb{R}^2)$. Denote its inverse in $QL^2(\mathbb{R}^2)$ by $D_0 = (QTQ)^{-1}$. Consider the operator in $L^2 = PL^2 \oplus QL^2$ defined by

$$S = \begin{bmatrix} P & -PTQD_0Q \\ -QD_0QTP & QD_0QTPTQD_0Q \end{bmatrix}$$

and $h(z) = \|q\|_{L^1}c(z) + \mathrm{trace}(PTP - PTQD_0QTP)$. Then by [26]

$$\begin{aligned} R_{h_q}(z) = {} & R_0(z) - h^{-1}(z)R_0(z)\sqrt{q}S\sqrt{q}R_0(z) - R_0(z)\sqrt{q}QD_0Q\sqrt{q}R_0(z) \\ & - R_0(z)\sqrt{q}\,\mathrm{O}(-z\log\sqrt{-z}\,)\sqrt{q}R_0(z). \end{aligned} \tag{6}$$

By direct computation

$$\begin{aligned} h^{-1}(z)R_0(z)\sqrt{q}S\sqrt{q}R_0(z) = {} & \frac{c^2(z)}{h(z)}\langle\cdot, 1\rangle\sqrt{q}S\sqrt{q}\langle\cdot, 1\rangle + \frac{c(z)}{h(z)}\langle\cdot, 1\rangle\sqrt{q}S\sqrt{q}G_0 + \frac{c(z)}{h(z)}G_0\sqrt{q}S\sqrt{q}\langle\cdot, 1\rangle \\ & + \frac{c(z)}{h(z)}G_0\sqrt{q}S\sqrt{q}G_0 + \mathrm{O}\big(-z\log\sqrt{-z}\,\big), \end{aligned}$$

where all terms, except the first on the right-hand side, admit continuous extension in $\overline{\mathbb{C}}_+$ at 0. We have $\langle\cdot, 1\rangle\sqrt{q}S\sqrt{q}\langle\cdot, 1\rangle = \|q\|_{L^1}P_0$ and so by (5)

$$R_0(z) - \frac{c^2(z)}{h(z)}\|q\|_{L^1}P_0$$

admits continuous extension in $\overline{\mathbb{C}}_+$ at 0. By direct computation

$$R_0(z)\sqrt{q}QD_0Q\sqrt{q}R_0(z) = G_0\sqrt{q}QD_0Q\sqrt{q}G_0 + \mathrm{O}\big(-z\log\sqrt{-z}\,\big)$$

admits continuous extension in $\overline{\mathbb{C}}_+$ at 0. So $R_{h_q}(z)$ admits continuous extension in $\overline{\mathbb{C}}_+$ at 0, and so on all $\overline{\mathbb{C}}_+$.

A consequence of Lemma 5.1 is the h_q smoothness in the sense of Kato [19] of multiplication operators involving rapidly decreasing functions ψ:

Lemma 5.2. *Let $\psi(x) \in L^\infty(\mathbb{R}^2) \cap L^{2,s}(\mathbb{R}^2)$ for $s \gg 1$ and q as in Lemma 5.1. Then the multiplication operator ψ is h_q smooth, that is, for a fixed $C > 0$*

$$\int_{\mathbb{R}} \|\psi R_{h_q}(\lambda + i\varepsilon)u\|_2^2 \, d\lambda < C\|u\|_2^2 \quad \text{for all } u \in L^2(\mathbb{R}^2) \text{ and } \varepsilon \neq 0.$$

This follows from one of the characterizations of H smoothness in the case H is selfadjoint, see Theorem 5.1 [19], specifically from the fact that by Lemma 5.1 we have that for $\psi_1, \psi_1 \in L^\infty \cap L^{2,s}$ for $s \gg 1$ there is a number $C > 0$ such that for all $z \notin \mathbb{R}$ we have $\|\psi_1 R_{h_q}(z)\psi_2\|_{L^2,L^2} < C$.

We consider now $H_q = \sigma_3(-\Delta + q + \omega)$ and consider our linearization H_ω. Write $H_\omega = H_q + (V_\omega - \sigma_3 q)$, and factorize $V_\omega - \sigma_3 q = B^*A$ with A, B smooth $|\partial_x^\beta A(x)| + |\partial_x^\beta B(x)| < Ce^{-\alpha|x|}$ $\forall x$, for some $\alpha, C > 0$ and for $|\beta| \leqslant N_0$, N_0 sufficiently large. We have $\sigma_1 H_q = -H_q\sigma_1$, $\sigma_1 H_\omega = -H_\omega\sigma_1$. We choose the factorization B^*A so that $\sigma_1 B^* = -B^*\sigma_1$, $\sigma_1 A = A\sigma_1$. By these equalities $\sigma_1 R_{H_q}(z) = -R_{H_q}(-z)\sigma_1$ and $\sigma_1 R_{H_\omega}(z) = -R_{H_\omega}(-z)\sigma_1$, so in some of the estimates below it is enough to consider $z \in \overline{\mathbb{C}_{++}}$ with $\mathbb{C}_{++} = \{z\colon \Im z > 0,\ \Re z > 0\}$.

Lemma 5.3. *For $z \in \overline{\mathbb{C}_+}$ the function $R_{H_q}^+(z)$ is well defined and satisfies the following properties*:

(1) *There exists $s_0 > 0$ and $C_0 > 0$ such that for $s \geqslant s_0$ the function $z \to R_{H_q}^+(z)$ is in $(L^\infty \cap C^0)(\overline{\mathbb{C}_+}, B(L^{2,s}, L^{2,-s}))$.*

(2) *For any $n_0 \in \mathbb{N}$ there exists $s_0 > 0$ such that for any $a_0 > 0$ there is a choice of $C > 0$ such that for $n \leqslant n_0$ and $\forall z \in \overline{\mathbb{C}_+} \cap \{z\colon \operatorname{dist}(z, \pm\omega) \geqslant a_0\}$,*

$$\left\| \frac{d^n}{dz^n} R_{H_q}^+(z) : L^{2,s}(\mathbb{R}^2) \to L^{2,-s}(\mathbb{R}^2) \right\| \leqslant C_0 \langle z \rangle^{-\frac{1}{2}(1+n)}.$$

(3) *For any $\psi(x) \in L^\infty(\mathbb{R}^2) \cap L^{2,s}(\mathbb{R}^2)$ for $s \gg 1$ the multiplication operator ψ is H_q smooth, that is, for a fixed $C > 0$*

$$\int_{\mathbb{R}} \|\psi R_{H_q}(\lambda + i\varepsilon)u\|_2^2 \, d\lambda < C\|u\|_2^2 \quad \text{for all } u \in L^2(\mathbb{R}^2) \text{ and } \varepsilon \neq 0.$$

(4) *Analogous statements hold for $z \in \overline{\mathbb{C}_-}$ and the function $R_{H_q}^-(z)$.*

Lemma 5.3 is a trivial consequence of Lemmas 5.1–5.2. The properties in Lemma 5.4 are partially inherited by H_ω. Let $Q_q^+(z) = AR_{H_q}^+(z)B^*$. Then for $z \in \mathbb{C}_+$

Lemma 5.4. *Fix an exponentially decreasing bounded function ψ. For $z \in \mathbb{C}_+$ the function $AR_{H_\omega}(z)\psi$ extends into a function $AR_{H_\omega}^+(z)\psi$ for $z \in \overline{\mathbb{C}_+}\backslash\sigma_d(H_\omega)$ with the following properties*:

(1) *$\forall a_0 > 0$ $\exists C_0 > 0$ such that for $X_{a_0} = \overline{\mathbb{C}_+} \cap \{z\colon \operatorname{dist}(z, \sigma_d(H_\omega)) \geqslant a_0\}$*

$$AR_{H_\omega}^+(z)\psi \in (L^\infty \cap C^0)(X_{a_0}, B(L^2, L^2));$$

(2) *For any $n_0 \in \mathbb{N}$ there exists $s_0 > 0$ such that for any $a_0 > 0$ there is a choice of $C > 0$ such that for $n \leqslant n_0$ and $\forall z \in X_{a_0} \cap \{z\colon \operatorname{dist}(z, \pm\omega) \geqslant a_0\}$,*

$$\left\| \frac{d^n}{dz^n} AR_{H_\omega}^+(z)\psi : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2) \right\| \leqslant C_0 \langle z \rangle^{-\frac{1}{2}(1+n)}.$$

(3) *There is a constant $C > 0$ such that*

$$\int \|AR_{H_\omega}(\lambda + i\varepsilon)u\|_2^2 \, d\lambda \leqslant C\|u\|_2^2 \quad \text{for all } u \in L_c^2(H_\omega) \text{ and } \varepsilon \neq 0.$$

(4) *Analogous statements hold for $z \in \overline{\mathbb{C}_-}$ and the function $R_{H_\omega}^-(z)$.*

Proof. Let us write $Q_q^+(z) = AR_{H_q}^+(z)B^*$ and for $z \in \mathbb{C}_+$

$$AR_{H_\omega}(z) = \big(1 + Q_q^+(z)\big)^{-1} AR_{H_q}(z). \tag{5}$$

By Lemma 5.3 we have $\lim_{z\to\infty} \|Q_q^+(z)\|_{L^2,L^2} = 0$. By analytic Fredholm theory $1 + Q_q^+(z)$ is not invertible only at the $z \in \overline{\mathbb{C}_+}$ where $\ker(1 + Q_q^+(z)) \neq 0$. This set has 0 measure in $\mathbb{R}$. By Lemma 2.4 [11] if at some $z \neq \pm\omega$ we have $\ker(1 + Q_q^+(z)) \neq 0$, then z is an eigenvalue. By hypothesis there are no eigenvalues in $\sigma_e(H_\omega)$. Hence we get claim (2).

Lemma 5.5. *If* $\ker(1 + Q_q^+(\omega)) \neq 0$ *then there exists* $g \in W^{2,\infty}(\mathbb{R}^2)$ *with* $g \neq 0$ *such that* $H_\omega g = \omega g$.

Let us assume Lemma 5.5. By hypothesis such g does not exist. This yields (1). By (5), claim (4) Lemma 5.4 and Neumann expansion we get (4). Next, apply (5) to $u \in L_c(H_\omega)$. $AR_{H_\omega}(z)u$ is an analytic function in z with values in $L^2(\mathbb{R}^2)$ for z near any isolated eigenvalue z_0 of H_ω because the natural projection of u in $N_g(H_\omega - z_0)$ is 0. Away from isolated eigenvalues of H_ω, $(1 + Q_q^+(z))^{-1}$ is uniformly bounded. Hence (3) in Lemma 5.3 implies (3) in Lemma 5.4. □

Proof of Lemma 5.5. Let $0 \neq \tilde g \in \ker(1 + Q_q^+(\omega))$. Then

$$B^*\tilde g + (V_\omega - q)R_{H_q}(\omega)B^*\tilde g = 0.$$

Set $g = R_{H_q}(\omega)B^*\tilde g$. Then $Ag = -\tilde g$ and so $g \neq 0$. By $g + R_{H_q}(\omega)(V_\omega - q)g = 0$ we have $g \in H^2_{\mathrm{loc}}(\mathbb{R}^2)$ and $H_\omega g = \omega g$. We want now to show that $g \in L^\infty(\mathbb{R}^2)$, contrary to the hypotheses. We have ${}^t g = (g_1, g_2)$ with $g_2 = (\Delta - q - 2\omega)^{-1}(B^*\tilde g)_2$, where $B^*\tilde g \in L^{2,s}(\mathbb{R}^2)$ for any s, so $g_2 \in H^2(\mathbb{R}^2)$. We have $g_1 = R_{h_q}^+(0)(B^*\tilde g)_1$ with $g_1 \in L^{2,-s}(\mathbb{R}^2)$ for sufficiently large s. We split $L^{2,\pm s} = L_r^{2,\pm s} \oplus (L_r^{2,\mp s})^\perp$ where $L_r^{2,\pm s}$ are the radial functions and we are considering the standard pairing $L^{2,s} \times L^{2,-s} \to \mathbb{C}$ given by $\int_{\mathbb{R}^2} f(x)g(x)\,dx$. We decompose $g_1 = g_{1r} + g_{1nr}$ with $g_{1r} \in L_r^{2,-s}$ and $g_{1nr} \in (L_r^{2,s})^\perp$. In $(L_r^{2,-s})^\perp \to (L_r^{2,s})^\perp$ we have $R_{h_q}^+(0) = G_0 - G_0 q(1 + QG_0qQ)^{-1}G_0$ with $Q = 1 - P$, for $P = P_0 q_0$, $q_0 = c_0^{-1}q$, $c_0 = \int_{\mathbb{R}^2} q\,dx$, $P_0 u = \int_{\mathbb{R}^2} u\,dx$. Then

$$g_{1nr} = G_0(B^*\tilde g)_{1nr} - G_0 q(1 + QG_0qQ)^{-1}G_0(B^*\tilde g)_{1nr}$$

and by asymptotic expansion for $|x| \to \infty$ we conclude that for some constants

$$\partial_x^\alpha\left(g_{1nr} - a - \frac{b_1x_1 + b_2x_2}{|x|^2}\right) = \mathrm{O}\big(|x|^{-1-\alpha-\epsilon}\big)$$

for some $\epsilon > 0$. Finally we look ar $\tilde g_{1r}$. We can consider solutions $\phi(r)$ and $\psi(r)$ of $h_q u = 0$ with: $\phi(0) = 1$ and $\phi_r(0) = 0$; $\psi(r_0) = 1$ and $|\psi(r)|$ bounded for $r \geqslant r_0$, $\psi(r_0) \approx c\log r$ with $c \neq 0$ for $r \to 0$. In terms of these two functions the kernel of $R_{h_q}^+(0)$ in $L^2((0,\infty), dr)$ is

$$R_{h_q}^+(0)(r_1, r_2) = \begin{cases} \dfrac{\phi(r_1)\psi(r_2)}{W(r_2)} & \text{if } r_1 < r_2, \\[2ex] \dfrac{\phi(r_2\psi(r_1)}{W(r_2)} & \text{if } r_1 > r_2, \end{cases}$$

with $W(r) = [\phi(\cdot), \psi(\cdot)](r) = c/r$ for some $c \neq 0$. We have

$$g_{1r}(r) = c^{-1}\psi(r)\int_0^r \phi(s)(B^*\tilde g)_{1r}(s)s\,ds + c^{-1}\phi(r)\int_r^{+\infty} \psi(s)(B^*\tilde g)_{1r}(s)s\,ds.$$

Then for $r \geqslant r_0$,

$$\begin{aligned} \big|g_{1r}(r)\big| &\leqslant \big|c^{-1}\psi(r)\big|\left|\int_0^r |\phi(t)(B^*\tilde g)_{1r}(t)\right| t\,dt + \big|c^{-1}\phi(r)\big|\int_r^{+\infty} \big|\psi(t)(B^*\tilde g)_{1r}(t)\big| t\,dt \\ &\lesssim \big\|\log\langle x\rangle\big\|_{L^{2,-s}(\mathbb{R}^2)}\|B^*\tilde g\|_{L^{2,s}(\mathbb{R}^2)} + \log(2+r)\|B^*\tilde g\|_{L^{2,s}(\{x\in\mathbb{R}^2: |x|\geqslant r\})} = \mathrm{O}(1). \end{aligned}$$

Then we conclude that we have a nonzero $g \in H^2_{\text{loc}}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ such that $H_\omega g = \omega g$. But this is contrary to the nonresonance hypothesis. $\square$

Analogous to Lemma 5.4 is:

Lemma 5.6. *Fix an exponentially decreasing bounded function ψ. For $z \in \mathbb{C}_+$ the function $BR_{H^*_\omega}(z)\psi$ extends into a function $BR^+_{H^*_\omega}(z)\psi$ for $z \in \overline{\mathbb{C}_+}\backslash\sigma_d(H_\omega)$ with the following properties*:

(1) *For any $a_0 > 0$ there exists $C_0 > 0$ such that $BR^+_{H^*_\omega}(z)\psi \in L^\infty(X_{a_0}, B(L^2, L^2))$ where*
$$X_{a_0} = \overline{\mathbb{C}_+} \cap \{z\colon \operatorname{dist}(z, \sigma_d(H_\omega)) \geqslant a_0\}.$$
(2) *For any $n_0 \in \mathbb{N}$ there exists $s_0 > 0$ such that for any $a_0 > 0$ there is a choice of $C > 0$ such that for $n \leqslant n_0$ and $\forall z \in X_{a_0} \cap \{z\colon \operatorname{dist}(z, \pm\omega) \geqslant a_0\}$,*
$$\left\| \frac{d^n}{dz^n} BR^+_{H^*_\omega}(z)\psi : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2) \right\| \leqslant C_0 \langle z \rangle^{-\frac{1}{2}(1+n)}.$$
(3) *There is a constant $C > 0$ such that*
$$\int \left\| BR_{H^*_\omega}(\lambda + i\varepsilon)u \right\|_2^2 d\lambda \leqslant C\|u\|_2^2 \quad \textit{for all } u \in L^2_c(H^*_\omega) \textit{ and } \varepsilon \neq 0.$$
(4) *Analogous statements hold for $z \in \overline{\mathbb{C}_-}$ and the function $R^-_{H^*_\omega}(z)$.*

From [19, Section 2] we conclude:

Lemma 5.7. *There are isomorphisms $\widetilde{W} : L^2 \to L^2_c(H_\omega)$ and $\widetilde{Z} : L^2_c(H_\omega) \to L^2$, inverses of each other, defined as follows*:
for $u \in L^2$, $v \in L^2_c(H^*_\omega)$,
$$\langle \widetilde{W}u, v\rangle = \langle u, v\rangle + \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \langle AR_{H_q}(\lambda + i\epsilon)u, BR_{H^*_\omega}(\lambda + i\epsilon)v\rangle d\lambda;$$
for $u \in L^2_c(H_\omega)$, $v \in L^2$,
$$\langle \widetilde{Z}u, v\rangle = \langle u, v\rangle + \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \langle AR_{H_\omega}(\lambda + i\epsilon)u, BR_{H_q}(\lambda + i\epsilon)v\rangle d\lambda.$$
We have $H_\omega \widetilde{W} = \widetilde{W} H_q$ and $H_q \widetilde{Z} = \widetilde{Z} H_\omega$, $e^{itH_\omega}\widetilde{W} = \widetilde{W} e^{itH_q}$ and $e^{itH_q}\widetilde{Z} = \widetilde{Z} e^{itH_\omega} P_c(H_\omega)$. The operators $\widetilde{W}$ and $\widetilde{Z}$ depend continuously on $\widetilde{A}$ and $\widetilde{B}^*$ and can be expressed as
$$\widetilde{W}u = \lim_{t \to +\infty} e^{itH_\omega} e^{-itH_q} u \quad \textit{for any } u \in L^2,$$
$$\widetilde{Z}u = \lim_{t \to +\infty} e^{itH_q} e^{-itH_\omega} \quad \textit{for any } u \in L^2(H_\omega).$$

In particular we remark:

Lemma 5.8. *We have for $C(\omega)$ upper semicontinuous in ω and*
$$\left\| e^{-itH_\omega} g \right\|_2 \leqslant C(\omega)\|g\|_2 \quad \textit{for any } g \in L^2_c(H_\omega).$$

Having proved that $e^{-itH_\omega}P_c(H_\omega)$ are bounded in L^2, we want to relate H_ω to $H_0 = \sigma_3(-\Delta + \omega)$. Write $H = H_0 + V_\omega$, $V_\omega = B^*A$. We have $\sigma_1 H_0 = -H_0\sigma_1$, $\sigma_1 H_\omega = -H_\omega\sigma_1$. We choose the factorization of V_ω so that $\sigma_1 B^* = B^*\sigma_1$, $\sigma_1 A = -A\sigma_1$. By these equalities $\sigma_1 R_{H_0}(z) = -R_{H_0}(-z)\sigma_1$ and $\sigma_1 R_{H_\omega}(z) = -R_{H_\omega}(-z)\sigma_1$. We have the following result about existence and completeness of wave operators:

Lemma 5.9. *The following limits are well defined*:

(1) $Wu = \lim_{t\to+\infty} e^{itH_\omega} e^{-itH_0} u$ *for any* $u \in L^2$,
(2) $Zu = \lim_{t\to+\infty} e^{itH_0} e^{-itH_\omega} u$ *for any* $u \in L^2_c(H_\omega)$.

$W(L^2) = L^2_c(H_\omega)$ *is an isomorphism with inverse* Z.

Proof. The existence of $P_c(H_\omega) \circ W$ follows from Cook's method and Lemma 5.8. By an elementary argument $Wu \in L^2_c(H_\omega)$ for any $u \in L^2$, so $W = P_c(H_\omega) \circ W$. We have $W = \widetilde{W} \circ W_1$ with

$$W_1 u = \lim_{t\to+\infty} e^{itH_q} e^{-itH_0} u \quad \text{for any } u \in L^2(\mathbb{R}^2),$$
$$\widetilde{W} u = \lim_{t\to+\infty} e^{itH_{\omega}\omega} e^{-itH_q} \quad \text{for any } u \in L^2.$$

By standard theory W_1 is an isometric isomorphism of $L^2(\mathbb{R}^2)$ into itself with inverse $Z_1 u = \lim_{t\to+\infty} e^{itH_0} e^{-itH_q} u$ and by Lemma 5.7 $\widetilde{W}$ is an isomorphism $L^2(\mathbb{R}^2) \to L^2_c(H_\omega)$ with inverse $\widetilde{Z}$. Then by product rule the limit in (2) exists and we have $Z = Z_1 \circ \widetilde{Z}$ with Z the inverse of W. □

Lemma 5.10. *For* $u \in L^{2,s}(\mathbb{R}^2)$ *with* $s > 1/2$ *we have*

$$Wu = u - \frac{1}{2\pi i} \int_{|\lambda|\geqslant\omega} R^-_{H_\omega}(\lambda) V_\omega \big[R^+_{H_0}(\lambda) - R^-_{H_0}(\lambda)\big] u \, d\lambda.$$

Proof. $Wu \in L^2(\mathbb{R}^2)$ by Lemma 5.9, but the above formula is meaningful in the larger space $L^{2,-s}(\mathbb{R}^2)$. For $v \in L^{2,s}(\mathbb{R}^2) \cap L^2_c(H^*_\omega)$ and for $\langle u, v\rangle_2 = \int_{\mathbb{R}^2} u \cdot \bar{v}\, dx$ the standard L^2 pairing, we have by Plancherel

$$\langle Wu, v\rangle_2 = \langle u, v\rangle_2 + \lim_{\epsilon\to0^+} \int_0^{+\infty} \big\langle V_\omega e^{-iH_0 t - \epsilon t} u, e^{-iH^*_\omega t - \epsilon t} v\big\rangle_2 \, dt$$
$$= \langle u, v\rangle + \lim_{\epsilon\to0^+} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \big\langle A R_{H_0}(\lambda + i\epsilon) u, B R_{H^*_\omega}(\lambda + i\epsilon) v\big\rangle_2 \, d\lambda.$$

By the orthogonality in $L^2(\mathbb{R})$ of boundary values of Hardy functions in $H^2(\mathbb{C}_+)$ and in $H^2(\mathbb{C}_-)$ we have for $\epsilon > 0$

$$\int_{-\infty}^{+\infty} \big\langle A R_{H_0}(\lambda + i\epsilon) u, B R_{H^*_\omega}(\lambda + i\epsilon) v\big\rangle_2 \, d\lambda = \int_{-\infty}^{+\infty} \big\langle A\big[R_{H_0}(\lambda + i\epsilon) - R_{H_0}(\lambda - i\epsilon)\big] u, B R_{H^*_\omega}(\lambda + i\epsilon) v\big\rangle_2 \, d\lambda.$$

By $u \in L^{2,s}(\mathbb{R}^2)$ and $v \in L^{2,s}(\mathbb{R}^2) \cap L^2_c(H^*_\omega)$ the limit in the right-hand side for $\epsilon \searrow 0$ exists and we have

$$\langle Wu, v\rangle_2 = \langle u, v\rangle_2 + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \big\langle A\big[R_{H_0}(\lambda + i0) - R_{H_0}(\lambda - i0)\big] u, B R_{H^*_\omega}(\lambda + i0) v\big\rangle_2 \, d\lambda$$
$$= \langle u, v\rangle_2 + \frac{1}{2\pi} \int_{|\lambda|\geqslant\omega} \big\langle A\big[R_{H_0}(\lambda + i0) - R_{H_0}(\lambda - i0)\big] u, B R_{H^*_\omega}(\lambda + i0) v\big\rangle_2 \, d\lambda.$$

This yields Lemma 5.10. □

The crucial part of our linear theory is the proof of the following analogue of [40]:

Lemma 5.11. *For any* $p \in (1, \infty)$ *the restrictions of* W *and* Z *to* $L^2 \cap L^p$ *extend into operators such that for* $C(\omega) < \infty$ *semicontinuous in* ω

$$\|W\|_{L^p(\mathbb{R}^2), L^p_c(H_\omega)} + \|Z\|_{L^p_c(H_\omega), L^p(\mathbb{R}^2)} < C(\omega).$$

In the next two sections we will consider W only, since the proof for Z is similar. The argument in the following two sections is a transposition of [40]. We consider diagonal matrices

$$E_+ = \mathrm{diag}(1,0) \quad \text{and} \quad E_- = \mathrm{diag}(0,1).$$

Keeping in mind Lemma 5.10, $\sigma_1 R(z) = -R(-z)\sigma_1$ for $R(z)$ equal to $R_{H_\omega}(z)$ or to $R_{H_0}(z)$ and $\sigma_1 L_c^2(H_\omega) = L_c^2(H_\omega)$, it is easy to conclude that the L^p boundness of W is equivalent to L^p boundness of

$$\begin{aligned} Uu &:= \int\limits_{\lambda\geqslant\omega} R_{H_\omega}^-(\lambda)V_\omega\big[R_{H_0}^+(\lambda) - R_{H_0}^-(\lambda)\big]u\,d\lambda \\ &= \int\limits_{\lambda\geqslant\omega} R_{H_\omega}^-(\lambda)V_\omega\big[R_0^+(\lambda) - R_0^-(\lambda)\big]E_+u\,d\lambda. \end{aligned}$$

As in [40] we deal separately with high,treated in Section 6, and low energies, treated in Section 7. We introduce cut-off functions $\psi_1(x) \in C_0^\infty(\mathbb{R})$, and $\psi_2(x) \in C^\infty(\mathbb{R})$, with $\psi_1(x) + \psi_2(x) = 1$, $\psi_1(-x) = \psi_1(x)$, $\psi_1(x) = 1$ for $|x| \leqslant C$ and $\psi_1(x) = 0$ or $|x| > 2C$ for some $C > \omega$.

6. L^p boundness of U: high energies

This part is almost the same of the corresponding part in [40]. For $\psi_1(x)$ the cutoff function introduced after Lemma 5.11, $\psi_1(H_0)$ is a convolution operator with symbol $\psi_1(|\xi|^2 + \omega)$. Both $\psi_1(H_0)$ and $\psi_2(H_0)$ are bounded operators in $L^p(\mathbb{R}^2)$ for any $p \in [1,\infty]$. In order to estimate the high frequency part (the so called high energy) $U\psi_2(H_0)$, we expand $R_{H_\omega}^-(\lambda)$ into the sum of few terms of Born series

$$R_{H_\omega}^-(\lambda) = R_{H_0}^-(\lambda) - R_{H_0}^-(\lambda)V_\omega R_{H_0}^-(\lambda) + R_{H_0}^-(\lambda)V_\omega R_{H_0}^-(\lambda)V_\omega R_{H_\omega}^-(\lambda),$$

getting by Lemma 5.10 the decomposition $U = U_1 + U_2 + U_3$ with

$$\begin{aligned} U_1u &= -\frac{1}{2\pi i}\int\limits_{\lambda\geqslant\omega} R_{H_0}^-(\lambda)V_\omega R_0^+(\lambda-\omega)E_+u\,d\lambda, \\ U_2u &= \frac{1}{2\pi i}\int\limits_{\lambda\geqslant\omega} R_{H_0}^-(\lambda)V_\omega R_{H_0}^-(\lambda)V_\omega R_0^+(\lambda-\omega)E_+u\,d\lambda, \\ U_3u &= -\frac{1}{2\pi i}\int\limits_{\lambda\geqslant\omega} R_{H_0}^-(\lambda)V_\omega R_{H_0}^-(\lambda)V_\omega R_{H_\omega}^-(\lambda)V_\omega R_0^+(\lambda-\omega)E_+u\,d\lambda. \end{aligned}$$

Lemma 6.1. *The operator $U_1\psi_2(H_0)$ is bounded in $L^p(\mathbb{R}^2)$ for all $1 < p < \infty$. Specifically for any $s > 1$ there exists a constant $C_s > 0$ so that for $T = U_1\psi_2(H_0)$*

$$\|Tu\|_{L^p} \leqslant C_s\big\|\langle x\rangle^s V_\omega\big\|_{L^2}\|u\|_{L^p} \quad \text{for all } u \in L^p\big(\mathbb{R}^2\big). \tag{1}$$

Proof. Recall $R_0(z) = (-\Delta - z)^{-1}$ and $R_{H_0}^\pm(z) = \mathrm{diag}(R_0^\pm(z-\omega), -R_0^\pm(z+\omega))$. For $u = (u_1,u_2)$, and for $\mathcal{F}$ the Fourier transform, we are reduced to operators of schematic form

$$\mathcal{F}(E_\pm U_1u)(\xi) = \int\limits_{\lambda\geqslant\omega} d\lambda \int\limits_{\mathbb{R}^2} \frac{1}{|\xi|^2 + \omega \mp \lambda + i0}\hat{u}_1(\xi-\eta)\delta\big(\lambda - \big(|\xi-\eta|^2 + \omega\big)\big)\widehat{V}(\eta)\,d\eta,$$

with $\widehat{V}$ the Fourier transform of the generic component of V_ω. Then

$$E_\pm U_1u = \int\limits_{\mathbb{R}^2} d\eta\,\widehat{V}(\eta)T_\eta^\pm u_{1\eta}$$

where $u_{1\eta}(x) = e^{ix\cdot\eta}u_1(x)$, $T_\eta^- u_{1\eta} = \frac{1}{4\pi}K_0(\sqrt{\eta^2/4+\omega}\,|\cdot|) * u_{1\eta}$ and by [39]

$$T_\eta^+ u_{1\eta}(x) = \frac{i}{2|\eta|}\int_0^\infty e^{it|\eta|}u_{1\eta}\big(x + t\eta/|\eta|\big)\,dt.$$

By [40] we have that $T = E_+U_1$ satisfies inequality (1) while for $T = E_-U_1$ we use

$$\big\|T_\eta^\pm u\big\|_{L^p} \leqslant \frac{1}{4\pi}\left\|K_0\left(\sqrt{\frac{\eta^2}{4}+\omega}\,|x|\right)\right\|_{L^1_x}\|u_1\|_{L^p} \leqslant C\langle\eta\rangle^{-1}\|u_1\|_{L^p}$$

and so $\|E_-U_1u\|_{L^p} \lesssim \|\widehat{V}(\eta)/\langle\eta\rangle\|_{L^1}\|u_1\|_{L^p}$. □

Lemma 6.2. *The operator $U_2\psi_2(H_0)$ is bounded in $L^p(\mathbb{R}^2)$ for all $1 < p < \infty$, moreover, there exists a constant $C_s > 0$ so that for $T = U_2\psi_2(H_0)$*

$$\|Tu\|_{L^p} \leqslant C_s\big\|\langle x\rangle^s V_\omega\big\|_{L^2}^2\|u\|_{L^p} \quad \text{for all } u \in L^p\big(\mathbb{R}^2\big) \tag{1}$$

is valid, provided $s > 1$.

Proof. By [39] and with the notation of Lemma 6.1 we are reduced to a combination of operators

$$I_{\pm,\pm}u = \int_{\mathbb{R}^2} d\eta_1 T_{\eta_1}^\pm \int_{\mathbb{R}^2} d\eta_2 \widehat{V}(\eta_1)\widehat{V}(\eta_2-\eta_1)T_{\eta_2}^\pm u_{1\eta_2}.$$

$Tf = I_{-,-}u$ satisfies inequality (1) by [40, Proposition 2.2]. The other cases follow from Lemma 6.1. For example, for $K(\eta_1,\eta_2) = \widehat{V}(\eta_1)\widehat{V}(\eta_2-\eta_1)$ and $\widetilde{K}(x,\eta_2) = \int d\eta\, e^{i\eta\cdot x}K(\eta,\eta_2)$,

$$\begin{aligned}\|I_{\pm,\pm}u\|_{L^p} &= \left\|\int_{\mathbb{R}^2} d\eta_2 \int_{\mathbb{R}^2} d\eta_1 K(\eta_1,\eta_2)T_{\eta_1}^- T_{\eta_2}^+ u_{1\eta_2}\right\|_{L^p} \\ &\leqslant \widehat{C}_s\int_{\mathbb{R}^2} d\eta_2\big\|\langle x\rangle^s\widetilde{K}(x,\eta_2)\big\|_{L^2_x}\|T_{\eta_2}^+u_{1\eta_2}\|_{L^p} \\ &\leqslant \widetilde{C}_s\int_{\mathbb{R}^2} d\eta_2\big\|\langle x\rangle^s\widetilde{K}(x,\eta_2)\big\|_{L^2_x}\langle\eta_2\rangle^{-1}\|u_1\|_{L^p}C_s\big\|\langle x\rangle^s V_\omega\big\|_{L^2}^2\|u_1\|_{L^p}. \qquad \square\end{aligned}$$

Lemma 6.3. *Set $T = U_3\psi_2(H_0)$. Then T is bounded in $L^p(\mathbb{R}^2)$ for all $1 \leqslant p \leqslant \infty$.*

Proof. Schematically

$$E_+U_3\psi_2(H_0)u = \int_{k\geqslant 0} R_0^-\big(k^2\big)VF\big(k^2+\omega\big)V\big[R_0^+\big(k^2\big) - R_0^-\big(k^2\big)\big]\psi_2(\lambda+\omega)u_1k\,dk,$$

with $F(k^2+\omega) = R_{H_0}^-(k)VR^-(k)$ and V the generic component of V_ω. By (3) Lemma 5.4 for $G_{k,y}^\pm(x) = e^{\mp ik|y|}G^\pm(x-y,k)$ with $G^\pm(x,k) = \pm\frac{i}{4}H_0^\pm(k|x|)$ we have the following analogue of inequality (3.5) [40]

$$\big|\partial_k^j\big\langle F\big(k^2+\omega\big)VG_{k,y}^\pm, VG_{k,x}^+\big\rangle\big| \leqslant \frac{C_j\|\langle x\rangle^s V_\omega\|_\infty^3}{k^3\sqrt{\langle x\rangle\langle y\rangle}} \tag{1}$$

and by [40, Proposition 3.1] this yields the desired result for $T = E_+U_3\psi_2(H_0)$. Since (1) continues to hold if we replace $G_{k,x}^+$ with $e^{-ik|x|}G_{k,x}$ with $G_{k,x}(y) = G(x-y,k)$, where $G(x,k) = K_0(\sqrt{k^2+\omega}|x|)$, we get also the desired result for $T = E_-U_3\psi_2(H_0)$. □

7. L^p boundness of U: low energies

Set

$$Tu := \int_{\lambda \geqslant \omega} R^-_{H_\omega}(\lambda) V_\omega \big[R_0^+(\lambda-\omega) - R_0^-(\lambda-\omega)\big]\psi_1(\lambda) E_+ u \, d\lambda.$$

We want to prove:

Lemma 7.1. *For any $p \in (1,\infty)$ the restriction of T on $L^2 \cap L^p$ extends into an operator such that $\|T\|_{L^p(\mathbb{R}^2),L^p(\mathbb{R}^2)} < C(\omega)$ for $C(\omega) < \infty$ semicontinuous in ω.*

Let $V_\omega = V = \{V_{\ell j}\colon \ell, j = 1,2\}$, $W = \{W_{\ell j}\colon \ell, j = 1,2\}$ with $W_{12} = W_{21} = 0$, $W_{22} = 1 \in \mathbb{R}$ and $W_{11}(x) = 1$ for $V_{11}(x) \geqslant 0$ and $W_{11}(x) = -1$ for $V_{11}(x) < 0$. Set $B^* = \langle x \rangle^{-N}$ for some large $N > 0$, and $A = \{A_{\ell j}\colon \ell, j = 1,2\}$ with $A_{11}(x) = |V_{11}(x)|$, $A_{12}(x) = W_{11}(x)V_{12}(x)$ and $A_{2j}(x) = V_{2j}(x)$. Then $W^2 = 1$, $B^*WA = V$. Let $k > 0$ be such that $k^2 = \lambda - \omega$ and set $M(k) = W + AR^-_{H_0}(\lambda)B^*$. Then

$$R^-_{H_\omega}(\lambda) = R^-_{H_0}(\lambda) - R^-_{H_0}(\lambda)B^*M^{-1}(k)AR^-_{H_0}(\lambda).$$

We have $M(k) = W + c^-(k)P + A\widetilde{G}_0B^* + \mathrm{O}(k^2\log k)$ where: $c^-(k) = a^- + b^-\log k$; P is a projection in L^2 defined by

$$P = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} \frac{\langle \cdot, B^*_{11}\rangle}{\|V_{11}\|_{L^1}};$$

$$\widetilde{G}_0 = \mathrm{diag}\bigg(-\frac{1}{2\pi}\log|x|*, -R_0(-2\omega)\bigg);$$

$$\big\| d^j/dk^j \, \mathrm{O}\big(k^2\log k\big)\big\|_{L^2,L^2} \leqslant Ck^{2-j}\langle \log k\rangle, \quad j = 0,1,2, \ 0 < k < c.$$

Let $Q = 1 - P$ and let $M_0 = W + A\widetilde{G}_0B^*$. Then QM_0Q is invertible in QL^2 if and only if ω is not a resonance or an eigenvalue for H_ω and in that case

$$M^{-1}(k) = g^{-1}(k)\big(P - PM_0QD_0Q - QD_0QM_0PM_0QD_0Q + QD_0Q + O\big(k^2\log k\big)\big)$$

with $g(k) = c^-\log k + d^-$ for $c^- \neq 0$ and $D_0 = (QM_0Q)^{-1}$ by [17]. We claim now that $QD_0Q - QWQ$ is a Hilbert–Schmidt operator. In fact, following the argument in Lemma 3 [18], we get that the operator $L = P + QM_0Q$ is invertible in QL^2, and $D_0 = QL^{-1}Q$. We have

$$L = W + \big[A\widetilde{G}_0B^* + P + PM_0P - PM_0Q - QM_0P\big].$$

Set $L := W(1+\widetilde{S})$, the operators P, PM_0P, PM_0Q, QM_0P are of rank one while $A\widetilde{G}_0B^*$ is a Hilbert–Schmidt operator. From the fact that W is invertible, we get that also $(1+\widetilde{S})$ is invertible. Moreover the identity $(1+\widetilde{S})^{-1} = 1 - \widetilde{S}(1+\widetilde{S})^{-1}$ yields

$$L^{-1} - W = -\widetilde{S}(1+\widetilde{S})^{-1}W,$$

that is the product of an Hilbert–Schmidt operator with one in $B(L^2(\mathbb{R}^2), L^2(\mathbb{R}^2))$. Finally, an application of the Theorem VI.22, Chapter VI, in [25], shows that $L^{-1} - W$ is of Hilbert–Schmidt type.

So we are reduced to the following list of operators:

$$T_0^+u := \int_0^\infty R_0^-\big(k^2\big)E_+V_\omega E_+\big[R_0^+\big(k^2\big) - R_0^-\big(k^2\big)\big]\psi_1(\lambda)uk\,dk,$$

and T_0^- defined as above but with $R_0^-(k^2)E_+$ replaced by $R_0(-k^2-2\omega)E_-$ which are bounded in L^p for $1 < p < \infty$ by Lemma 6.1;

$$T_1^+u := \int_0^\infty R_0^-\big(k^2\big)E_+N(k)\big[R_0^+\big(k^2\big) - R_0^-\big(k^2\big)\big]\psi_1(\lambda)E_+uk\,dk$$

with

$$\left\| d^j/dk^j N\left(k^2 \log k\right)\right\|_{L^{2,-s},L^{2,s}} \leqslant Ck^{2-j}\langle \log k\rangle, \quad j=0,1,2,\ 0<k<c$$

which is bounded in L^p for $1 \leqslant p \leqslant \infty$ by Proposition 4.1 [40];

$$T_2^+ u := \int_0^\infty R_0^-\left(k^2\right)E_+ B^*\left(d(k)F + L + W\right)A\left[R_0^+\left(k^2\right) - R_0^-\left(k^2\right)\right]\psi_1(\lambda)E_+ uk\,dk$$

with F a rank 3 operator, L a Hilbert–Schmidt operator in L^2, and $d(k) = g^{-1}(k)$. There are also operators T_j^-, for $j = 0, 1, 2$, defined as above but with $R_0^-(k^2)E_+$ replaced by $R_0(-k^2 - 2\omega)E_-$ and bounded in L^p. So $T_2^\pm = T_{2,1}^\pm d(\sqrt{-\Delta}) + T_{2,2}^\pm + T_{2,3}^\pm$ with $T_{2,j}^\pm$ for $j = 1, 2, 3$ operators bounded in L^p for $1 < p < \infty$ because of the following statement proved in [40] (the + case is exactly that in [40], and the – case can be proved following the same argument):

if K is an operator with integral kernel $K(x, y)$ such that for some $s > 1$

$$\|K\|_s := \int_{\mathbb{R}^2} dy \left(\int_{\mathbb{R}^2} dx \langle x\rangle^{2s}\left|K(x, x-y)\right|^2\right)^{\frac{1}{2}} < \infty$$

then the operators

$$Z^+ u := \int_0^\infty R_0^-\left(k^2\right)K\left[R_0^+\left(k^2\right) - R_0^-\left(k^2\right)\right]uk\,dk,$$

$$Z^- u := \int_0^\infty R_0\left(-k^2 + 2\omega\right)K\left[R_0^+\left(k^2\right) - R_0^-\left(k^2\right)\right]uk\,dk$$

are bounded in L^p for $1 < p < \infty$ with $\|Z^\pm\|_{L^p,L^p} < C_{s,p}\|K\|_s$. □

8. Proofs of Lemmas 3.2, 3.3 and 3.4

We mimic Mizumachi [22]. By the limiting absorption principle we have

$$P_c(\omega)e^{-itH_\omega} f = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{-it\lambda}(\lambda)P_c(\omega)\left[R_{H_\omega}^+(\lambda) - R_{H_\omega}^-(\lambda)\right]f\,d\lambda.$$

We consider a smooth function $\chi(x)$ satisfying $0 \leqslant \chi(x) \leqslant 1$ for $x \in \mathbb{R}$, $\chi(x) = 1$ if $x \geqslant 2$ and $\chi(x) = 0$ if $x \leqslant 1$. $\chi_M(x)$ is an even function satisfying $\chi_M(x) = \chi(x - M)$ for $x \geqslant 0$. Let $\tilde{\chi}_M(x) = 1 - \chi_M(x)$. We have:

Lemma 8.1. *For any fixed $s > 1$ there exists a positive $C(\omega)$ upper semicontinuous in ω, such that for any $u \in S(\mathbb{R}^2)$ we have*

$$\left\| R_{H_\omega}^\pm(\lambda) f\right\|_{L_\lambda^2(\sigma_c(H_\omega);L_x^{2,-s})} \leqslant C\|f\|_{L^2}.$$

First, we prove Lemma 3.2 assuming Lemma 8.1.

Proof of Lemma 3.2. We split

$$P_c(\omega)e^{-itH_\omega} f = P_c(\omega)e^{-itH_\omega}\chi_M(H_\omega) f + P_c(\omega)e^{-itH_\omega}\tilde{\chi}_M(H_\omega) f$$

with

$$P_c(\omega)\chi_M(H_\omega)e^{-itH_\omega}f = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{-it\lambda}\chi_M(\lambda)\big(R^+_{H_\omega}(\lambda) - R^-_{H_\omega}(\lambda)\big)P_c(\omega)f\,d\lambda,$$

$$P_c(\omega)e^{-itH_\omega}\tilde\chi_M(H_\omega)f = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{-it\lambda}\tilde\chi_M(\lambda)\big(R^+_{H_\omega}(\lambda) - R^-_{H_\omega}(\lambda)\big)P_c(\omega)f\,d\lambda.$$

Integrating by parts, in $S'_x(\mathbb{R}^2)$ for any $t \neq 0$ and $f \in S_x(\mathbb{R}^2)$

$$P_c(\omega)e^{-itH_\omega}f = \frac{(it)^{-j}}{2\pi i}\int_{-\infty}^{\infty} d\lambda\, e^{-it\lambda}\partial_\lambda^j P_c(\omega)\big\{\big(R^+_{H_\omega}(\lambda) - R^-_{H_\omega}(\lambda)\big)\chi_M(\lambda)\big\}f.$$

Since by (3) Lemma 5.4 for high energies we have

$$\big\|\partial_\lambda^j P_c(\omega)R^{\pm}_{H_\omega}(\lambda) : \langle x\rangle^{(j+1)/2+0}L^2 \to \langle x\rangle^{-(j+1)/2-0}L^2\big\| \lesssim \langle\lambda\rangle^{-(j+1)/2},$$

the above integral absolutely converges in $\langle x\rangle^{-(j+1)/2-0}L^2_x$ for $j \geqslant 2$. Let $g(t,x) \in S(\mathbb{R}\times\mathbb{R}^2)$. By Fubini and integration by parts, $j \geqslant 2$,

$$\begin{aligned}\big\langle \chi_M(H_\omega)e^{-itH_\omega}P_c(\omega)f, g\big\rangle_{t,x} &= \frac{1}{2\pi i}\int_{\mathbb{R}} dt\,(it)^{-j}\int_{\mathbb{R}} d\lambda\, e^{-it\lambda}\partial_\lambda^j\big\langle\chi_M(\lambda)\big(R^+_{H_\omega}(\lambda) - R^-_{H_\omega}(\lambda)\big)f, \bar g\big\rangle_x \\ &= \frac{1}{2\pi i}\int_{\mathbb{R}} d\lambda\left\langle \partial_\lambda^j\big\{\chi_M(\lambda)\big(R^+_{H_\omega}(\lambda) - R^-_{H_\omega}(\lambda)\big)\big\}P_c(\omega)f, \int_{\mathbb{R}} dt\,(-it)^{-j}\bar g(t)e^{it\lambda}\right\rangle_x \\ &= \frac{1}{\sqrt{2\pi} i}\int_{\mathbb{R}} d\lambda\big\langle\chi_M(\lambda)\big(R^+_{H_\omega}(\lambda) - R^-_{H_\omega}(\lambda)\big)P_c(\omega)f, \bar{\hat g}(\lambda)\big\rangle_x.\end{aligned}$$

Hence, by Fubini and Plancherel, we have

$$\begin{aligned}\big|\big\langle\chi_M(H_\omega)e^{-itH_\omega}P_c(\omega)f, g\big\rangle_{t,x}\big| &\leqslant (2\pi)^{-1/2}\big\|\chi_M(\lambda)\big(R^+_{H_\omega}(\lambda) - R^-_{H_\omega}(\lambda)\big)f\big\|_{L^2_\lambda(\sigma_c(H_\omega);L^{2,-s}_x)}\big\|\hat g(\lambda,\cdot)\big\|_{L^2_\lambda L^{2,s}_x} \\ &= (2\pi)^{-1/2}\big\|\chi_M(\lambda)\big(R^+_{H_\omega}(\lambda) - R^-_{H_\omega}(\lambda)\big)f\big\|_{L^2_\lambda(\sigma_c(H_\omega);L^{2,-s}_x)}\|g\|_{L^2_t L^{2,s}_x}.\end{aligned}$$

In a similar way we have

$$\big|\big\langle e^{-itH_\omega}\tilde\chi_M(H_\omega)f, g\big\rangle_{t,x}\big| \leqslant (2\pi)^{-1/2}\big(\big\|\tilde\chi_M(H_\omega)\big(R^+_{H_\omega}(\lambda) - R^-_{H_\omega}(\lambda)\big)f\big\|_{L^2_\lambda(\sigma_c(H_\omega);L^{2,-s}_x)}\|g\|_{L^2_t L^{2,s}_x},$$

therefore we achieve

$$\begin{aligned}\big|\big\langle e^{-itH_\omega}P_c(\omega)f, g\big\rangle_{t,x}\big| &\leqslant (2\pi)^{-1/2}\big(\big\|\chi_M(\lambda)\big(R_{H_\omega}(\lambda+i0) - R_{H_\omega}(\lambda-i0)\big)f\big\|_{L^2_\lambda(\sigma_c(H_\omega);L^{2,-s}_x)} \\ &\quad + \big\|\tilde\chi_M(\lambda)\big(R^+_{H_\omega}(\lambda) - R^-_{H_\omega}(\lambda)\big)f\big\|_{L^2_\lambda(\sigma_c(H_\omega);L^{2,-s}_x)}\|g\|_{L^2_t L^{2,s}_x}\end{aligned}$$

and by Lemma 8.1 this estimate yields Lemma 3.2. □

Proof of Lemma 3.3. By Plancherel's identity and Hölder inequalities we have

$$\begin{aligned}\left\|\int_0^t e^{-i(t-s)H_\omega}P_c(\omega)g(s,\cdot)\,ds\right\|_{L^{2,-s}_x L^2_t} &\leqslant \big\|R^+_{H_\omega}(\lambda)P_c(\omega)\hat\chi_{[0,+\infty)} *_\lambda \hat g(\lambda,x)\big\|_{L^{2,-s}_x L^2_\lambda} \\ &\leqslant \big\|\big\|R^+_{H_\omega}(\lambda)P_c(\omega)\big\|_{L^{2,s}_x, L^{2,-s}_x}\big\|\hat\chi_{[0,+\infty)} *_\lambda \hat g(\lambda,x)\big\|_{L^{2,s}_x}\big\|_{L^2_\lambda}.\end{aligned}$$

By Lemma 5.4 $\sup_{\lambda\geqslant\omega}\|R^+_{H_\omega}(\lambda)P_c(\omega)\|_{B(L^{2,s},L^{2,-s})}\lesssim\langle\lambda\rangle^{-1/2}$, and so

$$\sup_{\lambda\in\mathbb{R}}\big\|R^+_{H_\omega}(\lambda)P_c(\omega)\big\|_{B(L^{2,s}_x,L^{2,-s}_x)}\|g\|_{L^{2,s}_xL^2_t}\leqslant C\|g\|_{L^{2,s}_xL^2_t}.$$

The above inequalities yields Lemma 3.3. □

Proof of Lemma 3.4. Let (q,r) be admissible and let T be an operator defined by

$$Tg(t)=\int_{\mathbb{R}}ds e^{-i(t-s)H_\omega}P_c(\omega)g(s).$$

Using Lemmas 3.2 and 3.3 we get $f:=\int_{\mathbb{R}}ds e^{isH_\omega}P_c(\omega)g(s)\in L^2(\mathbb{R})$ and that there exists a $C>0$ such that

$$\big\|Tg(t)\big\|_{L^q_tL^r_x}\leqslant C\|g\|_{L^2_tL^{2,s}_x} \tag{1}$$

for every $g\in S(\mathbb{R}\times\mathbb{R}^2)$. Since $q>2$, it follows from Lemma 3.1 in [28] (see also [2]) and (1) that

$$\left\|\int_{s<t}ds e^{-i(t-s)H_\omega}P_c(\omega)g(s)\right\|_{L^q_tL^p_x}\lesssim\|g\|_{L^2_tL^{2,s}_x}.$$

This yields Lemma 3.4. □

To prove Lemma 8.1 observe that it is not restrictive to prove

$$\big\|R^\pm_{H_\omega}(\lambda)f\big\|_{L^2_\lambda((\omega,\infty);L^{2,-s}_x)}\leqslant C\|f\|_{L^2}. \tag{8.1}$$

Following the argument in [22, Section 4] we need the following:

Lemma 8.2. *There exists a positive constant C such that for $s>1$*

$$\big\|R^\pm_{H_0}(\lambda)f\big\|_{L^{2,-s}_xL^2_\lambda(\omega,\infty)}\leqslant C\|f\|_{L^2}.$$

Proof. $E_+R^\pm_{H_0}(\lambda)f=R^\pm_0(\lambda-\omega)E_+f$ and by Lemma 4.2 [22] we get

$$\big\|R^\pm_0(\lambda)E_+f\big\|_{L^{2,-s}_xL^2_\lambda(0,\infty)}\leqslant C\sup_x\big\|R^\pm_0(\lambda)E_+f\big\|_{L^2_\lambda(0,\infty)}\leqslant C\|E_+f\|_{L^2}. \tag{1}$$

We have $E_-R^\pm_{H_0}(\lambda)f=-R_0(-\omega-\lambda)E_-f=-\frac{-\Delta+\omega-\lambda}{-\Delta+2\omega+\lambda}R^+_0(\lambda-\omega)E_-f$. So by (1)

$$\begin{aligned}\big\|E_-R^\pm_{H_0}(\lambda)f\big\|_{L^{2,-s}_xL^2_\lambda(\omega,\infty)}&\leqslant\left\|\frac{-\Delta+\omega-\lambda}{-\Delta+\omega+\lambda}\right\|_{L^\infty_\lambda((\omega,\infty),B(L^{2,-s}_x,L^{2,-s}_x))}\big\|R^\pm_0(\lambda)E_-f\big\|_{L^{2,-s}_xL^2_\lambda(0,\infty)}\\&\leqslant C_1\big\|R^\pm_0(\lambda)E_-f\big\|_{L^{2,-s}_xL^2_\lambda(0,\infty)}\leqslant C_1C\|E_-f\|_{L^2}.\end{aligned}$$
□

Proof of inequality (8.1). We consider the operator $h_q=-\Delta+q(x)$ introduced in Section 5 and $H_q=\sigma_3(h_q+\omega)$. We claim that

$$\big\|R^\pm_{H_q}(\lambda)f\big\|_{L^2_\lambda((\omega,\infty),L^{2,-s}_x)}\leqslant C\|f\|_{L^2}. \tag{1}$$

Indeed $E_+R^\pm_{H_q}(\lambda)f=R^\pm_{h_q}(\lambda-\omega)E_+f$ and $\|R^\pm_{h_q}(\lambda)E_+f\|_{L^2_\lambda(0,\infty),L^{2,-s}_x)}\leqslant C\|f\|_{L^2}$ by [22, Lemma 4.1]. On the other hand

$$E_-R^\pm_{H_q}(\lambda)f=-R_{h_q}(-\lambda-\omega)E_-f=-R_0(-\lambda-\omega)E_-f+R_0(-\lambda-\omega)qR_{h_q}(-\lambda-\omega)E_-f.$$

The bound for the first term comes from Lemma 8.2 and

$$\begin{aligned}\big\|R_0(-\lambda-\omega)qR_{h_q}(-\lambda-\omega)E_-f\big\|_{L^{2,-s}_xL^2_\lambda}&\lesssim\big\|R_0(-\lambda-\omega)qR_{h_q}(-\lambda-\omega)E_-f\big\|_{L^\infty_xL^2_\lambda}\\&\lesssim\big\|qR_{h_q}(-\lambda-\omega)E_-f\big\|_{L^\infty_\lambda L^2_x}\leqslant C\|E_-f\|_{L^2_x}.\end{aligned}$$

Armed with inequality (1) we consider the identity

$$\begin{aligned} R^{\pm}_{H_\omega}(\lambda) &= \big(1 + R^{\pm}_{H_q}(\lambda)(V_\omega - \sigma_3 q)\big)^{-1} R^{\pm}_{H_q}(\lambda) \\ &= R^{\pm}_{H_q}(\lambda) - R^{\pm}_{H_q}(\lambda)(V_\omega - \sigma_3 q)\big(1 + R^{\pm}_{H_q}(\lambda)(V_\omega - \sigma_3 q)\big)^{-1} R^{\pm}_{H_q}(\lambda). \end{aligned} \tag{8.2}$$

By (1) it is enough to bound the last term in the last sum. This is bounded by

$$\begin{aligned} &\big\| R^{\pm}_{H_q}(\lambda)(V_\omega - \sigma_3 q)\big(1 + R^{\pm}_{H_q}(\lambda)(V_\omega - \sigma_3 q)\big)^{-1} R^{\pm}_{H_q}(\lambda) f\big\|_{L^2_\lambda L^{2,-s}_x} \\ &\quad \leqslant \big\| R^{\pm}_{H_q}(\lambda)(V_\omega - \sigma_3 q)\big(1 + R^{\pm}_{H_q}(\lambda)(V_\omega - \sigma_3 q)\big)^{-1}\big\|_{L^\infty_\lambda B(L^{2,-s}_x, L^{2,-s}_x)} \big\| R^{\pm}_{H_q}(\lambda) f\big\|_{L^2_\lambda L^{2,-s}_x} \\ &\quad \lesssim \big\| R^{\pm}_{H_q}(\lambda)\big\|_{L^\infty_\lambda (B(L^{2,s}_x, L^{2,-s}_x))} \big\|\big(1 + R^{\pm}_{H_q}(\lambda)(V_\omega - \sigma_3 q)\big)^{-1}\big\|_{L^\infty_\lambda B(L^{2,-s}_x, L^{2,-s}_x)} \|f\|_{L^2_x} \\ &\quad \lesssim \|f\|_{L^2_x} \end{aligned}$$

by (1) and by the fact that the above $L^\infty_\lambda(\omega, \infty)$ norms are bounded by Lemmas 5.1 and 5.4. □

9. Proof of Lemma 4.7

The proof is standard and analogous to [9, Lemma 5.8]. Recall:

Lemma 4.7. *We have for $\varphi(x)$ and $\varphi(t, x)$ Schwarz functions, for $t \in [0, \infty)$ and for fixed $s > 1$ sufficiently large*

$$\big\| e^{-iH_\omega t} R^{+}_{H_\omega}(\Lambda) P_c(\omega)\varphi\big\|_{L^2_t L^{2,-s}_x} < C(\Lambda, \omega)\big\|\varphi(x)\big\|_{L^{2,s}_x},$$

$$\left\| \int_0^t e^{-iH_\omega(t-\tau)} R^{+}_{H_\omega}(\Lambda) P_c(\omega)\varphi(\tau)\, d\tau \right\|_{L^2_t L^{2,-s}_x} < C(\Lambda, \omega)\big\|\varphi(t, x)\big\|_{L^2_t L^{2,s}_x}$$

with $C(\Lambda, \omega)$ upper semicontinuous in ω and in $\Lambda > \omega$.

Proof. We consider $\omega < a/ < a << \Lambda < b < \infty$ and the partition of unity $1 = g + \tilde g$ with $g \in C^\infty_0(\mathbb{R})$ with $g = 1$ in $[a, b]$ and $g = 0$ in $[a/2, 2b]$. By Lemma 3.2 we get

$$\begin{aligned} \big\| e^{-iH_\omega t} R^{+}_{H_\omega}(\Lambda) P_c(\omega)\tilde g(H_\omega)\varphi\big\|_{L^2_t L^{2,-s}_x} &\leqslant C(\omega)\big\| R^{+}_{H_\omega}(\Lambda) P_c(\omega)\tilde g(H_\omega)\varphi\big\|_{L^2_x} \\ &\leqslant C(\omega) c_0(a, b, \omega)\|\varphi\|_{L^2_x}. \end{aligned}$$

Similarly by the proof of Lemma 3.3, for any $s > 1$

$$\begin{aligned} &\left\| \int_0^t e^{-i(t-s)H_\omega} R^{+}_{H_\omega}(\Lambda) P_c(\omega)\tilde g(H_\omega)\varphi(s, \cdot)\, ds \right\|_{L^{2,-s}_x L^2_t} \\ &\quad \leqslant \big\| R^{+}_{H_\omega}(\lambda) R^{+}_{H_\omega}(\Lambda)\tilde g(H_\omega) P_c(\omega)\hat\chi_{[0,+\infty)} *_\lambda \hat\varphi(\lambda, x)\big\|_{L^{2,-s}_x L^2_\lambda} \\ &\quad \leqslant \big\| \big\| R^{+}_{H_\omega}(\lambda) R^{+}_{H_\omega}(\Lambda)\tilde g(H_\omega) P_c(\omega)\big\|_{L^{2,s}_x, L^{2,-s}_x} \big\|\hat\chi_{[0,+\infty)} *_\lambda \hat\varphi(\lambda, x)\big\|_{L^{2,s}_x}\big\|_{L^2_\lambda} \\ &\quad \leqslant C(s, a, b, \omega)\|\varphi\|_{L^{2,s}_x L^2_t} \end{aligned}$$

by $(\lambda - \Lambda) R^{+}_{H_\omega}(\lambda) R^{+}_{H_\omega}(\Lambda) = R^{+}_{H_\omega}(\lambda) - R^{+}_{H_\omega}(\Lambda)$, Lemma 5.4 and $|\lambda - \Lambda| \geqslant a \wedge b$. We consider now

$$\begin{aligned} &\langle x\rangle^{-\gamma} g(H_\omega) e^{-iH_\omega t} R_{H_\omega}(\Lambda + i\epsilon) P_c(H_\omega)\langle y\rangle^{-\gamma} \\ &\quad = e^{-i\Lambda t}\langle x\rangle^{-\gamma} \int_t^{+\infty} e^{-i(H_\omega - \Lambda - i\epsilon)s} g(H_\omega) P_c(H_\omega)\, ds \langle y\rangle^{-\gamma}. \end{aligned} \tag{9.1$_\epsilon$}$$

We claim the following:

Lemma 9.1. *There are functions $u(x,\xi)$ defined for $x \in \mathbb{R}^2$ and for $|\xi| \in [a/2, 2b]$ with values in $\mathbb{C}^2$ such that for any $\chi \in C_0^\infty(a/2, 2b)$ we have (for ${}^t u\sigma_3 f$ the product row column and ${}^t u$ the transpose of a column vector)*

$$\chi(H_\omega)f(x) = (2\pi)^{-2} \int_{\mathbb{R}^4} u(x,\xi)^t \bar{u}(y,\xi)\sigma_3 f(y)\chi\left(|\xi|^2 + \omega\right) d\xi\, dy. \tag{9.2}$$

There are constants $c_{\alpha\beta}$ such that

$$\left|\partial_x^\alpha \partial_\xi^\beta u(x,\xi)\right| \leqslant c_{\alpha\beta}\langle x\rangle^{|\beta|} \quad \text{for all } x \in \mathbb{R}^2 \text{ and } |\xi| \in [a/2, 2b]. \tag{9.3}$$

Let us assume Lemma 9.1. Then we can write the kernel of operator (9.1) as

$$\begin{aligned}
&\langle x\rangle^{-\gamma} g(H_\omega)e^{-iH_\omega t} R_{H_\omega}(\Lambda + i\epsilon)\langle y\rangle^{-\gamma} \\
&\quad = (\text{constant})\langle x\rangle^{-\gamma} \int_{\mathbb{R}^3} u(x,\xi)e^{-i(\sigma_3(\xi^2+\omega)-\Lambda-i\epsilon)s} g(\xi^2+\omega)^t \bar{u}(y,\xi)\, d\xi\, \langle y\rangle^{-\gamma}.
\end{aligned} \tag{9.4}$$

Estimates (9.3) and elementary integration by parts yields

$$\left|(9.4)\right| \leqslant c\langle x\rangle^{-\gamma+r}\langle y\rangle^{-\gamma+r} s^{-r} e^{-\epsilon t} \text{ and so} |(9.1)_{0+}| \leqslant c\langle x\rangle^{-\gamma+r}\langle y\rangle^{-\gamma+r}\langle t\rangle^{-r+1}.$$

For $\gamma > r+1$ and $r \geqslant 3$, we obtain

$$\left\|e^{-iH_\omega t} R_{H_\omega}^+(\Lambda)g(H_\omega)P_c(H_\omega)\varphi\right\|_{L_t^2((0,\infty),L^{2,-\gamma})} \leqslant C\left\|\varphi(x)\right\|_{L^{2,\gamma}}.$$

Similarly

$$\left\|\int_0^t e^{-i(t-s)H_\omega} R_{H_\omega}^+(\Lambda)P_c(\omega)g(H_\omega)\varphi(s,\cdot)\,ds\right\|_{L_t^2 L_x^{2,-\gamma}} \leqslant \left\|\int_0^t \langle t-s\rangle^{-2}\left\|\varphi(s,\cdot)\,ds\right\|_{L_x^{2,\gamma}}\right\|_{L_t^2} \leqslant C\|\varphi\|_{L_t^2 L_x^{2,\gamma}}.$$

We need now to prove Lemma 9.1. □

10. Proof of Lemma 9.1

First of all we explain how to define the $u(x,\xi)$. We set $V_\omega = B^*A$ with $A(x)$ and $B^*(x)$ rapidly decreasing and continuous. Then we have

Lemma 10.1. *For any $\lambda > \omega$ and any $\xi \in \mathbb{R}^2$ with $\lambda = \omega + |\xi|^2$, in $L^2(\mathbb{R}^2)$ the system*

$$\left(1 + AR_{H_0}^+(\lambda)B^*\right)\tilde{u} = Ae^{-i\xi\cdot x}\vec{e}_1 \tag{1}$$

admits exactly one solution $\tilde{u}(x,\xi) \in H^2$ such that for any $[a,b] \subset (1,\infty) \setminus \sigma_p(H)$ there is a fixed $C < \infty$ such that for any $\lambda \in [a,b]$ and any ξ as above we have

$$\left\|\tilde{u}(\cdot,\xi)\right\|_{H^2} \leqslant C. \tag{2}$$

Proof. $AR_{H_0}^+(\lambda)B^*$ is compact and $\ker(1 + AR_{H_0}^+(\lambda)B^*) = \{0\}$ for $\lambda > \omega$ by [11], since in that case $\lambda \notin \sigma_p(H_\omega)$. By Fredholm alternative we get existence and uniqueness of $\tilde{u}(x,\xi)$. Regularity theory and continuity of the coefficients of system (1) with respect to ξ yield (2). □

Let now ${}^t e_1 = (1,0)$ and $G_0(|x|,k) = \mathrm{diag}(\frac{i}{4}H_0^+(k|x|), -\frac{1}{2\pi}K_0(\sqrt{k^2+2\omega}|x|))$ for $k > 0$. We have $G_0(r,k) = \frac{i\sqrt{2}}{4\sqrt{i\pi kr}}e^{ikr}e_1 + \mathrm{O}(r^{-\frac{3}{2}})$ and $\partial_r G_0(r,k) = -k\frac{\sqrt{2k}}{4\sqrt{i\pi r}}e^{ikr}e_1 + \mathrm{O}(r^{-\frac{3}{2}})$. We set

$$u(x,\xi) = e^{-i\xi\cdot x}e_1 + v(x,\xi) = e^{-i\xi\cdot x}e_1 - R_{H_0}^+(\lambda)B^*\tilde{u}(\cdot,\xi).$$

Then $(H_\omega - \lambda)u(x,\xi) = B^*(Ae^{-i\xi\cdot x}e_1 - \tilde{u} - AR^+_{H_0}(\lambda)B^*\tilde{u}) = 0$. Notice $B^*\tilde{u} = V_\omega u$ so $v(x,\xi) = e^{-ix\cdot\xi}w(x,\xi)$ where $w(x,\xi)$ is the unique solution in L^2_{-s}, $s > 1$, of the integral equation

$$w(x,\xi) = -F(x,\xi) - \int_{\mathbb{R}^2} G_0\big(|x-z|,|\xi|\big)e^{i(x-z)\cdot\xi}V_\omega(z)w(z,\xi)\,dz, \tag{1}$$

with

$$F(x,\xi) = \int_{\mathbb{R}^2} G_0\big(|x-z|,|\xi|\big)V_\omega(z)e^{i(x-z)\cdot\xi}e_1\,dz.$$

It is elementary to show that, for $|\xi| \in [a,b]$, then $|\partial_x^\alpha\partial_\xi^\beta F(x,\xi)| \leqslant \tilde{c}_{\alpha\beta}\langle x\rangle^{|\beta|-1/2}$. By standard arguments and Lemmas 5.3 and 5.4 we have $|\partial_x^\alpha\partial_\xi^\beta w(x,\xi)| \leqslant \tilde{c}_{\alpha\beta}\langle x\rangle^{|\beta|}$. This yields (9.3). To get (9.2) we follow the presentation in Chapter 9 [32]. We denote by $R^\pm_{H_\omega}(x,y,k)$ the kernel of $R^\pm_{H_\omega}(k^2+\omega)$. We set

$$R^+_{H_\omega}(x,y,k) = G_0\big(|x-y|,k\big) + h(x,y,k)$$

with $h(\cdot,y,k) = -R^+_{H_0}(k^2+\omega)V_\omega G_0(|\cdot - y|,k)$. Let (r,Σ) be polar coordinates on the sphere S^1, then we claim:

Lemma 10.2. *Let $k > 0$. For $r \to \infty$ we have uniform convergence on compact sets of, with $u\cdot(1,0)$ the raw column product between column u and raw $(1,0)$,*

$$R^+_{H_\omega}(x,r\Sigma,k) = \frac{i\sqrt{2}}{4\sqrt{i\pi kr}}e^{ikr}u(x,k\Sigma)\cdot(1,0) + \mathrm{O}\big(r^{-2}\big), \tag{1}$$

$$\frac{\partial}{\partial r}R^+_{H_\omega}(x,r\Sigma,k) = -\frac{\sqrt{2}}{4\sqrt{i\pi kr}}ke^{ikr}u(x,k\Sigma)\cdot(1,0) + \mathrm{O}\big(r^{-2}\big), \tag{2}$$

$$R^+_{H_\omega}(r\Sigma,y,k) = \frac{i\sqrt{2}}{4\sqrt{i\pi kr}}e^{ikr}\begin{bmatrix}1\\0\end{bmatrix}{}^t u(y,k\Sigma)\sigma_3 + \mathrm{O}\big(r^{-2}\big), \tag{3}$$

$$\frac{\partial}{\partial r}R^+_{H_\omega}(r\Sigma,y,k) = -\frac{\sqrt{2}}{4\sqrt{i\pi kr}}ke^{ikr}\begin{bmatrix}1\\0\end{bmatrix}{}^t u(y,k\Sigma)\sigma_3 + \mathrm{O}\big(r^{-2}\big). \tag{4}$$

For $R^-_{H_\omega}(x,y,k)$ the asymptotic expansion follows from $R^-_{H_\omega}(x,y,k) = \overline{R^+_{H_\omega}}(x,y,k)$.

We write $R^+_{H_\omega}(x,r\Sigma,k) = G_0(|x-r\Sigma|,k) + h(x,r\Sigma,k)$ with

$$\begin{aligned}h(x,r\Sigma,k) &= -R^+_{H_0}\big(k^2+\omega\big)V_\omega G_0\big(|\cdot - r\Sigma|,k\big)\\ &= -R^+_{H_0}\big(k^2+\omega\big)\left[V_\omega(x)\left(\frac{i\sqrt{2}}{4\sqrt{i\pi kr}}e^{ikr}e^{-ik\Sigma\cdot x}\operatorname{diag}(1,0) + \mathrm{O}\big(r^{-3/2}\big)\right)\right].\end{aligned}$$

We have

$$\left\| V_\omega(x)G_0\big(|x-r\Sigma|,k\big) - V_\omega(x)\frac{i\sqrt{2}}{4\sqrt{i\pi kr}}e^{ikr}e^{-ik\Sigma\cdot x}\operatorname{diag}(1,0)\right\|_{L^{2,s}_x} = \mathrm{O}\big(r^{-3/2}\big).$$

From $v(x,\xi) = -R^+_{H_0}(k^2+\omega)V_\omega(x)e^{-ik\Sigma\cdot x}e_1$, with ${}^te_1 = (1,0)$ we get

$$v(x,\xi)^te_1 = -R^+_{H_0}\big(k^2+\omega\big)V_\omega(x)e^{-ik\Sigma\cdot x}\operatorname{diag}(1,0).$$

Then we conclude for any $s > 1$

$$\left\| h(x,r\Sigma,k) - \frac{i\sqrt{2}}{4\sqrt{i\pi kr}}v(x,k\Sigma)^te_1\right\|_{L^{2,-s}} = \mathrm{O}\big(r^{-3/2}\big)$$

and

$$\left\| R^+_{H_\omega}(x, r\Sigma, k) - \frac{i\sqrt{2}}{4\sqrt{i\pi kr}} u(x, k\Sigma)^t e_1 \right\|_{L^{2,-s}} = \mathrm{O}\big(r^{-3/2}\big).$$

Then point wise $h(x, r\Sigma, k + i0) - \frac{i\sqrt{2}}{4\sqrt{i\pi kr}} v(x, k\Sigma)^t e_1 = \mathrm{O}(r^{-3/2})$ and

$$R^+_{H_\omega}(x, r\Sigma, k) - \frac{i\sqrt{2}}{4\sqrt{i\pi kr}} u(x, k\Sigma)^t e_1 = \mathrm{O}\big(r^{-3/2}\big).$$

This yields (1) in Lemma 10.2. (2) can be obtained with a similar argument. (3) and (4) follow from (1) and (2) by

$$\sigma_3 R^\pm_{H_\omega}(x, y, k)\sigma_3 = R^\pm_{H^*_\omega}(x, y, k) = {}^t R^\mp_{H_\omega}(y, x, k).$$

By Lemma 3.5 for $v \in L^2(H_\omega) \cap C^\infty_1$ and for $\varphi \in C^\infty_0(\mathbb{R})$ supported in (ω, ∞) we have

$$\varphi(H_\omega)v(x) = \frac{2}{\pi} \int_0^\infty k\,dk \int_{\mathbb{R}^2} \varphi(k^2 + \omega) \Im R^+_{H_\omega}(x, y, k) v(y)\,dy.$$

We prove (here $u^t\bar{u}$ is a raw column product between column u and raw ${}^t\bar{u}$)

$$\Im R^+_{H_\omega}(x, y, k) = \frac{1}{8\pi} \int_{S^1} u(x, k\Sigma)^t \bar{u}(y, k\Sigma)\sigma_3\,d\Sigma, \tag{3}$$

where $d\Sigma$ is the standard measure on S^1. By the Green theorem for $S_R = \{z \in \mathbb{R}^2\colon |z| = R\}$, $|x| < R$, $|y| < R$ and $r = |z|$.

By Green theorem for $S_R = \{z \in \mathbb{R}^2\colon |z| = R\}$, $|x| < R$ and $|y| < R$,

$$\Im R^+_{H_\omega}(x, y, k) = \frac{1}{2i} \int_{S_R} I(x, y, z, k)\,d\ell(z),$$

$$I(x, y, z, k) := R^+_{H_\omega}(x, z, k)\sigma_3 \partial_{|z|} R^-_{H_\omega}(z, y, k) - \big(\partial_{|z|} R^+_{H_\omega}(x, z, k)\big)\sigma_3 R^-_{H_\omega}(z, y, k).$$

By Lemma 10.2

$$\begin{aligned} &\left| \Im R^+_{H_\omega}(x, y, k) - \frac{1}{8\pi} \int_{S^1} u(x, k\Sigma)^t \bar{u}(y, k\Sigma)\sigma_3\,d\Sigma \right| \\ &\quad = \left| \frac{R}{2i} \int_{S^1} I(x, y, r\Sigma, k)\Big|_{r=R} d\Sigma - \frac{1}{8\pi} \int_{S^1} u(x, k\Sigma)^t \bar{u}(y, k\Sigma)\sigma_3\,d\Sigma \right| \leqslant \mathrm{O}\big(R^{-\frac{3}{2}}\big). \end{aligned}$$

Therefore, taking $R \to +\infty$, we arrive at (3). Moreover, we obtain

$$\begin{aligned} \varphi(H_\omega)v(x) &= \frac{2}{\pi} \int_0^\infty k\,dk \int_{\mathbb{R}^2} \varphi(k^2 + \omega) \Im G(x, y, k) v(y)\,dy \\ &= \frac{1}{4\pi^2} \int_0^\infty k\,dk \int_{\mathbb{R}^2} \int_{S^1} u(x, k\Sigma)^t \bar{u}(y, k\Sigma)\sigma_3 v(y) \varphi\big(k^2 + \omega\big)\,d\Sigma\,dy \\ &= (2\pi)^{-2} \int_{\mathbb{R}^4} u(x, \xi)^t \bar{u}(y, \xi)\sigma_3 v(y) \varphi\big(|\xi|^2 + \omega\big)\,d\xi\,dy, \end{aligned}$$

that is the integral representation (9.2). This completes the proof of Lemma 9.1. □

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