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Null controllability of the complex Ginzburg–Landau equation

Contrôlabilité à zéro de l'équation de Ginzburg–Landau complexe

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Abstract

The paper investigates the boundary controllability, as well as the internal controllability, of the complex Ginzburg–Landau equation. Zero-controllability results are derived from a new Carleman estimate and an analysis based upon the theory of sectorial operators.

Résumé

Dans ce papier on étudie la contrôlabilité au bord, ainsi que la contrôlabilité interne, de l'équation de Ginzburg–Landau complexe. On obtient des résultats de contrôlabilité à zéro au moyen d'une inégalité de Carleman et d'une analyse basée sur la théorie des opérateurs sectoriels.

Keywords: Ginzburg–Landau equation; Null controllability; Carleman estimate

1. Introduction

The classical cubic Ginzburg–Landau (GL) equation

$$\partial_t u = (a + i\alpha)\Delta u + Ru - (b + i\beta)|u|^2 u, \quad t > 0, \ x \in \Omega \subset \mathbb{R}^N \tag{1.1}$$

plays an important role in the theory of amplitude equations, and provides a simple model of turbulence. Here, a and b denote some positive real numbers, that actually may be set to one by introducing the new variables $t' = at$, $u' = \sqrt{b/a}u$.

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The Cauchy problem for GL has been investigated in several papers (see e.g. [3,14,15,17,19]). It has been proved in [17] with $a = b = 1$ that (1.1) is globally well-posed in some uniformly local spaces $L^p_{lu}(\Omega)$ when $p > N$ and (i) $N \leqslant 2$; (ii) $N = 3$ and $|\alpha| < \sqrt{8}$ or $-(1 + \alpha\beta) < \sqrt{3}|\alpha - \beta|$; (iii) $N \geqslant 4$ and $|\alpha| < 2\sqrt{N-1}/(N-2)$. On the other hand, the well-posedness of the initial-boundary value problem for the cubic GL equation has been addressed in [5,11]. The quintic GL equation ($|u|^2u$ being replaced by $|u|^4u$) has been investigated in [3,14] for the well-posedness issue.

In this paper we are interested in the control properties of the GL equation. To our knowledge, only a few papers have been devoted to the control of that equation. In [6], the authors develop a numerical method to solve a constrained optimal control problem for a generalized GL equation. [1] is concerned with the stabilization of the linearized GL equation around an unstable equilibrium state. Finally, [9] contains a Carleman inequality for the operator $(a+ib)\partial_t + \sum_{j,k} \partial_k(a^{jk}\partial_j)$ (where (a^{jk}) is a smooth uniformly elliptic matrix) and a zero-controllability result for a *linear* PDE of GL type with an internal control.

The class of GL operators $\partial_t - (a + i\alpha)\Delta$ contains both the heat operator $\partial_t - \Delta$ and the Schrödinger operator $i\partial_t + \Delta$ in the limit $a \to 0$. One may wonder whether the control properties of the GL equation are similar to the ones for the heat equation, or for the Schrödinger operator. Also of interest is the study of the singular limit $\alpha \to 0$ (resp. $a \to 0$).

A first observation is that the GL operator $\partial_t - (a + i\alpha)\Delta$ is hypoelliptic (see [13]) when $a > 0$, so that the solutions of the linear GL equation are C^∞ smooth in the complement of the control region. As a consequence, no exact controllability result can be obtained in the Sobolev space $H^k(\Omega)$ for any $k \in \mathbb{Z}$.

This paper will actually demonstrate that the control properties of the GL equation are very similar to the ones for the (semilinear) heat equation. Zero-controllability results in the spirit of those in [8] will be established. Furthermore, it will become clear that the geometry of the control region play no role in the results.

The proof of the results will follow the general pattern exposed in [10]. A linearized equation is first proved to be zero controllable by means of some Carleman inequality. Then, a fixed-point argument is applied to extend the result to the nonlinear equation.

The Carleman estimate proved here is interesting in its own right. Indeed, it is more precise than the one in [9] as it contains in the left-hand side the terms u_t, Δu (exactly as for the heat equation). For the sake of clarity, the proof given here is divided into two parts: the first one provides an exact computation of a scalar product, and may as well be used for the Schrödinger equation (see [16]); the second one gives the estimates obtained thanks to the smoothing effects of the GL operator.

The fixed-point argument applied here proves to be more tricky than for the heat equation, as many classical properties of the heat equation (comparison principle, maximum principle, etc.) fail for GL. The smoothing effect needed to apply Schauder fixed-point theorem is carefully proved with the aid of the theory of sectorial operators in all the spaces $L^p(\Omega)$, $N < p < \infty$. Notice also that the use of that theory allows to give almost sharp results, as far as the regularity of the trajectories is concerned.

The paper is outlined as follows. The main results are stated in Section 2. Some background material on sectorial operators is provided in Section 3. Section 4 contains the proof of the main result (Theorem 2.1) and of the Carleman inequality (Proposition 4.3). Indications for the proof of the main corollary are given in Section 5. The annexe contains some elementary lemmas.

2. Main results

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a C^2 boundary $\partial\Omega$. Let $\Gamma_0 \subset \partial\Omega$ denote an arbitrary open set. We introduce the spaces

$$C_0(\Omega) = \{u \in C(\bar{\Omega});\ u = 0 \text{ on } \partial\Omega\}$$

and

$$X = \{u \in C(\bar{\Omega});\ u = 0 \text{ on } \partial\Omega \setminus \Gamma_0\}.$$

Let $f : \mathbb{C} \to \mathbb{C}$ be a function such that

$$f \in C(\mathbb{C};\mathbb{C}), \quad f(0) = 0, \quad \text{and} \quad f'(0) = \lim_{z\to 0} f(z)/z \text{ exists} \tag{2.1}$$

(i.e., f is differentiable at 0 in the complex sense). Clearly, $f(z) = Rz - (1+i\beta)|z|^2 z$ (resp. $f(z) = Rz - (1+i\beta)|z|^4 z$) are concerned.

We will consider first the following boundary control system

$$\partial_t u = (1 + i\alpha)\Delta u + f(u) \quad \text{in } \Omega, \tag{2.2}$$

$$u = 1_{\Gamma_0} h \quad \text{on } \partial\Omega, \tag{2.3}$$

$$u(0) = u_0, \tag{2.4}$$

where $1_{\Gamma_0}(x) = 1$ if $x \in \Gamma_0$, 0 otherwise.

One of the main contributions of the paper is to show that this boundary control system is locally null controllable.

Theorem 2.1. *Let $f : \mathbb{C} \to \mathbb{C}$ fulfilling* (2.1). *Then for any $T > 0$, the system* (2.2)–(2.4) *is locally null controllable in X. More precisely, there exist a number $R > 0$ such that for any $u_0 \in X$ with $\|u_0\|_{L^\infty(\Omega)} < R$, there exists a control input $h \in C(\partial\Omega \times [0, T])$ with $\mathrm{supp}(h) \subset \Gamma_0 \times (0, T)$ such that the system* (2.2)–(2.4) *admits a solution*

$$u \in L^2\big(0, T; H^1(\Omega)\big) \cap C\big([0, T]; X\big)$$

satisfying

$$u(T) = 0.$$

Moreover, the solution u and the control h satisfy $\sqrt{t}u \in L^2(0, T; H^2(\Omega))$, $\sqrt{t}\partial_t u \in L^2(0, T; L^2(\Omega))$, $\sqrt{t}h \in L^2(0, T; H^{\frac{3}{2}}(\partial\Omega))$ and for all $\nu < 2$, $u \in C((0, T]; C^\nu(\bar{\Omega}))$ and $h \in C((0, T]; C^\nu(\partial\Omega))$.

It is easy to see that the uniqueness of the solution of (2.2)–(2.4) in the above class holds provided that f is locally Lipschitz continuous.

Remark 2.2. As f is only assumed to be continuous, one cannot expect more than $u \in C((0, T], C^2(\bar{\Omega}))$. Since the trajectory u provided by Theorem 2.1 is in $C((0, T], C^\alpha(\bar{\Omega}))$ for any $\alpha < 2$, we conclude that the smoothness of the trajectory given in Theorem 2.1 is almost sharp. Smooth trajectories associated with smooth control inputs were given in [7, Theorem 4], but under the additional assumption that the nonlinear term $f(y)$ in the Fourier boundary conditions was of class C^3.

Corollary 2.3. *Assume that $f(z) = Rz + \mu|z|^{2\sigma} z$ with $\sigma \in \mathbb{R}^{+*}$, $R \in \mathbb{R}$ and $\mu \in \mathbb{C}$. Then the system* (2.2)–(2.4) *is locally null controllable in the space $L^p(\Omega)$ for any $p > \sigma N$, and in the Sobolev space $H^q(\Omega)$ for any $q > \frac{N}{2} - \frac{1}{\sigma}$ provided that $2\sigma \geqslant 1$ and $N\sigma \geqslant 1$.*

The space $H^q(\Omega)$ for $q < 0$ is defined as the dual space of the space $D((-\Delta_D)^{|q|/2})$ with respect to the pivot space $L^2(\Omega)$.

Remark 2.4.

(1) It can also be shown as in [8] that if $\lim_{|z|\to\infty} f(z)/(z \ln |z|) = 0$, then the system (2.2)–(2.4) is globally null controllable in X, i.e., R may be given any value in Theorem 2.1.

(2) An internal controllability result may be derived from Theorem 2.1 by an extension procedure similar to the one used in [8, Theorem 2.2]. Pick any open set $\omega \subset \Omega$ and let us consider the following forced initial-value problem.

$$\partial_t u = (1 + i\alpha)\Delta u + f(u) + 1_\omega h \quad \text{in } \Omega, \tag{2.5}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{2.6}$$

$$u(0) = u_0. \tag{2.7}$$

where f satisfies again (2.1). Then for any $T > 0$, the system (2.5)–(2.7) is locally null controllable in $C_0(\Omega)$. More precisely, there exists a number $R > 0$ such that for any $u_0 \in C_0(\Omega)$ with $\|u_0\|_{L^\infty(\Omega)} < R$, there exists a control input $h \in C(\bar{\Omega} \times [0, T])$ with $\mathrm{supp}\, h \subset \omega \times (0, T)$ such that the system (2.5)–(2.7) admits a solution

$$u \in L^2\big(0, T; H^1(\Omega)\big) \cap C\big([0, T]; C_0(\Omega)\big)$$

satisfying

$$u(T)=0.$$

Moreover, $\sqrt{t}u \in L^2(0,T;H^2(\Omega))$, $\sqrt{t}\partial_t u \in L^2(0,T;L^2(\Omega))$, $\sqrt{t}h \in L^2(0,T;H^1(\Omega))$, and for all $\nu < 2$, $u \in C((0,T];C^\nu(\bar{\Omega}))$ and $h \in C((0,T];C^{\nu-1}(\bar{\Omega}))$.

3. Background on sectorial operators

In this section, we recall some basic properties of a sectorial operator (the reader is referred to [12] for the details). Our focus is on the Ginzburg–Landau operator with Dirichlet boundary conditions.

We begin with a definition (see [12] or [2]). (Let us point out that an operator A is sectorial according to [12] if and only if $-A$ is sectorial according to [2].)

Definition 3.1. A closed operator A on a Banach space X is said to be *sectorial of angle* $\theta_0 \in (0,\pi/2]$ if there exists some number $a \in \mathbb{R}$ such that for any $\theta < \theta_0$, the set

$$\Sigma_{a,\theta} := \left\{\lambda \in \mathbb{C};\ \frac{\pi}{2}-\theta < \left|\arg(\lambda-a)\right| \leqslant \pi,\ \lambda \neq a\right\}$$

is contained in $\rho(A)$, and there exists some number $M_\theta \geqslant 1$ such that

$$\left\|(\lambda-A)^{-1}\right\| \leqslant \frac{M_\theta}{|\lambda-a|} \quad \text{for all } \lambda \in \Sigma_{a,\theta}.$$

Let A be a densely defined sectorial operator A of angle θ_0 in a Banach space X. We recall some of its well-known properties.

Proposition 3.2. *(See [2,12].) The operator $-A$ generates a (strongly continuous) analytic semigroup on X.*

Assume in addition that $\sigma(A) \subset \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda > \delta\}$ for some $\delta > 0$. Then we may define the operator A^γ for any $\gamma > 0$ as the inverse of $A^{-\gamma}$, where

$$A^{-\gamma} = \frac{1}{\Gamma(\gamma)} \int_0^\infty t^{\gamma-1} e^{-At}\, dt.$$

The following estimate [12, Theorem 1.4.3] reveals a strong smoothing effect.

Proposition 3.3. *For any $\gamma \geqslant 0$ there exists a constant $C_\gamma > 0$ such that*

$$\left\|A^\gamma e^{-At}\right\| \leqslant C_\gamma t^{-\gamma} e^{-\delta t} \quad \forall t > 0. \tag{3.1}$$

It is well known that the negative Laplacian with Dirichlet boundary conditions is sectorial in $L^p(\Omega)$ for any $p \in (1,+\infty)$. We extend that property to the operator $-(1+i\alpha)\Delta$.

Proposition 3.4. *Let $\alpha \in \mathbb{R}$ and $p \in [2,+\infty)$. Let A_p denote the operator $Au := -(1+i\alpha)\Delta u$ with domain $\mathcal{D}(A_p) := W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \subset L^p(\Omega)$. Then*

(i) *A_p is a densely defined sectorial operator on $L^p(\Omega)$, which generates an analytic semigroup $(e^{-A_p t})$ on $L^p(\Omega)$;*
(ii) *$\sigma(A_p) \subset \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \geqslant \delta\}$ for some $\delta > 0$ which does not depend on p.*

Proof. The first part is essentially [17, Theorem 4.2]. For the second part, it is well known that the spectrum of the operator $-\Delta$ with Dirichlet boundary conditions is a nondecreasing and unbounded sequence of positive real numbers $(\lambda_n)_{n\geqslant 0}$. Clearly, $\sigma(A_p) = \{(1+i\alpha)\lambda_n\} \subset \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \geqslant \lambda_0\}$. The result follows by taking $\delta = \lambda_0 > 0$. □

Notice that a similar result holds for $p=\infty$, provided that $L^\infty(\Omega)$ is replaced by $C_0(\Omega)$.

Proposition 3.5. *Let A denote the operator $Au := -(1+i\alpha)\Delta u$ with domain $\mathcal{D}(A) := \{u \in C_0(\Omega);\ \Delta u \in C_0(\Omega)\} \subset C_0(\Omega)$. Then A is a densely defined sectorial operator on $C_0(\Omega)$, which generates an analytic semigroup on $C_0(\Omega)$. Furthermore, $\sigma(A) \subset \{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda \geqslant \delta\}$ for some $\delta > 0$.*

Proof. According to [2, Theorem 6.1.9] the Dirichlet Laplacian

$$\mathcal{A}u := -\Delta u, \qquad D(\mathcal{A}) := \{u \in C_0(\Omega);\ \Delta u \in C(\bar{\Omega})\}$$

is sectorial on $C(\bar{\Omega})$ of angle $\pi/2$. Notice that $\mathcal{A}$ is not densely defined. To overcome that difficulty, we change $\mathcal{A}$ into $\tilde{\mathcal{A}}$, with

$$\tilde{\mathcal{A}}u := -\Delta u, \qquad D(\tilde{\mathcal{A}}) := \{u \in C_0(\Omega);\ \Delta u \in C_0(\Omega)\} \subset C_0(\Omega).$$

Then $\tilde{\mathcal{A}}$ is also clearly sectorial of angle $\pi/2$ on $C_0(\Omega)$. Therefore $A = -(1+i\alpha)\tilde{\mathcal{A}}$ is sectorial of angle $\pi/2 - |\theta|$ on $C_0(\Omega)$, where $1 + i\alpha =: \rho e^{i\theta}$, $|\theta| < \pi/2$. This is because for any $\lambda \in \mathbb{C}$ with $|\theta| + \varepsilon < |\arg(\lambda)| \leqslant \pi$, $\lambda \neq 0$,

$$\|(\lambda - A)^{-1}\| = \rho^{-1}\left\|\left(\tilde{\mathcal{A}} - \frac{|\lambda|}{\rho}e^{i(\arg(\lambda)-\theta)}\right)^{-1}\right\| \leqslant \frac{C}{|\lambda|}.$$

The proof is complete. □

The next result relates the domains of the operator A_p^γ to the classical Sobolev (or Hölder) spaces.

Proposition 3.6. *Let A_p be the sectorial operator defined in Proposition 3.4, where $p > N$, and let $\gamma \in [0,1]$ be a given number. Then*

$$D(A_p^\gamma) \subset W^{1,q}(\Omega) \quad \text{when } 1 - \frac{N}{q} < 2\gamma - \frac{N}{p}, q \geqslant p, \tag{3.2}$$

$$D(A_p^\gamma) \subset C^\nu(\bar{\Omega}) \quad \text{when } 0 \leqslant \nu < 2\gamma - \frac{N}{p}. \tag{3.3}$$

Proof. A sketch of the proof is given in [12, Theorem 1.6.1] for the negative Laplacian ($\alpha = 0$). As the result is crucial, we provide the details here. According to the Gagliardo–Nirenberg inequality, we have for $1 - \frac{N}{q} < \theta(2 - N/p) - (1-\theta)N/p = 2\theta - N/p$, $0 \leqslant \theta < 1$ and $q \geqslant p$

$$\|u\|_{W^{1,q}(\Omega)} \leqslant C\|u\|_{W^{2,p}(\Omega)}^\theta \|u\|_{L^p(\Omega)}^{1-\theta}.$$

As $\|u\|_{W^{2,p}(\Omega)} \leqslant C\|A_p u\|_{L^p(\Omega)}$ for $u \in D(A_p)$, we obtain that

$$\|u\|_{W^{1,q}(\Omega)} \leqslant C\|A_p u\|_{L^p(\Omega)}^\theta \|u\|_{L^p(\Omega)}^{1-\theta}. \tag{3.4}$$

For any $\gamma > \theta$ and $u \in D(A_p^{1+\gamma})$,

$$u = A_p^{-\gamma}A_p^\gamma u = \frac{1}{\Gamma(\gamma)}\int_0^\infty t^{\gamma-1}e^{-A_p t}A_p^\gamma u\,dt.$$

Applying (3.4) with $v = A_p^\gamma u$,

$$\begin{aligned}
\|u\|_{W^{1,q}(\Omega)} &\leqslant C\int_0^\infty t^{\gamma-1}\left\|A_p e^{-A_p t}v\right\|_{L^p(\Omega)}^\theta \left\|e^{-A_p t}v\right\|_{L^p(\Omega)}^{1-\theta} dt \\
&\leqslant C\left(\int_0^\infty t^{\gamma-1-\theta}e^{-\delta t}\,dt\right)\|v\|_{L^p(\Omega)} \\
&\leqslant C\|u\|_{D(A_p^\gamma)}
\end{aligned}$$

where (3.1) has been applied twice. It follows that $D(A_p^\gamma) \subset W^{1,q}(\Omega)$ continuously. The second statement of the proposition can be proved along the same lines in using the following Gagliardo–Nirenberg inequality

$$\|u\|_{C^\nu(\bar{\Omega})} \leqslant C\|u\|_{W^{2,p}(\Omega)}^{\theta}\|u\|_{L^p(\Omega)}^{1-\theta},$$

valid for $\nu < \theta(2 - N/p) - N(1-\theta)/p = 2\theta - N/p$. The proof is complete. □

4. Proof of Theorem 2.1

Let $\tilde{\Omega} \subset \mathbb{R}^N$ be a bounded smooth open set such that

$$\Omega \subset \tilde{\Omega} \quad \text{and} \quad \partial\Omega \cap \partial\tilde{\Omega} = \partial\Omega \setminus \Gamma_0.$$

Let $\Gamma : z \in X \mapsto \tilde{z} \in C_0(\tilde{\Omega})$ be a extension map such that $z \equiv \tilde{z}$ in $\bar{\Omega}$ and $\|\tilde{z}\|_{C(\bar{\tilde{\Omega}})} = \|z\|_X$. Introduce the spaces

$$V := C([0,T]; X), \qquad \tilde{V} := C([0,T]; C(\bar{\tilde{\Omega}})) \quad \text{and} \quad \tilde{V}_0 := C([0,T]; C_0(\tilde{\Omega}))$$

all being endowed with the uniform norm.

In what follows, the letter C will denote a positive constant which may vary from line to line, and which may depend on the geometry ($\tilde{\Omega}$, S^-, etc.) or on the time T, but *not* on the functions z, d, or on the number R.

To prove Theorem 2.1, we need first to establish the null-controllability of a corresponding linearized equation.

4.1. Null controllability of a linearized equation

Let $g \in C(\mathbb{C}; \mathbb{C})$ be defined as

$$g(z) = \begin{cases} -\frac{f(z)}{z} & \text{if } z \neq 0, \\ -f'(0) & \text{if } z = 0. \end{cases}$$

For any given $z \in V$, set $\tilde{z} := \Gamma(z) \in \tilde{V}_0$ and $d := g(\tilde{z}) \in \tilde{V}$. We are first concerned with the following "linearized" control problem:

For any given initial state $u_0 \in X$, find a control input h such that the solution $u = u(x,t)$ of

$$\begin{cases} \partial_t u - (1+i\alpha)\Delta u + du = 0 & \text{in } Q := \Omega \times (0,T), \\ u = 1_{\Gamma_0} h & \text{on } \Sigma := \partial\Omega \times (0,T), \\ u(0) = u_0 \end{cases} \tag{4.1}$$

satisfies

$$u(T) = 0. \tag{4.2}$$

This problem will be solved by adapting the method developed by Fursikov–Imanuvilov in [10] to prove the null controllability of semilinear parabolic equations.

Consider the initial-boundary-value problem (IBVP)

$$\begin{cases} \partial_t v - (1+i\alpha)\Delta v + dv = 0 & \text{in } \tilde{Q} := \tilde{\Omega} \times (0,T), \\ v = 0 & \text{on } \tilde{\Sigma} := \partial\tilde{\Omega} \times (0,T), \\ v(0) = \tilde{u}_0 := \Gamma(u_0) \end{cases} \tag{4.3}$$

in which the initial condition in (4.1) comes into play. We first show that this problem is well posed in $\tilde{V}$. As the result will be needed later with a forcing term, we incorporate it now.

Lemma 4.1. *Let $d \in \tilde{V}$, $F \in \tilde{V}$, and $v_0 \in C_0(\tilde{\Omega})$. Then the system*

$$\begin{cases} \partial_t v - (1+i\alpha)\Delta v + dv = F & \text{in } \tilde{Q}, \\ v = 0 & \text{on } \tilde{\Sigma}, \\ v(0) = v_0 \end{cases} \tag{4.4}$$

possesses a unique mild solution $v \in \tilde{V}_0$. Furthermore, we have that

$$\|v(t)\|_{L^\infty(\tilde{\Omega})} \leqslant C(\|v_0\|_{L^\infty(\tilde{\Omega})} + \|F\|_{\tilde{V}})e^{C\|d\|_{\tilde{V}}t} \quad \forall t \in [0,T] \tag{4.5}$$

and

$$\|v(t)\|_{D(\tilde{A}_p^\gamma)} \leqslant C\big(t^{-\gamma}\|v_0\|_{L^p(\tilde{\Omega})} + \|F\|_{\tilde{V}} + \|d\|_{\tilde{V}}\|v\|_{\tilde{V}}\big) \tag{4.6}$$

for any $p \in (N,\infty)$ and any $\gamma \in (0,1)$.

Proof. Rewrite the IBVP (4.4) in its integral form

$$v(t) = e^{-\tilde{A}t}v_0 + \int_0^t e^{-\tilde{A}(t-s)}\big[F(s) - d(s)v(s)\big]\,ds. \tag{4.7}$$

Here the operator $\tilde{A}$ is as defined in Section 2, the notation $\tilde{A}$ meaning that the functions on which it operates are defined on $\tilde{\Omega}$.

Define a map Γ on the space $\tilde{V}_0(T) = C([0,T]; C_0(\tilde{\Omega}))$ by

$$\Gamma(v)(t) = e^{-\tilde{A}t}v_0 + \int_0^t e^{-\tilde{A}(t-s)}\big[F(s) - d(s)v(s)\big]\,ds.$$

Then

$$\begin{aligned}\|\Gamma(v)(t)\|_{L^\infty(\tilde{\Omega})} &\leqslant \|e^{-\tilde{A}t}v_0\|_{L^\infty(\tilde{\Omega})} + \left\|\int_0^t e^{-\tilde{A}(t-s)}\big[F(s) - d(s)v(s)\big]\,ds\right\|_{L^\infty(\tilde{\Omega})} \\ &\leqslant C\|v_0\|_{L^\infty(\tilde{\Omega})} + Ct\big(\|F\|_{\tilde{V}_0(T)} + \|dv\|_{\tilde{V}_0(T)}\big).\end{aligned}$$

Let $R = 2C(\|v_0\|_{L^\infty(\tilde{\Omega})} + T\|F\|_{\tilde{V}_0(T)})$ and pick a time $T' < T$ such that $CT'\|d\|_{\tilde{V}(T)} < 1/2$. Then

$$\|v\|_{\tilde{V}_0(T')} \leqslant R \Rightarrow \|\Gamma(v)\|_{\tilde{V}_0(T')} \leqslant R,$$
$$\|\Gamma(v_1) - \Gamma(v_2)\|_{\tilde{V}_0(T')} \leqslant \frac{1}{2}\|v_1 - v_2\|_{\tilde{V}_0(T')}.$$

Thus, by the contraction mapping theorem, the map Γ admits a unique fixed-point in the ball $B_R(0) \subset \tilde{V}_0(T')$. Notice that T' does not depend on $\|v_0\|_{L^\infty(\tilde{\Omega})}$. The solution v to (4.4) may be extended from $[0,T']$ to the interval $[0,T]$ by a standard argument. Let us proceed to the proof of (4.5) and (4.6). It follows from (4.7) that

$$\|v(t)\|_{L^\infty(\tilde{\Omega})} \leqslant C\left(\|v_0\|_{L^\infty(\tilde{\Omega})} + \|F\|_{\tilde{V}} + \|d\|_{\tilde{V}}\int_0^t \|v(s)\|_{L^\infty(\tilde{\Omega})}\,ds\right).$$

Hence an application of Gronwall lemma gives (4.5). Finally, (4.6) follows from (4.5) and Lemma 6.3 presented in Section 6. The proof is complete. □

According to Lemma 4.1, the IBVP (4.3) admits a unique mild solution $v \in \tilde{V}_0$. Furthermore, according to Lemmas 6.1, 6.2 and 6.3, we have that

$$v \in L^2\big(0,T; H_0^1(\tilde{\Omega})\big), \qquad \sqrt{t}v \in L^2\big(0,T; H^2(\tilde{\Omega})\big), \quad \text{and} \quad \sqrt{t}\partial_t v \in L^2\big(0,T; L^2(\tilde{\Omega})\big)$$

with

$$\|v\|_{\tilde{V}} + \|v\|_{L^2(0,T;H^1(\Omega))} + \|\sqrt{t}v\|_{L^2(0,T;H^2(\tilde{\Omega}))} + \|\sqrt{t}\partial_t v\|_{L^2(0,T;L^2(\tilde{\Omega}))} \leqslant e^{C(1+\|d\|_{\tilde{V}})}\|u_0\|_X \tag{4.8}$$

and $v \in C((0,T]; D(\tilde{A}_p^\gamma))$ for all $p \in (N,+\infty)$ and all $\gamma \in [0,1)$ with

$$\|v(t)\|_{D(\tilde{A}_p^\gamma)} \leqslant C\big(t^{-\gamma}\|u_0\|_X + e^{C(1+\|d\|_{\tilde{V}})}\|u_0\|_X\big) \quad \forall t \in (0,T]. \tag{4.9}$$

Let ξ be any function of class C^∞ on $[0,T]$ such that

$$\xi(t) = \begin{cases} 1 & \text{for } t \leqslant \frac{T}{3}, \\ 0 & \text{for } t \geqslant \frac{2T}{3}. \end{cases} \tag{4.10}$$

Let $S^- \subset \partial\tilde\Omega$ be an open neighborhood of $\partial\Omega \setminus \Gamma_0$, with $S^- \neq \partial\tilde\Omega$, and let

$$S^+ := \partial\tilde\Omega \setminus S^-, \quad \tilde\Sigma := \partial\tilde\Omega \times (0,T), \quad \text{and} \quad \tilde\Sigma^\pm := S^\pm \times (0,T).$$

To define a solution u of (4.1)–(4.2), we seek an extension of it on $\tilde Q$, again denoted u, in the form

$$u(x,t) = \xi(t)v(x,t) + w(x,t),$$

where v denotes the solution of (4.3). Set

$$L = L(d) := \partial_t - (1+i\alpha)\Delta + d.$$

Then $0 = Lu = \xi'(t)v + Lw$ in Q. We are led to "define" w as a solution of the system

$$\begin{cases} Lw = \partial_t w - (1+i\alpha)\Delta w + dw = f & \text{in } \tilde Q, \\ w = 0 & \text{on } \tilde\Sigma^-, \\ w(0) = w(T) = 0 \end{cases} \tag{4.11}$$

with

$$f(x,t) := -\xi'(t)v. \tag{4.12}$$

Indeed, if w solves (4.11), then $u_{|Q}$ solves (4.1)–(4.2), the boundary control h being defined as the trace $u_{|\partial\Omega\times(0,T)}$.

Notice that

$$\operatorname{Supp} f \subset \bar{\tilde\Omega} \times \left[\frac{T}{3}, \frac{2T}{3}\right] \tag{4.13}$$

by (4.10), and that $f \in \tilde V \cap L^2(0,T;H^2(\tilde\Omega)) \cap H^1(0,T;L^2(\tilde\Omega)) \cap C([0,T];D(\tilde A_p^\gamma))$ for all $p > N$ and $\gamma < 1$. Furthermore, by (4.8)–(4.10), for any $p > N$ and any $\gamma < 1$ we have

$$\|f\|_{C([0,T];D(\tilde A_p^\gamma))} + \|f\|_{L^2(0,T;H^2(\tilde\Omega))} + \|f\|_{H^1(0,T;L^2(\tilde\Omega))} \leqslant e^{C(1+\|d\|_{\tilde V})}\|u_0\|_X. \tag{4.14}$$

To prove the existence of a solution $w \in L^2(\tilde Q)$ to (4.11) we need a Carleman estimate associated with the Ginzburg–Landau equation, which is stated and proved in the next subsection.

4.2. A Carleman estimate

Let n denote the unit outward normal vector to $\tilde\Omega$, and $\partial_n u = \partial u/\partial n$. The following lemma, which is essentially a result from [10], will be needed.

Lemma 4.2. *There exists a function $\psi \in C^\infty(\bar{\tilde\Omega})$ such that $\psi > 0$ and $\nabla\psi \neq 0$ on $\bar{\tilde\Omega}$ and $\partial_n\psi < 0$ on S^-.*

Proof. According to [10], there exists a function $\psi_1 \in C^2(\bar{\tilde\Omega})$ such that $\psi_1 > 0$ and $\nabla\psi_1 \neq 0$ on $\bar{\tilde\Omega}$ and $\partial_n\psi_1 \leqslant 0$ on S^-. We claim that there exists a function $\psi_2 \in C^2(\mathbb{R}^N)$ such that $\partial_n\psi_2 < 0$ on $\partial\tilde\Omega$. Notice that the lemma follows at once from the claim, by smoothing by mollification the function $\psi_1 + \varepsilon\psi_2$ where $\varepsilon > 0$ is a small enough number. It remains to prove the claim. Since $\tilde\Omega$ is of class C^2 and bounded, there exist some charts $H_i : U_i \to \mathbb{R}^N$, $i = 1,\dots,k$, such that (i) $(U_i)_{1\leqslant i\leqslant k}$ is an open covering of $\partial\tilde\Omega$; (ii) H_i is a bijection from U_i onto $B_1(0)$, $H_i \in C^2(U_i, B_1(0))$ and $H_i^{-1} \in C^2(B_1(0), U_i)$ for $i = 1,\dots,k$; (iii) $H_i(U_i \cap \tilde\Omega) = B_1(0) \cap \{x_N > 0\}$ and $H_i(U_i \cap \partial\tilde\Omega) = B_1(0) \cap \{x_N = 0\}$. Then we consider a partition of unity $(\theta_i)_{i=0,\dots,k}$ satisfying (i) $\theta_i \in C^\infty(\mathbb{R}^N)$ and $0 \leqslant \theta_i \leqslant 1$ for $i = 0,\dots,k$ and $\sum_{i=0}^k \theta_i = 1$ on $\mathbb{R}^N$; (ii) the support of θ_i is a compact set included in U_i for $i = 1,\dots,k$, and the support of θ_0 does not intersect $\partial\tilde\Omega$. Let us define ψ_2 as

$$\psi_2(x) = \sum_{i=1}^k \theta_i(x) p_N\big(H_i(x)\big)$$

where $p_N(x_1,\dots,x_N) = x_N$ is the projection along the x_N axis. Notice that $p_N(H_i(x)) = 0$ for $x \in U_i \cap \partial\tilde\Omega$, hence

$$\partial_n\psi_2(x) = \sum_{i=1}^k \theta_i(x)\partial_n\big[p_N\big(H_i(x)\big)\big]$$

Clearly, $\partial_n[p_N(H_i(x))] \leqslant 0$, since $p_N(H_i(x)) > 0$ for $x \in U_i \cap \tilde{\Omega}$. Actually, $\partial_n[p_N(H_i(x))] < 0$ in $U_i \cap \partial\tilde{\Omega}$, otherwise we should have $\nabla[p_N(H_i(x))] = 0$ and also $(\nabla p_N)(H_i(x)) = 0$, which is absurd. We conclude that $\partial_n \psi_2 < 0$ on $\partial\tilde{\Omega}$, as desired. □

Pick a function ψ as in Lemma 4.2. Replacing ψ by $\psi + C$, where $C > 0$ is a large enough number, we may as well assume that

$$\psi(x) > \frac{3}{4}\|\psi\|_{L^\infty(\tilde{\Omega})} \quad \forall x \in \tilde{\Omega}. \tag{4.15}$$

That property will be used later. Set $C_\psi = (3/2)\|\psi\|_{L^\infty(\tilde{\Omega})}$ and

$$\theta(x,t) := \frac{e^{\lambda\psi(x)}}{t(T-t)}, \quad \varphi(x,t) := \frac{e^{\lambda C_\psi} - e^{\lambda\psi(x)}}{t(T-t)}, \quad \forall (x,t) \in \tilde{\Omega} \times (0,T) \tag{4.16}$$

where λ denotes some positive number whose range will be specified later. We also introduce the set

$$\mathcal{Z} := \{q \in C^{2,1}(\bar{\tilde{\Omega}} \times [0,T]);\ q = 0 \text{ on } \tilde{\Sigma}\}.$$

The following result is a Carleman estimate for the Ginzburg–Landau equation.

Proposition 4.3. *Let $a > 0$ and $b \in \mathbb{R}$. Then there exist some constants $\lambda_0 \geqslant 1$, $s_0 \geqslant 1$, and $C_0 > 0$ such that for all $\lambda \geqslant \lambda_0$, $s \geqslant s_0$ and all $q \in \mathcal{Z}$ it holds*

$$\begin{aligned}&\int_0^T\!\!\int_{\tilde{\Omega}} \big[(s\theta)^{-1}\big(|\partial_t q|^2 + |\Delta q|^2\big) + \lambda^2(s\theta)|\nabla q|^2 + \lambda^4(s\theta)^3|q|^2\big]e^{-2s\varphi}\,dx\,dt + \int_0^T\!\!\int_{S^-} \lambda(s\theta)|\partial_n\psi||\partial_n q|^2 e^{-2s\varphi}\,d\sigma\,dt \\ &\quad\leqslant C_0\left(\int_0^T\!\!\int_{\tilde{\Omega}} \big|\partial_t q + (a+ib)\Delta q\big|^2 e^{-2s\varphi}\,dx\,dt + \int_0^T\!\!\int_{S^+} \lambda(s\theta)|\partial_n\psi||\partial_n q|^2 e^{-2s\varphi}\,d\sigma\,dt\right).\end{aligned} \tag{4.17}$$

Proof. In what follows, the letter C will denote a constant (independent of s, λ, q) which may vary from line to line. Let $q \in \mathcal{Z}$ be given. Set $u = e^{-s\varphi}q$ and $w = e^{-s\varphi}P(q) = e^{-s\varphi}P(e^{s\varphi}u)$, where P denotes the operator

$$P = \partial_t + (a+ib)\Delta.$$

Straightforward computations yield that

$$w = Mu := u_t + s\varphi_t u + (a+ib)\big(\Delta u + 2s\nabla\varphi\cdot\nabla u + s(\Delta\varphi)u + s^2|\nabla\varphi|^2 u\big)$$

with the convention that

$$z\cdot z' = \sum_{i=1}^N z_i z_i' \quad \text{for all } z = (z_1,\dots,z_N) \in \mathbb{C}^N,\ z' = (z_1',\dots,z_N') \in \mathbb{C}^N.$$

Let M_1 and M_2 denote respectively the (formal) adjoint and skew-adjoint parts of the operator M, i.e.,

$$M_1 u := a\Delta u + ib\big(2s\nabla\varphi\cdot\nabla u + s(\Delta\varphi)u\big) + s\varphi_t u + as^2|\nabla\varphi|^2 u, \tag{4.18}$$

$$M_2 u := u_t + ib\big(\Delta u + s^2|\nabla\varphi|^2 u\big) + a\big(2s\nabla\varphi\cdot\nabla u + s(\Delta\varphi)u\big). \tag{4.19}$$

Thus

$$\|w\|^2 = \|M_1 u + M_2 u\|^2 = \|M_1 u\|^2 + \|M_2 u\|^2 + 2\,\mathrm{Re}(M_1 u, M_2 u) \tag{4.20}$$

where $(u,v) := \int_0^T\int_{\tilde{\Omega}} u(x,t)\overline{v(x,t)}\,dx\,dt$ and $\|w\|^2 = (w,w)$. From now on, for the sake of brevity, we write $\iint u$ (resp. $\iint_{\tilde{\Sigma}^\pm} u$) instead of $\int_0^T\int_{\tilde{\Omega}} u(x,t)\,dx\,dt$ (resp. $\int_0^T\int_{S^\pm} u(x,t)\,d\sigma\,dt$).

STEP 1. EXACT COMPUTATION OF THE SCALAR PRODUCT IN (4.20).
The scalar product in (4.20) is decomposed into the sum of six integral terms, namely

$$2\,\mathrm{Re}(M_1u, M_2u) = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 \tag{4.21}$$

with

$$I_1 := 2\,\mathrm{Re}\iint a\Delta u(\bar{u}_t - ib\Delta\bar{u}), \tag{4.22}$$

$$I_2 := 2\,\mathrm{Re}\iint a\Delta u\big(-ibs^2|\nabla\varphi|^2\bar{u} + 2as\nabla\varphi\cdot\nabla\bar{u} + as(\Delta\varphi)\bar{u}\big), \tag{4.23}$$

$$I_3 := 2\,\mathrm{Re}\iint (ib)\big(2s\nabla\varphi\cdot\nabla u + s(\Delta\varphi)u\big)\big(\bar{u}_t - ib\big(\Delta\bar{u} + s^2|\nabla\varphi|^2\bar{u}\big)\big), \tag{4.24}$$

$$I_4 := 2\,\mathrm{Re}\iint (ib)\big(2s\nabla\varphi\cdot\nabla u + s(\Delta\varphi)u\big)a\big(2s\nabla\varphi\cdot\nabla\bar{u} + s(\Delta\varphi)\bar{u}\big), \tag{4.25}$$

$$I_5 := 2\,\mathrm{Re}\iint \big(s\varphi_t u + as^2|\nabla\varphi|^2u\big)(\bar{u}_t - ib\Delta\bar{u} + 2as\nabla\varphi\cdot\nabla\bar{u}), \tag{4.26}$$

$$I_6 := 2\,\mathrm{Re}\iint \big(s\varphi_t u + as^2|\nabla\varphi|^2u\big)\big(-ibs^2|\nabla\varphi|^2\bar{u} + as(\Delta\varphi)\bar{u}\big). \tag{4.27}$$

First, observe that

$$I_1 = -a\iint \partial_t|\nabla u|^2 = 0. \tag{4.28}$$

We obtain after some integrations by parts that

$$\begin{aligned} I_2 &= 2\,\mathrm{Re}\bigg\{iabs^2\iint \nabla u\cdot\big(\nabla|\nabla\varphi|^2\bar{u} + |\nabla\varphi|^2\nabla\bar{u}\big) + 2a^2s\iint \Delta u\nabla\varphi\cdot\nabla\bar{u} \\ &\quad - a^2s\iint \nabla u\cdot\big(\nabla(\Delta\varphi)\bar{u} + \Delta\varphi\nabla\bar{u}\big)\bigg\} \\ &= 2\,\mathrm{Re}\bigg\{iabs^2\iint \nabla u\cdot\nabla|\nabla\varphi|^2\bar{u} + 2a^2s\iint \Delta u\nabla\varphi\cdot\nabla\bar{u}\bigg\} + a^2s\iint\big(\Delta^2\varphi|u|^2 - 2\Delta\varphi|\nabla u|^2\big). \end{aligned} \tag{4.29}$$

Let us compute the integral term $J := \iint \Delta u(\nabla\varphi\cdot\nabla\bar{u})$. Using the convention of repeated indices, we obtain that

$$J = \iint \partial_j^2 u\partial_i\varphi\partial_i\bar{u} = -\iint \partial_j u(\partial_j\partial_i\varphi\partial_i\bar{u} + \partial_i\varphi\partial_j\partial_i\bar{u}) + \iint_{\tilde{\Sigma}} (\partial_j u)n_j\partial_i\varphi\partial_i\bar{u}.$$

Since $u = 0$ on $\tilde{\Sigma}$, $\nabla u = (\partial_n u)n$, we have $\nabla\varphi\cdot\nabla\bar{u} = \partial_n\varphi\partial_n\bar{u}$ and

$$\iint_{\tilde{\Sigma}} (\partial_j u)n_j\partial_i\varphi\partial_i\bar{u} = \iint_{\tilde{\Sigma}} \partial_n\varphi|\partial_n u|^2.$$

On the other hand

$$\begin{aligned} 2\,\mathrm{Re}\iint \partial_j u\partial_i\varphi\partial_j\partial_i\bar{u} &= \iint \partial_i\varphi\partial_i(\partial_j u\partial_j\bar{u}) \\ &= -\iint \partial_i^2\varphi|\partial_j u|^2 + \iint_{\tilde{\Sigma}} (\partial_i\varphi)n_i\partial_j u\partial_j\bar{u} \\ &= -\iint \Delta\varphi|\nabla u|^2 + \iint_{\tilde{\Sigma}} \partial_n\varphi|\partial_n u|^2. \end{aligned}$$

Thus

$$2\,\mathrm{Re}\,J = \iint \Delta\varphi|\nabla u|^2 - 2\iint \partial_j\partial_i\varphi\partial_j u\partial_i\bar{u} + \iint_{\tilde{\Sigma}} \partial_n\varphi|\partial_n u|^2 \tag{4.30}$$

and, consequently,

$$
\begin{aligned}
I_2 = {} & 2\,\mathrm{Re}\biggl\{iabs^2 \iint \bigl(\nabla u\cdot\nabla|\nabla\varphi|^2\bigr)\bar u\biggr\}\\
& + a^2 s\biggl(\iint \Delta^2\varphi|u|^2 - 4\iint \partial_j\partial_i\varphi\partial_j u\partial_i\bar u + 2\iint_{\tilde\Sigma} \partial_n\varphi|\partial_n u|^2\biggr).
\end{aligned}
\tag{4.31}
$$

We now turn to the computation of I_3. Rewrite I_3 as

$$
\begin{aligned}
I_3 = {} & 2\,\mathrm{Re}\biggl\{b^2\iint \bigl(2s\nabla\varphi\cdot\nabla u + s(\Delta\varphi)u\bigr)\bigl(\Delta\bar u + s^2|\nabla\varphi|^2\bar u\bigr)\biggr\} + 2\,\mathrm{Re}\biggl\{ib\iint \bigl(2s\nabla\varphi\cdot\nabla u + s(\Delta\varphi)u\bigr)\bar u_t\biggr\}\\
=: {} & I_3^1 + I_3^2.
\end{aligned}
$$

Expanding I_3^1, performing integrations by parts and using (4.30), we find that

$$
\begin{aligned}
I_3^1 = {} & 2\,\mathrm{Re}\biggl\{2b^2 sJ + b^2\iint s(\Delta\varphi)u\Delta\bar u + b^2\iint 2s^3\nabla\varphi\cdot\nabla u|\nabla\varphi|^2\bar u + b^2\iint s^3\Delta\varphi|\nabla\varphi|^2|u|^2\biggr\}\\
= {} & 2b^2 s(2\,\mathrm{Re}\,J) - 2b^2 s\,\mathrm{Re}\iint \bigl(\nabla(\Delta\varphi)u + \Delta\varphi\nabla u\bigr)\cdot\nabla\bar u \qquad (4.32)\\
& + 2b^2 s^3\iint |\nabla\varphi|^2\nabla\varphi\cdot\nabla|u|^2 + 2b^2 s^3\iint \Delta\varphi|\nabla\varphi|^2|u|^2\\
= {} & 2b^2 s\biggl(\iint \bigl(\Delta\varphi|\nabla u|^2 - 2\partial_j\partial_i\varphi\partial_j u\partial_i\bar u\bigr) + \iint_{\tilde\Sigma} \partial_n\varphi|\partial_n u|^2\biggr)\\
& + b^2 s\biggl(\iint \Delta^2\varphi|u|^2 - 2\iint \Delta\varphi|\nabla u|^2\biggr) - 2b^2 s^3\iint \bigl(\nabla|\nabla\varphi|^2\cdot\nabla\varphi\bigr)|u|^2. \qquad (4.33)
\end{aligned}
$$

On the other hand, integrating by parts with respect to x or t in I_3^2, we obtain

$$
\begin{aligned}
I_3^2 = {} & ib\iint \bigl(2s\nabla\varphi\cdot\nabla u + s(\Delta\varphi)u\bigr)\bar u_t - ib\iint \bigl(2s\nabla\varphi\cdot\nabla\bar u + s(\Delta\varphi)\bar u\bigr)u_t\\
= {} & -ib\iint \bigl(2s\nabla\varphi_t\cdot\nabla u + 2s\nabla\varphi\cdot\nabla u_t + s(\Delta\varphi_t)u + s(\Delta\varphi)u_t\bigr)\bar u\\
& + ib\iint \bigl(2s(\Delta\varphi)\bar u u_t + 2s\nabla\varphi\cdot\nabla u_t\bar u - s(\Delta\varphi)\bar u u_t\bigr)\\
= {} & ib\iint \bigl(s\nabla\varphi_t\cdot\nabla|u|^2 - 2s\nabla\varphi_t\cdot\nabla u\bar u\bigr)\\
= {} & ib\iint s\nabla\varphi_t\cdot(u\nabla\bar u - \bar u\nabla u).
\end{aligned}
\tag{4.34}
$$

Gathering together (4.33) and (4.34), we arrive at

$$
\begin{aligned}
I_3 = {} & b^2 s\biggl(\iint -4\partial_j\partial_i\varphi\partial_j u\partial_i\bar u + 2\iint_{\tilde\Sigma} \partial_n\varphi|\partial_n u|^2 + \iint \Delta^2\varphi|u|^2\biggr)\\
& -2b^2 s^3\iint \bigl(\nabla|\nabla\varphi|^2\cdot\nabla\varphi\bigr)|u|^2 + 2\,\mathrm{Re}\biggl\{ibs\iint \nabla\varphi_t\cdot(u\nabla\bar u)\biggr\}.
\end{aligned}
\tag{4.35}
$$

Note that

$$
I_4 = 0 \tag{4.36}
$$

obviously. Furthermore,

$$
\begin{aligned}
I_5 = {} & 2\,\mathrm{Re}\iint \bigl(s\varphi_t u + as^2|\nabla\varphi|^2 u\bigr)\bar u_t + 2\,\mathrm{Re}\iint \bigl(s\varphi_t u + as^2|\nabla\varphi|^2 u\bigr)(-ib\Delta\bar u + 2as\nabla\varphi\cdot\nabla\bar u)\\
=: {} & I_5^1 + I_5^2.
\end{aligned}
$$

Integrating by parts with respect to t in I_5^1 yields

$$I_5^1 = -\iint \left(s\varphi_{tt} + as^2\partial_t|\nabla\varphi|^2\right)|u|^2. \tag{4.37}$$

As for I_5^2, we have

$$\begin{aligned}
I_5^2 &= \iint \left(s\varphi_t + as^2|\nabla\varphi|^2\right)\left(-ibu\Delta\bar{u} + ib\bar{u}\Delta u + 2as\nabla\varphi\cdot\nabla|u|^2\right)\\
&= -\iint \left(s\nabla\varphi_t + as^2\nabla|\nabla\varphi|^2\right)\cdot\left(-ibu\nabla\bar{u} + ib\bar{u}\nabla u + 2as|u|^2\nabla\varphi\right) - \iint \left(s\varphi_t + as^2|\nabla\varphi|^2\right)(2as\Delta\varphi)|u|^2\\
&= -2\operatorname{Re}\left\{ib\iint \left(s\nabla\varphi_t + as^2\nabla|\nabla\varphi|^2\right)\bar{u}\nabla u\right\} - \iint \left\{2as^2\nabla\cdot(\varphi_t\nabla\varphi) + 2a^2s^3\nabla\cdot\left(|\nabla\varphi|^2\nabla\varphi\right)\right\}|u|^2.
\end{aligned} \tag{4.38}$$

Finally,

$$I_6 = 2\iint \left(s\varphi_t + as^2|\nabla\varphi|^2\right)(as\Delta\varphi)|u|^2. \tag{4.39}$$

Gathering together (4.31), (4.35), (4.37), (4.38), and (4.39), we infer that

$$\begin{aligned}
&2\operatorname{Re}(M_1u, M_2u)\\
&\quad= 2\operatorname{Re}\left\{iabs^2\iint \left(\nabla u\cdot\nabla|\nabla\varphi|^2\right)\bar{u}\right\} + a^2s\left(\iint \Delta^2\varphi|u|^2 - 4\iint \partial_j\partial_i\varphi\partial_j u\partial_i\bar{u} + 2\iint_{\tilde{\Sigma}} \partial_n\varphi|\partial_n u|^2\right)\\
&\qquad + b^2s\left(\iint -4\partial_j\partial_i\varphi\partial_j u\partial_i\bar{u} + 2\iint_{\tilde{\Sigma}} \partial_n\varphi|\partial_n u|^2 + \iint \Delta^2\varphi|u|^2\right) - 2b^2s^3\iint \left(\nabla|\nabla\varphi|^2\cdot\nabla\varphi\right)|u|^2\\
&\qquad + 2\operatorname{Re}\left\{ibs\iint \nabla\varphi_t\cdot(u\nabla\bar{u})\right\} - \iint \left(s\varphi_{tt} + as^2\partial_t|\nabla\varphi|^2\right)|u|^2 + 2\iint \left(s\varphi_t + as^2|\nabla\varphi|^2\right)(as\Delta\varphi)|u|^2\\
&\qquad - 2\operatorname{Re}\left\{ib\iint \left(s\nabla\varphi_t + as^2\nabla|\nabla\varphi|^2\right)\bar{u}\nabla u\right\} - \iint \left\{2as^2\nabla\cdot(\varphi_t\nabla\varphi) + 2a^2s^3\nabla\cdot\left(|\nabla\varphi|^2\nabla\varphi\right)\right\}|u|^2\\
&\quad= 2\left(a^2+b^2\right)s\left(-2\iint \partial_j\partial_i\varphi\partial_j u\partial_i\bar{u} + \iint_{\tilde{\Sigma}} \partial_n\varphi|\partial_n u|^2\right) + 4\operatorname{Re}\left\{ib\iint s\nabla\varphi_t\cdot(u\nabla\bar{u})\right\}\\
&\qquad + \iint |u|^2\left[s\left(\left(a^2+b^2\right)\Delta^2\varphi - \varphi_{tt}\right) - 2as^2\partial_t|\nabla\varphi|^2 - 2\left(a^2+b^2\right)s^3\nabla\varphi\cdot\nabla|\nabla\varphi|^2\right].
\end{aligned} \tag{4.40}$$

Consequently, (4.20) can be rewritten as

$$\begin{aligned}
\|w\|^2 &= \|M_1u\|^2 + \|M_2u\|^2\\
&\quad + 2\left(a^2+b^2\right)s\left(-2\iint \partial_j\partial_i\varphi\partial_j u\partial_i\bar{u} + \iint_{\tilde{\Sigma}} \partial_n\varphi|\partial_n u|^2\right) + 4\operatorname{Re}\left\{ib\iint s\nabla\varphi_t\cdot(u\nabla\bar{u})\right\}\\
&\quad + \iint |u|^2\left[s\left(\left(a^2+b^2\right)\Delta^2\varphi - \varphi_{tt}\right) - 2as^2\partial_t|\nabla\varphi|^2 - 2\left(a^2+b^2\right)s^3\nabla\varphi\cdot\nabla|\nabla\varphi|^2\right].
\end{aligned} \tag{4.41}$$

STEP 2. ESTIMATION OF THE TERMS IN (4.41).
We now have to bound from below the terms in the right-hand side of (4.41). We begin with the

Claim 1. *There exist two numbers $\lambda_1 \geqslant 1$ and $s_1 \geqslant 1$ and some constant $A > 0$ such that for all $\lambda \geqslant \lambda_1$ and all $s \geqslant s_1$,*

$$\begin{aligned}
&\iint |u|^2\left[s\left(\left(a^2+b^2\right)\Delta^2\varphi - \varphi_{tt}\right) - 2as^2\partial_t|\nabla\varphi|^2 - 2\left(a^2+b^2\right)s^3\nabla\varphi\cdot\nabla|\nabla\varphi|^2\right]\\
&\quad \geqslant A\lambda s^3\iint |u|^2|\nabla\varphi|^2.
\end{aligned} \tag{4.42}$$

Proof of Claim 1. Easy computations show that

$$\partial_i\varphi = -\frac{\lambda e^{\lambda\psi(x)}\partial_i\psi}{t(T-t)}, \qquad \partial_j\partial_i\varphi = -\frac{e^{\lambda\psi(x)}}{t(T-t)}\big(\lambda^2\partial_i\psi\partial_j\psi + \lambda\partial_j\partial_i\psi\big) \tag{4.43}$$

and

$$-\nabla|\nabla\varphi|^2\cdot\nabla\varphi = -2(\partial_j\partial_i\varphi)\partial_i\varphi\partial_j\varphi = 2\left(\frac{\lambda e^{\lambda\psi(x)}}{t(T-t)}\right)^3\big(\lambda|\nabla\psi|^4 + \partial_j\partial_i\psi\partial_i\psi\partial_j\psi\big).$$

Since $\nabla\psi \neq 0$ on $\bar{\tilde{\Omega}}$, it follows that for λ large enough, say $\lambda \geqslant \lambda_1$, we have

$$-\nabla|\nabla\varphi|^2\cdot\nabla\varphi \geqslant C\lambda|\nabla\varphi|^3.$$

According to (4.15),

$$\big|\Delta^2\varphi\big| + |\varphi_{tt}| + \big|\partial_t|\nabla\varphi|^2\big| \leqslant C\lambda|\nabla\varphi|^3,$$

we infer that for s large enough, say $s \geqslant s_1$, and for all $\lambda \geqslant \lambda_1$, we have that

$$s\big((a^2+b^2)\Delta^2\varphi - \varphi_{tt}\big) - 2as^2\partial_t|\nabla\varphi|^2 - 2(a^2+b^2)s^3\nabla\varphi\cdot\nabla|\nabla\varphi|^2 \geqslant A\lambda s^3|\nabla\varphi|^3 \tag{4.44}$$

for some constant $A > 0$. Integrating in (4.44) leads at once to (4.42). $\square$

Thus, using the fact that $\partial_n\varphi = -\lambda e^{\lambda\psi(x)}(\partial_n\psi)t^{-1}(T-t)^{-1} \geqslant 0$ on Σ^-, we conclude that

$$\begin{aligned}
&\|M_1u\|^2 + \|M_2u\|^2 + A\lambda s^3\iint|\nabla\varphi|^3|u|^2 + 2(a^2+b^2)s\iint_{\Sigma^-}|\partial_n\varphi||\partial_nu|^2\\
&\quad\leqslant \|w\|^2 + 2(a^2+b^2)s\iint_{\Sigma^+}|\partial_n\varphi||\partial_nu|^2 + 4(a^2+b^2)s\iint\partial_j\partial_i\varphi\partial_ju\partial_i\bar{u}\\
&\qquad - 4\operatorname{Re}\left\{ib\iint s\nabla\varphi_t\cdot(u\nabla\bar{u})\right\}.
\end{aligned} \tag{4.45}$$

To control the two last terms in (4.45) we need to prove the following claim.

Claim 2. *There exist some numbers $s_2 \geqslant s_1$, $\lambda_2 \geqslant \lambda_1$, and a positive constant C such that for all $\lambda \geqslant \lambda_2$, $s \geqslant s_2$*

$$\lambda s\iint|\nabla\varphi||\nabla u|^2 + \lambda s^{-1}\iint|\nabla\varphi|^{-1}|\Delta u|^2 \leqslant C\left(s^{-1}\|M_1u\|^2 + \lambda s^3\iint|\nabla\varphi|^3|u|^2\right). \tag{4.46}$$

Proof of Claim 2. By (4.18)

$$\begin{aligned}
a^2s^{-1}\iint|\nabla\varphi|^{-1}|\Delta u|^2 &= s^{-1}\iint|\nabla\varphi|^{-1}\big|M_1u - ib\big(2s\nabla\varphi\cdot\nabla u + s(\Delta\varphi)u\big) - s\varphi_tu - as^2|\nabla\varphi|^2u\big|^2\\
&\leqslant Cs^{-1}\iint|\nabla\varphi|^{-1}\big\{|M_1u|^2 + 4b^2s^2|\nabla\varphi|^2|\nabla u|^2 + s^2\big(b^2|\Delta\varphi|^2 + |\varphi_t|^2\big)|u|^2\\
&\quad + a^2s^4|\nabla\varphi|^4|u|^2\big\}\\
&\leqslant A'\left(\frac{\|M_1u\|^2}{\lambda s} + s^3\iint|\nabla\varphi|^3|u|^2 + s\iint|\nabla\varphi||\nabla u|^2\right)
\end{aligned} \tag{4.47}$$

for some constant $A' > 0$, where we used (4.15) in the last line to bound φ_t. On the other hand, for any $\varepsilon > 0$ small enough,

$$\begin{aligned}
\lambda s\iint|\nabla\varphi||\nabla u|^2 &= \lambda s\operatorname{Re}\left\{\iint|\nabla\varphi|(-\Delta u)\bar{u} - \iint\big(\nabla|\nabla\varphi|\cdot\nabla u\big)\bar{u}\right\}\\
&\leqslant \frac{\varepsilon\lambda}{2s}\iint|\nabla\varphi|^{-1}|\Delta u|^2 + \frac{\lambda s^3}{2\varepsilon}\iint|\nabla\varphi|^3|u|^2 + \frac{\lambda s}{2}\iint\Delta|\nabla\varphi||u|^2
\end{aligned}$$

$$\leqslant \frac{\varepsilon A'}{2a^2}\left(s^{-1}\|M_1u\|^2+\lambda s^3\iint |\nabla\varphi|^3|u|^2+\lambda s\iint |\nabla\varphi||\nabla u|^2\right) + C\lambda s^3\iint |\nabla\varphi|^3|u|^2 \tag{4.48}$$

by (4.47), provided that $s\geqslant s_2$, $\lambda\geqslant\lambda_2$. (Here, C depends on ε.)

Therefore, if we pick ε so that $(\varepsilon A')/(2a^2)<1/2$, we obtain that

$$\lambda s\iint |\nabla\varphi||\nabla u|^2\leqslant C\left(s^{-1}\|M_1u\|^2+\lambda s^3\iint |\nabla\varphi|^3|u|^2\right) \tag{4.49}$$

and also, by (4.47),

$$\lambda s^{-1}\iint |\nabla\varphi|^{-1}|\Delta u|^2\leqslant C\left(s^{-1}\|M_1u\|^2+\lambda s^3\iint |\nabla\varphi|^3|u|^2\right). \tag{4.50}$$

Then (4.46) follows from (4.49)–(4.50). $\square$

We infer from (4.45) and (4.46) that

$$\begin{aligned}&\|M_2u\|^2+\lambda s\iint |\nabla\varphi\|\nabla u|^2+\lambda s^{-1}\iint |\nabla\varphi|^{-1}|\Delta u|^2+\lambda s^3\iint |\nabla\varphi|^3|u|^2+s\iint_{\Sigma^-} |\partial_n\varphi\|\partial_n u|^2\\ &\quad\leqslant C\bigg(\|w\|^2+2\big(a^2+b^2\big)s\iint_{\Sigma^+} |\partial_n\varphi\|\partial_n u|^2+4\big(a^2+b^2\big)s\iint \partial_j\partial_i\varphi\partial_j u\partial_i\bar u\\ &\qquad -4\operatorname{Re}\left\{ib\iint s\nabla\varphi_t\cdot(u\nabla\bar u)\right\}\bigg).\end{aligned} \tag{4.51}$$

Since

$$\begin{aligned}s\iint \big|\nabla\varphi_t\cdot(u\nabla\bar u)\big|&\leqslant Cs\iint |\nabla\varphi|^2|\nabla u||u|\\ &\leqslant Cs^{3/2}\iint |\nabla\varphi|^3|u|^2+Cs^{1/2}\iint |\nabla\varphi||\nabla u|^2,\end{aligned}$$

the last term in (4.51) may be removed for s large enough. On the other hand, by (4.43)

$$\begin{aligned}s\iint \partial_j\partial_i\varphi\partial_j u\partial_i\bar u&\leqslant -s\lambda\iint \frac{e^{\lambda\psi(x)}}{t(T-t)}\partial_j\partial_i\psi\partial_j u\partial_i\bar u\\ &\leqslant Cs\iint |\nabla\varphi||\nabla u|^2.\end{aligned}$$

Therefore, for λ large enough,

$$\begin{aligned}&\|M_2u\|^2+\lambda s^3\iint |\nabla\varphi|^3|u|^2+\lambda s\iint |\nabla\varphi||\nabla u|^2+\lambda s^{-1}\iint |\nabla\varphi|^{-1}|\Delta u|^2+s\iint_{\Sigma^-} |\partial_n\varphi\|\partial_n u|^2\\ &\quad\leqslant C\bigg(\|w\|^2+s\iint_{\Sigma^+} |\partial_n\varphi\|\partial_n u|^2\bigg).\end{aligned} \tag{4.52}$$

Using (4.19) and (4.52), we see that for λ large enough

$$\begin{aligned}\lambda s^{-1}\iint |\nabla\varphi|^{-1}|u_t|^2&\leqslant C\lambda s^{-1}\iint |\nabla\varphi|^{-1}\big[|M_2u|^2+|\Delta u|^2+s^4|\nabla\varphi|^4|u|^2+s^2|\nabla\varphi|^2|\nabla u|^2+s^2|\Delta\varphi|^2|u|^2\big]\\ &\leqslant C\bigg(\|w\|^2+s\iint_{\Sigma^+} |\partial_n\varphi\|\partial_n u|^2\bigg).\end{aligned}$$

We conclude that there exist some numbers $s_3\geqslant 1$, $\lambda_3\geqslant 1$ such that for all $s\geqslant s_3$, $\lambda\geqslant\lambda_3$

$$\lambda s^3 \iint |\nabla\varphi|^3|u|^2 + \lambda s \iint |\nabla\varphi||\nabla u|^2 + \lambda s^{-1} \iint |\nabla\varphi|^{-1}\big(|\Delta u|^2 + |u_t|^2\big) + s \iint_{\Sigma^-} |\partial_n\varphi||\partial_n u|^2$$
$$\leqslant C\bigg(\|w\|^2 + s \iint_{\Sigma^+} |\partial_n\varphi||\partial_n u|^2\bigg). \tag{4.53}$$

Replacing u by $e^{-s\varphi}q$ in (4.53) yields (4.17). □

Remark 4.4. An *internal* Carleman estimate may be derived along the same lines. Let ω be any given open subset of $\tilde{\Omega}$, and let ψ denote now some function of class C^4 on $\bar{\tilde{\Omega}}$ such that (4.15) holds true, $\partial_n\psi \leqslant 0$ on $\partial\tilde{\Omega}$, and $\nabla\psi \neq 0$ on $\bar{\tilde{\Omega}} \setminus \omega$ (see [10]). The functions θ and φ being again defined in (4.16), we have the following Carleman estimate.

Proposition 4.5. *Let $a > 0$ and $b \in \mathbb{R}$. Then there exist some constants $\lambda_0 \geqslant 1$, $s_0 \geqslant 1$, and $C_0 > 0$ such that for all $\lambda \geqslant \lambda_0$, $s \geqslant s_0$ and all $q \in \mathcal{Z}$ it holds*

$$\int_0^T\!\!\int_{\tilde{\Omega}} \big[(s\theta)^{-1}\big(|\partial_t q|^2 + |\Delta q|^2\big) + \lambda^2(s\theta)|\nabla q|^2 + \lambda^4(s\theta)^3|q|^2\big]e^{-2s\varphi}\,dx\,dt$$
$$+ \int_0^T\!\!\int_{\partial\tilde{\Omega}} \lambda(s\theta)|\partial_n\psi||\partial_n q|^2 e^{-2s\psi}\,dx\,dt$$
$$\leqslant C_0\bigg(\int_0^T\!\!\int_{\tilde{\Omega}} \big|\partial_t q + (a+ib)\Delta q\big|^2 e^{-2s\varphi}\,dx\,dt + \int_0^T\!\!\int_{\omega} \lambda^4(s\theta)^3|q|^2 e^{-2s\varphi}\,dx\,dt\bigg).$$

Corollary 4.6. *Let $a > 0$, $b \in \mathbb{R}$ and $R > 0$. Introduce the set $\mathcal{Z}^+ := \{q \in \mathcal{Z};\ \partial_n q = 0 \text{ on } \tilde{\Sigma}^+\}$. Then there exist some numbers $\lambda_0 = \lambda_0(\tilde{\Omega}, T)$, $C_0 = C_0(\tilde{\Omega}, T)$, and $s_0 = s_0(\tilde{\Omega}, T, R)$ with $s_0 \leqslant C(1 + R^{2/3})$ where $C = C(\tilde{\Omega}, T)$, such that for all $\lambda \geqslant \lambda_0$, all $s \geqslant s_0$, all $d \in L^\infty(\tilde{Q})$ with $\|d\|_{L^\infty(\tilde{Q})} \leqslant R$, and all $q \in \mathcal{Z}^+$, it holds*

$$\int_0^T\!\!\int_{\tilde{\Omega}} \big[(s\theta)^{-1}\big(|\partial_t q|^2 + |\Delta q|^2\big) + \lambda^2(s\theta)|\nabla q|^2 + \lambda^4(s\theta)^3|q|^2\big]e^{-2s\varphi}\,dx\,dt$$
$$\leqslant C_0 \int_0^T\!\!\int_{\tilde{\Omega}} \big|\partial_t q + (a+ib)\Delta q + dq\big|^2 e^{-2s\varphi}\,dx\,dt. \tag{4.54}$$

Indeed,

$$\big|\partial_t q + (a+ib)\Delta q\big|^2 \leqslant 2\big|\partial_t q + (a+ib)\Delta q + dq\big|^2 + 2R^2|q|^2$$

and the last term is dominated by $[\lambda^4\theta^3]s^3|q|^2$ for $s \sim CR^{2/3}$.

The (formal) adjoint to the operator $Lw = \partial_t w - (1+i\alpha)\Delta w + dw$ is

$$L^*w = -\partial_t w - (1-i\alpha)\Delta w + \bar{d}w.$$

Let H denote the completion of the space $\mathcal{Z}^+$ for the Hilbertian norm $\|\cdot\|_H$ defined as

$$\|q\|_H^2 := \|L^*q\|_{L^2(\tilde{Q})}^2 = \iint_{\tilde{Q}} \big|\partial_t q + (1-i\alpha)\Delta q - \bar{d}q\big|^2\,dx\,dt. \tag{4.55}$$

The proof of the next result is only sketched.

Lemma 4.7. *Let $R>0$ and $d\in L^\infty(\tilde Q)$ with $\|d\|_{L^\infty(\tilde Q)}\leqslant R$, and let $s_0=s_0(\tilde\Omega,T,R)$, $\lambda_0=\lambda_0(\tilde\Omega,T)$, $C_0=C_0(\tilde\Omega,T)$ and ψ be as given in Corollary 4.6. Set $\zeta_0(x)=2s_0(e^{\lambda_0 C_\psi}-e^{\lambda_0\psi(x)})>Const.$ Then H is constituted of the functions $q\in L^1_{\mathrm{loc}}(\tilde Q)$ such that*

$$\iint\limits_{\tilde Q}\left\{\frac{1}{t^3(T-t)^3}|q|^2+\frac{1}{t(T-t)}|\nabla q|^2+t(T-t)\big(|\partial_t q|^2+|\Delta q|^2\big)\right\}\exp\left(-\frac{\zeta_0(x)}{t(T-t)}\right)dx\,dt$$
$$+\|q\|_H^2<\infty \tag{4.56}$$

and

$$q=0\quad on\ \tilde\Sigma,\qquad \partial_n q=0\quad on\ \tilde\Sigma^+. \tag{4.57}$$

Proof. (4.56) readily follows from Corollary 4.6. The boundary conditions (4.57) readily follow from (4.56) and from the density of $\mathcal{Z}^+$ in H. The proof of the fact that $\mathcal{Z}^+$ is dense in the space of the functions $q\in L^1_{\mathrm{loc}}(\tilde Q)$ satisfying (4.56) and (4.57) is left to the reader. □

Now we turn to the existence of solution for the system (4.11).

Theorem 4.8. *For any given $z\in V$, the system* (4.11) *admits a solution $w\in L^2(\tilde Q)$ satisfying*

$$\|w\|_{L^2(\tilde Q)}\leqslant e^{C(1+\|d\|_{\tilde V})}\|u_0\|_X \tag{4.58}$$

where we recall that $d=g(\tilde z)$.

Proof. We claim that the antilinear form

$$l(q)=\iint\limits_{\tilde Q} f\bar q\,dx\,dt$$

is well defined and continuous on H. Indeed, using (4.14), (4.12), (4.13), (4.54) and the fact that $s_0\leqslant C(1+\|d\|_{\tilde V}^{2/3})$, we get

$$\big|l(q)\big|\leqslant e^{C(1+\|d\|_{\tilde V}^{2/3})}\|f\|_{L^2(\tilde\Omega\times(T/3,2T/3))}\|q\|_H\leqslant e^{C(1+\|d\|_{\tilde V})}\|u_0\|_X\|q\|_H. \tag{4.59}$$

It follows from the Riesz representation theorem that there exists a unique $p\in H$ such that

$$\big(L^*p,L^*q\big)_{L^2(\tilde Q)}=l(q)\quad \forall q\in H. \tag{4.60}$$

Set $w=L^*p$. Clearly, $w\in L^2(\tilde Q)$. Taking $q=p$ in (4.60) and using (4.59) we obtain

$$\|w\|^2_{L^2(\tilde Q)}=\|p\|_H^2=l(p)\leqslant e^{C(1+\|d\|_{\tilde V})}\|u_0\|_X\|p\|_H$$

from which (4.58) follows. Choosing any $q\in C_0^\infty(\tilde Q)$ as a test function in (4.60) yields

$$\langle Lw,q\rangle_{\mathcal{D}'(\tilde Q),\mathcal{D}(\tilde Q)}=\langle f,q\rangle_{\mathcal{D}'(\tilde Q),\mathcal{D}(\tilde Q)}.$$

Thus

$$Lw=\partial_t w-(1+i\alpha)\Delta w+dw=f. \tag{4.61}$$

Notice that $w\in H^1(0,T;H^{-2}(\tilde\Omega))$ since $w\in L^2(\tilde Q)$ and $\partial_t w=(1+i\alpha)\Delta w-dw+f$ belong to $L^2(0,T;H^{-2}(\tilde\Omega))$. In particular, both $w(0)$ and $w(T)$ belong to the space $H^{-2}(\tilde\Omega)$. Taking q in (4.60) in the form $q(x,t):=q_1(x)q_2(t)$ with $q_1\in\mathcal{D}(\tilde\Omega)$ and $q_2\in C^1([0,T])$, we obtain

$$\iint_{\tilde{Q}} f\bar{q}\,dx\,dt = \iint_{\tilde{Q}} w\big(-\partial_t\bar{q} - (1+i\alpha)\Delta\bar{q} + d\bar{q}\big)\,dx\,dt$$
$$= \int_0^T \big\langle \partial_t w - (1+i\alpha)\Delta w + dw, q\big\rangle\,dt - \big[\langle w, q\rangle\big]_0^T$$
$$= \iint_{\tilde{Q}} f\bar{q}\,dx\,dt - \big[\langle w, q_1q_2\rangle\big]_0^T \tag{4.62}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing $\langle\cdot,\cdot\rangle_{H^{-2}(\tilde{\Omega}),H_0^2(\tilde{\Omega})}$. Since q_1 and q_2 can be chosen arbitrarily, we conclude that $w(0) = w(T) = 0$ in $H^{-2}(\tilde{\Omega})$.

Finally we verify that $w = 0$ on $\partial\Omega \setminus \Gamma_0$. From $\Delta w = (1+i\alpha)^{-1}(\partial_t w + dw - f)$, we readily infer that $w \in H^{-1}(0,T;H^2(\tilde{\Omega}))$ and that $w_{|\partial\tilde{\Omega}} \in H^{-1}(0,T;H^{3/2}(\partial\tilde{\Omega}))$. Pick now $q \in \mathcal{Z}^+$ in the form $q(x,t) = q_1(x)q_2(t)$ with $q_1 \in C^2(\bar{\tilde{\Omega}})$ fulfilling (4.57), $q_2 \in \mathcal{D}(0,T)$, and apply (4.60) once again. We obtain

$$\iint_{\tilde{Q}} f\bar{q}\,dx\,dt = \int_0^T\!\!\int_{\tilde{\Omega}} w\big(-\partial_t\bar{q} - (1+i\alpha)\Delta\bar{q} + d\bar{q}\big)\,dx\,dt$$
$$= \int_0^T\!\!\int_{\tilde{\Omega}} (Lw)\bar{q}\,dx\,dt - (1+i\alpha)\langle w, \partial_n q\rangle \tag{4.63}$$

where $\langle\cdot,\cdot\rangle$ stands for the duality pairing between $H^{-1}(0,T;H^{3/2}(\partial\tilde{\Omega}))$ and $H_0^1(0,T;H^{-3/2}(\partial\tilde{\Omega}))$. Since $\partial_n q$ was arbitrary on $\tilde{\Sigma}^-$, we infer that $w = 0$ on $\tilde{\Sigma}^-$ and, in particular, $w = 0$ on $(\partial\Omega \setminus \Gamma_0) \times (0,T)$. □

At this stage, we know that the control problem (4.1)–(4.2) has (at least) one solution $u \in L^2(\Omega \times (0,T))$, namely the restriction of $\xi v + w$ to Q. To apply a fixed point argument we need more regularity on u. This is done in the following subsection.

4.3. Regularity of w

The function w inherits additional regularity properties due to the fact that the operators A_p are sectorial. Some of them are gathered in the following proposition, whose proof is inspired in part from the one given in [8].

Proposition 4.9. *Under the assumptions of Theorem* 4.8*, we have for any* $\nu < 2$

$$w \in L^2\big(0,T;H^2(\Omega)\big) \cap H^1\big(0,T;L^2(\Omega)\big) \cap C\big([0,T];C^\nu(\bar{\Omega})\big)$$

and

$$\|w\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t w\|_{L^2(Q)} + \|w\|_{C([0,T];C^\nu(\bar{\Omega}))} \leqslant e^{C(1+\|d\|_{\tilde{Y}})}\|u_0\|_X. \tag{4.64}$$

Proof. Let $\eta_0 \in C^\infty(\bar{\tilde{\Omega}})$ denote a cut-off function such that $0 \leqslant \eta_0 \leqslant 1$, $\eta_0 = 1$ in a neighborhood of Ω, $\eta_0 = 0$ in a neighborhood of S^+. Set $w_0 := \eta_0 w$. Then w_0 solves

$$\begin{cases} \partial_t w_0 - (1+i\alpha)\Delta w_0 + dw_0 = f_0 & \text{in } \tilde{Q}, \\ w_0 = 0 & \text{on } \tilde{\Sigma}, \\ w_0(0) = 0 \quad \text{and} \quad w_0(T) = 0, \end{cases}$$

where $f_0 := \eta_0 f - (1+i\alpha)[2\nabla\eta_0 \cdot \nabla w + (\Delta\eta_0)w]$.

Since $f_0 \in L^2(0,T;H^{-1}(\tilde{\Omega}))$ and $dw_0 \in L^2(\tilde{Q})$, it follows from Lemma 6.1 in Annexe that $w_0 \in L^2(0,T;H_0^1(\tilde{\Omega})) \cap C([0,T];L^2(\tilde{\Omega}))$. Furthermore, using the definition of f_0, (4.14), (4.58), and (6.4), we infer that

$$\|w_0\|_{L^2(0,T;H_0^1(\tilde{\Omega}))} + \|w_0\|_{C([0,T];L^2(\tilde{\Omega}))} \leqslant e^{C(1+\|d\|_{\tilde{Y}})}\|u_0\|_X. \tag{4.65}$$

Let $\eta_1 \in C^\infty(\bar{\tilde{\Omega}})$ be a second cut-off function such that $0 \leqslant \eta_1 \leqslant 1$, $\eta_1 = 1$ in a neighborhood of Ω, and $\operatorname{supp} \eta_1 \subset \{\eta_0 = 1\}$. Observe that $\eta_1 = \eta_1 \eta_0$. Let $w_1 := \eta_1 w = \eta_1 w_0$. Then w_1 solves

$$\begin{cases} \partial_t w_1 - (1 + i\alpha)\Delta w_1 + d w_1 = f_1 & \text{in } \tilde{Q}, \\ w_1 = 0 & \text{on } \tilde{\Sigma}, \\ w_1(0) = 0 \quad \text{and} \quad w_1(T) = 0, \end{cases}$$

where $f_1 := \eta_1 f - (1+i\alpha)[2\nabla\eta_1 \cdot \nabla w_0 + (\Delta\eta_1) w_0] \in L^2(\tilde{Q})$. We infer from Lemma 6.2 that $w_1 \in L^2(0, T; H^2(\tilde{\Omega})) \cap C([0, T]; H_0^1(\tilde{\Omega}))$ with

$$\|w_1\|_{L^2(0,T;H^2(\tilde{\Omega}))} + \|w_1\|_{C([0,T];H_0^1(\tilde{\Omega}))} \leqslant e^{C(1+\|d\|_{\tilde{V}})} \|u_0\|_X. \tag{4.66}$$

Using (4.61), we infer that

$$w \in L^2\big(0, T; H^2(\Omega)\big) \cap C\big([0, T]; H^1(\Omega)\big) \cap H^1\big(0, T; L^2(\Omega)\big)$$

with

$$\|w\|_{L^2(0,T;H^2(\Omega))} + \|w\|_{C([0,T];H^1(\Omega))} + \|\partial_t w\|_{L^2(Q)} \leqslant e^{C(1+\|d\|_{\tilde{V}})} \|u_0\|_X. \tag{4.67}$$

In particular, it follows that $w_{|\partial\Omega} \in L^2(0, T; H^{\frac{3}{2}}(\partial\Omega))$. To prove the continuity of w as a function of (x, t), we define by induction a sequence of cut-off functions $(\eta_k)_{k\geqslant 2}$, with $\eta_k \in C^\infty(\bar{\tilde{\Omega}})$, $0 \leqslant \eta_k \leqslant 1$, $\eta_k = 1$ in a neighborhood of Ω, and $\operatorname{supp} \eta_k \subset \{\eta_{k-1} = 1\}$. Then $w_k := \eta_k w = \eta_k w_{k-1}$ satisfies

$$\begin{cases} \partial_t w_k - (1 + i\alpha)\Delta w_k + d w_k = f_k & \text{in } \tilde{Q}, \\ w_k = 0 & \text{on } \tilde{\Sigma}, \\ w_k(0) = 0 \quad \text{and} \quad w_k(T) = 0, \end{cases}$$

where $f_k := \eta_k f - (1 + i\alpha)[2\nabla\eta_k \cdot \nabla w_{k-1} + (\Delta\eta_k) w_{k-1}]$. We notice that $f_2 = \eta_2 f - (1 + i\alpha)[2\nabla\eta_2 \cdot \nabla w_1 + (\Delta\eta_2) w_1] \in L^2(0, T; H^1(\tilde{\Omega})) \cap L^\infty(0, T; L^2(\tilde{\Omega}))$. (Recall that the function f, given by (4.11), satisfies $f \in L^2(0, T; H^2(\tilde{\Omega})) \cap C([0, T]; D(\tilde{A}_p^\gamma))$ for all $p > N$, $\gamma < 1$.)

Let p_0 defined by $1/p_0 = 1/2^* = 1/2 - 1/N$, so that $H^1(\tilde{\Omega}) \subset L^{p_0}(\tilde{\Omega})$ by virtue of the classical Sobolev embedding

$$W^{1,p}(\tilde{\Omega}) \subset L^{p^*}(\tilde{\Omega}) \quad \text{for } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N} > 0. \tag{4.68}$$

Thus $f_2 \in L^2(0, T; L^{p_0}(\tilde{\Omega})) \cap L^\infty(0, T; L^2(\tilde{\Omega}))$ which implies, by interpolation,

$$f_2 \in L^r\big(0, T; L^p(\tilde{\Omega})\big)$$

for any $r > 2$ and some $p(r) < p_0$, with $p(r) \to p_0$ as $r \to 2$. Let $\tilde{A}_p$ be the operator introduced in Section 2 (with $\tilde{\Omega}$ instead of Ω). We claim that for some $\gamma > 1/2$

$$w_2 \in C\big([0, T]; D\big(\tilde{A}_p^\gamma\big)\big).$$

Indeed,

$$\begin{aligned} \big\|\tilde{A}_p^\gamma w_2(t)\big\|_{L^p(\tilde{\Omega})} &\leqslant \int_0^t \big\|\tilde{A}_p^\gamma e^{-\tilde{A}_p(t-s)}[f_2 - d w_2](s)\big\|_{L^p(\tilde{\Omega})}\, ds \\ &\leqslant C \int_0^t (t-s)^{-\gamma} \|f_2 - d w_2\|_{L^p(\tilde{\Omega})}\, ds \\ &\leqslant C \left(\int_0^t s^{-\gamma\frac{r}{r-1}}\, ds \right)^{1-\frac{1}{r}} \|f_2 - d w_2\|_{L^r(0,T;L^p(\tilde{\Omega}))} \\ &\leqslant C_\gamma \|f_2 - d w_2\|_{L^r(0,T;L^p(\tilde{\Omega}))} \end{aligned}$$

when $1/2 < \gamma < 1 - 1/r$. Since $\gamma > 1/2$, it follows from Proposition 3.6 that $D(A_p^\gamma) \subset W^{1,p}(\tilde{\Omega})$. Thus

$$w_2 \in C\big([0,T]; W^{1,p}(\tilde{\Omega})\big) \quad \text{and} \quad f_3 \in C\big([0,T]; L^p(\tilde{\Omega})\big) \quad \forall p < p_0.$$

Taking $\gamma < 1$ close to 1, we obtain

$$\begin{aligned}\big\|\tilde{A}_p^\gamma w_3(t)\big\|_{L^p(\tilde{\Omega})} &\leqslant C\int_0^t (t-s)^{-\gamma}\|f_3 - dw_3\|_{L^p(\tilde{\Omega})} \\ &\leqslant C_\gamma \|f_3 - dw_3\|_{C([0,T];L^p(\tilde{\Omega}))}.\end{aligned}$$

Hence $w_3 \in C([0,T]; D(\tilde{A}_p^\gamma))$ for any $p < p_0$ and any $\gamma < 1$. Define the number p_1 by

$$\frac{1}{p_1} = \frac{1}{p_0} - \frac{1}{N}.$$

Then $D(\tilde{A}_p^\gamma) \subset W^{1,q}(\tilde{\Omega})$ for some $q < p_1$ such that $q \to p_1$ when $p \to p_0$ and $\gamma \to 1$. Therefore,

$$w_3 \in C\big([0,T]; W^{1,p}(\tilde{\Omega})\big) \quad \text{and} \quad f_4 \in C\big([0,T]; L^p(\tilde{\Omega})\big) \quad \forall p < p_1.$$

Define the sequence (p_k) inductively by

$$\frac{1}{p_k} = \frac{1}{p_{k-1}} - \frac{1}{N}, \quad k \geqslant 2.$$

Then the above argument shows that $f_{k+3} \in C([0,T]; L^p(\tilde{\Omega}))$ for any $p < p_k$ while $p_{k-1} < N$. Let k be the last index for which $p_{k-1} \leqslant N$. Then $p_k > N$ and

$$w_{k+3} \in C\big([0,T]; D\big(\tilde{A}_p^\gamma\big)\big) \quad \forall p < p_k,\ \forall \gamma < 1.$$

It follows that

$$w_{k+3} \in C\big([0,T]; W^{1,p}(\tilde{\Omega})\big) \quad \text{and} \quad f_{k+4} \in C\big([0,T]; L^p(\tilde{\Omega})\big) \quad \forall p < \infty.$$

Therefore, $w_{k+4} \in C([0,T]; D(\tilde{A}_p^\gamma))$ for all $p < \infty$ and all $\gamma < 1$. Using (3.3), we obtain that for any $\nu < 2$, $w_{k+4} \in C([0,T]; C^\nu(\bar{\tilde{\Omega}}))$, and that

$$\|w\|_{C([0,T];C^\nu(\bar{\Omega}))} \leqslant e^{C(1+\|d\|_{\tilde{V}})}\|u_0\|_X \quad \forall \nu < 2. \qquad \square$$

Gathering together the results in Theorem 4.8, Proposition 4.9, and (4.8)–(4.9), we have established the following result.

Theorem 4.10. *For any $u_0 \in X$ and any $d \in \tilde{V}$, one can assign a solution $\{u,h\}$ to the null controllability problem*

$$\begin{cases} \partial_t u - (1+i\alpha)\Delta u + du = 0 & \text{in } \Omega \times (0,T), \\ u = 1_{\Gamma_0} h & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0, \quad u(T) = 0, \end{cases}$$

in such a way that $u \in V \cap L^2(0,T;H^1(\Omega))$, $h = u_{|\partial\Omega} \in C(\partial\Omega \times [0,T])$ with $\operatorname{supp} h \subset \Gamma_0 \times (0,T)$, $\sqrt{t}u \in L^2(0,T;H^2(\Omega))$, $\sqrt{t}\partial_t u \in L^2(Q)$, $\sqrt{t}h \in L^2(0,T;H^{\frac{3}{2}}(\partial\Omega))$, and for any $\nu < 2$, $u \in C((0,T];C^\nu(\bar{\Omega}))$, $h \in C((0,T];C^\nu(\partial\Omega))$.

We are now in a position to complete the proof of Theorem 2.1.

4.4. The fixed-point argument

Let $V = \{z \in C(\bar{Q});\ z = 0 \text{ in } (\partial\Omega \setminus \Gamma_0) \times [0,T]\}$ be endowed with the $L^\infty(Q)$ norm (recall that $Q = \Omega \times (0,T)$), and let Λ denote the map from V into itself, defined as follows: for any $z \in V$, $\Lambda(z)$ is the restriction to $\bar{Q}$ of the function $u(x,t) = \xi(t)v(x,t) + w(x,t)$, where v and w are defined in (4.3) (with $d := g(\tilde{z})$) and in (4.11), respectively. The goal is to prove that Λ has a fixed point in some closed ball $B_r(0)$ by using Schauder fixed-point theorem. We first check that for any $r > 0$, $\Lambda(B_r(0))$ is relatively compact in V.

Proposition 4.11. *For any $r > 0$, the closure of $\Lambda(B_r(0))$ in V is compact.*

Proof. For any $z \in V$, we have that $u = \Lambda(z) = (\xi v + w)_{|\bar{Q}} \in V$ by virtue of Lemma 4.1 and Proposition 4.9. Let $\mathcal{V} := \{v_{|\bar{Q}}; z \in B_r(0)\}$ and $\mathcal{W} := \{w_{|\bar{Q}}; z \in B_r(0)\}$. We have that $\Lambda(B_r(0)) \subset \xi\mathcal{V} + \mathcal{W}$. It is sufficient to prove that both $\mathcal{V}$ and $\mathcal{W}$ are relatively compact in V. This is done in two claims.

Claim 3. *The closure of $\mathcal{V}$ in V is compact.*

Let (z^n) be a given sequence in $B_r(0)$, and let (v^n) be the corresponding sequence in $\mathcal{V}$. Let $\tau \in (0, T]$, and pick $p > N$, $\gamma \in (0, 1)$, and $\nu > 0$ such that $\nu < 2\gamma - N/p$. By (4.8), (4.9) (with $d = g(\tilde{z}^n)$) and (3.3), we see that the sequence (v^n) is bounded in the space $C([\tau, T]; C^\nu(\bar{\tilde{\Omega}})) \cap H^1(\tau, T; L^2(\tilde{\Omega}))$. Since the first embedding in

$$C^\nu(\bar{\tilde{\Omega}}) \subset C(\bar{\tilde{\Omega}}) \subset L^2(\tilde{\Omega})$$

is compact, we infer from Aubin's lemma (see e.g. [20]) that one can choose a subsequence $(v^{n'})$ which converges uniformly on $C([\tau, T]; C(\bar{\tilde{\Omega}}))$ to a function v. Using a diagonal process, we may assume that $v \in C((0, T]; C(\bar{\tilde{\Omega}}))$ and that $v^{n'} \to v$ uniformly on $\bar{\tilde{\Omega}} \times [\tau, T]$ for all $\tau > 0$. In particular, the sequence $(v^{n'})$ is uniformly equicontinuous on $\bar{\tilde{\Omega}} \times [\tau, T]$ for each $\tau > 0$. We claim that this is also true on $\bar{\tilde{\Omega}} \times [0, T]$. Indeed, pick any $\varepsilon > 0$. Since

$$v^n(t) = e^{-\tilde{A}t}\tilde{u}_0 - \int_0^t e^{-\tilde{A}(t-s)} g(\tilde{z}^n) v^n \, ds,$$

we infer from (4.5) (applied with $d = g(\tilde{z}^n)$, $F = 0$) and the fact that $z^n \in B_r(0)$ that there exists a constant $C > 0$ independent of n such that

$$\left\| v^n(t) - e^{-\tilde{A}t}\tilde{u}_0 \right\|_{L^\infty(\tilde{\Omega})} \leqslant Ct.$$

On the other hand, by continuity of the map $t \mapsto e^{-\tilde{A}t}\tilde{u}_0$, we may find a number $\delta_0 > 0$ such that $\|e^{-\tilde{A}t}\tilde{u}_0 - e^{-\tilde{A}s}\tilde{u}_0\|_{L^\infty(\tilde{\Omega})} \leqslant \varepsilon/3$ for $0 \leqslant s \leqslant t \leqslant \delta_0$. Therefore, if $0 \leqslant s \leqslant t \leqslant \delta = \min(\delta_0, \varepsilon/(3C))$, then

$$\left\| v^n(t) - v^n(s) \right\|_{L^\infty(\tilde{\Omega})} \leqslant \varepsilon.$$

As the estimate $\|v^{n'}(t) - v^{n'}(s)\|_{L^\infty(\tilde{\Omega})} \leqslant \varepsilon$ is still valid on $[\delta/2, T]$ for all n' if $|s - t|$ is small enough, we conclude that the sequence $(v^{n'})$ is uniformly equicontinuous on $\bar{\tilde{\Omega}} \times [0, T]$. Since that sequence is also bounded by (4.5), we infer from Ascoli theorem that it possesses a convergent subsequence in $\tilde{V}$, hence in V.

We now turn to the relative compactness of $\mathcal{W}$.

Claim 4. *The closure of $\mathcal{W}$ in V is compact.*

This follows from (4.64) and Aubin's lemma. The proof of Proposition 4.11 is therefore complete. □

Remark 4.12. We stress that, in contrast to [8], no regularity assumption on u_0 is needed to prove the relative compactness of $\Lambda(B_r(0))$. Accordingly, we are able to design a "smooth" control input h steering the state u_0 to 0 in Theorem 2.1.

Let us now establish the continuity of Λ.

Proposition 4.13. *If $z^n \to z$ in V, then $\Lambda(z^n) \to \Lambda(z)$ in V.*

Proof. Note first that $\tilde{z}^n \to \tilde{z}$ in $\tilde{V}$. Set $u^n = \Lambda(z^n) = \xi v^n + w^n$ for each n, and $u = \Lambda(z) = \xi v + w$. According to Proposition 4.11, a subsequence $(u^{n'})$ converges uniformly on $\bar{Q}$ to a function $\hat{u} \in C(\bar{Q})$. To prove that $u = \hat{u}$, it is sufficient to show that $u^n \to u$ in $L^2(Q)$.

Claim 5. *$v^n \to v$ in $\tilde V$.*

Let $\varepsilon^n = v^n - v$. Then ε^n solves the system

$$\begin{cases} \partial_t \varepsilon^n - (1+i\alpha)\Delta\varepsilon^n + g(\tilde z)\varepsilon^n = (g(\tilde z) - g(\tilde z^n))v^n & \text{in } \tilde Q, \\ \varepsilon^n = 0 & \text{on } \tilde\Sigma, \\ \varepsilon^n(0) = 0. \end{cases} \tag{4.69}$$

By (4.8), $\|v^n\|_{\tilde V} \leqslant \text{Const}$. Hence $(g(\tilde z) - g(\tilde z^n))v^n \to 0$ in $\tilde V$ as $n \to \infty$, and therefore $\varepsilon^n \to 0$ in $\tilde V$ by virtue of (4.5). The claim is proved.

Let $R \geqslant \sup_{n\geqslant 0} \|g(\tilde z^n)\|_{L^\infty(\tilde Q)}$ and $L_n q = L(z^n)q = \partial_t q - (1+i\alpha)\Delta q + g(z^n)q$ for any n. In addition, let H_n denote the completion of $\mathcal Z^+$ for the norm defined in (4.55) (with d replaced by $g(\tilde z^n)$ and let $s_0 = s_0(\tilde\Omega, T, R)$, $\lambda_0 = \lambda_0(\tilde\Omega, T)$, and $C_0 = C_0(\tilde\Omega, T)$ be as given in Corollary 4.6 for $a + ib = 1 - i\alpha$. Set $f^n(x,t) = -\xi'(t)v^n(x,t)$. It follows from Claim 5 that

$$f^n \to f \quad \text{in } L^2(\tilde Q). \tag{4.70}$$

By construction $w^n = L_n^* p^n$, where p^n is the unique function in H_n satisfying

$$\iint_{\tilde Q} \left(L_n^* p^n\right)\left(\overline{L_n^* q}\right) dx\,dt = \iint_{\tilde Q} f^n \bar q\, dx\,dt \quad \text{for all } q \in \mathcal Z^+. \tag{4.71}$$

By (4.58),

$$\|w^n\|_{L^2(\tilde Q)} \leqslant e^{C(1+\|g(\tilde z^n)\|_{\tilde V})} \|f^n\|_{L^2(\tilde Q)} \leqslant \text{Const}.$$

Using (4.54),

$$\iint_{\tilde Q} \left\{t^{-3}(T-t)^{-3}\left|p^n\right|^2 + t^{-1}(T-t)^{-1}\left|\nabla p^n\right|^2 + t(T-t)\left(\left|\partial_t p^n\right|^2 + \left|\Delta p^n\right|^2\right\} \exp\left(-\frac{\zeta_0(x)}{t(T-t)}\right) dx\,dt \leqslant \text{Const}.$$

Therefore, there exist a function $\hat p \in L^2_{\text{loc}}(0, T, H^2(\tilde\Omega))$ and a sequence $n' \to +\infty$ such that

$$p^{n'} \rightharpoonup \hat p \quad \text{in } L^2\left(\tilde Q, t^{-3}(T-t)^{-3} \exp\left(-\frac{\zeta_0(x)}{t(T-t)}\right) dx\,dt\right), \tag{4.72}$$

$$\nabla p^{n'} \rightharpoonup \nabla \hat p \quad \text{in } L^2\left(\tilde Q, t^{-1}(T-t)^{-1} \exp\left(-\frac{\zeta_0(x)}{t(T-t)}\right) dx\,dt\right), \tag{4.73}$$

$$\Delta p^{n'} \rightharpoonup \Delta \hat p \quad \text{in } L^2\left(\tilde Q, t(T-t) \exp\left(-\frac{\zeta_0(x)}{t(T-t)}\right) dx\,dt\right), \tag{4.74}$$

$$\partial_t p^{n'} \rightharpoonup \partial_t \hat p \quad \text{in } L^2\left(\tilde Q, t(T-t) \exp\left(-\frac{\zeta_0(x)}{t(T-t)}\right) dx\,dt\right). \tag{4.75}$$

Let $\hat w := L^* \hat p = -\partial_t \hat p - (1-i\alpha)\Delta\hat p + \overline{g(\tilde z)}\hat p$.

Claim 6. *$w^{n'} \to \hat w$ in $\mathcal D'(\tilde Q)$.*

Since $p^{n'} \to \hat p$ in $\mathcal D'(\tilde Q)$, we only have to check that $\overline{g(\tilde z^{n'})}p^{n'} \to \overline{g(\tilde z)}\hat p$ in $\mathcal D'(\tilde Q)$. This is true since $g(\tilde z^{n'}) \to g(\tilde z)$ in $\tilde V$ and $p^{n'} \rightharpoonup \hat p$ in $L^2_{\text{loc}}(\tilde Q)$.

As $\|w^n\|_{L^2(\tilde\Omega)} \leqslant \text{Const.}$, we infer that $\hat w \in L^2(\tilde Q)$ and that

$$w^{n'} \rightharpoonup \hat w \quad \text{in } L^2(\tilde Q).$$

It remains to show that $\hat{w} = w$. To this end, write $w = L^* p$, where $p \in H$ is the unique function in H satisfying

$$\iint_{\tilde{Q}} L^* p \overline{L^* q}\, dx\, dt = \iint_{\tilde{Q}} f \bar{q}\, dx\, dt \quad \forall q \in \mathcal{Z}^+.$$

Claim 7. $\hat{p} = p$.

Since in $L^2(\tilde{Q})$ $L^*_{n'} p^{n'} = w^{n'} \rightharpoonup \hat{w} = L^* \hat{p}$, $L^*_{n'} q \to L^* q$ (for $\tilde{z}^n \to \tilde{z}$ in $L^\infty(\tilde{Q})$), and $f^n \to f$ (according to Claim 5), we can pass to the limit in (4.71) to obtain

$$\iint_{\tilde{Q}} L^* \hat{p} \overline{L^* q}\, dx\, dt = \iint_{\tilde{Q}} f \bar{q}\, dx\, dt \quad \forall q \in \mathcal{Z}^+. \tag{4.76}$$

We need to show that $\hat{p} \in H$. The condition (4.56) is clearly satisfied. The boundary conditions $\hat{p}_{|_{\tilde{\Sigma}}} = 0$, $\partial_n \hat{p}_{|_{\tilde{\Sigma}^+}} = 0$ follow from (4.72)–(4.74). We have thus proved that $\hat{p} \in H$. Moreover, it follows from (4.76) that $\hat{p} = p$. Hence $\hat{w} = w$. A standard argument shows that the convergence in (4.72)–(4.75) holds for the whole sequence (p^n), and that $w^n \rightharpoonup w$ in $L^2(\tilde{Q})$. Finally,

$$\|w^n\|^2_{L^2(\tilde{Q})} = \iint_{\tilde{Q}} \left|L^*_n p^n\right|^2 dx\, dt = \iint_{\tilde{Q}} f^n \overline{p^n}\, dx\, dt \to \iint_{\tilde{Q}} f \bar{p}\, dx\, dt = \|w\|^2_{L^2(\tilde{Q})}$$

thanks to (4.70), (4.72), and the fact that supp $f^n \subset \bar{\tilde{\Omega}} \times [\frac{T}{3}, \frac{2T}{3}]$. Consequently, $w^n \to w$ and $u^n \to u$ in $L^2(\tilde{Q})$. The proof of Proposition 4.13 is complete. □

We are now in a position to apply the fixed point argument. Take $r > 0$ and pick any $z \in V$ with $\|z\|_V \leqslant r$. Let $R = \sup\{|g(\xi)|; |\xi| \leqslant r\}$. Set $u = \Lambda(z) \in V$. By (4.8) and (4.64),

$$\begin{aligned}
\|u\|_V &\leqslant \left\|\xi(t)v\right\|_V + \|w\|_V \\
&\leqslant e^{C(1+\|g(\tilde{z})\|_{\tilde{V}})}\|u_0\|_X \\
&\leqslant e^{C(1+R)}\|u_0\|_X.
\end{aligned}$$

Therefore, if $\|u_0\|_X$ is small enough, the closed ball $B_r(0) = \{z \in V; \|z\|_V \leqslant r\}$ is mapped into itself by the application $\Lambda : z \mapsto u$. On the other hand, $\Lambda(B_r(0))$ is relatively compact according to Proposition 4.11, and Λ is continuous according to Proposition 4.13. By virtue of Schauder fixed-point theorem, Λ has a fixed point $z = u$ in $B_r(0)$. The regularity of u is the one depicted in Theorem 4.10.

5. Proof of Corollary 2.3

To prove Corollary 2.3, we follow the argument developed in [17, Proof of Theorem 4.3] to prove that the GL equation with homogeneous boundary condition ($h = 0$) is locally well-posed in $L^p(\Omega)$ for $p > \sigma N$ when $f(z) = Rz + \mu|z|^{2\sigma}z$. More precisely, given any $T > 0$, we may find a number $r_0 > 0$ such that for $\|u_0\|_{L^p(\Omega)} \leqslant r < r_0$, the solution of (2.2)–(2.4) with $h \equiv 0$ exists on $[0, T/2]$ and satisfies

$$\left\|u\left(\frac{T}{2}\right)\right\|_{C_0(\bar{\Omega})} < G(r)$$

with $G(r) \to 0$ as $r \to 0$. Next, we choose $r > 0$ so that $G(r)$ is less than the number $R(T/2)$ given in Theorem 2.1. Applying the control h given by Theorem 2.1 on the interval $[T/2, T]$, we conclude that $u(T) = 0$.

When the initial state $u_0 \in H^q(\Omega)$ for some $q > \frac{N}{2} - \frac{1}{\sigma}$ and $\sigma \geqslant \sup\{2^{-1}, N^{-1}\}$, we proceed in a similar way in using a local well-posedness result in $H^q(\Omega)$, namely [15, Theorem 4]. However, the result in [15] was stated and proved only when $\Omega = \mathbb{T}^N$, the N-dimensional torus. The proof, based upon the knowledge of the corresponding Green function, essentially established that the semigroup $S(t)$ associated with the operator $Lu = (1 + i\alpha)\Delta u + Ru$ was a smoothing operator from H^q to L^2, and from L^2 to L^p for some $p > \sigma N(2\sigma + 1)$. We provide here alternative proofs of the corresponding estimates for any bounded, smooth open set $\Omega \subset \mathbb{R}^N$ and for the Dirichlet boundary conditions. Notice that we can take $R = 0$ without loss of generality.

Lemma 5.1. *Let $S(t)$ be the semigroup associated to the (formal) operator $Lu = (1 + i\alpha)\Delta u$ with Dirichlet boundary conditions. Let $q < 0$ and $p \in (2, \infty)$. Then the following estimates hold true for $t \in (0, T)$*

$$\|S(t)w\|_{L^2} \leqslant Ct^{\frac{q}{2}}\|w\|_{H^q}, \tag{5.1}$$

$$\|S(t)w\|_{L^p} \leqslant Ct^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{p})}\|w\|_{L^2}. \tag{5.2}$$

(Here, H^q denotes the dual space of $D(A_2^{|q|/2})$ for $\alpha = 0$ with respect to the pivot space $L^2(\Omega)$.)

Proof. Let $(e_n)_{n\geqslant 0}$ be a Hilbert basis in $L^2(\Omega)$ constituted of eigenfunctions of the Dirichlet Laplacian; that is, $-\Delta e_n(x) = \lambda_n e_n(x)$, $e_n \in H^\infty(\Omega) \cap H_0^1(\Omega)$, and $0 < \lambda_n \nearrow \infty$ as $n \to \infty$. If $w = \sum_{n\geqslant 0} c_n e_n$, then $\|w\|_{H^q}^2 \sim \sum_{n\geqslant 0}(1 + \lambda_n)^q |c_n|^2$. Furthermore, $S(t)w = \sum_{n\geqslant 0} e^{-(1+i\alpha)\lambda_n t} c_n e_n$. It follows that

$$\|S(t)w\|_{L^2}^2 = \sum_{n\geqslant 0} e^{-2\lambda_n t}|c_n|^2 \leqslant C \sup_{n\geqslant 0}\big[e^{-2\lambda_n t}(1 + \lambda_n)^{-q}\big]\|w\|_{H^q}^2.$$

The function $s \geqslant 0 \to e^{-2st}(1 + s)^{-q}$ is easily found to be bounded by $[|q|/(2et)]^{|q|}e^{2t}$, hence (5.1) follows at once.

To prove (5.2), we first notice that, by (3.1) (applied with $\gamma = 1/2$ and $p = 2$),

$$\|S(t)w\|_{H^1} \leqslant C\frac{e^{-\delta t}}{\sqrt{t}}\|w\|_{L^2}. \tag{5.3}$$

Then (5.3), combined with the Gagliardo–Nirenberg inequality

$$\|S(t)w\|_{L^p} \leqslant C\|S(t)w\|_{L^2}^{\theta}\|S(t)w\|_{H^1}^{1-\theta}$$

where $\theta = 1 - \frac{N}{2} + \frac{N}{p}$, gives

$$\|S(t)w\|_{L^p} \leqslant Ce^{-\delta t}t^{-\frac{N}{2}(\frac{1}{2}-\frac{1}{p})}\|w\|_{L^2}.$$

With (5.1) and (5.2) at hand, the local well-posedness in H^q of the GL equation follows exactly as in [15]. Furthermore, the solution enters the space $C_0(\Omega)$ at once, so that Theorem 2.1 may be used as above. □

6. Annexe

Let $\Omega \subset \mathbb{R}^N$ and $\alpha \in \mathbb{R}$. We collect a series of simple regularity results for the forced initial-value problem

$$\partial_t v - (1 + i\alpha)\Delta v = f \quad \text{in } \Omega \times (0, T), \tag{6.1}$$

$$v = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{6.2}$$

$$v(0) = v_0. \tag{6.3}$$

Lemma 6.1. *If $f \in L^2(0, T; H^{-1}(\Omega))$ and $v_0 \in L^2(\Omega)$, then $v \in L^2(0, T; H_0^1(\Omega)) \cap C([0, T]; L^2(\Omega))$ with*

$$\|v\|_{L^2(0,T;H_0^1(\Omega))} + \|v\|_{C([0,T];L^2(\Omega))} \leqslant C\big(\|f\|_{L^2(0,T;H^{-1}(\Omega))} + \|v_0\|_{L^2(\Omega)}\big). \tag{6.4}$$

Proof. Simple exercise. □

Lemma 6.2. *If $f \in L^2(Q)$ and $v_0 \in L^2(\Omega)$, then $\sqrt{t}v \in L^2(0, T; H^2(\Omega)) \cap C((0, T]; H_0^1(\Omega))$ with*

$$\|\sqrt{t}v\|_{L^2(0,T;H^2(\Omega))} + \|\sqrt{t}v\|_{L^\infty(0,T;H_0^1(\Omega))} \leqslant C\big(\|f\|_{L^2(Q)} + \|v_0\|_{L^2(\Omega)}\big). \tag{6.5}$$

If moreover $v_0 \in H_0^1(\Omega)$, then $v \in L^2(0, T; H^2(\Omega)) \cap C([0, T]; H_0^1(\Omega))$ with

$$\|v\|_{L^2(0,T;H^2(\Omega))} + \|v\|_{L^\infty(0,T;H_0^1(\Omega))} \leqslant C\big(\|f\|_{L^2(Q)} + \|v_0\|_{L^2(\Omega)}\big). \tag{6.6}$$

Proof. Simple exercise. □

Lemma 6.3. *Let $p \in (N, \infty)$ and $\gamma \in (0,1)$ be given numbers, and let A_p be the sectorial operator defined in Section 3. If $f \in L^\infty(0,T;L^p(\Omega))$ and $v_0 \in L^p(\Omega)$, then $v \in C((0,T];D(A_p^\gamma))$ and we have for some constant $C = C(\gamma, p, T)$*

$$\|v(t)\|_{D(A_p^\gamma)} \leqslant C\left(t^{-\gamma}\|v_0\|_{L^p(\Omega)} + \|f\|_{L^\infty(0,T;L^p(\Omega))}\right) \quad \forall t \in (0,T]. \tag{6.7}$$

Proof. According to Duhamel formula

$$v(t) = e^{-A_p t}v_0 + \int_0^t e^{-A_p(t-s)} f(s)\,ds,$$

hence

$$\begin{aligned}\|v(t)\|_{D(A_p^\gamma)} &\leqslant \|A_p^\gamma e^{-A_p t}v_0\|_{L^p(\Omega)} + \int_0^t \|A_p^\gamma e^{-A_p(t-s)} f(s)\|_{L^p(\Omega)}\,ds \\ &\leqslant Ct^{-\gamma}\|v_0\|_{L^p(\Omega)} + \int_0^t C(t-s)^{-\gamma}\|f(s)\|_{L^p(\Omega)}\,ds \\ &\leqslant C\left(t^{-\gamma}\|v_0\|_{L^p(\Omega)} + \frac{T^{1-\gamma}}{1-\gamma}\|f\|_{L^\infty(0,T;L^p(\Omega))}\right).\end{aligned}$$

The proof is complete. □

Note added in proof

The results in this paper have been announced in [18]. After this paper was submitted, the authors learned that J.L. Boldrini, E. Fernandez-Cara and S. Guerrero have obtained a result similar to Theorem 2.1 in [4].

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References

[1] O.M. Aamo, A. Smyshlyaev, M. Krstić, Boundary control of the linearized Ginzburg–Landau model of vortex shedding, SIAM J. Control Optim. 43 (6) (2005) 1953–1971 (electronic).
[2] W. Arendt, C.J.K. Batty, M. Hieber, F. Neubrander, Vector-Valued Laplace Transforms and Cauchy Problems, Monographs in Mathematics, vol. 96, Birkhäuser Verlag, Basel, 2001.
[3] M. Bartuccelli, P. Constantin, C.R. Doering, J.D. Gibbon, M. Gisselfält, On the possibility of soft and hard turbulence in the complex Ginzburg–Landau equation, Physica D 44 (3) (1990) 421–444.
[4] J.L. Boldrini, E. Fernández-Cara, S. Guerrero, On the controllability of the Ginzburg–Landau equation, in preparation.
[5] C. Bu, An initial-boundary value problem for the Ginzburg–Landau equation, Appl. Math. Lett. 5 (5) (1992) 31–34.
[6] Z. Chen, K.-H. Hoffmann, Numerical solutions of an optimal control problem governed by a Ginzburg–Landau model in superconductivity, Numer. Funct. Anal. Optim. 19 (7–8) (1998) 737–757.
[7] A. Doubova, E. Fernández-Cara, M. González-Burgos, On the controllability of the heat equation with nonlinear boundary Fourier conditions, J. Differential Equations 196 (2) (2004) 385–417.
[8] E. Fernández-Cara, Null controllability of the semilinear heat equation, ESAIM Control Optim. Calc. Var. 2 (1997) 87–103 (electronic).
[9] X. Fu, A weighted identity for partial differential operators of second order and its applications, C. R. Acad. Sci. Paris, Ser. I 342 (8) (2006) 579–584.

[10] A.V. Fursikov, O.Yu. Imanuvilov, Controllability of Evolution Equations, Lecture Notes Series, vol. 34, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.
[11] H. Gao, C. Bu, Dirichlet inhomogeneous boundary value problem for the $n+1$ complex Ginzburg–Landau equation, J. Differential Equations 198 (1) (2004) 176–195.
[12] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Mathematics, vol. 840, Springer-Verlag, Berlin, 1981.
[13] L. Hörmander, The Analysis of Linear Partial Differential Operators. II, Differential Operators with Constant Coefficients, Classics in Mathematics, Springer-Verlag, Berlin, 2005. Reprint of the 1983 original.
[14] C.D. Levermore, M. Oliver, The complex Ginzburg–Landau equation as a model problem, in: Dynamical Systems and Probabilistic Methods in Partial Differential Equations, Berkeley, CA, 1994, in: Lectures in Appl. Math., vol. 31, Amer. Math. Soc., Providence, RI, 1996, pp. 141–190.
[15] C.D. Levermore, M. Oliver, Distribution-valued initial data for the complex Ginzburg–Landau equation, Comm. Partial Differential Equations 22 (1–2) (1997) 39–48.
[16] A. Mercado, A. Osses, L. Rosier, Inverse problems for the Schrödinger equation via Carleman inequalities with degenerate weights, Inverse Problems 24 (1) (2008), pp. 015017, 18.
[17] A. Mielke, The complex Ginzburg–Landau equation on large and unbounded domains: sharper bounds and attractors, Nonlinearity 10 (1) (1997) 199–222.
[18] L. Rosier, B.-Y. Zhang, Controllability of the Ginzburg–Landau equation, C. R. Acad. Sci. Paris, Ser. I 346 (3–4) (2008) 167–172.
[19] G. Schneider, Global existence via Ginzburg–Landau formalism and pseudo-orbits of Ginzburg–Landau approximations, Commun. Math. Phys. 164 (1) (1994) 157–179.
[20] J. Simon, Compact sets in the space $L^p(0,T;B)$, Ann. Mat. Pura Appl. (4) 146 (1987) 65–96.