

Transition layer for the heterogeneous Allen–Cahn equation

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Abstract

We consider the equation

$$\varepsilon^2 \Delta u = (u - a(x))(u^2 - 1) \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \quad (1)$$

where Ω is a smooth and bounded domain in \mathbb{R}^n , ν the outer unit normal to $\partial\Omega$, and a a smooth function satisfying $-1 < a(x) < 1$ in $\overline{\Omega}$. We set K , Ω_+ and Ω_- to be respectively the zero-level set of a , $\{a > 0\}$ and $\{a < 0\}$. Assuming $\nabla a \neq 0$ on K and $a \neq 0$ on $\partial\Omega$, we show that there exists a sequence $\varepsilon_j \rightarrow 0$ such that Eq. (1) has a solution u_{ε_j} which converges uniformly to ± 1 on the compact sets of Ω_{\pm} as $j \rightarrow +\infty$. This result settles in general dimension a conjecture posed in [P. Fife, M.W. Greenlee, Interior transition layers of elliptic boundary value problem with a small parameter, *Russian Math. Surveys* 29 (4) (1974) 103–131], proved in [M. del Pino, M. Kowalczyk, J. Wei, Resonance and interior layers in an inhomogeneous phase transition model, *SIAM J. Math. Anal.* 38 (5) (2007) 1542–1564] only for $n = 2$.

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1. Introduction

Given a smooth bounded domain Ω of \mathbb{R}^n ($n \geq 2$), we consider the following problem

$$\begin{cases} \varepsilon^2 \Delta u = h(x, u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \quad (2)$$

where ε is a small parameter, ν the unit outer normal vector to $\partial\Omega$ and h a smooth function such that the equation $h(x, t) = 0$ admits two different stable solutions $t_1 \neq t_2$ for any $x \in \overline{\Omega}$. Using matched asymptotics, Fife and Greenlee in [19] proved under some hypothesis on h the existence of a solution of (2) which converges uniformly to t_i in the compact subsets of Ω_i , $i = 1, 2$, where Ω_1 and Ω_2 are two subdomains of Ω such that $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$.

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In this paper we consider the *model* heterogeneous case $h(x, u) = (u - a(x))(u^2 - 1)$, for a smooth function a satisfying $-1 < a(x) < 1$ on $\overline{\Omega}$ and $\nabla a \neq 0$ on the set $K = \{a(x) = 0\}$, with $K \cap \partial\Omega = \emptyset$. We prove the existence of a new type of solution of (2) for any $n \geq 2$ settling in full generality a result previously proved in [15] for the particular case $n = 2$.

Let us describe the result in more detail: in the case $h(x, u) = (u - a(x))(u^2 - 1)$ problem (2) becomes

$$\begin{cases} \varepsilon^2 \Delta u = (u - a(x))(u^2 - 1) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \quad (3)$$

In particular, when $a \equiv 0$, (3) is nothing but the Allen–Cahn equation in material sciences (see [6])

$$\begin{cases} \varepsilon^2 \Delta u + u - u^3 = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \quad (4)$$

Here the function $u(x)$ represents a continuous realization of the phase present in a material confined to the region Ω at the point x . Of particular interest are the solutions which, except for a narrow region, take values close to $+1$ or -1 . Such solutions are called *transition layers*, and have been studied by many authors, see for instance [4,8,20,23,24,30–32,34,35,37–40,43], and the references therein for these and related issues.

In this paper, we are interested in transition layers for the *heterogeneous* equation (3). Define

$$K = \{x \in \Omega : a(x) = 0\}.$$

We assume that K is a smooth closed hypersurface of Ω which separates the domain into two disjoint components

$$\Omega = \Omega_- \cup K \cup \Omega_+, \quad (5)$$

with

$$a(x) < 0 \quad \text{in } \Omega_-, \quad a(x) > 0 \quad \text{in } \Omega_+, \quad \nabla a \neq 0 \quad \text{on } K. \quad (6)$$

We then define the Euler functional $J_\varepsilon(u)$ associated to (3) in Ω as

$$J_\varepsilon(u) = \frac{\varepsilon^2}{2} \int_{\Omega} |\nabla u|^2 + \int_{\Omega} F(x, u) dx, \quad (7)$$

where

$$F(x, u) := \int_{-1}^u (s - a(x))(s^2 - 1) ds.$$

The solution constructed by Fife and Greenlee in [19] (adapted to our choice of the function h) consists in adding an interior transition layer correction to expressions of the form $t_i + \varepsilon t_i^1 + \varepsilon^2 t_i^2$, which approximate the solution u in the regions Ω_i (notice that with our choice of the function h , we have $\Omega_1 = \Omega_+$, $\Omega_2 = \Omega_-$, $t_1 \equiv -1$ and $t_2 \equiv 1$). This allowed Fife and Greenlee to construct an approximation U_ε which yields an exact solution to (11) using a classical implicit function argument. No restrictions on ε are required, and the solution satisfies

$$u_\varepsilon \rightarrow -1 \quad \text{in } \Omega_+ \quad \text{and} \quad u_\varepsilon \rightarrow 1 \quad \text{in } \Omega_- \quad \text{as } \varepsilon \rightarrow 0. \quad (8)$$

Super-subsolutions were later used by Angenent, Mallet-Paret and Peletier in the one-dimensional case (see [7]) for construction and classification of stable solutions. Radial solutions were found variationally by Alikakos and Simpson in [5]. These results were extended by del Pino in [12] for general (even non-smooth) interfaces in any dimension, and further constructions have been done recently by Dancer and Yan [11] and Do Nascimento [16]. In particular, it was proved in [11] that solutions with the asymptotic behavior like (8) are typically minimizer of J_ε . Related results can be found in [1,2].

On the other hand, a solution exhibiting a transition layer in the *opposite direction*, namely

$$u_\varepsilon \rightarrow +1 \quad \text{in } \Omega_+, \quad u_\varepsilon \rightarrow -1 \quad \text{on } \Omega_- \quad \text{as } \varepsilon \rightarrow 0 \quad (9)$$

has been believed to exist for many years. Hale and Sakamoto [21] established the existence of this type of solution in the one-dimensional case, while this was done for the radial case in [13], see also [10]. The layer with the asymptotics

in (9) in this scalar problem is meaningful in describing pattern-formation for reaction–diffusion systems such as Gierer–Meinhardt with saturation, see [13,18,36,41,42] and the references therein.

For one-dimensional or radial problems it is possible to use finite-dimensional reductions, which basically consist in determining the location of the transition layer. In this kind of approach, the same technique works for both the asymptotic behaviors in (8) and (9): the only difference is the sign of the small eigenvalue (of order ε) arising from the approximate degeneracy of the equation (when we tilt the solutions perpendicularly to the interface). This makes the former solution stable and the latter unstable.

On the other hand, one faces a dramatically different situation in higher-dimensional, non-symmetric cases. This is clearly seen already linearizing around a spherically symmetric solution of (1) (with profile as in (9)), as bifurcations of non-radial solutions along certain infinite discrete set of values for $\varepsilon \rightarrow 0$ take place, as established by Sakamoto in [42]. This reveals that the radial solution has Morse index which changes with ε (precisely diverges as $\varepsilon \rightarrow 0$, as shown in [17]). This poses a serious difficulty for a general construction. A phenomenon of this type was previously observed in the one-dimensional case by Alikakos, Bates and Fusco [3] in finding solutions with any prescribed Morse index.

In [15], del Pino, Kowalczyk and the third author considered the two-dimensional case, constructing transition layer solutions with asymptotics as in (9), while in this paper we extend that result to any dimension. Our main theorem is the following.

Theorem 1.1. *Let Ω be a smooth bounded domain of \mathbb{R}^n ($n \geq 2$) and assume that $a: \bar{\Omega} \rightarrow (-1, 1)$ is a smooth function. Define K , Ω_+ and Ω_- to be respectively the zero-set, the positive set and the negative set of a . Assume that $\nabla a \neq 0$ on K and that $K \cap \partial\Omega = \emptyset$. Then there exists a sequence $\varepsilon_j \rightarrow 0$ such that problem (3) has a solution u_{ε_j} which approach 1 in Ω_+ and -1 in Ω_- . Precisely, parameterizing a point x near K by $x = (\bar{y}, \bar{\zeta})$, with $\bar{y} \in K$ and $\bar{\zeta} = d(x, K)$ (with sign, positive in Ω_+), u_{ε_j} admits the following behavior*

$$u_{\varepsilon_j}(\bar{y}, \bar{\zeta}) = H\left(\frac{\bar{\zeta}}{\varepsilon_j} + \Phi(\bar{y})\right) + O(\varepsilon_j) \quad \text{as } j \rightarrow +\infty.$$

Here Φ is a smooth function defined on K and $H(\zeta)$ is the unique heteroclinic solution of

$$H'' + H - H^3 = 0, \quad H(0) = 0, \quad H(\pm\infty) = \pm 1. \tag{10}$$

As in [14,15,26–28,31] and other results for singularly perturbed (or geometric) problems, the existence is proved only along a sequence $\varepsilon_j \rightarrow 0$ (actually it can be obtained for ε in a sequence of intervals (a_j, b_j) approaching zero, but not for any small ε). This is caused by a resonance phenomenon we are going to discuss below, explaining the ideas of the proof.

To describe the reasons which cause the main difficulty in proving Theorem 1.1, we first scale problem (3) using the change of variable $x \mapsto \varepsilon x$, so Eq. (3) becomes

$$\begin{cases} \Delta u = (u - a(\varepsilon x))(u^2 - 1) & \text{in } \Omega_\varepsilon, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{11}$$

where $\Omega_\varepsilon = \frac{1}{\varepsilon}\Omega$. Near the hypersurface $K_\varepsilon := \frac{1}{\varepsilon}K$, we can choose scaled coordinates (y, ζ) in Ω_ε with $y \in K_\varepsilon$ and $\zeta = \text{dist}(x, K_\varepsilon)$ (with sign), see Subsection 2.2, and we let \tilde{u}_ε denote the scaling of u_ε to Ω_ε : with these notations we have that $\tilde{u}_\varepsilon(y, \zeta) = u_\varepsilon(y, \varepsilon\zeta) \simeq H(\zeta)$. The function $H(\zeta) = H(\text{dist}(x, K_\varepsilon))$ for $x \in \Omega_\varepsilon$ can be then considered as a first order approximate solution to (11), so it is natural to use local inversion arguments near this function in order to find true solutions. For this purpose it is necessary to understand the spectrum of the linearization of (11) at approximate solutions.

Letting L_ε be the linearization of (11) at \tilde{u}_ε , it turns out that L_ε admits a sequence of small positive eigenvalues of order ε . Using asymptotic expansions (see Section 3, and in particular formula (72)), one can see that this family behaves qualitatively like $\varepsilon - \varepsilon^2\lambda_j$, where the λ_j 's are the eigenvalues of the Laplace–Beltrami operator of K . By the Weyl's asymptotic formula, we have that $\lambda_j \simeq j^{\frac{2}{n-1}}$ as $j \rightarrow +\infty$, therefore we have an increasing number of positive eigenvalues, many of which accumulate to zero and sometimes, depending on the value of ε , we even have the presence of a kernel: this clearly causes difficulties if one wants to apply local inversion arguments. Notice that,

by the above qualitative formula, the average spectral gap of resonant eigenvalues is of order $\varepsilon^{\frac{n+1}{2}}$. For the case $n = 2$ (considered in [15]) this gap is relatively large, so it was possible to show invertibility using direct estimates on the eigenvalues. However in higher dimension this is not possible anymore, and one needs to apply different arguments.

To overcome this problem, we use an approach introduced in [28,29] (see also [25–27]) to handle similar resonance phenomena for another class of singularly perturbed equations. The main idea consists in looking at the eigenvalues (of the linearized problem) as functions of the parameter ε , and estimate their derivatives with respect to ε . This can be rigorously done employing a classical theorem due to T. Kato, see Proposition 3.3, and by characterizing the eigenfunctions corresponding to resonant modes. Using this result we get invertibility along a suitable sequence $\varepsilon_j \rightarrow 0$, and the norm of the inverse operator along this sequence has an upper bound of order $\varepsilon_j^{-\frac{n+1}{2}}$ (consistently with the above heuristic evaluation of the spectral gaps). This loss of uniform bounds as $j \rightarrow +\infty$ should be expected, since more and more eigenvalues are accumulating near zero. However, we are able to deal with this further difficulty by choosing approximate solutions with a sufficiently high accuracy.

Fixing an integer $k \geq 1$ and using the coordinates introduced after (11), from the fact that a vanishes on K , one can consider the Taylor expansion

$$a(\varepsilon y, \varepsilon \zeta) = \varepsilon \zeta b(\varepsilon y) + \sum_{l=2}^k (\varepsilon \zeta)^l b_l(\varepsilon y) + \tilde{b}(y, \zeta) \quad \text{with } |\tilde{b}(y, \zeta)| \leq C_k |\varepsilon \zeta|^{k+1},$$

and look at an approximate solution of the form

$$u_{k,\varepsilon}(y, \zeta) = H(\zeta - \Phi(\varepsilon y)) + \sum_{i=1}^k \varepsilon^i h_i(\varepsilon y, \zeta - \Phi(\varepsilon y)),$$

for a smooth function $\Phi(\varepsilon y) = \Phi_0(\varepsilon y) + \sum_{i=1}^{k-1} \varepsilon^i \Phi_i(\varepsilon y)$ defined on K and some corrections h_i defined on $K \times \mathbb{R}_+$. Using similar Taylor expansions of the Laplace–Beltrami operator in the above coordinates, see Section 2.2, the couple (h_j, Φ_{j-1}) for $j \geq 1$ can be determined via equations of the form

$$\begin{cases} \mathcal{L}_0 h_1 = -\kappa(\varepsilon y) H'(s) + (s + \Phi_0) b(\varepsilon y) (1 - H^2(s)), \\ \mathcal{L}_0 h_j = \Phi_{j-1} b(\varepsilon y) (1 - H^2(s)) + \mathfrak{F}_k(s, \Phi_0, \dots, \Phi_{j-2}, h_1, \dots, h_{j-1}, b_1, \dots, b_j), \quad \text{for } j \geq 2, \end{cases} \quad (12)$$

where $\mathcal{L}_0 u = u'' + (1 - 3H^2)u$, \mathfrak{F}_k is a smooth function of its arguments, and $s = \zeta - \Phi(\varepsilon y)$. (12) is always solvable in h_j by the Fredholm alternative if we choose properly the functions Φ_l .

Such an accurate approximate solution allows us, using the above characterization of the spectrum of the linearized operator and the bound on its inverse, to apply the contraction mapping theorem and find true solutions. Specifically for the homogeneous Allen–Cahn equation, a related method was used in [31] to study the effect of $\partial\Omega$ on the structure of solutions to (4). Some common arguments are here simplified, and we believe our approach could also be used to handle general non-linearities as in [19].

The paper is organized in the following way: in Section 2 we collect some preliminary results concerning the profile H , we expand the Euclidean metric and the Laplace–Beltrami operator in suitable coordinates near K_ε , and recall some well-known spectral results. In Section 3 we first construct approximate solutions, and then derive some spectral properties of the linearized operator characterizing the resonant eigenfunctions: this is a crucial step to apply Kato’s theorem. Finally, Section 4 is devoted to the proof of our main result.

2. Notation and preliminaries

In this section we first collect some notation and conventions. Then, we list some properties of the heteroclinic solution H , and we expand the metric and the Laplace–Beltrami operator in a local normal coordinates. Finally we recall some results in spectral theory like the Weyl asymptotic formula.

Notation and convention We shall always use the convention that capital letters like A, B, \dots will vary between 1 and n , while indices like i, j, \dots will run between 1 and $n - 1$. We adopt the standard geometric convention of summing over repeated indices.

(y_1, \dots, y_{n-1}) will denote coordinates in \mathbb{R}^{n-1} , and they will also be written as $y = (y_1, \dots, y_{n-1})$, while coordinates in \mathbb{R}^n will be written $x = (y, \zeta) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

The hypersurface K will be parameterized with local coordinates $\bar{y} = (\bar{y}_1, \dots, \bar{y}_{n-1})$. It will be convenient to define its dilation $K_\varepsilon := \frac{1}{\varepsilon}K$ which will be parameterized by coordinates (y_1, \dots, y_{n-1}) related to the \bar{y} 's simply by $\bar{y} = \varepsilon y$.

Derivatives with respect to the variables \bar{y} , y or ζ will be denoted by $\partial_{\bar{y}}$, ∂_y , ∂_ζ and for brevity we shall sometimes use the notations ∂_i for ∂_{y_i} . When dealing with functions depending just on the variable ζ we will write H', h', \dots instead of $\partial_\zeta H, \partial_\zeta h, \dots$.

In a local system of coordinates, \bar{g}_{ij} are the components of the metric on K naturally induced by \mathbb{R}^{n-1} . Similarly, \bar{g}_{AB} are the entries of the metric on Ω in a neighborhood of the hypersurface K . κ_i^j will denote the components of the mean curvature operator of K into \mathbb{R}^{n-1} .

For a real positive variable r and an integer m , $O(r^m)$ (resp. $o(r^m)$) will denote a function for which $|\frac{O(r^m)}{r^m}|$ remains bounded (resp. $|\frac{o(r^m)}{r^m}|$ tends to zero) when r tends to zero. For brevity, we might also write $O(1)$ (resp. $o(1)$) for a quantity which stays bounded (resp. tends to zero) as ε tends to zero.

2.1. Some analytic properties of the hetheroclinic solution H

In this subsection we collect some useful properties of the hetheroclinic solution H to (10). Note first that H can be explicitly determined by

$$H(\zeta) = \tanh\left(\frac{\sqrt{2}}{2}\zeta\right), \tag{13}$$

and moreover the following estimates hold

$$\begin{cases} H(\zeta) - 1 = -A_0 e^{-\sqrt{2}|\zeta|} + O(e^{-(2\sqrt{2})|\zeta|}) & \text{for } \zeta \rightarrow +\infty; \\ H(\zeta) + 1 = A_0 e^{-\sqrt{2}|\zeta|} + O(e^{-(2\sqrt{2})|\zeta|}) & \text{for } \zeta \rightarrow -\infty; \\ H'(\zeta) = \sqrt{2}A_0 e^{-\sqrt{2}|\zeta|} + O(e^{-(2\sqrt{2})|\zeta|}) & \text{for } |\zeta| \rightarrow +\infty, \end{cases} \tag{14}$$

where $A_0 > 0$ is a fixed constant. We have the following well-known result (we refer to Lemma 4.1 in [33] for the proof).

Lemma 2.1. *Consider the following eigenvalue problem*

$$\phi'' + (1 - 3H^2)\phi = \Lambda\phi, \quad \phi \in H^1(\mathbb{R}). \tag{15}$$

Then, letting Λ_j be the eigenvalues arranged in non-increasing order (counted with multiplicity) and ϕ_j be the corresponding eigenfunctions, one has that

$$\Lambda_1 = 0, \quad \phi_1 = cH'; \quad \Lambda_2 < 0. \tag{16}$$

As a consequence (by Fredholm's alternative), given any function $\psi \in L^2(\mathbb{R})$ satisfying $\int_{\mathbb{R}} \psi H' = 0$, the following problem has a unique solution ϕ

$$\phi'' + (1 - 3H^2)\phi = \psi \quad \text{in } \mathbb{R}, \quad \int_{\mathbb{R}} H'\phi = 0. \tag{17}$$

Furthermore, there exists a positive constant C such that $\|\phi\|_{H^1(\mathbb{R})} \leq C\|\psi\|_{L^2(\mathbb{R})}$.

We collect next some useful formulas: first of all we notice that

$$H' = \frac{1}{\sqrt{2}}(1 - H^2) \quad \text{and} \quad H'' = -\sqrt{2}HH'. \tag{18}$$

Moreover, setting

$$\mathcal{L}_0 u = u'' + (1 - 3H^2)u, \tag{19}$$

we have that

$$\mathcal{L}_0(HH') = -3\sqrt{2}H(H')^2. \tag{20}$$

2.2. Geometric background

In this subsection we expand the coefficients of the metric in local normal coordinates. We then derive as a consequence an expansion for the Laplace–Beltrami operator. First of all, it is convenient to scale by $\frac{1}{\varepsilon}$ the coordinates in Eq. (3) to obtain

$$\begin{cases} \Delta u = (u - a(\varepsilon x))(u^2 - 1) & \text{in } \Omega_\varepsilon, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{21}$$

where we have set $\Omega_\varepsilon = \frac{1}{\varepsilon}\Omega$. Following the same notation we also set $K_\varepsilon = \frac{1}{\varepsilon}K$, and for $\gamma \in (0, 1)$ we define

$$S_\varepsilon = \{x \in \Omega_\varepsilon : \text{dist}(x, K_\varepsilon) < \varepsilon^{-\gamma}\}; \quad I_\varepsilon = [-\varepsilon^{-\gamma}, \varepsilon^{-\gamma}].$$

We parameterize elements $x \in S_\varepsilon$ using their closest point y in K_ε and their distance ζ (with sign, positive in the dilation of Ω_+). Precisely, we choose a system of coordinates \bar{y} on K , and denote by $\mathbf{n}(\bar{y})$ the (unique) unit normal vector to K (at the point with coordinates \bar{y}) pointing towards Ω_- . Choosing also coordinates y on K_ε such that $\bar{y} = \varepsilon y$, we define the diffeomorphism $\Gamma_\varepsilon : K_\varepsilon \times I_\varepsilon \rightarrow S_\varepsilon$ by

$$\Gamma_\varepsilon(y, \zeta) = y + \zeta \mathbf{n}(\varepsilon y). \tag{22}$$

We let the upper-case indices A, B, C, \dots run from 1 to n , and the lower-case indices i, j, l, \dots run from 1 to $n - 1$. Using some local coordinates $(y_i)_{i=1, \dots, n-1}$ on K_ε , and letting φ_ε be the corresponding immersion into \mathbb{R}^n , we have

$$\begin{cases} \frac{\partial \Gamma_\varepsilon}{\partial y_i}(y, \zeta) = \frac{\partial \varphi_\varepsilon}{\partial y_i}(y) + \varepsilon \zeta \frac{\partial \mathbf{n}}{\partial y_i}(\varepsilon y) = \frac{\partial \varphi_\varepsilon}{\partial y_i}(y) + \varepsilon \zeta \kappa_i^j(\varepsilon y) \frac{\partial \varphi_\varepsilon}{\partial y_j}(y), & \text{for } i = 1, \dots, n - 1; \\ \frac{\partial \Gamma_\varepsilon}{\partial \zeta}(y, \zeta) = \mathbf{n}(\varepsilon y), \end{cases}$$

where (κ_i^j) are the coefficients of the mean-curvature operator on K . Let also \bar{g}_{ij} be the coefficients of the metric on K_ε in the above coordinates y . Then, letting $g = g_\varepsilon$ denote the metric on Ω_ε induced by \mathbb{R}^n , we have

$$g_{AB} = \left(\frac{\partial \Gamma_\varepsilon}{\partial x_A}, \frac{\partial \Gamma_\varepsilon}{\partial x_B} \right) = \begin{pmatrix} (g_{ij}) & 0 \\ 0 & 1 \end{pmatrix}, \tag{23}$$

where

$$\begin{aligned} g_{ij} &= \left(\frac{\partial \varphi_\varepsilon}{\partial y_i}(y) + \varepsilon \zeta \kappa_i^k(\varepsilon y) \frac{\partial \varphi_\varepsilon}{\partial x_k}(y), \frac{\partial \varphi_\varepsilon}{\partial y_j}(y) + \varepsilon \zeta \kappa_j^l(\varepsilon y) \frac{\partial \varphi_\varepsilon}{\partial x_l}(y) \right) \\ &= \bar{g}_{ij} + \varepsilon \zeta (\kappa_i^k \bar{g}_{kj} + \kappa_j^l \bar{g}_{il}) + \varepsilon^2 \zeta^2 \kappa_i^k \kappa_j^l \bar{g}_{kl}. \end{aligned}$$

Note that also the inverse matrix $\{g^{AB}\}$ decomposes as

$$g^{AB} = \begin{pmatrix} (g^{ij}) & 0 \\ 0 & 1 \end{pmatrix}.$$

From the above decomposition of g_{AB} (and g^{AB}) and for u defined on S_ε , one has

$$\begin{aligned} \Delta_g u &= g^{AB} u_{AB} + \frac{1}{\sqrt{\det g}} \partial_A (g^{AB} \sqrt{\det g}) u_B \\ &= u_\zeta \zeta + g^{ij} u_{ij} + \frac{1}{\sqrt{\det g}} \partial_\zeta (\sqrt{\det g}) u_\zeta + \frac{1}{\sqrt{\det g}} \partial_i (g^{ij} \sqrt{\det g}) u_j. \end{aligned} \tag{24}$$

We have, formally

$$\det g = \det(\bar{g}^{-1} g) \det \bar{g} = (\det \bar{g})(1 + \varepsilon \zeta \text{tr}(\bar{g}^{-1} \alpha)) + o(\varepsilon), \tag{25}$$

where

$$\alpha_{ij} = \kappa_i^k \bar{g}_{kj} + \kappa_j^l \bar{g}_{il}.$$

There holds

$$(\bar{g}^{-1} \alpha)_{is} = \bar{g}^{sj} \alpha_{ij} = \bar{g}^{sj} (\kappa_i^k \bar{g}_{kj} + \kappa_j^l \bar{g}_{il}),$$

and hence

$$\text{tr}(\bar{g}^{-1}\alpha) = \bar{g}^{ij}(\kappa_i^k \bar{g}_{kj} + \kappa_j^l \bar{g}_{il}) = 2\bar{g}^{ij}\kappa_i^k \bar{g}_{kj} = 2\kappa_i^i. \tag{26}$$

We recall that the quantity κ_i^i represents the mean curvature of K , and in particular it is independent of the choice of coordinates.

We note that the metric g_{AB} can be expressed in function of the metric \bar{g}_{ij} , the operator κ_j^i , and the variable $\varepsilon\zeta$. Hence, fixing an integer k and using a Taylor expansion, we can write

$$\frac{1}{\sqrt{\det g}} \partial_\zeta \sqrt{\det g} = \sum_{\ell=1}^k \varepsilon^\ell \zeta^{\ell-1} \tilde{G}_\ell(\varepsilon y) + \tilde{G}(\varepsilon y, \zeta), \tag{27}$$

where $\tilde{G}_\ell : K \rightarrow \mathbb{R}$ are smooth functions, and \tilde{G} satisfies

$$\|\tilde{G}(\cdot, \zeta)\|'_{C^m(K)} \leq C_{k,m} |\zeta|^k \varepsilon^{k+1}, \quad \zeta \in I_\varepsilon, \tag{28}$$

where $C_{k,m}$ is a constant depending only on K , k , and m . Again (and in the following), when we write $\|\cdot\|'$ we keep the variable ζ fixed. In particular, from the above computations it follows that

$$\tilde{G}_1(\varepsilon y) = \kappa(\varepsilon y) := \kappa_i^i(\varepsilon y). \tag{29}$$

We need now a similar expansion for the operator Δ_g : fixing the variable $\zeta \in I_\varepsilon$, the metric $g(y, \zeta) = g_\varepsilon(y, \zeta)$ induces a metric $\hat{g}_{\varepsilon,\zeta}$ on K in the following way. Consider the homothety $i_\varepsilon : K \rightarrow K_\varepsilon$. We define $\hat{g}_{\varepsilon,\zeta}$ to be

$$\hat{g}_{\varepsilon,\zeta} = \varepsilon^2 i_\varepsilon^* g_\varepsilon(\cdot, \zeta),$$

where i_ε^* denotes the pull-back operator. Basically, i_ε^* are freezing the variable ζ and letting y vary. Fixing an integer k , for any smooth function $v : K \rightarrow \mathbb{R}$ we have the expansion below, which follows from (24), reasoning as for (27)

$$\Delta_{\hat{g}_{\varepsilon,\zeta}} v = \sum_{\ell=0}^k (\varepsilon\zeta)^\ell L_\ell v + \tilde{L}_{\varepsilon,k+1} v = \Delta_K v + \sum_{\ell=1}^k (\varepsilon\zeta)^\ell L_\ell v + \varepsilon^{k+1} \tilde{L}_{\varepsilon,k+1} v. \tag{30}$$

Here $\{L_i\}_i, \tilde{L}_{\varepsilon,k+1}$ are linear second-order differential operators acting on y and satisfying

$$\|L_i v\|'_{C^m(K)} \leq C_m \|v\|'_{C^{m+2}(K)}; \quad \|\tilde{L}_{\varepsilon,k+1} v\|'_{C^m(K)} \leq C_m |\zeta|^{k+1} \|v\|'_{C^{m+2}(K)} \tag{31}$$

for all smooth v , where C_m is a constant depending only on K , k , and m .

Consider now a function $u : S_\varepsilon \rightarrow \mathbb{R}$ of the form

$$u(y, \zeta) = \tilde{u}(\varepsilon y, \zeta), \quad y \in K_\varepsilon, \zeta \in I_\varepsilon. \tag{32}$$

Then, scaling in the variable y , we have

$$\Delta_{g_\varepsilon} u(y, \zeta) = \tilde{u}_{\zeta\zeta}(\varepsilon y, \zeta) + \frac{1}{\sqrt{\det g}} \partial_\zeta (\sqrt{\det g}) \tilde{u}_\zeta(\varepsilon y, \zeta) + \varepsilon^2 \Delta_{\hat{g}_{\varepsilon,\zeta}} \tilde{u}(\varepsilon y, \zeta).$$

Using the expansions (27), (30) together with (29), the latter equation becomes

$$\begin{aligned} \Delta_{g_\varepsilon} u(y, \zeta) &= \tilde{u}_{\zeta\zeta}(\varepsilon y, \zeta) + \left(\varepsilon\kappa + \sum_{\ell=2}^k \varepsilon^\ell \zeta^{\ell-1} \tilde{G}_\ell \right) \tilde{u}_\zeta(\varepsilon y, \zeta) + \tilde{G}(\varepsilon y, \zeta) \tilde{u}_\zeta(\varepsilon y, \zeta) \\ &\quad + \varepsilon^2 \Delta_K \tilde{u} + \sum_{\ell=1}^k \varepsilon^{2+\ell} \zeta^\ell L_\ell \tilde{u}(\varepsilon y, \zeta) + \varepsilon^{k+3} \tilde{L}_{\varepsilon,k+1} \tilde{u}(\varepsilon y, \zeta). \end{aligned} \tag{33}$$

2.3. Spectral analysis

We define the scaled Euler functional $\bar{J}_\varepsilon(u)$ in Ω_ε by

$$\bar{J}_\varepsilon(u) = \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla u|^2 + \int_{\Omega_\varepsilon} F(\varepsilon x, u) dx, \quad \text{with } F(x, u) := \int_{-1}^u (s - a(x))(s^2 - 1) ds. \quad (34)$$

We set for brevity

$$b(y) := \partial_{\mathbf{n}} a(y, 0) \quad (35)$$

and we notice that by our choice of \mathbf{n} , we have $b > 0$ on K . Now, we let φ_j and λ_j be the eigenfunctions and the eigenvalues (with weight b) of

$$-\Delta_K \varphi_j = \lambda_j b(\bar{y}) \varphi_j.$$

The λ_j 's can be obtained for example using the Rayleigh quotient: precisely if M_j denotes the family of j -dimensional subspaces of $H^1(K)$, then one has

$$\lambda_j = \inf_{M \in M_j} \sup_{\varphi \in M, \varphi \neq 0} \frac{\int_K |\nabla_K \varphi|^2}{\int_K b(\bar{y}) \varphi^2} = \sup_{M \in M_{j-1}} \inf_{\varphi \perp M, \varphi \neq 0} \frac{\int_K |\nabla_K \varphi|^2}{\int_K b(\bar{y}) \varphi^2}, \quad (36)$$

where \perp denotes the orthogonality with respect to the L^2 scalar product with weight b . We can estimate the λ_j using a standard Weyl's asymptotic formula [9], one has

$$\lambda_j \simeq C_{K,b} j^{\frac{2}{n-1}} \quad \text{as } j \rightarrow +\infty, \quad (37)$$

for some constant $C_{K,b}$ depending only on K and b .

3. Asymptotic analysis

This section is devoted to the construction of approximate solutions to (21), and of approximate eigenfunctions (and eigenvalues) in the ζ component (see the coordinates introduced in (22)) of the relative linearized equation. Then we characterize, via Fourier analysis, the profile of resonant eigenfunctions in both the variables y and ζ .

3.1. Approximate solutions and eigenfunctions

In this section, given any integer $k \geq 1$, we construct an approximate solution $u_{k,\varepsilon}$ to problem (21), which solves the equation up to an error of order ε^{k+1} . Using the above parametrization (y, ζ) in S_ε , we make the following ansatz

$$u_{k,\varepsilon}(y, \zeta) = H(\zeta - \Phi(\varepsilon y)) + \sum_{i=1}^k \varepsilon^i h_i(\varepsilon y, \zeta - \Phi(\varepsilon y)) \quad \text{in } S_\varepsilon, \quad (38)$$

where H is the heteroclinic solution of (10) and where $\Phi(\varepsilon y) = \Phi_0(\varepsilon y) + \sum_{i=1}^{k-1} \varepsilon^i \Phi_i(\varepsilon y)$ for some smooth functions Φ_j defined on K . The corrections h_i and Φ_i are to be constructed recursively in the index i , depending on the Taylor expansion of a and the geometry of K . Since all the h_i 's will turn out to have an exponential decay in ζ , $u_{k,\varepsilon}$ can be easily extended (via some cutoff functions) to an approximate solution in the whole Ω_ε , see (102) below.

We first determine h_1 by solving the equation up to an error of order ε^2 . To this aim, we expand the function a in powers of ε as (notice that $a(\varepsilon y, 0) \equiv 0$)

$$a(\varepsilon y, \varepsilon \zeta) = \varepsilon b(\varepsilon y) \zeta + \sum_{\ell=2}^k (\varepsilon \zeta)^\ell b_\ell(\varepsilon y) + \tilde{b}(y, z), \quad (39)$$

where $\tilde{b}(y, z)$ is smooth and satisfies

$$|\tilde{b}(y, z)| \leq C_k |\varepsilon z|^{k+1}.$$

Using the above expansion of the metric coefficients and the Laplace–Beltrami operator, see in particular (33), setting $s = \zeta - \Phi$, we obtain that the term (formally) of order ε in the equation is identically zero if and only if the correction h_1 satisfies

$$\mathcal{L}_0 h_1 := (h_1)_{ss} + (1 - 3H(s)^2)h_1 = -\kappa(\varepsilon y)H'(s) + (s + \Phi_0)b(\varepsilon y)(1 - H^2(s)). \tag{40}$$

By the asymptotics in (14), the right-hand side is of class L^2 in \mathbb{R} and, by Lemma 2.1, (40) is solvable provided the latter is orthogonal in L^2 to the function $H'(s)$. Since $H'(s)$ is even in s and since $b(\varepsilon y) > 0$, this is possible choosing $\Phi_0(\varepsilon y)$ so that

$$b(\varepsilon y)\Phi_0(\varepsilon y) = \kappa(\varepsilon y) \frac{\int_{-\infty}^{+\infty} (H')^2(s) ds}{\int_{-\infty}^{+\infty} H'(s)(1 - H^2(s)) ds} = \frac{\sqrt{2}}{3} \kappa(\varepsilon y). \tag{41}$$

Moreover one can prove, using standard ODE estimates, that h_1 has the following (regularity properties and) decay at infinity

$$|\partial_s^l \partial_y^m h_1(\varepsilon y, s)| \leq C_m \varepsilon^m (1 + |s|) e^{-\sqrt{2}|s|}, \quad l = 0, 1, 2, \quad m = 0, 1, 2, \dots, \tag{42}$$

where C_m depends only on m, a and K .

To obtain the other corrections Φ_i and h_i one can proceed by induction, assuming that $N \geq 2$, that $\Phi_0, \dots, \Phi_{N-2}$ and h_1, \dots, h_{N-1} have been determined, and that $(h_i)_{i \leq N-1}$ satisfy

$$|\partial_s^l \partial_y^m h_i(\varepsilon y, s)| \leq C_m \varepsilon^m (1 + |s|^{d_i}) e^{-\sqrt{2}|s|}, \quad i \leq N - 1, \quad l = 0, 1, 2, \quad m = 0, 1, 2, \dots, \tag{43}$$

where C_m depends only on m, a, K and d_i only on i . When we expand Eq. (21) for $u = u_{N,\varepsilon}$ in power series of ε , the couple (h_N, Φ_{N-1}) can be found reasoning as for (h_1, Φ_0) : indeed, considering the coefficient of ε^N in this expansion, one can easily see that h_N satisfies an equation on the form

$$\mathcal{L}_0 h_N = \Phi_{N-1} b(\varepsilon y)(1 - H^2(s)) + \mathfrak{F}_N(s, \Phi_0, \dots, \Phi_{N-2}, h_1, \dots, h_{N-1}, b_1, \dots, b_N), \tag{44}$$

where \mathfrak{F}_N is a smooth function of its argument. Reasoning as for h_1 this equation is solvable provided the right-hand side is L^2 -orthogonal to the function $H'(s)$. This is indeed true choosing Φ_{N-1} so that

$$b(\varepsilon y)\Phi_{N-1}(\varepsilon y) = \frac{-\int_{-\infty}^{+\infty} H'(s)\mathfrak{F}_N(s, \dots) ds}{\int_{-\infty}^{+\infty} H'(s)(1 - H^2(s)) ds}.$$

Furthermore, one can show that h_N satisfies regularity and decay estimates as in (43). Reasoning as in Section 3 of [29] one can check that the above formal estimates can be made rigorous, and that the exponential decay of the corrections yields the following result.

Proposition 3.1. *Given any integer $k \geq 1$ there exist a function $u_{k,\varepsilon} : S_\varepsilon \rightarrow \mathbb{R}$ which solves Eq. (21) up to an error of order ε^{k+1} . Precisely, setting*

$$\mathfrak{S}_\varepsilon(u) = \Delta u - (u - a(\varepsilon x))(u^2 - 1), \tag{45}$$

there exist a polynomial $P_k(\zeta)$ such that

$$|\mathfrak{S}_\varepsilon(u_{k,\varepsilon}(\varepsilon y, \zeta))| \leq \varepsilon^{k+1} P_k(\zeta) e^{-\sqrt{2}|\zeta|} \quad \text{in } S_\varepsilon. \tag{46}$$

Moreover, the following estimate holds

$$|\partial_s^l \partial_y^m u_{k,\varepsilon}(\varepsilon y, \zeta)| \leq C_m \varepsilon^m P_k(\zeta) e^{-\sqrt{2}|\zeta|}, \quad l = 0, 1, 2, \quad m = 0, 1, 2, \dots, \tag{47}$$

where C_m is a constant depending only on m, a and K .

We will look at solutions u of (21) as small corrections of $u_{k,\varepsilon}$ (suitably extended to Ω_ε via some cutoffs in ζ , see (102) below), namely of the form

$$u = u_{k,\varepsilon} + w,$$

for w small in a sense to be specified later. The equation $\mathfrak{S}_\varepsilon(u) = 0$ is then equivalent to

$$L_\varepsilon(w) + \mathfrak{S}_\varepsilon(u_{k,\varepsilon}) + \mathcal{N}_\varepsilon(w) = 0,$$

where L_ε is nothing but the linearized operator at the approximate solution $u_{k,\varepsilon}$

$$L_\varepsilon w := \Delta_{g_\varepsilon} w + (1 - 3u_{k,\varepsilon}^2)w + 2a(\varepsilon x)u_{k,\varepsilon} w, \tag{48}$$

and where \mathcal{N}_ε is the remainder given by non-linear terms in \mathfrak{S}_ε , namely

$$\mathcal{N}_\varepsilon(w) := -(3u_{k,\varepsilon} - a(\varepsilon x))w^2 - w^3. \tag{49}$$

It is also convenient to define the following linear operator

$$\mathbb{L}_\varepsilon w := w_\zeta \zeta + \frac{1}{\sqrt{\det g}} \partial_\zeta (\sqrt{\det g}) w_\zeta + (1 - 3u_{k,\varepsilon}^2)w + 2a(\varepsilon y, \zeta)u_{k,\varepsilon} w. \tag{50}$$

In particular, using the expansion (33), the operators L_ε and \mathbb{L}_ε are related by the following formula: setting $w(y, \zeta) = \tilde{w}(\varepsilon y, \zeta)$ one has

$$L_\varepsilon w = \mathbb{L}_\varepsilon \tilde{w} + \varepsilon^2 \Delta_K \tilde{w} + \varepsilon^3 \hat{\mathfrak{L}}_{3,\varepsilon} \tilde{w}, \tag{51}$$

where $\hat{\mathfrak{L}}_{3,\varepsilon}$ consists of the last two terms in (33) (replacing u with w). Precisely, $\hat{\mathfrak{L}}_{3,\varepsilon}$ is a linear differential operator of second order acting on the variables \bar{y} , which for every integer m satisfies

$$\|\hat{\mathfrak{L}}_{3,\varepsilon} v\|'_{C^m(K)} \leq \hat{P}(\zeta) \|v\|'_{C^{m+2}(K)}. \tag{52}$$

Here $\hat{P}(\zeta)$ is a polynomial in ζ with fixed degree, and coefficients depending only on m .

We want next to derive some formal estimates on the following eigenvalue problem

$$\mathbb{L}_\varepsilon v = \mu v \quad \text{in } I_\varepsilon, \tag{53}$$

with zero Dirichlet boundary conditions. It follows from Lemma 2.1 that the eigenvalues either stay bounded away from zero, or converge to zero as $\varepsilon \rightarrow 0$: we are interested in the latter case. We argue heuristically expanding (53) at first order in ε . In the limit $\varepsilon \rightarrow 0$, we have $\mu = 0$ with corresponding eigenfunction H' , therefore it is natural to look for approximate eigenfunctions of the form

$$\Psi = H' + \varepsilon H_1, \tag{54}$$

and eigenvalues $\mu = \varepsilon \bar{\mu} + o(\varepsilon)$. We impose that H_1 is orthogonal to H' in $L^2(\mathbb{R})$. Therefore the approximate eigenvalue equation formally becomes

$$\mathcal{L}_0 H_1 = -2b(\varepsilon y)(s + \Phi_0)HH' - H''\kappa(\varepsilon y) + 6HH'h_1 + \bar{\mu}H' + o(1). \tag{55}$$

As for (40), solvability is guaranteed provided the right-hand side is orthogonal in L^2 to H' . Using the oddness of H , formulas (18), (20), (40) and the self-adjointness of \mathcal{L}_0 we find that orthogonality is equivalent to

$$\bar{\mu} = 4b(\varepsilon y) \frac{\int_{\mathbb{R}} s H(s)(H'(s))^2 ds}{\int_{\mathbb{R}} (H'(s))^2 ds} = \sqrt{2} b(\varepsilon y).$$

With this choice of $\bar{\mu}$, the function H_1 is defined as the unique solution of

$$\mathcal{L}_0 H_1 = -2b(\varepsilon y)(s + \Phi_0)HH' - H''\kappa(\varepsilon y) + 6HH'h_1 + \sqrt{2}b(\varepsilon y)H'.$$

From the exponential decay of H' and h_1 (see (42)) we deduce that H_1 satisfies estimates similar to (43).

Using this fact and the rigorous expansions in (27), (39), we then derive the following estimate

$$\mathbb{L}_\varepsilon(\Psi) = \varepsilon \bar{\mu} H' + \varepsilon^2 R_\varepsilon(\varepsilon y, \zeta), \tag{56}$$

where the error term R_ε satisfies

$$|R_\varepsilon(\varepsilon y, \zeta)| \leq P(\zeta) e^{-\sqrt{2}|\zeta|} \tag{57}$$

for some polynomial $P(\zeta)$. Also, from the regularity and the decay of H_1 we have

$$|\partial_s^l \partial_y^m \Psi(\varepsilon y, s)| \leq P(\zeta) \varepsilon^{m+1} e^{-\sqrt{2}|\zeta|}, \quad l = 0, 1, 2, \quad m = 0, 1, 2, \dots \tag{58}$$

3.2. Characterization of resonant eigenfunctions

We characterize next the eigenfunctions of L_ε , see (48), corresponding to small eigenvalues. Let us recall first the definition of φ_j and λ_j in Section 2.3.

Lemma 3.2. *Let $\lambda_\varepsilon = O(\varepsilon^2)$ be an eigenvalue of the linearized operator L_ε in S_ε with eigenfunction ϕ and weight b , namely*

$$L_\varepsilon \phi = \lambda_\varepsilon b \phi \quad \text{in } S_\varepsilon$$

(and with zero Dirichlet boundary conditions). Let us write the eigenfunction ϕ as

$$\phi(y, \zeta) = \varphi(\varepsilon y) \Psi(\varepsilon y, \zeta) + \phi^\perp(y, \zeta),$$

with Ψ defined in (54) and with ϕ^\perp satisfying the following orthogonality condition (we are freezing the y variables in the volume element)

$$\int_{I_\varepsilon} \Psi(\varepsilon y, \zeta) \phi^\perp(y, \zeta) dV_{g_\varepsilon}(\zeta) = 0 \quad \text{for every } y \in K_\varepsilon. \tag{59}$$

Then one has $\|\phi^\perp\|_{H^1(S_\varepsilon)} = o(\varepsilon) \|\phi\|_{H^1(S_\varepsilon)}$ as ε tends to zero.

Proof. We notice first that, since $\Psi = H' + o(1)$ in $H^1(\mathbb{R})$, the operator \mathcal{L}_0 is negative definite on ϕ^\perp by Lemma 2.1. Therefore, using the estimates on the metric g_ε in Section 2, we find easily that there exist a constant $C > 0$ such that

$$\int_{S_\varepsilon} \phi^\perp(y, \zeta) L_\varepsilon \phi^\perp(y, \zeta) dV_{g_\varepsilon}(y, \zeta) \leq -\frac{1}{C} \|\phi^\perp\|_{H^1(S_\varepsilon)}^2. \tag{60}$$

Let us write the eigenvalue equation $L_\varepsilon \phi = \lambda_\varepsilon b \phi$ as

$$L_\varepsilon \phi^\perp = -L_\varepsilon(\varphi \Psi) + \lambda_\varepsilon b \phi^\perp + \lambda_\varepsilon b \varphi \Psi.$$

Multiplying by ϕ^\perp , integrating over S_ε and using (59) we obtain

$$\int_{S_\varepsilon} \phi^\perp(y, \zeta) L_\varepsilon \phi^\perp(y, \zeta) dV_{g_\varepsilon} = - \int_{S_\varepsilon} \phi^\perp(y, \zeta) L_\varepsilon(\varphi(\varepsilon y) \Psi(\varepsilon y, \zeta)) dV_{g_\varepsilon} + \lambda_\varepsilon \int_{S_\varepsilon} b(\varepsilon y) \phi^\perp(y, \zeta)^2 dV_{g_\varepsilon}.$$

By (60) (and the smallness of λ_ε) it then follows

$$\|\phi^\perp\|_{H^1(S_\varepsilon)}^2 \leq C \left| \int_{S_\varepsilon} \phi^\perp(y, \zeta) L_\varepsilon(\varphi(\varepsilon y) \Psi(\varepsilon y, \zeta)) dV_{g_\varepsilon} \right|. \tag{61}$$

Now by (51) and (56) we can write $L_\varepsilon(\varphi \Psi)$ as

$$\begin{aligned} L_\varepsilon(\varphi \Psi) &= \mathbb{L}_\varepsilon(\varphi \Psi) + \varepsilon^2 \Delta_K(\varphi \Psi) + \varepsilon^3 \hat{\mathcal{L}}_{3,\varepsilon}(\varphi \Psi) \\ &= \varphi(\varepsilon \sqrt{2} b(\varepsilon y) H' + \varepsilon^2 R_\varepsilon(\varepsilon y, \zeta)) + \varepsilon^2 \Delta_K(\varphi \Psi) + \varepsilon^3 \hat{\mathcal{L}}_{3,\varepsilon}(\varphi \Psi). \end{aligned} \tag{62}$$

Then, again by (59), we have

$$\begin{aligned} \left| \int_{S_\varepsilon} \phi^\perp(y, \zeta) L_\varepsilon(\varphi(\varepsilon y) \Psi(\varepsilon y, \zeta)) dV_{g_\varepsilon} \right| &\leq \varepsilon^2 \left| \int_{S_\varepsilon} \phi^\perp(y, \zeta) \Delta_K(\varphi(\varepsilon y) \Psi(\varepsilon y, \zeta)) dV_{g_\varepsilon} \right| \\ &\quad + \left| \int_{S_\varepsilon} (\varepsilon^2 \tilde{R}_\varepsilon(\varepsilon y, \zeta) \varphi(\varepsilon y) + \varepsilon^3 \hat{\mathcal{L}}_{3,\varepsilon}(\varphi(\varepsilon y) \Psi(\varepsilon y, \zeta))) \phi^\perp dV_{g_\varepsilon} \right|, \end{aligned} \tag{63}$$

where \tilde{R}_ε is as in (57). We first estimate the second term: since $\hat{\mathcal{L}}_{3,\varepsilon}$ is a second order operator in \bar{y} satisfying the bound (52), by an integration by parts we find

$$\varepsilon^3 \left| \int_{S_\varepsilon} \hat{\mathcal{L}}_{3,\varepsilon}(\varphi(\varepsilon y)\Psi(\varepsilon y, \zeta))\phi^\perp dV_{g_\varepsilon}(y, \zeta) \right| \leq C\varepsilon^2 \int_{K_\varepsilon \times I_\varepsilon} \hat{P}(\zeta) |\nabla_{\bar{y}}(\varphi(\varepsilon y)\Psi(\varepsilon y, \zeta))| |\nabla_y \phi^\perp| dV_{g_\varepsilon}(y, \zeta).$$

Therefore, using the Hölder inequality, the change of variables $\bar{y} = \varepsilon y$, (58) and the estimate on \tilde{R}_ε we find

$$\left| \int_{S_\varepsilon} (\varepsilon^2 \tilde{R}_\varepsilon(\varepsilon y, \zeta)\varphi(\varepsilon y) + \varepsilon^3 \hat{\mathcal{L}}_{3,\varepsilon}(\varphi(\varepsilon y)\Psi(\varepsilon y, \zeta)))\phi^\perp dV_{g_\varepsilon}(y, \zeta) \right| \leq C \frac{\varepsilon^2}{\varepsilon^{\frac{n-1}{2}}} \|\phi^\perp\|_{H^1(S_\varepsilon)} \|\varphi\|_{H^1(K)}, \tag{64}$$

for some positive constant C . It remains to estimate the first term in (63). To this aim we decompose φ as (see the above notation)

$$\varphi(\varepsilon y) = \sum_j \alpha_j \varphi_j(\varepsilon y), \tag{65}$$

for some real numbers α_j . One can write

$$\begin{aligned} \int_{S_\varepsilon} \phi^\perp(y, \zeta) \Delta_K(\varphi(\varepsilon y)\Psi(\varepsilon y, \zeta)) dV_{g_\varepsilon}(y, \zeta) &= \int_{S_\varepsilon} \phi^\perp(y, \zeta) \Psi(\varepsilon y, \zeta) \Delta_K \varphi(\varepsilon y) dV_{g_\varepsilon}(y, \zeta) \\ &+ \int_{S_\varepsilon} \phi^\perp(y, \zeta) \varphi(\varepsilon y) \Delta_K \Psi(\varepsilon y, \zeta) dV_{g_\varepsilon}(y, \zeta) \\ &+ 2 \int_{S_\varepsilon} \phi^\perp(y, \zeta) \nabla_K \Psi(\varepsilon y, \zeta) \cdot \nabla_K \varphi(\varepsilon y) dV_{g_\varepsilon}(y, \zeta). \end{aligned}$$

The first term vanishes by (59). Hence, using the Hölder inequality, (58) and reasoning as for (64) we obtain

$$\left| \int_{S_\varepsilon} \phi^\perp(y, \zeta) \Delta_K(\varphi(\varepsilon y)\Psi(y, \zeta)) dV_{g_\varepsilon}(y, \zeta) \right| \leq C \frac{1}{\varepsilon^{\frac{n-1}{2}}} \|\phi^\perp\|_{H^1(S_\varepsilon)} \|\varphi\|_{H^1(K)},$$

which by estimates (63) and (64) implies

$$\left| \int_{S_\varepsilon} \phi^\perp(y, \zeta) L_\varepsilon(\varphi(y)\Psi(y, \zeta)) dV_{g_\varepsilon}(y, \zeta) \right| \leq C \frac{\varepsilon^2}{\varepsilon^{\frac{n-1}{2}}} \|\phi^\perp\|_{H^1(S_\varepsilon)} \|\varphi\|_{H^1(K)}. \tag{66}$$

Using then (61) and the latter equation together with the Weyl’s asymptotic formula (see Subsection 2.3), we then obtain

$$\|\phi^\perp\|_{H^1(S_\varepsilon)}^2 \leq \frac{C}{\varepsilon^{n-1}} \left(\sum_j \alpha_j^2 (\varepsilon^4 + \varepsilon^4 j^{\frac{2}{n-1}}) \right). \tag{67}$$

Now, we rewrite the eigenvalue equation as

$$(\varepsilon\sqrt{2}b\varphi + \varepsilon^2 \Delta_K \varphi)\Psi = \lambda_\varepsilon b\phi^\perp + \lambda_\varepsilon b\varphi\Psi - \varepsilon^2 \varphi \Delta_K \Psi - 2\varepsilon^2 \nabla_K \varphi \cdot \nabla_K \Psi - \varepsilon^2 \tilde{R}_\varepsilon \varphi - \varepsilon^3 \hat{\mathcal{L}}_{3,\varepsilon}(\varphi\Psi) - L_\varepsilon \phi^\perp.$$

We use again the above decomposition $\varphi(\varepsilon y) = \sum_j \alpha_j \varphi_j(\varepsilon z)$, we define the integer j_ε (depending on ε) to be the first j such that $\varepsilon^2 \lambda_j > \varepsilon^{\frac{1}{2}}$. We multiply this time the last equation by $\sum_{j \geq j_\varepsilon} \alpha_j \varphi_j \Psi$ and we integrate over S_ε . By the self-adjointness of L_ε , (58), (59) and a similar argument as for (64), incorporating the term involving $\hat{\mathcal{L}}_{3,\varepsilon}$ into the left-hand side we obtain that

$$\begin{aligned}
 (1 + O(\varepsilon)) \frac{1}{\varepsilon^{n-1}} \sum_{j \geq j_\varepsilon} \varepsilon^2 \alpha_j^2 \lambda_j &\leq C(\varepsilon^2 + |\lambda_\varepsilon|) \left(\frac{1}{\varepsilon^{n-1}} \sum_{j \geq j_\varepsilon} \alpha_j^2 \right)^{\frac{1}{2}} \left(\frac{1}{\varepsilon^{n-1}} \sum_j \alpha_j^2 \right)^{\frac{1}{2}} \\
 &+ C\varepsilon \left(\frac{1}{\varepsilon^{n-1}} \sum_j \alpha_j^2 \right)^{\frac{1}{2}} \left(\frac{1}{\varepsilon^{n-1}} \sum_{j \geq j_\varepsilon} \varepsilon^2 \lambda_j \alpha_j^2 \right)^{\frac{1}{2}} + \left| \int_{S_\varepsilon} \phi^\perp L_\varepsilon \left(\sum_{j \geq j_\varepsilon} \alpha_j \varphi_j \Psi \right) dV_{g_\varepsilon} \right|.
 \end{aligned} \tag{68}$$

The last term can be estimated reasoning as for (67): since $\lambda_j \gg 1$ for $j \geq j_\varepsilon$, we find that

$$\left| \int_{S_\varepsilon} \phi^\perp L_\varepsilon \left(\sum_{j \geq j_\varepsilon} \alpha_j \varphi_j \Psi \right) dV_{g_\varepsilon} \right| \leq C\varepsilon \|\phi^\perp\|_{H^1(S_\varepsilon)} \left(\frac{1}{\varepsilon^{n-1}} \sum_{j \geq j_\varepsilon} \varepsilon^2 \lambda_j \alpha_j^2 \right)^{\frac{1}{2}}.$$

From the above estimates and the fact that $\lambda_\varepsilon = O(\varepsilon^2)$ we obtain

$$\left(\frac{1}{\varepsilon^{n-1}} \sum_{j \geq j_\varepsilon} \varepsilon^2 \lambda_j \alpha_j^2 \right)^{\frac{1}{2}} \leq C\varepsilon \left(\left(\frac{1}{\varepsilon^{n-1}} \sum_j \alpha_j^2 \right)^{\frac{1}{2}} + \|\phi^\perp\|_{L^2(S_\varepsilon)} \right). \tag{69}$$

Then, writing $\varphi\Psi$ as

$$\varphi\Psi = \sum_{j < j_\varepsilon} \alpha_j \varphi_j \Psi + \sum_{j \geq j_\varepsilon} \alpha_j \varphi_j \Psi,$$

using (67) and (69) we find

$$\begin{aligned}
 \|\phi^\perp\|_{H^1(S_\varepsilon)} &\leq C(\varepsilon^2 + \varepsilon^{\frac{5}{4}}) \|\varphi\Psi\|_{L^2(S_\varepsilon)} + C\varepsilon \left(\frac{1}{\varepsilon^{n-1}} \sum_{j \geq j_\varepsilon} \varepsilon^2 \lambda_j \alpha_j^2 \right)^{\frac{1}{2}} \\
 &\leq C\varepsilon^{\frac{5}{4}} \|\varphi\Psi\|_{L^2(S_\varepsilon)} + C\varepsilon^2 (\|\varphi\Psi\|_{L^2(S_\varepsilon)} + \|\phi^\perp\|_{H^1(S_\varepsilon)}).
 \end{aligned}$$

This implies $\|\phi^\perp\|_{H^1(S_\varepsilon)} \leq C\varepsilon^{\frac{5}{4}} \|\varphi\Psi\|_{L^2(S_\varepsilon)}$, and noticing that $\|\phi\|_{L^2(S_\varepsilon)}^2 = \|\varphi\Psi\|_{L^2(S_\varepsilon)}^2 + \|\phi^\perp\|_{L^2(S_\varepsilon)}^2$, we achieve the desired estimate. \square

Our next task is to estimate the derivatives of small eigenvalues of the linearized operator L_ε with respect to the parameter ε . This will allow us to obtain invertibility of L_ε for a suitable family of small ε . The prove of the main result can be then obtained by a direct application of the contraction mapping theorem. Using a result by T. Kato, see [22], page 445, which can be applied by the symmetry of L_ε and elliptic regularity results (these ensure that the eigenvalues of L_ε are *stable* and *semi-simple*, according to the definitions in [22]), we have the following proposition.

Proposition 3.3. *For $k \in \mathbb{N}$, $k \geq 1$, let $u_{k,\varepsilon}$ be given by Proposition 3.1, and let L_ε be defined in (48). Then the eigenvalues λ_ε of the problem*

$$\begin{cases} L_\varepsilon u = \lambda_\varepsilon b u & \text{in } S_\varepsilon; \\ u = 0 & \text{on } \partial S_\varepsilon, \end{cases} \tag{70}$$

are differentiable with respect to ε , and they satisfy the following estimates

$$T_{\lambda_\varepsilon, \varepsilon}^1 \leq \frac{\partial \lambda_\varepsilon}{\partial \varepsilon} \leq T_{\lambda_\varepsilon, \varepsilon}^2. \tag{71}$$

Here we have set

$$\begin{aligned}
 T_{\lambda, \varepsilon}^1 &= \inf_{u \in H_\lambda, u \neq 0} \frac{\int_{S_\varepsilon} (-\frac{2}{\varepsilon} |\nabla_{g_\varepsilon} u|^2 - 6u_{k,\varepsilon} v_{k,\varepsilon} u^2) dV_{g_\varepsilon}}{\int_{S_\varepsilon} b u^2 dV_{g_\varepsilon}}; \\
 T_{\lambda, \varepsilon}^2 &= \sup_{u \in H_\lambda, u \neq 0} \frac{\int_{S_\varepsilon} (-\frac{2}{\varepsilon} |\nabla_{g_\varepsilon} u|^2 - 6u_{k,\varepsilon} v_{k,\varepsilon} u^2) dV_{g_\varepsilon}}{\int_{S_\varepsilon} b u^2 dV_{g_\varepsilon}},
 \end{aligned}$$

with $v_{k,\varepsilon} = \frac{\partial u_{k,\varepsilon}}{\partial \varepsilon}$, while H_λ stands for the eigenspace for (70) corresponding to the eigenvalue λ .

We next give a further characterization of some eigenfunctions of L_ε , in addition to the ones in Lemma 3.2, concerning in particular the function φ .

Lemma 3.4. *Suppose the assumptions of Lemma 3.2 hold true. Then, normalizing ϕ by $\|\phi\|_{H^1(S_\varepsilon)} = 1$, decomposing φ as in (65) and setting*

$$\lambda_{j,\varepsilon} = \sqrt{2}\varepsilon - \varepsilon^2\lambda_j, \tag{72}$$

as $\varepsilon \rightarrow 0$ we have that

$$\frac{1}{\varepsilon^{n-1}} \sum_{|\lambda_{j,\varepsilon}| \geq \varepsilon^{\frac{5}{4}}} \alpha_j^2 = o(1); \quad \frac{1}{\varepsilon^{n-1}} \sum_{|\lambda_{j,\varepsilon}| \geq \varepsilon^{\frac{5}{4}}} \lambda_{j,\varepsilon} \alpha_j^2 = o(\varepsilon).$$

Proof. We define the sets

$$A_{1,\varepsilon} = \{j \in \mathbb{N}: \lambda_{j,\varepsilon} > \varepsilon^{\frac{5}{4}}\}; \quad A_{2,\varepsilon} = \{j \in \mathbb{N}: \lambda_{j,\varepsilon} < -\varepsilon^{\frac{5}{4}}\},$$

and the functions

$$\begin{aligned} \bar{\varphi}_1(\varepsilon y) &= \sum_{j \in A_{1,\varepsilon}} \alpha_j \varphi_j(\varepsilon y); & \bar{\varphi}_2(\varepsilon y) &= \sum_{j \in A_{2,\varepsilon}} \alpha_j \varphi_j(\varepsilon y); \\ \phi_1 &= \bar{\varphi}_1(\varepsilon y) \Psi(\varepsilon y, \zeta); & \phi_2 &= \bar{\varphi}_2(\varepsilon y) \Psi(\varepsilon y, \zeta). \end{aligned}$$

As one can easily see, from the estimates on g_ε in Subsection 2.2 and from the decay of Ψ , see (58), as $\varepsilon \rightarrow 0$ there holds

$$\|\varphi \Psi\|_{H^1(S_\varepsilon)}^2 = \frac{1 + o(1)}{\varepsilon^{n-1}} \left(C_0 \int_K \varphi^2 + C_1 \int_K |\nabla \varphi|^2 \right), \tag{73}$$

where $C_0 = \int_{\mathbb{R}} (H')^2 + (H'')^2$ and $C_1 = \int_{\mathbb{R}} (H')^2$. Similar formulas hold true for ϕ_1 and ϕ_2 , and hence these two functions stay uniformly bounded in $H^1(S_\varepsilon)$ as ε tends to zero.

We multiply next the equation in (70) by ϕ_l , $l = 1, 2$: from the orthogonality of $\bar{\varphi}_1$ and $\bar{\varphi}_2$ on K (with weight b), (59), an integration by parts and the above arguments we get

$$O(\varepsilon^2) \|\phi_l\|_{L^2(S_\varepsilon)}^2 + O(\varepsilon^3) \|\varphi \Psi\|_{L^2(S_\varepsilon)} \|\phi_l\|_{L^2(S_\varepsilon)} = \int_{S_\varepsilon} \phi_l L_\varepsilon \phi \, dV_{g_\varepsilon} = \int_{S_\varepsilon} (\varphi \Psi + \phi^\perp) L_\varepsilon \phi_l.$$

Using (66) (replacing φ by $\bar{\varphi}_l$) we deduce

$$\left| \int_{S_\varepsilon} \phi^\perp(y, \zeta) L_\varepsilon (\bar{\varphi}_l(y) \Psi(y, \zeta)) \, dV_{g_\varepsilon}(y, \zeta) \right| \leq C \frac{\varepsilon^2}{\varepsilon^{\frac{n-1}{2}}} \|\phi^\perp\|_{H^1(S_\varepsilon)} \|\bar{\varphi}_l\|_{H^1(K)}, \quad \text{for } l = 1, 2.$$

Also, from the expression of L_ε (see (62)), the subsequent estimates and some straightforward computations one finds

$$\begin{aligned} \int_{S_\varepsilon} \varphi \Psi L_\varepsilon \phi_l &= -\frac{1 + o(1)}{\varepsilon^{n-1}} \sum_{j \in A_{1,\varepsilon}} \alpha_j^2 \lambda_{j,\varepsilon} + O(\varepsilon^2) \left(\frac{1}{\varepsilon^{n-1}} \sum_{j \in A_{1,\varepsilon}} \alpha_j^2 \right)^{\frac{1}{2}} \|\phi\|_{H^1(S_\varepsilon)} \\ &\quad + O(\varepsilon) \left(\frac{1}{\varepsilon^{n-1}} \sum_{j \in A_{1,\varepsilon}} \alpha_j^2 \varepsilon^2 \lambda_j \right)^{\frac{1}{2}} \|\phi\|_{H^1(S_\varepsilon)}. \end{aligned}$$

Then from the last three formulas, Lemma 3.2 and the normalization on ϕ we obtain

$$\frac{1}{\varepsilon^{n-1}} \sum_{j \in A_{1,\varepsilon}} \alpha_j^2 \lambda_{j,\varepsilon} = O(\varepsilon^2) \left(\frac{1}{\varepsilon^{n-1}} \sum_{j \in A_{1,\varepsilon}} \alpha_j^2 \right)^{\frac{1}{2}} + O(\varepsilon) \left(\frac{1}{\varepsilon^{n-1}} \sum_{j \in A_{1,\varepsilon}} \alpha_j^2 \varepsilon^2 \lambda_j \right)^{\frac{1}{2}}, \tag{74}$$

for $l = 1, 2$.

We next further split the sets $A_{l,\varepsilon}$ as $A_{l,\varepsilon} = \hat{A}_{l,\varepsilon} \cup \tilde{A}_{l,\varepsilon}$, where

$$\hat{A}_{l,\varepsilon} = \{j \in A_{l,\varepsilon} : |\varepsilon^2 \lambda_j| < \varepsilon^{\frac{3}{4}}\}; \quad \tilde{A}_{l,\varepsilon} = A_{l,\varepsilon} \setminus \hat{A}_{l,\varepsilon},$$

so from (74) we have clearly

$$\begin{aligned} \frac{1}{\varepsilon^{n-1}} \sum_{j \in A_{l,\varepsilon}} \alpha_j^2 \lambda_{j,\varepsilon} &= O(\varepsilon^2) \left(\frac{1}{\varepsilon^{n-1}} \sum_{j \in A_{l,\varepsilon}} \alpha_j^2 \right)^{\frac{1}{2}} + O(\varepsilon) \left(\frac{1}{\varepsilon^{n-1}} \sum_{j \in \hat{A}_{l,\varepsilon}} \alpha_j^2 \varepsilon^2 \lambda_j \right)^{\frac{1}{2}} \\ &\quad + O(\varepsilon) \left(\frac{1}{\varepsilon^{n-1}} \sum_{j \in \tilde{A}_{l,\varepsilon}} \alpha_j^2 \varepsilon^2 \lambda_j \right)^{\frac{1}{2}}. \end{aligned} \tag{75}$$

Obviously, since $\lambda_{j,\varepsilon} = \sqrt{2\varepsilon} - \varepsilon^2 \lambda_j$, for $j \in \tilde{A}_{l,\varepsilon}$ the ratio $\varepsilon^2 \lambda_j / \lambda_{j,\varepsilon}$ stays uniformly bounded from above and below by positive constants. Therefore, using the elementary inequality $|xy| \leq \delta |x|^2 + \frac{1}{\delta} |y|^2$ with δ small and fixed, we can absorb the last term into the left-hand side of the latter formula, obtaining an error of the form $O(\varepsilon^2 / \delta)$. Therefore, using also the definition of $\hat{A}_{l,\varepsilon}$ we find

$$\frac{1}{\varepsilon^{n-1}} \sum_{j \in A_{l,\varepsilon}} \alpha_j^2 \lambda_{j,\varepsilon} = O(\varepsilon^{\frac{11}{8}}) \left(\frac{1}{\varepsilon^{n-1}} \sum_{j \in A_{l,\varepsilon}} \alpha_j^2 \right)^{\frac{1}{2}} + O(\varepsilon^2).$$

By our normalization on ϕ , see also the comments after (73), the argument inside the last bracket is uniformly bounded as $\varepsilon \rightarrow 0$, and hence we obtain the second assertion of the lemma.

To obtain also the first one we notice that $|\lambda_{j,\varepsilon}| \geq \varepsilon^{\frac{5}{4}}$ for $j \in A_{l,\varepsilon}$, so we find

$$\frac{1}{\varepsilon^{n-1}} \sum_{j \in A_{l,\varepsilon}} \alpha_j^2 = O(\varepsilon^{\frac{11}{8} - \frac{5}{4}}) = O(\varepsilon^{\frac{1}{8}}) = o(1)$$

as $\varepsilon \rightarrow 0$. This concludes the proof. \square

Now, using the above lemma, we can estimate the derivatives of small eigenvalues of L_ε with respect to ε . Precisely, we have the following result.

Lemma 3.5. *Let λ be as in Lemma 3.2. Then, for ε sufficiently small λ is differentiable with respect to ε , and there exists a negative constant $c_{K,b}$, depending only on K and b , such that its derivative (which is possibly a multi-valued function) satisfies*

$$\left| \frac{\partial \lambda}{\partial \varepsilon} - c_{K,b} \right| = o(1) \quad \text{as } \varepsilon \rightarrow 0.$$

Proof. The proof is based on Lemma 3.2, Proposition 3.3 and Lemma 3.4. Since we want to apply formula (71) (in our previous notation) to the function

$$u = \phi = \left(\sum_{|\lambda_{j,\varepsilon}| \geq \varepsilon^{\frac{5}{4}}} \alpha_j \varphi_j(\varepsilon z) \right) \Psi + \phi^\perp = \varphi(H' + \varepsilon H_1) + \phi^\perp,$$

we need to estimate the two quantities

$$\int_{S_\varepsilon} \left(-\frac{2}{\varepsilon} |\nabla_{g_\varepsilon} u|^2 - 6u_{k,\varepsilon} v_{k,\varepsilon} u^2 \right) dV_{g_\varepsilon}; \quad \int_{S_\varepsilon} b u^2 dV_{g_\varepsilon}. \tag{76}$$

Here the function $v_{k,\varepsilon}$ is defined as $v_{k,\varepsilon} = \frac{\partial \tilde{u}_{k,\varepsilon}}{\partial \varepsilon}(\varepsilon \cdot)$, where $\tilde{u}_{k,\varepsilon} : \varepsilon S_\varepsilon \rightarrow \mathbb{R}$ is the scaling $u_{k,\varepsilon}(x) = \tilde{u}_{k,\varepsilon}(\varepsilon x)$. We claim that, normalizing u with $\|u\|_{H^1(S_\varepsilon)} = 1$ (this condition was required in Lemma 3.4), the following estimates hold

$$\int_{S_\varepsilon} \left(-\frac{2}{\varepsilon} |\nabla_{g_\varepsilon} u|^2 - 6u_{k,\varepsilon} v_{k,\varepsilon} u^2 \right) dV_{g_\varepsilon} = \frac{c}{\varepsilon^{n-1}} \int_K b(\bar{y}) \varphi^2 + o(1); \quad (77)$$

$$\int_{S_\varepsilon} bu^2 dV_{g_\varepsilon} = \frac{C_1}{\varepsilon^{n-1}} \int_K b(\bar{y}) \varphi^2 + o(1) \quad (78)$$

as $\varepsilon \rightarrow 0$, where $c < 0$ and where C_1 is defined after (73). This together with (71) would conclude the proof of the lemma. \square

Proof of (77) and (78). First of all, recall that by our normalization and by Lemma 3.2 we have that $\|\phi^\perp\|_{H^1(S_\varepsilon)} = o(\varepsilon)$. Therefore, using the expansions for g_ε in Subsection 2.2, some integration by parts, (58) and the estimates in Subsection 3.1 one finds

$$\begin{aligned} & \int_{S_\varepsilon} \left(-\frac{2}{\varepsilon} |\nabla_{g_\varepsilon} u|^2 - 6u_{k,\varepsilon} v_{k,\varepsilon} u^2 \right) dV_{g_\varepsilon} \\ &= \frac{2}{\varepsilon} \int_{S_\varepsilon} \varphi(H' + \varepsilon H_1) [\varphi(H''' + \varepsilon H_1'') + \varepsilon \kappa \varphi(H'' + \varepsilon H_1') + \varepsilon^2 \Delta_K \varphi(H' + \varepsilon H_1)] (1 + \varepsilon \kappa \zeta) \\ & \quad + 6 \int_{S_\varepsilon} \varphi^2(H + \varepsilon h_1) \left[\frac{\zeta - \Phi_0}{\varepsilon} H' + (\zeta - \Phi_0) h_1' \right] (H'^2 + 2\varepsilon H' H_1) (1 + \varepsilon \kappa \zeta) + o(1). \end{aligned} \quad (79)$$

Since the arguments of H , H' , h_1 and H_1 are all translated by Φ_0 in ζ , with the change of variables $s = \zeta - \Phi_0$ and some elementary estimates (which use the exponential decay of H' , h_1 and H_1) we find

$$\begin{aligned} & \int_{S_\varepsilon} \left(-\frac{2}{\varepsilon} |\nabla_{g_\varepsilon} u|^2 - 6u_{k,\varepsilon} v_{k,\varepsilon} u^2 \right) dV_{g_\varepsilon} = \frac{1}{\varepsilon} \int_{K_\varepsilon \times \mathbb{R}} \varphi^2(2H' H''' + 6s H H'^3) \\ & \quad + 4 \int_{K_\varepsilon \times \mathbb{R}} \varphi^2 H_1 (H''' + 3s H H'^2) + 6 \int_{K_\varepsilon \times \mathbb{R}} \varphi^2 (s h_1 (H')^3 + s h_1' H (H')^2) \\ & \quad + 2 \int_{K_\varepsilon \times \mathbb{R}} \varepsilon \varphi \Delta_K \varphi H'^2 + 2 \int_{K_\varepsilon \times \mathbb{R}} \kappa (s + \Phi_0) H' H''' \varphi^2 + 6 \int_{K_\varepsilon \times \mathbb{R}} s (s + \Phi_0) \kappa H H'^3 \varphi^2 + o(1). \end{aligned} \quad (80)$$

In the latter formula all the arguments now are simply in s , with no more translation. Using Eq. (10), the oddness of H together with some integration by parts, it is easy to see that the term of order $\frac{1}{\varepsilon}$ in the above expression is identically equal to zero. Let us consider now the terms of order 0 which involve H_1 . Using the self-adjointness of the operator \mathcal{L}_0 and the following elementary identity

$$\mathcal{L}_0 \left(-\frac{1}{\sqrt{2}} s H H' \right) = H''' + 3s H H'^2,$$

we can write that

$$4 \int_{K_\varepsilon \times \mathbb{R}} \varphi^2 H_1 (H''' + 3s H H'^2) = -\frac{4}{\sqrt{2}} \int_{K_\varepsilon \times \mathbb{R}} \varphi^2 s H H' \mathcal{L}_0 H_1.$$

Similarly, the terms of order 0 which involve h_1 can be written (up to some integration by parts in the variable s) as

$$\begin{aligned} 6 \int_{K_\varepsilon \times \mathbb{R}} \varphi^2 (s h_1 (H')^3 + s h_1' H (H')^2) &= -6 \int_{K_\varepsilon \times \mathbb{R}} \varphi^2 (2s H H' H'' + H (H')^2) h_1 \\ &= -6 \int_{K_\varepsilon \times \mathbb{R}} \varphi^2 H (H')^2 h_1 + 12\sqrt{2} \int_{K_\varepsilon \times \mathbb{R}} \varphi^2 s (H H')^2 h_1 \end{aligned}$$

$$= \frac{2}{\sqrt{2}} \int_{K_\varepsilon \times \mathbb{R}} \varphi^2 H H' \mathcal{L}_0(h_1) + \frac{24}{\sqrt{2}} \int_{K_\varepsilon \times \mathbb{R}} \varphi^2 s (H H')^2 h_1. \tag{81}$$

Here again we have used the self-adjointness of the operator \mathcal{L}_0 and the identities

$$H'' = -\sqrt{2} H H' \quad \text{and} \quad \mathcal{L}_0\left(\frac{2}{\sqrt{2}} H H'\right) = -6 H H'^2.$$

Regrouping the above terms, and using the oddness of H one finds

$$\begin{aligned} & \int_{S_\varepsilon} \left(-\frac{2}{\varepsilon} |\nabla_{g_\varepsilon} u|^2 - 6u_{k,\varepsilon} v_{k,\varepsilon} u^2 \right) dV_{g_\varepsilon} \\ &= -\frac{4}{\sqrt{2}} \int_{K_\varepsilon \times \mathbb{R}} \varphi^2 s H H' (\mathcal{L}_0 H_1 - 6 H H' h_1) + \frac{2}{\sqrt{2}} \int_{K_\varepsilon \times \mathbb{R}} \varphi^2 H H' \mathcal{L}_0(h_1) + 2 \int_{K_\varepsilon \times \mathbb{R}} \varepsilon \varphi \Delta_K \varphi H'^2 \\ & \quad + 2 \int_{K_\varepsilon \times \mathbb{R}} \kappa (s + \Phi_0) H' H''' \varphi^2 + 6 \int_{K_\varepsilon \times \mathbb{R}} s (s + \Phi_0) \kappa H H'^3 \varphi^2 + o(1) \\ &= -\frac{4}{\sqrt{2}} \int_{K_\varepsilon \times \mathbb{R}} \varphi^2 s H H' (\mathcal{L}_0 H_1 - 6 H H' h_1) + \frac{2}{\sqrt{2}} \int_{K_\varepsilon \times \mathbb{R}} \varphi^2 H H' \mathcal{L}_0(h_1) \\ & \quad + 2 \int_{K_\varepsilon \times \mathbb{R}} \varepsilon \varphi \Delta_K \varphi H'^2 + 2 \int_{K_\varepsilon \times \mathbb{R}} \kappa \Phi_0 \varphi^2 (3s H H'^3 + H' H''') + o(1). \end{aligned} \tag{82}$$

Now, using Eqs. (40) and (55) we arrive at

$$\begin{aligned} \int_{S_\varepsilon} \left(-\frac{2}{\varepsilon} |\nabla_{g_\varepsilon} u|^2 - 6u_{k,\varepsilon} v_{k,\varepsilon} u^2 \right) dV_{g_\varepsilon} &= \frac{4}{\sqrt{2}} \int_{K_\varepsilon \times \mathbb{R}} \varphi^2 s H H' (2b(\varepsilon y)(s + \Phi_0) H H' + H'' \kappa(\varepsilon y) - \sqrt{2} b(\varepsilon y) H') \\ & \quad + \frac{2}{\sqrt{2}} \int_{K_\varepsilon \times \mathbb{R}} \varphi^2 H H' (-\kappa(\varepsilon y) H' + (s + \Phi_0) b(\varepsilon y) (1 - H^2)) \\ & \quad + 2 \int_{K_\varepsilon \times \mathbb{R}} \varepsilon \varphi \Delta_K \varphi H'^2 + 2 \int_{K_\varepsilon \times \mathbb{R}} \kappa \Phi_0 \varphi^2 (3s H H'^3 + H' H''') + o(1). \end{aligned}$$

Using again the fact that H is odd, by the vanishing of the last integral (as one can easily check) we obtain

$$\begin{aligned} \int_{S_\varepsilon} \left(-\frac{2}{\varepsilon} |\nabla_{g_\varepsilon} u|^2 - 6u_{k,\varepsilon} v_{k,\varepsilon} u^2 \right) dV_{g_\varepsilon} &= \frac{4}{\sqrt{2}} \int_{K_\varepsilon \times \mathbb{R}} \varphi^2 s H H' (2b(\varepsilon y) s H H' - \sqrt{2} b(\varepsilon y) H') \\ & \quad + \frac{2}{\sqrt{2}} \int_{K_\varepsilon \times \mathbb{R}} \varphi^2 H H' s b(\varepsilon y) (1 - H^2) + 2 \int_{K_\varepsilon \times \mathbb{R}} \varepsilon \varphi \Delta_K \varphi H'^2 + o(1). \end{aligned}$$

By an explicit computation of the integral we find

$$\int_{S_\varepsilon} \left(-\frac{2}{\varepsilon} |\nabla_{g_\varepsilon} u|^2 - 6u_{k,\varepsilon} v_{k,\varepsilon} u^2 \right) dV_{g_\varepsilon} = \left(\frac{8}{45} \pi^2 - \frac{2}{3} \right) \frac{1}{\varepsilon^{n-1}} \int_K b \varphi^2 + \frac{4}{3} \frac{\varepsilon \sqrt{2}}{\varepsilon^{n-1}} \int_K \varphi \Delta_K \varphi + o(1),$$

so by Lemma 3.4 and some easy estimates, we find

$$\int_{S_\varepsilon} \left(-\frac{2}{\varepsilon} |\nabla_{g_\varepsilon} u|^2 - 6u_{k,\varepsilon} v_{k,\varepsilon} u^2 \right) dV_{g_\varepsilon} = \left(\frac{8}{45} \pi^2 - \frac{10}{3} \right) \frac{1}{\varepsilon^{n-1}} \int_K b \varphi^2 + o(1).$$

Then we obtain (77) taking $c = (\frac{8}{45} \pi^2 - \frac{10}{3}) < 0$. To prove (78) it is sufficient to use Lemma 3.2, the estimates on g_ε in Subsection 2.2 and the decay of H_1 , see (54). \square

4. Proof of Theorem 1.1

In this section we first prove the invertibility of the linearized operator L_ε using Lemma 3.5 and choosing carefully the parameter ε . Below, $H_0^2(S_\varepsilon)$ stands for the functions in $H^2(S_\varepsilon)$ with null trace on ∂S_ε .

Proposition 4.1. *Let $k \geq 1$, let $u_{k,\varepsilon}$ be the approximate solution defined in Proposition 3.1, and let L_ε be the linearized operator at $u_{k,\varepsilon}$, see (48). Then there exist a sequence $\varepsilon_j \rightarrow 0$ such that $L_{\varepsilon_j} : H_0^2(S_{\varepsilon_j}) \rightarrow L^2(S_{\varepsilon_j})$ is invertible and its inverse $L_{\varepsilon_j}^{-1} : L^2(S_{\varepsilon_j}) \rightarrow H_0^2(S_{\varepsilon_j})$ satisfies*

$$\|L_{\varepsilon_j}^{-1}\|_{\mathcal{L}(L^2(S_{\varepsilon_j}); H_0^2(S_{\varepsilon_j}))} \leq C\varepsilon_j^{-\frac{n+1}{2}}, \quad \text{for all } j \in \mathbb{N}.$$

Proof. The proof is similar in spirit to the one of Proposition 4.5 in [29]. As we will see, in order to study the spectral gap of L_ε it suffices to find an asymptotic estimate on the number N_ε of positive eigenvalues of L_ε and to apply then Lemma 3.5. We denote by $\tilde{\lambda}_{1,\varepsilon} \geq \tilde{\lambda}_{2,\varepsilon} \geq \dots \geq \tilde{\lambda}_{j,\varepsilon} \geq \dots$ the eigenvalues of L_ε , counted with multiplicity. The j -th eigenvalue $\tilde{\lambda}_{j,\varepsilon}$ can be estimated using the classical Courant–Fisher formula

$$\tilde{\lambda}_{j,\varepsilon} = \sup_{M \in M_j} \inf_{u \in M, u \neq 0} \frac{\int_{S_\varepsilon} u L_\varepsilon u dV_{g_\varepsilon}}{\int_{S_\varepsilon} b u^2 dV_{g_\varepsilon}}; \quad \tilde{\lambda}_{j,\varepsilon} = \inf_{M \in M_{j-1}} \sup_{u \perp M, u \neq 0} \frac{\int_{S_\varepsilon} u L_\varepsilon u dV_{g_\varepsilon}}{\int_{S_\varepsilon} b u^2 dV_{g_\varepsilon}}. \quad (83)$$

Here M_l represents the family of l -dimensional subspaces of $H_0^2(S_\varepsilon)$, and the symbol \perp denotes orthogonality with respect to the L^2 scalar product with weight b . Notice that the inf and sup are reversed compared to (36) since the principal part of the operator has the opposite sign.

We can find a lower bound of N_ε using the first formula in (83). Indeed, given a fixed $\delta > 0$, let j_ε be the largest integer j for which $\lambda_{j,\varepsilon} \geq \delta\varepsilon$. From (37) and (72) we find that

$$j_\varepsilon \simeq \left(\frac{\sqrt{2} - \delta}{C_{K,b\varepsilon}} \right)^{\frac{n-1}{2}} \quad \text{as } \varepsilon \rightarrow 0. \quad (84)$$

We can take a test function ϕ like $\varphi\Psi$ with $\varphi = \sum_{l=1}^{j_\varepsilon} \alpha_l \varphi_l$. Actually, since we want to work in the space $H_0^2(S_\varepsilon)$ we need to add a suitable cutoff function in ζ . However, by the exponential decay of Ψ , see (58), these will generate error terms exponentially small in ε . Therefore, for convenience of the exposition, we will omit these corrections.

By (62) we have that

$$L_\varepsilon \phi = (\varepsilon\sqrt{2}b\varphi + \varepsilon^2 \Delta_K \varphi)\Psi + \varepsilon^2 \varphi \Delta_K \Psi + 2\varepsilon^2 \langle \nabla_K \varphi, \nabla_K \Psi \rangle + \varepsilon^2 \tilde{R}_\varepsilon \varphi + \varepsilon^3 \hat{\mathcal{L}}_{3,\varepsilon}(\varphi\Psi),$$

where $\hat{\mathcal{L}}_{3,\varepsilon}$ and \tilde{R}_ε satisfy respectively the estimates (52) and (57). Reasoning as for (64), (68) we find

$$\int_{S_\varepsilon} \phi L_\varepsilon \phi dV_{g_\varepsilon} \geq \frac{1}{\varepsilon^{n-1}} \sum_l [(1 + o(1))\lambda_{l,\varepsilon} + O(\varepsilon^2)] \alpha_l^2; \quad \int_{S_\varepsilon} b \phi^2 dV_{g_\varepsilon} = \frac{1 + o(1)}{\varepsilon^{n-1}} \sum_l \alpha_l^2. \quad (85)$$

Defining $M = \text{span}\{\varphi_l \Psi, l \leq j_\varepsilon\}$, by the first formula in (83) and our choice of j_ε we have that

$$\tilde{\lambda}_{j_\varepsilon,\varepsilon} \geq \inf_{u \in M, u \neq 0} \frac{\int_{S_\varepsilon} u L_\varepsilon u dV_{g_\varepsilon}}{\int_{S_\varepsilon} b u^2 dV_{g_\varepsilon}} \geq 0, \quad \text{as } \varepsilon \rightarrow 0.$$

From (84) and the last formula we then find the following lower bound

$$N_\varepsilon \geq (1 + o(1)) \left(\frac{\sqrt{2} - \delta}{C_{K,b\varepsilon}} \right)^{\frac{n-1}{2}} \quad \text{as } \varepsilon \rightarrow 0. \quad (86)$$

A similar upper bound can be obtained using the second formula in (83): again, given a fixed $\delta > 0$, let \tilde{j}_ε be the smallest integer j for which $\lambda_{j,\varepsilon} \leq -\delta\varepsilon$. Still from (37) and (72) it follows that

$$\tilde{j}_\varepsilon \simeq \left(\frac{\sqrt{2} + \delta}{C_{K,b\varepsilon}} \right)^{\frac{n-1}{2}} \quad \text{as } \varepsilon \rightarrow 0. \quad (87)$$

Now let $\phi \in H_0^2(S_\varepsilon)$ be an arbitrary function orthogonal to $\tilde{M} := \text{span}\{\varphi_l \Psi, l \leq \tilde{j}_\varepsilon - 1\}$, and let us write it in the form $\phi = \varphi \Psi + \phi^\perp$ with ϕ^\perp as in Lemma 3.2.

We write as before $\varphi = \sum_l \alpha_l \varphi_l$, and split it as sum of the following two functions

$$\bar{\varphi}_1 = \sum_{l \leq \tilde{j}_\varepsilon - 1} \alpha_l \varphi_l, \quad \bar{\varphi}_2 = \sum_{l \geq \tilde{j}_\varepsilon} \alpha_l \varphi_l.$$

Using the second formula in (83) we have that

$$\tilde{\lambda}_{j,\varepsilon} \leq \sup_{u \perp \tilde{M}, u \neq 0} \frac{\int_{S_\varepsilon} u L_\varepsilon u dV_{g_\varepsilon}}{\int_{S_\varepsilon} b u^2 dV_{g_\varepsilon}}. \tag{88}$$

By the definition of ϕ^\perp , the expansions of the metric g_ε in Subsection 2.2 and (58) one finds

$$\begin{aligned} \|\phi\|_{L^2(S_\varepsilon)}^2 &= \|\varphi \Psi\|_{L^2(S_\varepsilon)}^2 + \|\phi^\perp\|_{L^2(S_\varepsilon)}^2 = (1 + o(1))(\|\bar{\varphi}_1 \Psi\|_{L^2(S_\varepsilon)}^2 + \|\bar{\varphi}_2 \Psi\|_{L^2(S_\varepsilon)}^2) + \|\phi^\perp\|_{L^2(S_\varepsilon)}^2 \\ &= (1 + o(1)) \frac{1}{\varepsilon^{n-1}} \sum_{l \leq \tilde{j}_\varepsilon - 1} \alpha_l^2 + (1 + o(1)) \frac{1}{\varepsilon^{n-1}} \sum_{l \geq \tilde{j}_\varepsilon} \alpha_l^2 + \|\phi^\perp\|_{L^2(S_\varepsilon)}^2 \quad \text{as } \varepsilon \rightarrow 0. \end{aligned} \tag{89}$$

Using also the estimates in the proof of Lemma 3.4, multiplying ϕ by $b(\varepsilon y) \bar{\varphi}_1 \Psi$, using the orthogonality to \tilde{M} and integrating one can easily prove that

$$\|\bar{\varphi}_1 \Psi\|_{L^2(S_\varepsilon)} \leq o(1) \|\phi\|_{L^2(S_\varepsilon)}, \quad \text{as } \varepsilon \rightarrow 0. \tag{90}$$

This, together with (62), the fact that $|\lambda_{j,\varepsilon}| \leq C\varepsilon$ for $j \leq \tilde{j}_\varepsilon - 1$ and some easy computations imply

$$\int_{S_\varepsilon} (\bar{\varphi}_1 \Psi) L_\varepsilon (\bar{\varphi}_1 \Psi) dV_{g_\varepsilon} = o(\varepsilon) \|\phi\|_{L^2(S_\varepsilon)}^2. \tag{91}$$

Now, by the self-adjointness of L_ε we can write

$$\begin{aligned} \int_{S_\varepsilon} \phi L_\varepsilon \phi dV_{g_\varepsilon} &= \int_{S_\varepsilon} (\bar{\varphi}_1 \Psi) L_\varepsilon (\bar{\varphi}_1 \Psi) dV_{g_\varepsilon} + \int_{S_\varepsilon} (\bar{\varphi}_2 \Psi) L_\varepsilon (\bar{\varphi}_2 \Psi) dV_{g_\varepsilon} + \int_{S_\varepsilon} \phi^\perp L_\varepsilon \phi^\perp dV_{g_\varepsilon} \\ &\quad + 2 \int_{S_\varepsilon} (\bar{\varphi}_1 \Psi) L_\varepsilon (\bar{\varphi}_2 \Psi) dV_{g_\varepsilon} + 2 \int_{S_\varepsilon} (\bar{\varphi}_1 \Psi) L_\varepsilon \phi^\perp dV_{g_\varepsilon} + 2 \int_{S_\varepsilon} (\bar{\varphi}_2 \Psi) L_\varepsilon \phi^\perp dV_{g_\varepsilon}. \end{aligned} \tag{92}$$

Let us first estimate the last two terms: from (66) we obtain

$$\int_{S_\varepsilon} (\bar{\varphi}_2 \Psi) L_\varepsilon \phi^\perp dV_{g_\varepsilon} \leq \frac{C\varepsilon^2}{\varepsilon^{\frac{n-1}{2}}} \|\phi^\perp\|_{H^1(S_\varepsilon)} \|\bar{\varphi}_2\|_{H^1(K)} \leq \frac{C\varepsilon^2}{\varepsilon^{\frac{n-1}{2}}} \|\phi^\perp\|_{H^1(S_\varepsilon)} \left(\sum_{l \geq \tilde{j}_\varepsilon} (1 + \lambda_l) \alpha_l^2 \right)^{\frac{1}{2}}. \tag{93}$$

Similarly, using the fact that $|\lambda_l| \leq \frac{C}{\varepsilon}$ for $l \leq \tilde{j}_\varepsilon - 1$ we have

$$\|\bar{\varphi}_1\|_{H^1(K)} \leq \left(\sum_{l \leq \tilde{j}_\varepsilon - 1} \lambda_l \alpha_l^2 \right)^{\frac{1}{2}} \leq \frac{1}{\varepsilon^{\frac{1}{2}}} \|\bar{\varphi}_1\|_{L^2(K)},$$

and hence

$$\int_{S_\varepsilon} (\bar{\varphi}_1 \Psi) L_\varepsilon \phi^\perp dV_{g_\varepsilon} \leq \frac{C\varepsilon^2}{\varepsilon^{\frac{n-1}{2}}} \|\phi^\perp\|_{H^1(S_\varepsilon)} \|\bar{\varphi}_1\|_{H^1(K)} \leq \frac{C\varepsilon^{\frac{3}{2}}}{\varepsilon^{\frac{n-1}{2}}} \|\phi^\perp\|_{H^1(S_\varepsilon)} \|\bar{\varphi}_1\|_{L^2(K)}. \tag{94}$$

On the other hand, a similar argument as for the first formula in (85) yields

$$\int_{S_\varepsilon} (\bar{\varphi}_2 \Psi) L_\varepsilon (\bar{\varphi}_2 \Psi) dV_{g_\varepsilon} = \frac{1}{\varepsilon^{n-1}} \sum_{l \geq \tilde{j}_\varepsilon} [(1 + o(1)) \lambda_{l,\varepsilon} + O(\varepsilon^2)] \alpha_l^2. \tag{95}$$

Moreover by the negative definiteness of L_ε on ϕ^\perp , see (60), we have that

$$\int_{S_\varepsilon} \phi^\perp L_\varepsilon \phi^\perp dV_{g_\varepsilon} \leq -C^{-1} \|\phi^\perp\|_{H^1(S_\varepsilon)}^2 \tag{96}$$

for some fixed constant C . It remains to estimate the term $\int_{S_\varepsilon} (\bar{\varphi}_1 \Psi) L_\varepsilon (\bar{\varphi}_2 \Psi) dV_{g_\varepsilon}$. Using again (62), (58), the fact that $\int_K b\varphi_i \varphi_j = 0$ for $i \leq \tilde{j}_\varepsilon - 1, j \geq \tilde{j}_\varepsilon$, and some integration by parts we get

$$\begin{aligned} \frac{\varepsilon^{n-1}}{C} \int_{S_\varepsilon} (\bar{\varphi}_1 \Psi) L_\varepsilon (\bar{\varphi}_2 \Psi) dV_{g_\varepsilon} &\leq \varepsilon^3 \int_K |\nabla \bar{\varphi}_1| |\nabla \bar{\varphi}_2| + \varepsilon^2 \int_K (|\nabla \bar{\varphi}_1| |\bar{\varphi}_2| + |\bar{\varphi}_1| |\nabla \bar{\varphi}_2| + |\bar{\varphi}_1| |\bar{\varphi}_2|) \\ &\leq \varepsilon^3 \left(\sum_{l \leq \tilde{j}_\varepsilon - 1} \lambda_l \alpha_l^2 \right)^{\frac{1}{2}} \left(\sum_{l \geq \tilde{j}_\varepsilon} \lambda_l \alpha_l^2 \right)^{\frac{1}{2}} + \varepsilon^2 \left(\sum_{l \leq \tilde{j}_\varepsilon - 1} \alpha_l^2 \right)^{\frac{1}{2}} \left(\sum_{l \geq \tilde{j}_\varepsilon} \lambda_l \alpha_l^2 \right)^{\frac{1}{2}} \\ &\quad + \varepsilon^2 \left(\sum_{l \leq \tilde{j}_\varepsilon - 1} \lambda_l \alpha_l^2 \right)^{\frac{1}{2}} \left(\sum_{l \geq \tilde{j}_\varepsilon} \alpha_l^2 \right)^{\frac{1}{2}} + \varepsilon^2 \left(\sum_{l \leq \tilde{j}_\varepsilon - 1} \alpha_l^2 \right)^{\frac{1}{2}} \left(\sum_{l \geq \tilde{j}_\varepsilon} \alpha_l^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{97}$$

We claim next that the terms in the right-hand side of (92) can be combined yielding

$$\int_{S_\varepsilon} \phi L_\varepsilon \phi dV_{g_\varepsilon} \leq \frac{1}{C} \left(\frac{1}{\varepsilon^{n-1}} \sum_{l \geq \tilde{j}_\varepsilon} \lambda_{l,\varepsilon} \alpha_l^2 - \|\phi^\perp\|_{H^1(S_\varepsilon)}^2 \right) + o(\varepsilon) \|\phi\|_{L^2(S_\varepsilon)}^2. \tag{98}$$

To prove this, we show that the main terms in (92) are the ones given in (95), (96), while all the others, listed in the left-hand sides of formulas (90), (93), (94) and (97) can be absorbed into the formers by the elementary inequality $|ab| \leq a^2 + b^2$. For example, for any small constant β (independent of ε) we can write

$$\frac{C\varepsilon^2}{\varepsilon^{\frac{n-1}{2}}} \|\phi^\perp\|_{H^1(S_\varepsilon)} \left(\sum_{l \geq \tilde{j}_\varepsilon} (1 + \lambda_l) \alpha_l^2 \right)^{\frac{1}{2}} \leq C\beta \|\phi^\perp\|_{H^1(S_\varepsilon)}^2 + \frac{C}{\beta} \frac{1}{\varepsilon^{n-1}} \sum_{l \geq \tilde{j}_\varepsilon} \varepsilon^4 (1 + \lambda_l) \alpha_l^2.$$

Taking β sufficiently small, and noticing that $\varepsilon^4 \lambda_l \geq \varepsilon^2 \lambda_{l,\varepsilon} + O(\varepsilon^3)$, from (89) we deduce our claim for this term. For the others, one reasons similarly, taking also (90) and the choice of \tilde{j}_ε into account. Now (89), (90), (98) and again the choice of \tilde{j}_ε imply

$$\int_{S_\varepsilon} \phi L_\varepsilon \phi dV_{g_\varepsilon} \leq -\frac{\delta\varepsilon}{C\varepsilon^{n-1}} \sum_{l \geq \tilde{j}_\varepsilon} \alpha_l^2 - \frac{1}{C} \|\phi^\perp\|_{H^1(S_\varepsilon)}^2 \leq 0.$$

Therefore, by (87) we find the following upper bound on N_ε

$$N_\varepsilon \leq (1 + o(1)) \left(\frac{\sqrt{2} + \delta}{C_{K,b}\varepsilon} \right)^{\frac{n-1}{2}} \text{ as } \varepsilon \rightarrow 0.$$

Since δ is arbitrary, the last estimate and (86) imply

$$N_\varepsilon \sim C_{1,K} \varepsilon^{-\frac{n-1}{2}} \text{ as } \varepsilon \rightarrow 0, \tag{99}$$

where we have set $C_{1,K} = (\sqrt{2}/C_{K,b})^{\frac{n-1}{2}}$.

Next, for $l \in \mathbb{N}$, let $\varepsilon_l = 2^{-l}$. From (99) we get

$$N_{\varepsilon_{l+1}} - N_{\varepsilon_l} \sim C_{1,K} (2^{(l+1)\frac{n-1}{2}} - 2^{l\frac{n-1}{2}}) = C_{1,K} (2^{\frac{n-1}{2}} - 1) \varepsilon_l^{-\frac{n-1}{2}}. \tag{100}$$

On the other hand it follows from Lemma 3.5 that the eigenvalues of L_ε which are bounded (in absolute value) by $O(\varepsilon^2)$ are decreasing in ε . Equivalently, by the last equation, the number of eigenvalues which become positive, when ε decreases from ε_l to ε_{l+1} , is of order $\varepsilon_l^{-\frac{n-1}{2}}$. Now we define

$$\mathcal{A}_l = \{ \varepsilon \in (\varepsilon_{l+1}, \varepsilon_l) : \ker L_\varepsilon \neq \emptyset \}; \quad \mathcal{B}_l = (\varepsilon_{l+1}, \varepsilon_l) \setminus \mathcal{A}_l.$$

By (100) and the monotonicity (in ε) of the *small* eigenvalues, we deduce that $\text{card}(\mathcal{A}_l) < C\varepsilon_l^{-\frac{n-1}{2}}$, and hence there exists an interval (a_l, b_l) such that

$$(a_l, b_l) \subseteq \mathcal{B}_l; \quad |b_l - a_l| \geq C^{-1} \frac{\text{meas}(\mathcal{B}_l)}{\text{card}(\mathcal{A}_l)} \geq C^{-1} \varepsilon_l^{\frac{n+1}{2}}. \tag{101}$$

From Lemma 3.5 we deduce that $L_{\frac{a_l+b_l}{2}}$ is invertible and

$$\|L_{\frac{a_l+b_l}{2}}^{-1}\|_{\mathfrak{L}(L^2(S_\varepsilon); H_0^2(S_\varepsilon))} \leq \frac{C}{\varepsilon_l^{\frac{n+1}{2}}}.$$

This concludes the proof taking $\varepsilon_j = \frac{a_j+b_j}{2}$. \square

Proposition 4.1 gives us a *localized* version of the invertibility result we need. To have a global one in the whole domain Ω_ε , we define a smooth cutoff function χ_ε by

$$\begin{cases} \chi_\varepsilon(t) = 1, & \text{for } t \leq \frac{1}{2}\varepsilon^{-\gamma}, \\ \chi_\varepsilon(t) = 0, & \text{for } t \geq \frac{3}{4}\varepsilon^{-\gamma}, \\ |\chi'_\varepsilon| \leq C\varepsilon^\gamma & \text{and } |\chi''_\varepsilon| \leq C\varepsilon^{2\gamma}. \end{cases}$$

We next set

$$\hat{u}_{k,\varepsilon}(y, \zeta) = \begin{cases} 1 + \chi_\varepsilon(\zeta)(u_{k,\varepsilon}(y, \zeta) - 1), & \text{in } \Omega_+, \\ -1 - \chi_\varepsilon(\zeta)(u_{k,\varepsilon}(y, \zeta) + 1), & \text{in } \mathbb{R}^n \setminus \Omega_+. \end{cases} \tag{102}$$

Now by (14) and (47) we know that $|u_{k,\varepsilon}|$ is exponentially close to 1 and its derivative are exponentially small for $|\zeta|$ sufficiently large. This and (46) imply that $\|\mathfrak{S}_\varepsilon(\hat{u}_{k,\varepsilon})\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon^{k+1-\frac{n-1}{2}}$ and $\|\mathfrak{S}_\varepsilon(\hat{u}_{k,\varepsilon})\|_{L^\infty(\Omega_\varepsilon)} \leq C\varepsilon^{k+1}$, see (45), where C depends only on K, b and k .

Let us denote by \hat{L}_ε the linearized operator at $\hat{u}_{k,\varepsilon}$ in Ω_ε . Given a smooth positive extension \hat{b} of b to $\overline{\Omega}$, we consider next the eigenvalue problem

$$\begin{cases} \hat{L}_\varepsilon u = \lambda \hat{b}(\varepsilon \cdot) u & \text{in } \Omega_\varepsilon; \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{103}$$

and we denote its eigenvalues by $\hat{\lambda}_{j,\varepsilon}$, counted in decreasing order with their multiplicity.

By (48), asymptotically away from K_ε , the eigenfunctions u satisfy

$$\Delta u - (2 \pm 2a(\varepsilon x) - \lambda \hat{b})u = 0 \quad \text{in } (\Omega_\pm)_\varepsilon,$$

where $(\Omega_\pm)_\varepsilon$ stands for the $\frac{1}{\varepsilon}$ dilation of Ω_\pm . Since we are assuming $a \in (-1, 1)$ in $\overline{\Omega}$, if λ is bounded from below by $-\varepsilon$, the coefficient of u in the above equation is negative. Hence, reasoning as in [28], Lemma 5.1, one can prove that u has an exponential decay away from K_ε .

Moreover, an argument based on the Courant–Fisher method, see Proposition 5.6 in [28], shows that there exists a constant C depending only on Ω, K, a and γ such that

$$|\hat{\lambda}_{j,\varepsilon} - \tilde{\lambda}_{j,\varepsilon}| \leq C e^{-\frac{C}{\varepsilon}} \quad \text{provided } \hat{\lambda}_{j,\varepsilon} \geq -\varepsilon \text{ or } \tilde{\lambda}_{j,\varepsilon} \geq -\varepsilon.$$

Here $\tilde{\lambda}_{j,\varepsilon}$ are the eigenvalues of L_ε in S_ε , see the proof of Proposition 4.1.

This and Proposition 4.1 allow us to prove the following result which guarantee the invertibility of the linearized operator for the range of the parameter ε constructed above.

Corollary 4.1. *Fix $k \in \mathbb{N}$ and let $\hat{u}_{k,\varepsilon}, \hat{L}_\varepsilon$ be as above. Define $H_v^2(\Omega_\varepsilon)$ to be the subset of $H^2(\Omega_\varepsilon)$ consisting of the functions with zero normal derivative at $\partial\Omega_\varepsilon$. Then for a suitable sequence $\varepsilon_j \rightarrow 0$, the operator $\hat{L}_{\varepsilon_j} : H_v^2(\Omega_{\varepsilon_j}) \rightarrow L^2(\Omega_{\varepsilon_j})$ is invertible and the inverse operator satisfies $\|\hat{L}_{\varepsilon_j}^{-1}\|_{\mathfrak{L}(L^2(\Omega_{\varepsilon_j}); H_v^2(\Omega_{\varepsilon_j}))} \leq \frac{C}{\varepsilon_j^{\frac{n+1}{2}}}$, for all $j \in \mathbb{N}$.*

Using the above results, we are in position to prove our main result, Theorem 1.1.

Proof of Theorem 1.1. Let ε_j be as in Corollary 4.1. We look for a solution u_ε of the equation $\mathfrak{S}_\varepsilon(u) = 0$ of the form

$$u_\varepsilon = \hat{u}_{k,\varepsilon} + w, \quad w \in H_v^2(\Omega_\varepsilon).$$

For $\varepsilon = \varepsilon_j$, define the function $\hat{F}_\varepsilon : H_v^2(\Omega_\varepsilon) \cap L^\infty(\Omega_\varepsilon) \rightarrow H_v^2(\Omega_\varepsilon) \cap L^\infty(\Omega_\varepsilon)$ by

$$\hat{F}_\varepsilon(w) := -\hat{L}_\varepsilon^{-1}[\mathfrak{S}_\varepsilon(\hat{u}_{k,\varepsilon}) - (3\hat{u}_{k,\varepsilon} - a)w^2 - w^3]. \quad (104)$$

We have that

$$\mathfrak{S}_\varepsilon(\hat{u}_{k,\varepsilon} + w) = 0 \iff \hat{F}_\varepsilon(w) = w. \quad (105)$$

We want to prove that \hat{F}_ε is a contraction in some closed ball of $H_v^2(\Omega_\varepsilon) \cap L^\infty(\Omega_\varepsilon)$. We first define the norm $\|\cdot\|$ as $\|w\| = \|w\|_{H_v^2(\Omega_\varepsilon)} + \|w\|_{L^\infty(\Omega_\varepsilon)}$. Then, for $r > 0$, we introduce the set

$$B_r = \{w \in H_v^2(\Omega_\varepsilon) \cap L^\infty(\Omega_\varepsilon) : \|w\| \leq r\}.$$

Applying a standard elliptic regularity theorem and using Corollary 4.1 one can prove that there exists positive constants C (depending on Ω , K and a) and d (depending on the dimension n) such that

$$\begin{aligned} \|\hat{F}_\varepsilon(w)\| &\leq C\varepsilon^{-d}(\varepsilon^{k+1-\frac{n-1}{2}} + \|w\|^2); \\ \|\hat{F}_\varepsilon(w_1) - \hat{F}_\varepsilon(w_2)\| &\leq C\varepsilon^{-d}(\|w_1\| + \|w_2\|)(\|w_1 - w_2\|), \end{aligned} \quad (106)$$

for $\varepsilon = \varepsilon_j$ and $w, w_1, w_2 \in H_v^2(\Omega_\varepsilon) \cap L^\infty(\Omega_\varepsilon)$. Now setting $r = \varepsilon^l$, we can choose first k sufficiently large, depending on d, n and then l depending on d and k so that \hat{F}_ε is a contraction in the ball B_r for $\varepsilon = \varepsilon_j$ sufficiently small. A solution of (105) can be then found using the contraction mapping theorem and its properties follows from the construction of $u_{k,\varepsilon}$. This concludes the proof.

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