# Solutions of an elliptic system with a nearly critical exponent 

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$$
\begin{aligned}
& \text { Abstract } \\
& \text { Consider the problem } \\
& \quad-\Delta u_{\varepsilon}=v_{\varepsilon}^{p}, \quad v_{\varepsilon}>0 \text { in } \Omega, \\
& -\Delta v_{\varepsilon}=u_{\varepsilon}^{q_{\varepsilon}}, \quad u_{\varepsilon}>0 \text { in } \Omega, \\
& u_{\varepsilon}=v_{\varepsilon}=0 \quad \text { on } \partial \Omega,
\end{aligned}
$$

where $\Omega$ is a bounded convex domain in $\mathbb{R}^{N}, N>2$, with smooth boundary $\partial \Omega$. Here $p, q_{\varepsilon}>0$, and

$$
\varepsilon:=\frac{N}{p+1}+\frac{N}{q_{\varepsilon}+1}-(N-2) .
$$

This problem has positive solutions for $\varepsilon>0$ (with $p q_{\varepsilon}>1$ ) and no non-trivial solution for $\varepsilon \leqslant 0$. We study the asymptotic behavior of least energy solutions as $\varepsilon \rightarrow 0^{+}$. These solutions are shown to blow-up at exactly one point, and the location of this point is characterized. In addition, the shape and exact rates for blowing up are given.
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## Résumé

Considérons le problème

$$
\begin{aligned}
& -\Delta u_{\varepsilon}=v_{\varepsilon}^{p}, \quad v_{\varepsilon}>0 \mathrm{en} \Omega, \\
& -\Delta v_{\varepsilon}=u_{\varepsilon}^{q_{\varepsilon}}, \quad u_{\varepsilon}>0 \mathrm{en} \Omega, \\
& u_{\varepsilon}=v_{\varepsilon}=0 \quad \operatorname{sur} \partial \Omega,
\end{aligned}
$$

où $\Omega$ est un domaine convexe et borné de $\mathbb{R}^{N}, N>2$, avec la frontière régulière $\partial \Omega$. Ici $p, q_{\varepsilon}>0$, et

$$
\varepsilon:=\frac{N}{p+1}+\frac{N}{q_{\varepsilon}+1}-(N-2) .
$$

Ce problème a des solutions positives pour $\varepsilon>0$ (avec $p q_{\varepsilon}>1$ ) et aucune solution non-triviale pour $\varepsilon \leqslant 0$. Nous étudions le comportement asymptotique de solutions d'énergie minimale quand $\varepsilon \rightarrow 0^{+}$. Ces solutions explosent en un seul point, et la position de ce point est caracterisée. De plus, le profil et les vitesses exactes d'explosion sont donnés.
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## 1. Introduction

We consider the elliptic system

$$
\begin{align*}
& -\Delta u_{\varepsilon}=v_{\varepsilon}^{p}, \quad v_{\varepsilon}>0 \text { in } \Omega  \tag{1.1}\\
& -\Delta v_{\varepsilon}=u_{\varepsilon}^{q_{\varepsilon}}, \quad u_{\varepsilon}>0 \text { in } \Omega  \tag{1.2}\\
& u_{\varepsilon}=v_{\varepsilon}=0 \quad \text { on } \partial \Omega \tag{1.3}
\end{align*}
$$

where $\Omega$ is a bounded convex domain in $\mathbb{R}^{N}, N>2$, with smooth boundary $\partial \Omega$. Here $p, q_{\varepsilon}>0$, and

$$
\begin{equation*}
\varepsilon:=\frac{N}{p+1}+\frac{N}{q_{\varepsilon}+1}-(N-2) . \tag{1.4}
\end{equation*}
$$

When $\varepsilon \leqslant 0$, there is no solution for (1.1)-(1.3), see [19] and [23]. On the other hand when $\varepsilon>0$, we can prove existence of solutions obtained by the variational method. In fact, for $\varepsilon>0$, the embedding $W^{2, \frac{p+1}{p}}(\Omega) \hookrightarrow L^{q_{\varepsilon}+1}(\Omega)$ is compact for any $q_{\varepsilon}+1>(p+1) / p$, that is $p q_{\varepsilon}>1$. Using this, it is not difficult to show that there exists a function $\bar{u}_{\varepsilon}$ positive solution of the variational problem

$$
\begin{equation*}
S_{\varepsilon}(\Omega)=\inf \left\{\left.\|\Delta u\|_{L^{\frac{p+1}{p}}(\Omega)} \right\rvert\, u \in W^{2, \frac{p+1}{p}}(\Omega),\|u\|_{L^{q_{\varepsilon}+1}(\Omega)}=1\right\} \tag{1.5}
\end{equation*}
$$

see for example [24]. This solution satisfies $-\Delta \bar{u}_{\varepsilon}=\bar{v}_{\varepsilon}^{p},-\Delta \bar{v}_{\varepsilon}=S_{\varepsilon}(\Omega) \bar{u}_{\varepsilon}^{q_{\varepsilon}}$, in $\Omega$ and $\bar{u}_{\varepsilon}=\bar{v}_{\varepsilon}=0$ on $\partial \Omega$. After changing to suitable multiples of $\bar{u}_{\varepsilon}$ and $\bar{v}_{\varepsilon}$, we obtain $u_{\varepsilon}$ and $v_{\varepsilon}$ solving (1.1)-(1.3). The pair ( $u_{\varepsilon}, v_{\varepsilon}$ ) is called a least energy solution of (1.1)-(1.3), which by regularity belongs to $C^{2}(\bar{\Omega}) \times C^{2}(\bar{\Omega})$. For others existence results, we refer to [4,7,9,16], and [20].

We observe that we can write the system (1.1)-(1.3) only in terms of $u_{\varepsilon}$, that is

$$
\begin{align*}
& -\Delta\left(-\Delta u_{\varepsilon}\right)^{1 / p}=u_{\varepsilon}^{q_{\varepsilon}}, \quad u_{\varepsilon}>0 \text { in } \Omega,  \tag{1.6}\\
& u_{\varepsilon}=\Delta u_{\varepsilon}=0 \quad \text { on } \partial \Omega . \tag{1.7}
\end{align*}
$$

Therefore, we refer to $u_{\varepsilon}$ as the least energy solution of (1.6)-(1.7).
Concerning least energy solutions of (1.6)-(1.7), in [24] it was proved that $S_{\varepsilon}(\Omega) \rightarrow S$ as $\varepsilon \downarrow 0$, where $S$ is independent of $\Omega$ and moreover is the best Sobolev constant for the inequality

$$
\begin{equation*}
\|u\|_{L^{q+1}\left(\mathbb{R}^{N}\right)} \leqslant S^{-\frac{p}{p+1}}\|\Delta u\|_{L^{\frac{p+1}{p}}}{\left(\mathbb{R}^{N}\right)} \tag{1.8}
\end{equation*}
$$

with $p, q, N$ satisfying

$$
\begin{equation*}
\frac{N}{p+1}+\frac{N}{q+1}-(N-2)=0 . \tag{1.9}
\end{equation*}
$$

This shows that the sequence $\left\{u_{\varepsilon}\right\}_{\varepsilon>0}$ of least energy solutions of (1.6)-(1.7) satisfy

$$
\begin{equation*}
S_{\varepsilon}(\Omega)=\frac{\int_{\Omega}\left|\Delta u_{\varepsilon}\right|^{\frac{p+1}{p}} \mathrm{~d} x}{\left\|u_{\varepsilon}\right\|_{L^{\frac{p+1}{p+1}(\Omega)}}^{p}}=S+\mathrm{o}(1) \quad \text { as } \varepsilon \rightarrow 0 . \tag{1.10}
\end{equation*}
$$

Relation (1.9) defines a curve in $\mathbb{R}_{+}^{2}$, for the variables $p$ and $q$. This curve is the so-called Sobolev Critical Hyperbola and replaces the notion of critical exponent in the scalar case. This hyperbola first appeared independently in [4] and [20] and later in [7] and [16].

In this article, we shall study in detail the asymptotic behaviors of the variational solution $u_{\varepsilon}$, of (1.6)-(1.7) as $\varepsilon \downarrow 0$, that is, as $q_{\varepsilon}$ approaches from below to $q$, in the Sobolev Critical Hyperbola (1.9).

By the symmetry of the hyperbola, we assume without restriction that

$$
\begin{equation*}
\frac{2}{N-2}<p \leqslant p^{*}:=\frac{N+2}{N-2} \tag{1.11}
\end{equation*}
$$

For each fixed value of $p$, the strict inequality gives a lower bound for the dimension, i.e. $N>\max \{2,2(p+1) / p\}$.
The asymptotic behaviors of Eqs. (1.6)-(1.7) as $\varepsilon \downarrow 0$ has already been studied for the cases $p=p^{*}$ and $p=1$. Next we recall some of these results and explain the relation with ours.

The case $p=p^{*}$ is equivalent to consider the single equation

$$
-\Delta u_{\varepsilon}=u_{\varepsilon}^{p^{*}-\varepsilon} \quad \text { in } \Omega, \quad \text { and } \quad u_{\varepsilon}=0 \quad \text { on } \partial \Omega .
$$

This problem was studied in [1,10,15,21]. There, exact rates of blow-up were given and the location of blow-up points were characterized. One key ingredient was the Pohozaev identity and the observation that the solution $u_{\varepsilon}$, scaled in the form $\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{-1} u_{\varepsilon}$ converges to $U$ solution of

$$
\begin{align*}
& -\Delta U=U^{p^{*}}, \quad U(y)>0 \quad \text { for } y \in \mathbb{R}^{N},  \tag{1.12}\\
& U(0)=1, \quad U \rightarrow 0, \quad \text { as }|y| \rightarrow \infty, \tag{1.13}
\end{align*}
$$

which is unique, explicit, and radially symmetric. For the location of the blow-up and the shape of the solution away of the singularity, it was proved that a scaled $u_{\varepsilon}$, given by $\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)} u_{\varepsilon}$, converges to the Green's function $G$, solution of $-\Delta G(x, \cdot)=\delta_{x}$ in $\Omega, G(x, \cdot)=0$ on $\partial \Omega$. The location of blowing-up points are the critical points of $\phi(x):=g(x, x)$ (in fact their minima, see [10]), where $g(x, y)$ is the regular part of $G(x, y)$, i.e.

$$
g(x, y)=G(x, y)-\frac{1}{(N-2) \sigma_{N}|x-y|^{N-2}} .
$$

In [6], a similar result was proven in the case $p=1(N>4)$. There the problem is reduced to study (1.12)-(1.13) with the operator $\Delta^{2}$ instead of $-\Delta$. Both cases $p=p^{*}$ and $p=1$ give the same blow-up rate

$$
\varepsilon\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{2} \rightarrow C \quad \text { as } \varepsilon \rightarrow 0^{+}
$$

for some explicit $C:=C(p, N, \Omega)>0$. We can ask ourselves if this behaviors is universal, i.e. holds for all $2 /(N-2)<p \leqslant p^{*}$. We will see later that this is only a coincide; a general result for the blow-up rate is given in Theorems 1.2.

Mimicking the above argument, we will study the asymptotic behaviors of the solution $u_{\varepsilon}$ of (1.6)-(1.7) as $\varepsilon \downarrow 0$. We shall show that $\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{-1} u_{\varepsilon}$ converges, as $\varepsilon \downarrow 0$, to the solution $U$ of the problem

$$
\begin{align*}
& -\Delta U=V^{p}, \quad V(y)>0 \text { for } y \in \mathbb{R}^{N},  \tag{1.14}\\
& -\Delta V=U^{q}, \quad U(y)>0 \text { for } y \in \mathbb{R}^{N},  \tag{1.15}\\
& U(0)=1, \quad U \rightarrow 0, \quad V \rightarrow 0 \quad \text { as }|y| \rightarrow \infty \tag{1.16}
\end{align*}
$$

In [5], it was proved that $U$ and $V$ are radially symmetric, if $p \geqslant 1$ and $U \in L^{q+1}\left(\mathbb{R}^{N}\right)$ and $V \in L^{p+1}\left(\mathbb{R}^{N}\right)$. These last integrability conditions hold when considering least energy solutions, see details in Section 2. Thus $U(r):=U(y)$ and $V(r):=V(y)$ with $r=|y|$, moreover $U$ and $V$ are unique, and decreasing in $r$, see [17,24]. There exist no explicit form of $(U, V)$ for all $p>2 /(N-2)$, however to carry out the analysis it is sufficient to know the asymptotic behaviors of $(U, V)$ as $r \rightarrow \infty$, which was studied in [17]. They found

$$
\lim _{r \rightarrow \infty} r^{N-2} V(r)=a \text { and } \begin{cases}\lim _{r \rightarrow \infty} r^{N-2} U(r)=b & \text { if } p>\frac{N}{N-2}  \tag{1.17}\\ \lim _{r \rightarrow \infty} \frac{r^{N-2}}{\log r} U(r)=b & \text { if } p=\frac{N}{N-2} \\ \lim _{r \rightarrow \infty} r^{p(N-2)-2} U(r)=b & \text { if } \frac{2}{N-2}<p<\frac{N}{N-2}\end{cases}
$$

In the following, we restrict further the value of $p \leqslant p^{*}$ from below:

$$
\begin{equation*}
p \geqslant 1 \quad \text { for } N>4 \quad \text { and } \quad p>\frac{2}{N-2} \quad \text { for } N=3,4 . \tag{1.18}
\end{equation*}
$$

This restriction is needed since, in different parts of the coming proofs, we use that $p \geqslant 1$, condition automatically satisfies when $N=3,4$. We believe however that this restriction is only technical and we conjecture that the results of this paper also hold for $2 /(N-2)<p<1$. In addition, we numerically found (radial) solutions in this range that satisfy Theorem 1.2.

The aim of this paper is to show the following results.
Theorem 1.1. Let $u_{\varepsilon}$ be a least energy solution of (1.6)-(1.7) and (1.18). Then
(a) there exists $x_{0} \in \Omega$ such that, after passing to a subsequence, we have
(i) $u_{\varepsilon} \rightarrow 0 \in C^{1}\left(\Omega \backslash\left\{x_{0}\right\}\right)$,
(ii) $v_{\varepsilon}=\left|\Delta u_{\varepsilon}\right|^{1 / p} \rightarrow 0 \in C^{1}\left(\Omega \backslash\left\{x_{0}\right\}\right)$
as $\varepsilon \rightarrow 0^{+}$and

$$
\text { (iii) }\left|\Delta u_{\varepsilon}\right|^{(p+1) / p} \rightarrow\|V\|_{L^{p+1}\left(\mathbb{R}^{N}\right)^{N+1}}^{p+1} \delta_{x_{0}} \quad \text { as } \varepsilon \rightarrow 0^{+}
$$

in the sense of distributions.
(b) $x_{0}$ is a critical point of

$$
\begin{array}{ll}
\phi(x):=g(x, x) & \text { if } p \in[N /(N-2),(N+2) /(N-2)] \quad \text { and } \\
\tilde{\phi}(x):=\tilde{g}(x, x) & \text { if } p \in(2 /(N-2), N /(N-2)) \tag{1.20}
\end{array}
$$

for $x \in \Omega$. The function $\tilde{g}(x, y)$ is defined for $p \in(2 /(N-2), N /(N-2))$ by

$$
\tilde{g}(x, y)=\widetilde{G}(x, y)-\frac{1}{(p(N-2)-2)(N-p(N-2))(N-2)^{p} \sigma_{N}^{p}|x-y|^{p(N-2)-2}}
$$

$$
\text { where }-\Delta \widetilde{G}(x, \cdot)=G^{p}(x, \cdot) \text { in } \Omega, \widetilde{G}(x, \cdot)=0 \text { on } \partial \Omega \text {. }
$$

This result gives a description of the function whose critical points are the blow-up points. We remark that for $p \in[N /(N-2),(N+2) /(N-2)]$, the critical points remain unchanged and equal to the case of a single equation. Note that if we consider a domain different from a ball, the critical points may change with $p$ in the region $2 /(N-2)<$ $p \leqslant N /(N-2)$.

We observe that regularity of $\tilde{\phi}$ is needed to compute its critical points in (b). We show next that $\tilde{\phi}$ is regular. By definition of $\widetilde{G}$, we have

$$
\begin{equation*}
\lim _{y \rightarrow x}|x-y|^{(p-1)(N-2)} \Delta \tilde{g}(x, y)=-\frac{p g(x, x)}{\left((N-2) \sigma_{N}\right)^{p-1}} \tag{1.21}
\end{equation*}
$$

for $x \in \Omega$. Thus $-\Delta \tilde{g}(x, \cdot) \in L_{\tilde{q}}^{q}(\Omega)$ for any $q \in(N / 2, N /(p(N-2)-N+2))$. This implies, by regularity, that $\tilde{g}(x, \cdot) \in L^{\infty}(\Omega)$ and therefore $\tilde{\phi}(x)=\tilde{g}(x, x), x \in \Omega$ is bounded. In addition, we define

$$
\begin{equation*}
\hat{g}(x, y)=\tilde{g}(x, y)+\frac{p g(x, x)|x-y|^{N-p(N-2)}}{(N-p(N-2))(2 N-p(N-2)-2)\left((N-2) \sigma_{N}\right)^{p-1}} \tag{1.22}
\end{equation*}
$$

and we have for any $x \in \Omega$ that

$$
\begin{equation*}
\lim _{y \rightarrow x}|x-y|^{(p-2)(N-2)} \Delta \hat{g}(x, y)=-\frac{p(p-1) g(x, x)}{\left((N-2) \sigma_{N}\right)^{p-2}} \tag{1.23}
\end{equation*}
$$

Thus $\hat{g}(x, y)$ is regular in $y$ for $x$ fixed. Since $N>p(N-2)$, we take first $y=x$ in (1.22) and then the gradient and we find $\nabla_{x} \tilde{g}(x, x)=\nabla_{x} \hat{g}(x, x)$. Hence $\tilde{\phi}(x)$ is regular.

The next two theorems make more precise the behavior of solutions. First we give the rate of blow-up of the maximum of the solutions.

Theorem 1.2. Let the assumptions of Theorem 1.1 be satisfied. Then

$$
\begin{cases}\lim _{\varepsilon \rightarrow 0^{+}} \varepsilon\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{\frac{N}{p(N-2)-2}+1}=S^{\frac{1-p q}{p(q+1)}}\|U\|_{L^{q}\left(\mathbb{R}^{N}\right)}^{q}\|V\|_{L^{p}\left(\mathbb{R}^{N}\right)}^{p}\left|\phi\left(x_{0}\right)\right| & \text { if } p>\frac{N}{N-2}, \\ \lim _{\varepsilon \rightarrow 0^{+}} \varepsilon \frac{\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{\frac{N}{N-2}+1}}{\log \left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}=\frac{p+1}{N-2} a^{\frac{N}{N-2}} S^{\frac{1-p q}{p(q+1)}}\|U\|_{L^{q}\left(\mathbb{R}^{N}\right)}^{q}\left|\phi\left(x_{0}\right)\right|} & \text { if } p=\frac{N}{N-2}, \\ \lim _{\varepsilon \rightarrow 0^{+}} \varepsilon\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{p+1}=S^{\frac{1-p q}{p(q+1)}}\|U\|_{L^{q}\left(\mathbb{R}^{N}\right)}^{q(p+1)}\left|\tilde{\phi}\left(x_{0}\right)\right| & \text { if } p<\frac{N}{N-2} .\end{cases}
$$

This theorem gives three regimes of blow-up depending on $p$. In the case $p>N /(N-2)$, the blow-up rate decreases as $p \downarrow N /(N-2)$ reaching a minimum at $p=N /(N-2)$. There we find a regime with a logarithmic correction. When $p<N /(N-2)$ the blow-up rate increases as $p \downarrow 2 /(N-2)$.

Observe that taking $p=q=p^{*}$, we recover the results in [15,21], that is

$$
\begin{equation*}
\varepsilon\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{2} \rightarrow C \quad \text { as } \varepsilon \rightarrow 0^{+} \tag{1.24}
\end{equation*}
$$

for some explicitly given $C>0$. See also [1] for the case when $\Omega$ is a ball.
When $N>4$, we can take $p=1$, i.e. $q=(N+4) /(N-4)$, recovering the result in $[2,6]$, where they prove that (1.24) holds for some $C>0$.

The previous theorem is a consequence of the following result, where the behaviors of solutions away from the singularity is given. Here the three regimes also appear and the behaviors of solutions are now given in terms of the Green's function.

Theorem 1.3. Let the assumptions of Theorem 1.1 be satisfied. Then

$$
\begin{align*}
& \lim _{\varepsilon \rightarrow 0^{+}}\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)} v_{\varepsilon}(x)=\|U\|_{L^{q}\left(\mathbb{R}^{N}\right)}^{q} G\left(x, x_{0}\right), \quad \text { and }  \tag{1.25}\\
& \begin{cases}\lim _{\varepsilon \rightarrow 0^{+}}\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{\frac{N}{p(N-2)-2}} u_{\varepsilon}(x)=\|V\|_{L^{p}\left(\mathbb{R}^{N}\right)}^{p} G\left(x, x_{0}\right) \quad \text { if } p>\frac{N}{N-2}, \\
\lim _{\varepsilon \rightarrow 0^{+}} \frac{\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{\frac{N}{N-2}} u_{\varepsilon}(x)=\frac{p+1}{N-2} a \frac{N}{N-2} G\left(x, x_{0}\right)}{} \quad \text { if } p=\frac{N}{N-2}, \\
\lim _{\varepsilon \rightarrow 0^{+}(\Omega)}\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{p} u_{\varepsilon}(x)=\|U\|_{L^{q}\left(\mathbb{R}^{N}\right)}^{p q} \widetilde{G}\left(x, x_{0}\right) & \text { if } p<\frac{N}{N-2},\end{cases} \tag{1.26}
\end{align*}
$$

where all the convergences are in $C^{1, \alpha}(\omega)$ with $\omega$ any subdomain of $\Omega$ not containing $x_{0}$. For $p<N /(N-2)$, the convergence in (1.26) can be improved to $C^{3, \alpha}(\omega)$.

Let us examine the limit $p \downarrow 2 /(N-2)$. In this limit, the exponent of $\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}$ in Theorem 1.2 tends to $N /(N-2)$. Next we consider the corresponding behaviors of $v_{\varepsilon}$. Let $x_{\varepsilon} \in \Omega$ such that $u_{\varepsilon}\left(x_{\varepsilon}\right)=\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}$. Using (2.7), (2.9) at $y=0$, and the convergence (2.13), we find

$$
\lim _{\varepsilon \rightarrow 0} v_{\varepsilon}\left(x_{\varepsilon}\right)=V(0) \lim _{\varepsilon \rightarrow 0}\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{\frac{N}{p(N-2)-2}} .
$$

This gives

$$
\lim _{\varepsilon \rightarrow 0} \varepsilon\left[v_{\varepsilon}\left(x_{\varepsilon}\right)\right]^{\frac{(p+1)[p(N-2)-2]}{N}}=S^{\frac{1-p q}{p(p+q)}}\left[\|U\|_{L^{q}\left(\mathbb{R}^{N}\right)}^{q} V(0) \frac{(p(N-2)-2)}{N}\right]^{p+1}\left|\tilde{\phi}\left(x_{0}\right)\right|
$$

for $p<N /(N-2)$. Note that the exponent of $v_{\varepsilon}\left(x_{\varepsilon}\right)$ tends to $0^{+}$as $p \downarrow 2 /(N-2)$. Recently in [13] the author studied the limiting case $p=2 /(N-2)$, and found that a positive solution $u$ of $-\Delta(-\Delta u)^{(N-2) / 2}=u^{q}$ in $\Omega$, with $u=\Delta u=0$ on $\partial \Omega$, remains bounded and develops peak(s) as $q \rightarrow \infty$.

We also consider the problem

$$
\begin{align*}
& -\Delta\left(-\Delta u_{\varepsilon}\right)^{1 / p}=u_{\varepsilon}^{q}+\varepsilon u_{\varepsilon}, \quad u_{\varepsilon}>0 \text { in } \Omega  \tag{1.27}\\
& u_{\varepsilon}=\Delta u_{\varepsilon}=0 \quad \text { on } \partial \Omega \tag{1.28}
\end{align*}
$$

for $\varepsilon>0$. The existence of positive solutions for this problem can be found in [16] and in [20] for the case of a ball. See [14] for related results when $p=1$. Similarly to the problem (1.6)-(1.7), we can define the least energy solutions for (1.27)-(1.28). Next, we will see a strong link in behaviors between the solutions of the two problems as $\varepsilon \rightarrow 0^{+}$.

The following theorem gives the behaviors of least energy solutions of (1.27)-(1.28) as $\varepsilon \rightarrow 0^{+}$. The blow up rates depend now on the integrability of $U^{2}$ in $\mathbb{R}^{N}$, consequently we divide the result in five cases. The first three cases are the analogous of Theorem 1.2 and there $\|U\|_{L^{2}\left(\mathbb{R}^{N}\right)}<\infty$ holds. The last two cases are the limiting cases, where we do not have integrability, but we can use the asymptotic behaviors of $U(y)$ as $|y| \rightarrow \infty$.

Theorem 1.4. Let $u_{\varepsilon}$ be a least energy solution of (1.27)-(1.28) and (1.18). Then the conclusions of Theorem 1.1 and Theorem 1.3 hold, and for $N>4$ we have

$$
\begin{align*}
& \lim _{\varepsilon \rightarrow 0^{+}} \varepsilon\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{\frac{2(N-3) p-3}{(N-2) p-2}}=\|U\|_{L^{2}\left(\mathbb{R}^{N}\right)}^{-2}\|U\|_{L^{q}\left(\mathbb{R}^{N}\right)}^{q}\|V\|_{L^{p}\left(\mathbb{R}^{N}\right)}^{p}\left|\phi\left(x_{0}\right)\right| \quad \text { if } p>\frac{N}{N-2},  \tag{1.29}\\
& \lim _{\varepsilon \rightarrow 0^{+}} \varepsilon \frac{\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{2 \frac{(N-3) p-3}{(N-2) p-2}}}{\log \left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}}=\frac{p+1}{N-2} a^{\frac{N}{N-2}}\|U\|_{L^{2}\left(\mathbb{R}^{N}\right)}^{-2}\|U\|_{L^{q}\left(\mathbb{R}^{N}\right)}^{q}\left|\phi\left(x_{0}\right)\right| \quad \text { if } p=\frac{N}{N-2},  \tag{1.30}\\
& \lim _{\varepsilon \rightarrow 0^{+}} \varepsilon\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{\frac{2[(N-3) p-3]-N}{(N-2) p-2}+p}=\|U\|_{L^{2}\left(\mathbb{R}^{N}\right)}^{-2}\|U\|_{L^{q}\left(\mathbb{R}^{N}\right)}^{q(p+1)}\left|\tilde{\phi}\left(x_{0}\right)\right| \quad \text { if } \frac{N+4}{2(N-2)}<p<\frac{N}{N-2} \\
& \quad \text { and } p>\frac{\sqrt{(N-4)^{2}+(N-2)(N+6)}-(N-4)}{N-2} \tag{1.31}
\end{align*}
$$

for $N \geqslant 8$, we have

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0^{+}} \varepsilon\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{\frac{N-8}{2(2)}} \log \left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}=b^{-2}\|U\|_{L^{q}\left(\mathbb{R}^{N}\right)}\left|\tilde{\phi}\left(x_{0}\right)\right| \quad \text { if } p=\frac{N+4}{2(N-2)} \tag{1.32}
\end{equation*}
$$

and for $N=4$, we have

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0^{+}} \varepsilon \log \left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}=b^{-2}\|U\|_{L^{q}\left(\mathbb{R}^{N}\right)}^{q}\|V\|_{L^{p}\left(\mathbb{R}^{N}\right)}^{p}\left|\phi\left(x_{0}\right)\right| \quad \text { if } p=q=p^{*}=3 . \tag{1.33}
\end{equation*}
$$

We can check in the three first cases that the corresponding exponents of $\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}$ are positive. In particular the case (1.31) with $p=1$, yields

$$
\lim _{\varepsilon \rightarrow 0^{+}} \varepsilon\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{\frac{2(N-8)}{N-4}}=\|U\|_{L^{2}\left(\mathbb{R}^{N}\right)}^{-2}\|U\|_{L^{\frac{N+4}{N-4}\left(\mathbb{R}^{N}\right)}}^{\frac{2 N+8}{N-4}}\left|\tilde{\phi}\left(x_{0}\right)\right| \quad \text { for } N>8
$$

In (1.32) the exponent of $\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}$ is non-negative, in fact in the limiting case $N=8$ and $p=1$, we have

$$
\lim _{\varepsilon \rightarrow 0^{+}} \varepsilon \log \left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}=b^{-2}\|U\|_{L^{3}\left(\mathbb{R}^{N}\right)}\left|\tilde{\phi}\left(x_{0}\right)\right|
$$

Note that for $p=1$, the function $U$ is known so the constants in the last two cases can be calculated explicitly.
The cases (1.29) (for $p=p^{*}$ ) and (1.33) have been found in [15]. Note that in these cases $U$ is a known function and equal to $V$, so the constants can be computed.

## 2. Preliminaries

Before proving the main theorem, we need some properties of $u_{\varepsilon}$. Using that $u_{\varepsilon}$ is a solution of (1.1)-(1.3), we have

$$
\int_{\Omega}\left(\Delta u_{\varepsilon}\right)^{\frac{p+1}{p}} \mathrm{~d} x=\int_{\Omega} v_{\varepsilon} \Delta u_{\varepsilon} \mathrm{d} x=\int_{\Omega} u_{\varepsilon} \Delta v_{\varepsilon} \mathrm{d} x=\int_{\Omega} u_{\varepsilon}^{q_{\varepsilon}+1} \mathrm{~d} x
$$

Then $[S+\mathrm{o}(1)]\left\|u_{\varepsilon}\right\|_{L^{q_{\varepsilon}+1}(\Omega)}^{\frac{p+1}{p}}=\left\|u_{\varepsilon}\right\|_{L^{q_{\varepsilon}+1}(\Omega)}^{q_{\varepsilon}+1}$ implies

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \int_{\Omega} u_{\varepsilon}^{q_{\varepsilon}+1} \mathrm{~d} x=S^{\frac{p q-1}{p(q+1)}} \tag{2.1}
\end{equation*}
$$

Lemma 2.1. The minimizing sequence $u_{\varepsilon}$ of (1.10) is such that

$$
\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)} \rightarrow \infty
$$

moreover $\left\|\left(-\Delta u_{\varepsilon}\right)^{1 / p}\right\|_{L^{\infty}(\Omega)}=\left\|v_{\varepsilon}\right\|_{L^{\infty}(\Omega)} \rightarrow \infty$ as $\varepsilon \rightarrow 0$.
Proof. If $\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)} \rightarrow \infty$ then by regularity, we find $\left\|v_{\varepsilon}\right\|_{L^{\infty}(\Omega)} \rightarrow \infty$, see [12, Theorem 3.7]. Now, assume that $\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)} \leqslant M$ and $\left\|v_{\varepsilon}\right\|_{L^{\infty}(\Omega)} \leqslant M$, by elliptic regularity, we have that

$$
\left\|v_{\varepsilon}\right\|_{C^{2+\alpha}(\bar{\Omega})} \leqslant M \quad \text { and } \quad\left\|u_{\varepsilon}\right\|_{C^{2+\alpha}(\bar{\Omega})} \leqslant M
$$

with $\alpha \in(0,1)$ and some constant $M$. This implies that there exists $u^{*}, v^{*} \in C^{2}(\bar{\Omega})$, such that

$$
u_{\varepsilon} \rightarrow u^{*} \quad \text { in } C^{2}(\bar{\Omega}), \quad v_{\varepsilon} \rightarrow v^{*} \quad \text { in } C^{2}(\bar{\Omega}) \quad \text { as } \varepsilon \rightarrow 0
$$

Hence $u^{*}$ satisfies

$$
0 \neq \int_{\Omega}\left(\Delta u^{*}\right)^{\frac{p+1}{p}} \mathrm{~d} x=S\left[\int_{\Omega}\left(u^{*}\right)^{q+1} \mathrm{~d} x\right]^{\frac{(p+1)}{p(q+1)}}
$$

which contradicts that $S$ cannot be achieved by a minimizer in a bounded domain, see [24]. In other words there exists no non-trivial solution for

$$
\begin{align*}
& -\Delta u^{*}=\left(v^{*}\right)^{p}, \quad v>0 \text { in } \Omega,  \tag{2.2}\\
& -\Delta v^{*}=\left(u^{*}\right)^{q}, \quad u>0 \text { in } \Omega,  \tag{2.3}\\
& u^{*}=v^{*}=0 \quad \text { on } \partial \Omega \tag{2.4}
\end{align*}
$$

in a convex bounded domain, with $p, q$ satisfying (1.9), see [19,23].
To simplify notation, we denote

$$
\alpha=\frac{N}{q+1} \quad \text { and } \quad \beta=\frac{N}{p+1},
$$

so the Sobolev Critical Hyperbola (1.9) takes the form $\alpha+\beta=N-2$.
For any $\varepsilon>0$, let $\left(u_{\varepsilon}, v_{\varepsilon}\right)$ be a solution of (1.1)-(1.3). By the Pohozaev identity, see [19] or [23], we have for any $\tilde{\alpha}, \tilde{\beta} \in \mathbb{R}$ that

$$
\begin{align*}
& \left(\frac{N}{q_{\varepsilon}+1}-\tilde{\alpha}\right) \int_{\Omega} u_{\varepsilon}^{q_{\varepsilon}+1} \mathrm{~d} x+\left(\frac{N}{p+1}-\tilde{\beta}\right) \int_{\Omega} v_{\varepsilon}^{p+1} \mathrm{~d} x+(N-2-\tilde{\alpha}-\tilde{\beta}) \int_{\Omega}\left(\nabla u_{\varepsilon}, \nabla v_{\varepsilon}\right) \mathrm{d} x \\
& \quad=-\int_{\partial \Omega}\left(\nabla u_{\varepsilon}, n\right)\left(\nabla v_{\varepsilon}, x-y\right) \mathrm{d} s \tag{2.5}
\end{align*}
$$

We can choose $\tilde{\alpha}+\tilde{\beta}=N-2, \tilde{\alpha}=\alpha$ and so $\tilde{\beta}=\beta$. This implies that

$$
\begin{equation*}
\varepsilon \int_{\Omega} u_{\varepsilon}^{q_{\varepsilon}+1} \mathrm{~d} x=-\int_{\partial \Omega} \frac{\partial u_{\varepsilon}}{\partial n} \frac{\partial v_{\varepsilon}}{\partial n}(n, x-y) \mathrm{d} s \tag{2.6}
\end{equation*}
$$

Since $u_{\varepsilon}$ becomes unbounded as $\varepsilon \rightarrow 0$ we choose $\mu=\mu(\varepsilon)$ and $x_{\varepsilon} \in \Omega$ such that

$$
\begin{equation*}
u_{\varepsilon}\left(x_{\varepsilon}\right)=\mu^{-\alpha_{\varepsilon}}=\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)} \tag{2.7}
\end{equation*}
$$

where $\alpha_{\varepsilon}=N /\left(q_{\varepsilon}+1\right)$. Note that $\mu \rightarrow 0$ as $\varepsilon \rightarrow 0$.
First we claim that $x_{\varepsilon}$ stays away from the boundary. This is a consequence of the moving plane method and interior estimates $[8,11]$ : let $\phi_{1}$ the positive eigenfunction of $\left(-\Delta, H_{0}^{1}(\Omega)\right)$, normalized to $\max _{x \in \Omega} \phi_{1}(x)=1$. Since $p \geqslant 1$, multiplying by $\phi_{1}$ we obtain

$$
\begin{aligned}
& \lambda_{1} \int_{\Omega} u_{\varepsilon} \phi_{1}=\int_{\Omega} v_{\varepsilon}^{p} \phi_{1} \geqslant 2 \lambda_{1} \int_{\Omega} v_{\varepsilon} \phi_{1}-C \int_{\Omega} \phi_{1}, \\
& \lambda_{1} \int_{\Omega} v_{\varepsilon} \phi_{1}=\int_{\Omega} u_{\varepsilon}^{q_{\varepsilon}} \phi_{1} \geqslant 2 \lambda_{1} \int_{\Omega} u_{\varepsilon} \phi_{1}-C \int_{\Omega} \phi_{1}
\end{aligned}
$$

for some $C=C\left(p, q, \lambda_{1}\right)>0$. Hence $\int_{\Omega} u_{\varepsilon} \phi_{1} \leqslant\left(C / \lambda_{1}\right) \int_{\Omega} \phi_{1}$ which implies $\int_{\Omega^{\prime}} u_{\varepsilon} \leqslant C\left(\Omega^{\prime}\right)$ with $\Omega^{\prime} \subset \Omega$ and $\int_{\Omega^{\prime}} v_{\varepsilon} \leqslant C\left(\Omega^{\prime}\right)$. Using the moving planes method [11], we find that there exist $t_{0} \alpha>0$ such that

$$
u_{\varepsilon}(x-t \nu) \quad \text { and } \quad v_{\varepsilon}(x-t \nu) \quad \text { are non-decreasing for } t \in\left[0, t_{0}\right]
$$

$\nu \in \mathbb{R}^{N}$ with $|\nu|=1$, and $(\nu, n(x)) \geqslant \alpha$ and $x \in \partial \Omega$. Therefore we can find $\gamma, \delta$ such that for any $x \in\{z \in \bar{\Omega}$ : $d(z, \partial \Omega)<\delta\}=\Omega_{\delta}$ there exists a measurable set $\Gamma_{x}$ with (i) meas $\left(\Gamma_{x}\right) \geqslant \gamma$, (ii) $\Gamma_{x} \subset \Omega \backslash \bar{\Omega}_{\delta / 2}$, and (iii) $u_{\varepsilon}(y) \geqslant$ $u_{\varepsilon}(x)$ and $v_{\varepsilon}(y) \geqslant v_{\varepsilon}(x)$ for any $y \in \Gamma_{x}$. Then for any $x \in \Omega_{\delta}$, we have

$$
\begin{aligned}
& u_{\varepsilon}(x) \leqslant \frac{1}{\operatorname{meas}\left(\Gamma_{x}\right)} \int_{\Gamma_{x}} u_{\varepsilon}(y) \mathrm{d} y \leqslant \frac{1}{\gamma} \int_{\Omega_{\delta}} u_{\varepsilon} \leqslant C\left(\Omega_{\delta}\right), \quad \text { and } \\
& v_{\varepsilon}(x) \leqslant \frac{1}{\operatorname{meas}\left(\Gamma_{x}\right)} \int_{\Gamma_{x}} v_{\varepsilon}(y) \mathrm{d} y \leqslant \frac{1}{\gamma} \int_{\Omega_{\delta}} v_{\varepsilon} \leqslant C\left(\Omega_{\delta}\right) .
\end{aligned}
$$

Hence if $u_{\varepsilon}\left(x_{\varepsilon}\right) \rightarrow \infty$, this implies that $x_{\varepsilon}$ will stay out of $\Omega_{\delta}$ a neighborhood of the boundary. This proves the claim.
Let $x_{\varepsilon} \rightarrow x_{0} \in \Omega$. We define a family of rescaled functions

$$
\begin{align*}
& u_{\varepsilon, \mu}(y)=\mu^{\alpha_{\varepsilon}} u_{\varepsilon}\left(\mu^{1-\varepsilon / 2} y+x_{\varepsilon}\right),  \tag{2.8}\\
& v_{\varepsilon, \mu}(y)=\mu^{\beta} v_{\varepsilon}\left(\mu^{1-\varepsilon / 2} y+x_{\varepsilon}\right) \tag{2.9}
\end{align*}
$$

and find using the definitions of $\varepsilon, \alpha_{\varepsilon}$ and $\beta$, that

$$
\begin{align*}
& -\Delta u_{\varepsilon, \mu}=v_{\varepsilon, \mu}^{p} \mu^{\alpha_{\varepsilon}+2-\varepsilon-p \beta}=v_{\varepsilon, \mu}^{p} \quad \text { in } \Omega_{\varepsilon}  \tag{2.10}\\
& -\Delta v_{\varepsilon, \mu}=u_{\varepsilon, \mu}^{q_{\varepsilon}} \mu^{\beta+2-\varepsilon-q_{\varepsilon} \alpha_{\varepsilon}}=u_{\varepsilon, \mu}^{q_{\varepsilon}} \quad \text { in } \Omega_{\varepsilon}  \tag{2.11}\\
& u_{\varepsilon, \mu}=v_{\varepsilon, \mu}=0 \quad \text { on } \partial \Omega_{\varepsilon} . \tag{2.12}
\end{align*}
$$

By equicontinuity and using Arzela-Ascoli, we have that

$$
\begin{equation*}
u_{\varepsilon, \mu} \rightarrow U \quad \text { and } \quad v_{\varepsilon, \mu} \rightarrow V \quad \text { as } \varepsilon \rightarrow 0 \tag{2.13}
\end{equation*}
$$

in $C^{2}(K)$ for any $K$ compact in $\mathbb{R}^{N}$, where ( $U, V$ ) satisfies (1.14)-(1.16). Now extending $u_{\varepsilon, \mu}$ and $v_{\varepsilon, \mu}$ by zero outside $\Omega_{\varepsilon}$ and using (2.1), by the argument in [22] or [24], we have that $u_{\varepsilon, \mu} \rightarrow \bar{U}$ strongly (up to a subsequence) in $W^{2, \frac{p+1}{p}}\left(\mathbb{R}^{N}\right)$. In the limit $\bar{U} \in L^{q+1}\left(\mathbb{R}^{N}\right)$ and $\bar{V}:=(-\Delta \bar{U})^{\frac{1}{p}} \in L^{p+1}\left(\mathbb{R}^{N}\right)$, and they satisfy (1.14)-(1.16). Since $p \geqslant 1$, the solution $(\bar{U}, \bar{V})$ is unique and radially symmetric, see [5]. In addition the radial solutions are unique [17,24], so $\bar{U} \equiv U$ and $\bar{V} \equiv V$, consequently

$$
\begin{equation*}
\int_{\mathbb{R}^{N}}\left[u_{\varepsilon, \mu}-U\right]^{q+1}(y) \mathrm{d} y \rightarrow 0, \quad \int_{\mathbb{R}^{N}}\left[v_{\varepsilon, \mu}-V\right]^{p+1}(y) \mathrm{d} y \rightarrow 0 . \tag{2.14}
\end{equation*}
$$

Lemma 2.2. There exists $\delta>0$ such that

$$
\delta \leqslant \mu^{\varepsilon} \leqslant 1
$$

Proof. Since $\mu \rightarrow 0$, we have $\mu^{\varepsilon} \leqslant 1$. By (2.14), we get $\int_{B_{1}} u_{\varepsilon, \mu}^{q_{\varepsilon}+1} \mathrm{~d} x \geqslant M$, but

$$
\begin{equation*}
M \leqslant \int_{B_{1}} u_{\varepsilon, \mu}^{q_{\varepsilon}+1} \mathrm{~d} x=\mu^{\varepsilon N / 2} \int_{\left|y-x_{\varepsilon}\right| \leqslant \mu^{1-\varepsilon / 2}} u_{\varepsilon}^{q_{\varepsilon}+1}(y) \mathrm{d} y \leqslant \mu^{\varepsilon N / 2} \int_{\Omega} u_{\varepsilon}^{q_{\varepsilon}+1}(y) \mathrm{d} y . \tag{2.15}
\end{equation*}
$$

Using the convergence (2.1), we obtain the result.
Lemma 2.3. There exists $K>0$ such that the solution $\left(u_{\varepsilon, \mu}, v_{\varepsilon, \mu}\right)$ satisfies

$$
\begin{equation*}
u_{\varepsilon, \mu}(y) \leqslant K U(y), \quad v_{\varepsilon, \mu}(y) \leqslant K V(y) \quad \forall y \in \mathbb{R}^{N} . \tag{2.16}
\end{equation*}
$$

We prove this lemma in Section 4.
Lemma 2.4. There exists a constant $C>0$ such that

$$
\varepsilon \leqslant C \mu^{N-2} h(\mu) \quad \text { with } h(\mu)= \begin{cases}1 & \text { for } p>N /(N-2),  \tag{2.17}\\ |\log (\mu)| & \text { for } p=N /(N-2), \\ \mu^{(p(N-2)-N)} & \text { for } p<N /(N-2)\end{cases}
$$

Proof. We will establish the following

$$
\int_{\partial \Omega} \frac{\partial u_{\varepsilon}}{\partial n} \frac{\partial v_{\varepsilon}}{\partial n}(n, x) \mathrm{d} x \leqslant C \mu^{N-2} h(\mu)
$$

and from here the result follows applying (2.6). We claim that

$$
\left|\frac{\partial u_{\varepsilon}}{\partial n}\right| \leqslant C \mu^{\alpha_{\varepsilon}}, \quad\left|\frac{\partial v_{\varepsilon}}{\partial n}\right| \leqslant C \mu^{\beta} h(\mu) .
$$

In the following $M$ is a positive constant that can vary from line to line and we shall use systematically Lemma 2.2.
For $p>N /(N-2)$, using that $-p \beta+N=\beta$, we have

$$
\int_{\Omega} v_{\varepsilon}^{p}(x) \mathrm{d} x \leqslant M \mu^{-p \beta+N(1-\varepsilon / 2)} \int_{\mathbb{R}^{N}} V^{p}(y) \mathrm{d} y \leqslant M \mu^{\beta}
$$

and by (2.16) there exists $M>0$ such that

$$
\begin{equation*}
v_{\varepsilon}^{p}(x) \leqslant M \frac{\mu^{\beta+p(N-2)-N-p(N-2) \varepsilon / 2}}{\left|x-x_{0}\right|^{p(N-2)}} \tag{2.18}
\end{equation*}
$$

for $x \neq x_{0}$. Using that $\beta<\beta+p(N-2)-N$, by Lemma 5.1 we find $\left|\partial v_{\varepsilon} / \partial n\right| \leqslant C \mu^{\beta}$. For $u_{\varepsilon}$, using that $-q_{\varepsilon} \alpha_{\varepsilon}+$ $N=\alpha_{\varepsilon}$,

$$
\int_{\Omega} u_{\varepsilon}^{q_{\varepsilon}} \mathrm{d} x \leqslant M \mu^{-q_{\varepsilon} \alpha_{\varepsilon}+N(1-\varepsilon / 2)} \int_{\mathbb{R}^{N}} U^{q}(y) \mathrm{d} y \leqslant M \mu^{\alpha_{\varepsilon}}
$$

and by (2.16) there exist $M>0$ such

$$
\begin{equation*}
u_{\varepsilon}^{q_{\varepsilon}}(x) \leqslant M \frac{\mu^{-q_{\varepsilon} \alpha_{\varepsilon}+q_{\varepsilon}(N-2)-q_{\varepsilon}(N-2) \varepsilon / 2}}{\left|x-x_{0}\right|^{q_{\varepsilon}(N-2)}} \tag{2.19}
\end{equation*}
$$

for $x \neq x_{0}$. Using that $\alpha_{\varepsilon}<\alpha_{\varepsilon}-N+q_{\varepsilon}(N-2)$, by Lemma 5.1, we obtain $\left|\partial u_{\varepsilon} / \partial n\right| \leqslant C \mu^{\alpha_{\varepsilon}}$.
For $p<N /(N-2)$, we have

$$
\begin{align*}
\int_{\Omega} v_{\varepsilon}^{p} \mathrm{~d} x & \leqslant M \mu^{-p \beta+p(N-2)(1-\varepsilon / 2)} \lim _{\mu \rightarrow 0} \frac{1}{\mu^{(p(N-2)-N)(1-\varepsilon / 2)}} \int_{B_{1 / \mu^{1-\varepsilon / 2}\left(x_{\varepsilon}\right)}} V^{p}(y) \mathrm{d} y  \tag{2.20}\\
& \leqslant M \mu^{\beta+(p(N-2)-N)} \tag{2.21}
\end{align*}
$$

and pointwise for $v_{\varepsilon}$, we have (2.18) for $x \neq x_{0}$. Now for $u_{\varepsilon}$, we have

$$
\int_{\Omega} u_{\varepsilon}^{q_{\varepsilon}} \leqslant M \mu^{-q_{\varepsilon} \alpha_{\varepsilon}+N(1-\varepsilon / 2)} \int_{\mathbb{R}^{N}} U^{q}(y) \mathrm{d} y \leqslant M \mu^{\alpha_{\varepsilon}}
$$

and by (2.16) there exist $M>0$ such that

$$
\begin{equation*}
u_{\varepsilon}^{q_{\varepsilon}}(x) \leqslant M \frac{\mu^{-q_{\varepsilon} \alpha_{\varepsilon}+q_{\varepsilon}(p(N-2)-2)-q_{\varepsilon}(p(N-2)-2) \varepsilon / 2}}{\left|x-x_{0}\right|_{\varepsilon}^{q_{\varepsilon}(p(N-2)-2)}} \tag{2.22}
\end{equation*}
$$

for $x \neq x_{0}$. From these estimates we prove the claim applying Lemma 5.1 and noting that $\alpha_{\varepsilon}<\alpha_{\varepsilon}-N+q_{\varepsilon}(p(N-$ 2) -2$)+(p+1) \varepsilon / \alpha_{\varepsilon}$. For the case $p=N /(N-2)$, we proceed as before noting that

$$
\int_{\Omega} v_{\varepsilon}^{p} \mathrm{~d} x \leqslant M \mu^{-p \beta+N(1-\varepsilon / 2)}|\log (\mu)| \lim _{\mu \rightarrow 0} \frac{1}{|\log (\mu)|} \int_{B_{1 / \mu^{1-\varepsilon / 2}\left(x_{\varepsilon}\right)}} V^{p}(y) \mathrm{d} y \leqslant M|\log (\mu)| \mu^{\beta}
$$

and for $x \neq x_{0}$ we have (2.18). Similarly to (2.22), we obtain that for $x \neq x_{0}$, there exist $M>0$ such that

$$
\begin{equation*}
u_{\varepsilon}^{q_{\varepsilon}}(x) \leqslant M \frac{\mu^{-q_{\varepsilon} \alpha_{\varepsilon}+q_{\varepsilon}(N-2)-q_{\varepsilon}(N-2) \varepsilon / 2}}{\left|x-x_{0}\right|_{\varepsilon}(N-2)} \log \left(\left|x-x_{0}\right| \mu^{-1+\varepsilon / 2}\right)^{q_{\varepsilon}} . \tag{2.23}
\end{equation*}
$$

Using this and proceeding as before we prove the claim and the lemma follows.

## Lemma 2.5.

$$
\left|\mu^{\varepsilon}-1\right|=\mathrm{O}\left(\mu^{N-2} h(\mu) \log \mu\right)
$$

Proof. By the theorem of the mean $\left|\mu^{\varepsilon}-1\right|=\left|\mu^{s \varepsilon} \varepsilon \log \mu\right|$ for some $s \in(0,1)$ and therefore (2.17) gives the result.

## 3. Proof of the theorems

We shall give only the proof of Theorems 1.1, 1.2 and 1.3. The proof of Theorem 1.4 is almost identical to the first three theorems. In fact the main difference is the Pohozaev identity (2.6), which now reads

$$
\begin{equation*}
\varepsilon \int_{\Omega} u_{\varepsilon}^{2} \mathrm{~d} x=-\int_{\partial \Omega} \frac{\partial u_{\varepsilon}}{\partial n} \frac{\partial v_{\varepsilon}}{\partial n}(n, x-y) \mathrm{d} s \tag{3.1}
\end{equation*}
$$

Proof of Theorem 1.3. We start by proving the case $p>N /(N-2)$. We have

$$
\begin{align*}
& -\Delta\left(\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{\beta / \alpha} u_{\varepsilon}\right)=\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{\beta / \alpha} v_{\varepsilon}^{p} \quad \text { in } \Omega,  \tag{3.2}\\
& -\Delta\left(\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)} v_{\varepsilon}\right)=\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{q_{\varepsilon}} \quad \text { in } \Omega,  \tag{3.3}\\
& u_{\varepsilon}=v_{\varepsilon}=0 \quad \text { on } \partial \Omega . \tag{3.4}
\end{align*}
$$

We integrate the right-hand side of (3.2)

$$
\int_{\Omega}\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{\beta / \alpha} v_{\varepsilon}^{p} \mathrm{~d} x=\mu^{-(p+1) \beta+N+N \varepsilon / 2} \int_{\Omega_{\varepsilon}} v_{\varepsilon, \mu}^{p}(y) \mathrm{d} y
$$

But $N-(p+1) \beta=0$, so using (2.16) by dominated convergence and Lemma 2.5 , we get

$$
\lim _{\varepsilon \rightarrow 0} \int_{\Omega}\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{\beta / \alpha} v_{\varepsilon}^{p} \mathrm{~d} x=\int_{\mathbb{R}^{N}} V^{p}(y) \mathrm{d} y=\|V\|_{L^{p}\left(\mathbb{R}^{N}\right)}<\infty
$$

Similarly, now using

$$
\begin{equation*}
\int_{\Omega}\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)} u_{\varepsilon}^{q_{\varepsilon}} \mathrm{d} x=\mu^{-\left(q_{\varepsilon}+1\right) \alpha_{\varepsilon}+N+N \varepsilon / 2} \int_{\Omega_{\varepsilon}} u_{\varepsilon, \mu}^{q_{\varepsilon}} \mathrm{d} x \rightarrow\|U\|_{L^{q}\left(\mathbb{R}^{N}\right)}<\infty \tag{3.5}
\end{equation*}
$$

as $\varepsilon \rightarrow 0$. Also using the bound (2.16), we find

$$
\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{\beta / \alpha} v_{\varepsilon}^{p}(x) \leqslant \frac{M \mu^{-(p+1) \beta+p(N-2)-p(N-2) \varepsilon / 2}}{\left|x-x_{0}\right|^{p(N-2)}}
$$

for $x \neq x_{0}$ and some $M>0$. But $-(p+1) \beta+p(N-2)>0$ and Lemma 2.2 then $\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{\beta / \alpha} v_{\varepsilon}^{p}(x) \rightarrow 0$ for $x \neq x_{0}$. Also we have

$$
\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)} u_{\varepsilon}^{q_{\varepsilon}}(x) \leqslant \frac{M \mu^{-\left(q_{\varepsilon}+1\right) \alpha_{\varepsilon}+q_{\varepsilon}(N-2)-q_{\varepsilon}(N-2) \varepsilon / 2}}{\left.\left|x-x_{0}\right|_{\varepsilon}\right|_{\varepsilon}(N-2)}
$$

for $x \neq x_{0}$ and some $M>0$. But $-\left(q_{\varepsilon}+1\right) \alpha_{\varepsilon}+q_{\varepsilon}(N-2)>0$ and Lemma 2.2 then $\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)} u_{\varepsilon}^{q_{\varepsilon}}(x) \rightarrow 0$ for $x \neq x_{0}$.

From here we have

$$
-\Delta\left(\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{\beta / \alpha} u_{\varepsilon}\right) \rightarrow\|V\|_{L^{p}\left(\mathbb{R}^{N}\right)}^{p} \delta_{x=x_{0}} \quad \text { and } \quad-\Delta\left(\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)} v_{\varepsilon}\right) \rightarrow\|U\|_{L^{q}\left(\mathbb{R}^{N}\right)}^{q} \delta_{x=x_{0}}
$$

in the sense of distributions in $\Omega$, as $\varepsilon \rightarrow 0$. Let $\omega$ be any neighborhood of $\partial \Omega$ not containing $x_{0}$. By regularity theory, see Lemma 5.1, we find

$$
\left\|\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{\beta / \alpha} u_{\varepsilon}\right\|_{C^{1, \alpha}(w)} \leqslant C\left[\| \| u_{\varepsilon}\left\|_{L^{\infty}(\Omega)}^{\beta / \alpha} v_{\varepsilon}^{p}\right\|_{L^{1}(\Omega)}+\| \| u_{\varepsilon}\left\|_{L^{\infty}(\Omega)}^{\beta / \alpha} v_{\varepsilon}^{p}\right\|_{L^{\infty}(w)}\right]
$$

and a similar bound for $\left\|\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)} v_{\varepsilon}\right\|_{C^{1, \alpha}(w)}$. Consequently

$$
\begin{equation*}
\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{\beta / \alpha} u_{\varepsilon} \rightarrow\|V\|_{L^{p}\left(\mathbb{R}^{N}\right)}^{p} G \quad \text { in } C^{1, \alpha}(w) \text { as } \varepsilon \rightarrow 0 \tag{3.6}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)} v_{\varepsilon} \rightarrow\|U\|_{L^{q}\left(\mathbb{R}^{N}\right)}^{q} G \quad \text { in } C^{1, \alpha}(w) \text { as } \varepsilon \rightarrow 0 . \tag{3.7}
\end{equation*}
$$

For the case $p<N /(N-2)$, we proceed as before and we have (3.5) and the bound

$$
\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)} u_{\varepsilon}^{q_{\varepsilon}}(x) \leqslant \frac{M \mu^{-\left(q_{\varepsilon}+1\right) \alpha_{\varepsilon}+q_{\varepsilon}(p(N-2)-2)-q_{\varepsilon}(p(N-2)-2) \varepsilon / 2}}{\left|x-x_{0}\right|_{\varepsilon}(p(N-2)-2)}
$$

for $x \neq x_{0}$ and some $M>0$. Using that $-\left(q_{\varepsilon}+1\right) \alpha_{\varepsilon}+q(p(N-2)-2)=2(p+1)>0$ and Lemma 2.2, we get $\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{\beta / \alpha} u_{\varepsilon}^{q_{\varepsilon}}(x) \rightarrow 0$ for $x \neq x_{0}$ and hence

$$
\begin{equation*}
\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)} v_{\varepsilon} \rightarrow\|U\|_{L^{q}\left(\mathbb{R}^{N}\right)}^{q} G \quad \text { in } C^{1, \alpha}(w) \text { as } \varepsilon \rightarrow 0 . \tag{3.8}
\end{equation*}
$$

Now we claim that

$$
\begin{equation*}
\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{\frac{1}{\alpha}(\beta+p(N-2)-N)} u_{\varepsilon} \rightarrow\|U\|_{L^{q}\left(\mathbb{R}^{N}\right)}^{p q} \widetilde{G} \quad \text { in } C^{1, \alpha}(w) \text { as } \varepsilon \rightarrow 0 . \tag{3.9}
\end{equation*}
$$

We have

$$
-\Delta\left(\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{\frac{1}{\alpha}(\beta+p(N-2)-N)} u_{\varepsilon}\right)=\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{\frac{1}{\alpha}(\beta+p(N-2)-N)} v_{\varepsilon}^{p}=\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{p} v_{\varepsilon}^{p} .
$$

Since the last term converges to $\left(\|U\|_{L^{q}\left(\mathbb{R}^{N}\right)}^{q} G\right)^{p}$ in $C^{1, \alpha}(\omega)$ as $\varepsilon \rightarrow 0$ and $p \geqslant 1$, we have

$$
\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{\frac{1}{\alpha}(\beta+p(N-2)-N)} u_{\varepsilon} \rightarrow\|U\|_{L^{q}\left(\mathbb{R}^{N}\right)}^{p q} \widetilde{G} \quad \text { in } C^{3, \alpha}(w) \text { as } \varepsilon \rightarrow 0 .
$$

For the remaining case $p=N /(N-2)$, we have as $\varepsilon \rightarrow 0$, the convergence

$$
\int_{\Omega} \frac{\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{\beta / \alpha}}{\left|\log \left(\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}\right)\right|} v_{\varepsilon}^{p} \mathrm{~d} x=\frac{\mu^{-(p+1) \beta+N+N \varepsilon / 2}}{\alpha_{\varepsilon}|\log (\mu)|} \int_{\Omega_{\varepsilon}} v_{\varepsilon, \mu}^{p} \mathrm{~d} y \rightarrow \frac{1}{\alpha} \lim _{r \rightarrow \infty} V(r)^{\frac{N}{N-2}} r^{N}=\frac{a^{\frac{N}{N-2}}}{\alpha}
$$

and the pointwise bound for $x \neq x_{0}$

$$
\frac{\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{\beta / \alpha}}{\left|\log \left(\left\|u_{\varepsilon}\right\|^{\beta / \alpha}\right)\right|} v_{\varepsilon}^{p}(x) \leqslant \frac{M \mu^{-p(N-2) \varepsilon / 2}}{\log (\mu)\left|x-x_{0}\right|^{p(N-2)}} .
$$

By Lemma 2.2,

$$
\frac{\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{\beta / \alpha}}{\left|\log \left(\left\|u_{\varepsilon}\right\|^{\beta / \alpha}\right)\right|} v_{\varepsilon}^{p}(x) \rightarrow 0
$$

for $x \neq x_{0}$. Writing

$$
-\Delta\left(\frac{\left\|u_{\varepsilon}\right\|_{L^{\infty}}^{\beta / \alpha}}{\left|\log \left(\left\|u_{\varepsilon}\right\|^{\beta / \alpha}\right)\right|^{2}} u_{\varepsilon}\right)=\frac{\left\|u_{\varepsilon}\right\|_{L^{\infty}}^{\beta / \alpha}}{\left|\log \left(\left\|u_{\varepsilon}\right\|^{\beta / \alpha}\right)\right|} v_{\varepsilon}^{p},
$$

we observe that the last term converges to $\delta_{x=x_{0}}$. By Lemma 5.1, we have

$$
\frac{\left\|u_{\varepsilon}\right\|_{L^{\infty}}^{\beta / \alpha}}{\left|\log \left(\left\|u_{\varepsilon}\right\|^{\beta / \alpha}\right)\right|} u_{\varepsilon} \rightarrow \frac{a^{N /(N-2)}}{\alpha} G \quad \text { in } C^{1, \alpha}(w) \text { as } \varepsilon \rightarrow 0
$$

and clearly we have (3.7) using (2.23). This completes the proof of the theorem.
Proof of Theorem 1.2. For $p>N /(N-2)$ we have

$$
\varepsilon\left\|u_{\varepsilon}\right\|^{(N-2) / \alpha} \int_{\Omega} u_{\varepsilon}^{q_{\varepsilon}+1} \mathrm{~d} x=\int_{\partial \Omega}\left(\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{\beta / \alpha} \nabla u_{\varepsilon}, n\right)\left(\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)} \nabla v_{\varepsilon}, n\right)(n, x-y) \mathrm{d} s
$$

By (3.6) and (3.7),

$$
\lim _{\varepsilon \rightarrow 0} \varepsilon\left\|u_{\varepsilon}\right\|^{(N-2) / \alpha} \int_{\Omega} u_{\varepsilon}^{q_{\varepsilon}+1} \mathrm{~d} x=\|V\|_{L^{p}\left(\mathbb{R}^{N}\right)}^{p}\|U\|_{L^{q}\left(\mathbb{R}^{N}\right)}^{q} \int_{\partial \Omega} \frac{\partial G\left(x, x_{0}\right)}{\partial n} \frac{\partial G\left(x, x_{0}\right)}{\partial n}\left(n, x-x_{0}\right) \mathrm{d} s .
$$

Also for the case $p<N /(N-2)$, using

$$
\varepsilon\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{\frac{1}{\alpha}(p(N-2)-2)} \int_{\Omega} u_{\varepsilon}^{q_{\varepsilon}+1} \mathrm{~d} x=\int_{\partial \Omega}\left(\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{\frac{1}{\alpha}(\beta+p(N-2)-N)} \nabla u_{\varepsilon}, n\right)\left(\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)} \nabla v_{\varepsilon}, n\right)(n, x-y) \mathrm{d} s
$$

and (3.9) and (3.8), we get

$$
\lim _{\varepsilon \rightarrow 0} \varepsilon\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{\frac{1}{\alpha}(p(N-2)-2)} \int_{\Omega} u_{\varepsilon}^{q_{\varepsilon}+1} \mathrm{~d} x=\|U\|_{L^{q}\left(\mathbb{R}^{N}\right)}^{q(p+1)} \int_{\partial \Omega} \frac{\partial \widetilde{G}\left(x, x_{0}\right)}{\partial n} \frac{\partial G\left(x, x_{0}\right)}{\partial n}\left(n, x-x_{0}\right) \mathrm{d} s .
$$

The case $p=N /(N-2)$ is analogous.
The proof of the theorems follows from the next lemma.
Lemma 3.1. We have the following identities
(i) $\int_{\partial \Omega} \frac{\partial G\left(x, x_{0}\right)}{\partial n} \frac{\partial G\left(x, x_{0}\right)}{\partial n}\left(n, x-x_{0}\right) \mathrm{d} s=-(N-2) g\left(x_{0}, x_{0}\right)$
and
(ii) $\int_{\partial \Omega} \frac{\partial \widetilde{G}\left(x, x_{0}\right)}{\partial n} \frac{\partial G\left(x, x_{0}\right)}{\partial n}\left(n, x-x_{0}\right) \mathrm{d} s=-\frac{N}{q+1} \tilde{g}\left(x_{0}, x_{0}\right)$.

Proof. (i) was proven in [3], see also [15]. To prove (ii) we follow a similar procedure. From [19,23], for any $y \in \mathbb{R}^{N}$, we have the following identity

$$
\begin{aligned}
\int_{\Omega^{\prime}} & \Delta u(x-y, \nabla v)+\Delta v(x-y, \nabla u)-(N-2)(\nabla u, \nabla v) \mathrm{d} x \\
& =\int_{\partial \Omega^{\prime}} \frac{\partial u}{\partial n}(x-y, \nabla v)+\frac{\partial v}{\partial n}(x-y, \nabla u)-(\nabla u, \nabla v)(x-y, n) \mathrm{d} s,
\end{aligned}
$$

where $\Omega^{\prime}=\Omega \backslash B_{r}$ with $r>0$. For a system $-\Delta v=0$ and $-\Delta u=v^{p}$, in $\Omega^{\prime}$, the identity takes the form

$$
\begin{align*}
\int_{\Omega^{\prime}} \frac{N}{p+1} v^{p+1}-\bar{a} v^{p+1} \mathrm{~d} x= & \int_{\partial \Omega^{\prime}} \frac{1}{p+1} v^{p+1}(x-y, n) \mathrm{d} s+\int_{\partial \Omega^{\prime}} \frac{\partial u}{\partial n}[(x-y, \nabla v)+\bar{a} v] \\
& +\frac{\partial v}{\partial n}[(x-y, \nabla u)+\bar{b} u]-(\nabla u, \nabla v)(x-y, n) \mathrm{d} s \tag{3.10}
\end{align*}
$$

with $\bar{a}+\bar{b}=N-2$. Let $y=0$, choose $\bar{a}=N /(p+1)$ and take $v=G(x, 0)$ and $u=\widetilde{G}(x, 0)$. Using that $u=v=0$ on $\partial \Omega$, and so $\nabla u=(\nabla u, n) n$ and $\nabla v=(\nabla v, n) n$ on $\partial \Omega$, we obtain

$$
\begin{aligned}
\int_{\partial \Omega} \frac{\partial \widetilde{G}}{\partial n} \frac{\partial \widetilde{G}}{\partial n}(x, n) \mathrm{d} s= & \int_{\partial B_{r}} \frac{1}{p+1} G^{p+1}(x, n)+\frac{\partial \widetilde{G}}{\partial n}\left[(x, \nabla G)+\frac{N}{p+1} G\right] \mathrm{d} s \\
& +\int_{\partial B_{r}} \frac{\partial G}{\partial n}\left[(x, \nabla \widetilde{G})+\frac{N}{q+1} \widetilde{G}\right]-(\nabla \widetilde{G}, \nabla G)(x, n) \mathrm{d} s
\end{aligned}
$$

Let $k=p(N-2)$ and $\Gamma=\sigma_{N}(N-2)$. For $|x|=r$, we have

$$
\begin{aligned}
& \nabla \widetilde{G}=-\frac{1}{\Gamma^{p}(N-k)}|x|^{-k} x+\nabla \tilde{g}, \quad \nabla G=-\frac{1}{\sigma_{N}}|x|^{-N} x+\nabla g, \\
& \frac{\partial \widetilde{G}}{\partial n}=-\frac{1}{\Gamma^{p}(N-k)}|x|^{1-k}+(\nabla \tilde{g}, n), \quad \frac{\partial G}{\partial n}=-\frac{1}{\sigma_{N}}|x|^{1-N}+(\nabla g, n), \\
& (x, \nabla \widetilde{G})+\frac{N}{q+1} \widetilde{G}=\left(\frac{N}{(q+1)(k-2)}-1\right) \frac{1}{\Gamma^{p}(N-k)}|x|^{2-k}+(x, \nabla \tilde{g})+\frac{N}{q+1} \tilde{g}, \\
& (x, \nabla G)+\frac{N}{p+1} G=\left(\frac{N}{p+1}-(N-2)\right) \frac{1}{\Gamma}|x|^{2-N}+(x, \nabla g)+\frac{N}{p+1} g, \\
& (\nabla \widetilde{G}, \nabla G)=\frac{|x|^{-k-N+2}}{\sigma_{N} \Gamma^{p}(N-k)}-\frac{(\nabla g, x)}{\Gamma^{p}(N-k)}|x|^{(2-N) p}-\frac{(\nabla \tilde{g}, x)}{\sigma_{N}}|x|^{-N}+(\nabla \tilde{g}, \nabla g)
\end{aligned}
$$

and

$$
\frac{1}{p+1} G^{p+1}=\frac{1}{p+1}\left[\frac{1}{\Gamma^{p}}|x|^{-k}-\Delta \tilde{g}\right]\left[\frac{1}{\Gamma}|x|^{2-N}+g\right] .
$$

From here, we check that the terms with $|x|^{3-N-k}$ cancel out, other integrals tend to 0 since the integrands are $\mathrm{o}\left(|x|^{1-N}\right)$, and only one term of order $|x|^{1-N}$ remain, giving

$$
\int_{\partial \Omega} \frac{\partial \widetilde{G}}{\partial n} \frac{\partial G}{\partial n}(x, n) \mathrm{d} s=-\lim _{r \rightarrow 0} \frac{1}{\sigma_{N} r^{N-1}} \int_{\partial B_{r}} \frac{N}{q+1} \tilde{g} \mathrm{~d} s=-\frac{N}{q+1} \tilde{g}(0,0)
$$

Proof of Theorem 1.1. (a) The part (ii) follows from Lemma 5.1,

$$
\left\|\left|\Delta u_{\varepsilon}\right|^{1 / p}\right\|_{C^{1, \alpha}(\omega)} \leqslant\left\|u_{\varepsilon}^{q_{\varepsilon}}\right\|_{L^{1}(\Omega)}+\left\|u_{\varepsilon}^{q_{\varepsilon}}\right\|_{L^{\infty}(\omega)}
$$

and estimates (2.19), (2.22), and (2.23). Part (i) follows from

$$
\left\|u_{\varepsilon}\right\|_{C^{1, \alpha}(\omega)} \leqslant\left\|v_{\varepsilon}^{p}\right\|_{L^{1}(\Omega)}+\left\|v_{\varepsilon}^{p}\right\|_{L^{\infty}(\omega)}
$$

and estimate (2.18). Finally (iii) follows combining (ii) with the convergence

$$
\int_{\mathbb{R}^{N}}\left|\Delta u_{\varepsilon}\right|^{(p+1) / p} \mathrm{~d} x=\int_{\mathbb{R}^{N}} v_{\varepsilon}^{p+1} \mathrm{~d} x \rightarrow\|V\|_{L^{p+1}\left(\mathbb{R}^{N}\right)}^{p+1}
$$

as $\varepsilon \rightarrow 0$. This completes part (a).
For part (b), note that from (2.6), we have the vectorial equality $\int_{\partial \Omega}\left(\nabla u_{\varepsilon}, \nabla v_{\varepsilon}\right) n \mathrm{~d} s=0$. In the limit for $p \geqslant$ $N /(N-2)$, we get

$$
\begin{equation*}
\int_{\partial \Omega}\left(\nabla G\left(x, x_{0}\right), \nabla G\left(x, x_{0}\right)\right) n \mathrm{~d} s=0 \tag{3.11}
\end{equation*}
$$

and similarly for $p<N /(N-2)$, we obtain

$$
\begin{equation*}
\int_{\partial \Omega}\left(\nabla \widetilde{G}\left(x, x_{0}\right), \nabla G\left(x, x_{0}\right)\right) n \mathrm{~d} s=0 \tag{3.12}
\end{equation*}
$$

But we have the following result.
Lemma 3.2. For every $x_{0} \in \Omega$

$$
\begin{equation*}
\int_{\partial \Omega}\left(\nabla G\left(x, x_{0}\right), n\right)\left(\nabla G\left(x, x_{0}\right), n\right) n \mathrm{~d} s=-\nabla \phi\left(x_{0}\right) \tag{3.13}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{\partial \Omega}\left(\nabla \widetilde{G}\left(x, x_{0}\right), n\right)\left(\nabla\left(\Delta \widetilde{G}\left(x, x_{0}\right)\right)^{1 / p}, n\right) n \mathrm{~d} s=-\nabla \tilde{\phi}\left(x_{0}\right) \tag{3.14}
\end{equation*}
$$

Hence combining (3.11) with (3.13), and (3.12) with (3.14), we complete the proof of part (b) and the theorem is proven.

Proof of Lemma 3.2. Equality (3.13) was proved in [3] and [15]. To prove (3.14), by (3.10) we have

$$
\int_{\partial \Omega} \frac{\partial \widetilde{G}}{\partial n} \frac{\partial G}{\partial n} n \mathrm{~d} s=\int_{\partial B_{r}}\left\{\frac{1}{p+1} G^{p+1} n+\frac{\partial \widetilde{G}}{\partial n} \nabla G+\frac{\partial G}{\partial n} \nabla \widetilde{G}-(\nabla \widetilde{G}, \nabla G) n\right\} \mathrm{d} s
$$

Using $\int_{\partial B_{r}} n=0$, we get

$$
\begin{align*}
\int_{\partial \Omega} \frac{\partial \widetilde{G}}{\partial n} \frac{\partial G}{\partial n} n \mathrm{~d} s= & \frac{1}{(p+1) r^{N-1}} \int_{\partial B_{r}}\left\{\frac{1}{\Gamma^{p}} r^{N-k-1} g-\Delta \tilde{g} \frac{1}{\Gamma} r-\Delta \tilde{g} g r^{N-1}\right\} n \mathrm{~d} s \\
& +\frac{1}{r^{N-1}} \int_{\partial B_{r}}\{(\nabla \tilde{g}, n) \nabla g+(\nabla g, n) \nabla \tilde{g}-(\nabla \tilde{g}, \nabla g) n\} r^{N-1} \mathrm{~d} s \\
& -\frac{1}{r^{N-1}} \int_{\partial B_{r}}\left\{\frac{1}{\sigma_{N}} \nabla \tilde{g}+\frac{r^{N-k}}{\Gamma^{p}(N-k)} \nabla g\right\} \mathrm{d} s . \tag{3.15}
\end{align*}
$$

We use the regular $\hat{g}(x, 0)$ instead of $\tilde{g}(x, 0)$. Thus

$$
\begin{align*}
& \nabla \hat{g}(x, 0)=\nabla \tilde{g}(x, 0)+\frac{p g(0,0)}{\Gamma^{p-1}(2 N-k-2)}|x|^{N-k-2} x,  \tag{3.16}\\
& \Delta \hat{g}(x, 0)=\Delta \tilde{g}(x, 0)+\frac{p g(0,0)}{\Gamma^{p-1}|x|^{N-k-2}} . \tag{3.17}
\end{align*}
$$

But $g(x, 0)=g(0,0)+(\nabla g(0,0), x)+\mathrm{o}\left(|x|^{2}\right)$ and

$$
\int_{\partial B_{r}} r^{-k} g(x, 0) n \mathrm{~d} s=\int_{\partial B_{1}} r^{N-k-1} g(0,0) n \mathrm{~d} s+\int_{\partial B_{1}} r^{N-k}(\nabla g(0,0), y) n \mathrm{~d} s+\mathrm{o}\left(r^{N-k+1}\right),
$$

where $y=x / r$. Clearly the first integral in the r.h.s is zero and the other terms tends to zero as $r \rightarrow 0$. Hence

$$
\begin{equation*}
\lim _{r \rightarrow 0} \frac{1}{r^{N-1}} \int_{\partial B_{r}} r^{N-k-1} g(x, 0) n \mathrm{~d} s=0 \tag{3.18}
\end{equation*}
$$

We replace (3.16) and (3.17) in (3.15), to obtain an identity without $\tilde{g}$. Using the limit (3.18) and that $\hat{g}$ and $g$ are regular, we obtain

$$
\int_{\partial \Omega} \frac{\partial \widetilde{G}}{\partial n} \frac{\partial G}{\partial n} n \mathrm{~d} s=\lim _{r \rightarrow 0} \frac{1}{r^{N-1}} \int_{\partial B_{r}} \frac{1}{\sigma_{N}} \nabla \hat{g} \mathrm{~d} s=\nabla \hat{g}(0,0)=\nabla \tilde{\phi}(0)
$$

where the last equality follows by the observation after Theorem 1.1.

## 4. Proof of Lemma 2.3

Let us recall the problem (2.10)-(2.12),

$$
\begin{array}{ll}
-\Delta u_{\varepsilon, \mu}=v_{\varepsilon, \mu}^{p} & \text { in } \Omega_{\varepsilon}, \\
-\Delta v_{\varepsilon, \mu}=u_{\varepsilon, \mu}^{q_{\varepsilon}} & \text { in } \Omega_{\varepsilon}, \\
u_{\varepsilon, \mu}=v_{\varepsilon, \mu}=0 & \text { on } \partial \Omega_{\varepsilon}, \tag{4.3}
\end{array}
$$

where $\Omega_{\varepsilon}=\left(\Omega-x_{\varepsilon}\right) / \mu^{1-\varepsilon / 2}$. Let $\bar{R}>0$. We define $\sigma(p):=2+N-p(N-2)$, and the scalar function

$$
J(|y|):= \begin{cases}1 & \text { if } \sigma(p)<2, \\ |\log (|y| / \bar{R})| & \text { if } \sigma(p)=2, \\ |y|^{2-\sigma(p)} & \text { if } \sigma(p)>2\end{cases}
$$

Note that $\sigma(p) \in[0, N)$ for $p \in(2 /(N-2),(N+2) /(N-2)]$ and $\sigma(q) \leqslant 0$. We consider the transformations

$$
z_{\varepsilon}(y)=|y|^{2-N} v_{\varepsilon, \mu}\left(\frac{y}{|y|^{2}}\right) \quad \text { and } \quad w_{\varepsilon}(y)=\frac{|y|^{2-N}}{J(|y|)} u_{\varepsilon, \mu}\left(\frac{y}{|y|^{2}}\right)
$$

in $\Omega_{\varepsilon}^{*}$, the image of $\Omega_{\varepsilon}$ under $x \mapsto x /|x|^{2}$.
The next lemma is equivalent to Lemma 2.3, using the asymptotic behaviors (1.17).
Lemma 4.1. Let $\left(w_{\varepsilon}, z_{\varepsilon}\right)$ solving

$$
\begin{align*}
& -\Delta J(|y|) w_{\varepsilon}=|y|^{-\sigma(p)} z_{\varepsilon}^{p} \quad \text { in } \Omega_{\varepsilon}^{*}  \tag{4.4}\\
& -\Delta z_{\varepsilon}=|y|^{-\sigma(q)+\left(q_{\varepsilon}-q\right)(N-2)}\left[J(|y|) w_{\varepsilon}\right]^{q_{\varepsilon}} \quad \text { in } \Omega_{\varepsilon}^{*}  \tag{4.5}\\
& w_{\varepsilon}=z_{\varepsilon}=0 \quad \text { on } \partial \Omega_{\varepsilon}^{*} . \tag{4.6}
\end{align*}
$$

Then for any fixed $R \in(0, \bar{R})$, we have

$$
\left\|w_{\varepsilon}\right\|_{L^{\infty}\left(\Omega_{\varepsilon}^{R}\right)}+\left\|z_{\varepsilon}\right\|_{L^{\infty}\left(\Omega_{\varepsilon}^{R}\right)} \leqslant C
$$

where $\Omega_{\varepsilon}^{R}=\Omega_{\varepsilon}^{*} \cap B_{R}$, and $C=C(R)$ independent of $\varepsilon>0$ provided $\varepsilon$ is sufficiently small.
Proof. Given $R>0$, let $w_{0}$ and $z_{0}$ be solutions of

$$
\Delta J(|y|) w_{0}=0 \quad \text { in } \Omega_{\varepsilon}^{R} \quad \text { and } \quad w_{0}=0 \quad \text { on } \partial \Omega_{\varepsilon}^{*}, \quad w_{0}=w_{\varepsilon} \quad \text { on } \partial B_{R},
$$

and

$$
\Delta z_{0}=0 \quad \text { in } \Omega_{\varepsilon}^{R} \quad \text { and } \quad z_{0}=0 \quad \text { on } \partial \Omega_{\varepsilon}^{*}, \quad z_{0}=z_{\varepsilon} \quad \text { on } \partial B_{R} .
$$

By the convergence in compact sets of $w_{\varepsilon}$ and $z_{\varepsilon}$, see (2.13), we have $\left|z_{\varepsilon}\right|+\left|\nabla z_{\varepsilon}\right|+\left|w_{\varepsilon}\right|+\left|\nabla w_{\varepsilon}\right| \leqslant C$ in $|y|=R$ for $C$ independent of $\varepsilon$. Therefore by the maximum principle, we get

$$
\left|J w_{0}\right|+\left|\nabla\left(J w_{0}\right)\right|+\left|z_{0}\right|+\left|\nabla z_{0}\right| \leqslant C \quad \text { in } \Omega_{\varepsilon}^{R} .
$$

Define $\tilde{w}=w_{\varepsilon}-w_{0}$ and $\tilde{z}=z_{\varepsilon}-z_{0}$. We now write

$$
\begin{array}{ll}
-\Delta J(|y|) \tilde{w}=a(y) z_{\varepsilon} & \text { in } \Omega_{\varepsilon}^{R}, \\
-\Delta \tilde{z}=b(y) J(|y|) w_{\varepsilon} & \text { in } \Omega_{\varepsilon}^{R}, \\
\tilde{w}=\tilde{z}=0 \quad \text { on } \partial \Omega_{\varepsilon}^{R}, & \tag{4.9}
\end{array}
$$

where $a(y)=|y|^{-\sigma(p)} z_{\varepsilon}^{p-1}$ and $b(y)=|y|^{-\sigma(q)+\left(q_{\varepsilon}-q\right)(N-2)}\left[J(|y|) w_{\varepsilon}\right]^{q_{\varepsilon}-1}$. Clearly by the maximum principle $\tilde{w} \geqslant 0$ and $\tilde{z} \geqslant 0$.

Let $P(y)=a(y)$ and

$$
Q(y)= \begin{cases}\frac{1}{M} b(y) & \text { for } y \in B_{R} \backslash \bar{B}_{r} \\ b(y) & \text { for } B_{r}\end{cases}
$$

where $r \in(0, R)$ and $M>1$ both independent of $\varepsilon$ and to be determined later. Then

$$
b(y) J(|y|) w_{\varepsilon}=Q(y) J(|y|) w_{\varepsilon}+f(y),
$$

where

$$
f(y)=(b(y)-Q(y)) J(|y|) w_{\varepsilon}= \begin{cases}0 & \text { for } y \in \Omega_{\varepsilon} \cap B_{r}, \\ \left(1-\frac{1}{M}\right) b(y) J(|y|) w_{\varepsilon} & \text { for } y \in B_{R} \backslash \bar{B}_{r} .\end{cases}
$$

It is clear that $f \in L^{\infty}\left(\Omega_{\varepsilon}^{R}\right)$, in fact $\|f\|_{L^{\infty}\left(\Omega_{\varepsilon}^{R}\right)} \leqslant(1-1 / M) r^{-(2+N)}$ by using that $w_{\varepsilon}(y) \leqslant C r^{\sigma(p)-N}$ for $|y| \geqslant r$, when $p<N /(N-2)$, and $w_{\varepsilon}(y) \leqslant C r^{2-N}$ for $|y| \geqslant r$ when $p>N /(N-2)$. A similar bound is obtained for $p=N /(N-2)$. Then we transform (4.7)-(4.8) in the system

$$
-\Delta J \tilde{w}=P z_{\varepsilon} \quad \text { in } \Omega_{\varepsilon}^{R}, \quad-\Delta \tilde{z}=Q J w_{\varepsilon}+f \quad \text { in } \Omega_{\varepsilon}^{R} .
$$

We define $\eta_{2}(y)=\chi_{w_{\varepsilon} \leqslant 2 \tilde{w}}(y)$ and $\eta_{1}(y)=\chi_{z_{\varepsilon} \leqslant 2 \tilde{z}}(y)$ for $y \in \Omega_{\varepsilon}^{R}$, we find

$$
-\Delta J \tilde{w} \leqslant 2 \eta_{1} P \tilde{z}+f_{1} \quad \text { in } \Omega_{\varepsilon}^{R}, \quad-\Delta \tilde{z} \leqslant 2 \eta_{2} Q J \tilde{w}+f_{2} \quad \text { in } \Omega_{\varepsilon}^{R} .
$$

Here $f_{1}=\left(1-\eta_{1}\right) P z_{\varepsilon}=\chi_{z_{\varepsilon} \leqslant 2 z_{0}} P z_{\varepsilon} \leqslant 2 P z_{0}$ and $f_{2}=f+\left(1-\eta_{2}\right) Q J w_{\varepsilon}$ where $\left(1-\eta_{2}\right) Q J w_{\varepsilon} \leqslant 2 Q J w_{0}$. We write the system in the form

$$
\begin{align*}
& -\Delta J \tilde{w} \leqslant 2 \eta_{1} P|y|^{\gamma}|y|^{-\gamma} \tilde{z}+f_{1} \quad \text { in } \Omega_{\varepsilon}^{R},  \tag{4.10}\\
& -|y|^{-\gamma} \Delta \tilde{z} \leqslant 2 \eta_{2} Q|y|^{\gamma} J \tilde{w}+f_{2}|y|^{-\gamma} \quad \text { in } \Omega_{\varepsilon}^{R},  \tag{4.11}\\
& \tilde{w}=\tilde{z}=0 \quad \text { on } \partial \Omega_{\varepsilon}^{R} . \tag{4.12}
\end{align*}
$$

Let $u(y) \mapsto 2 \eta_{2} Q|y|^{-\gamma} u(y)$ and $u(y) \mapsto 2 \eta_{1} P|y|^{\gamma} u(y)$ be the multiplication operators $\mathcal{P}$ and $\mathcal{Q}$ respectively. Note that a multiplication operator $\mathcal{C}$ with corresponding function $c(y) \in L^{s}\left(\Omega_{\varepsilon}^{R}\right)$ is bounded from $L^{s_{1}}\left(\Omega_{\varepsilon}^{R}\right)$ to $L^{s_{2}}\left(\Omega_{\varepsilon}^{R}\right)$ with $1 / s_{2}=1 / s_{1}+1 / s$.

Formally we define $-L$ as the operator $u(y) \mapsto-|y|^{-\gamma} \Delta\left(|y|^{\gamma} u(y)\right)$. More precisely, in the appendix, we define $(-\Delta)^{-1}$ and $(-L)^{-1}$, which by the Hardy-Littlewood-Sobolev inequality are bounded, independently of $\varepsilon$, from $L^{m_{1}}\left(\Omega_{\varepsilon}^{R}\right)$ to $L^{m_{2}}\left(\Omega_{\varepsilon}^{R}\right)$ with $1 / m_{1}=1 / m_{2}+2 / N$. Note that the image of these operators is a function with zeroDirichlet boundary condition, so they are positive. Then we can write

$$
J \tilde{w} \leqslant(-\Delta)^{-1} \mathcal{P}(-L)^{-1}\left(\mathcal{Q}(J \tilde{w})+|y|^{-\gamma} f_{2}\right)+(-\Delta)^{-1} f_{1} .
$$

Denoting by $K=(-\Delta)^{-1} \mathcal{P}(-L)^{-1} \mathcal{Q}$ and $h=K|y|^{-\gamma} f_{2}+(-\Delta)^{-1} f_{1}$ we have

$$
(I-K) J \tilde{w} \leqslant h
$$

The proof is complete finding $m$ large enough such that $h \in L^{m}\left(\Omega_{\varepsilon}^{R}\right)$ and $(I-K)$ is invertible from $L^{m}\left(\Omega_{\varepsilon}^{R}\right)$ to $L^{m}\left(\Omega_{\varepsilon}^{R}\right)$.

We can estimate $Q(y)|y|^{-\gamma}$ in $L^{q+1}\left(\Omega_{\varepsilon}^{R}\right)$, for $\gamma=2 \sigma(p) /(p+1) \geqslant 0$, and note that $\gamma=-\sigma(q) /(q+1)$ using the Sobolev Hyperbola. Since $v_{\varepsilon, \mu} \rightarrow V$ in $L^{q+1}\left(\mathbb{R}^{N}\right)$, we have

$$
\int_{\Omega_{\varepsilon}^{*}}\left[J(|y|) w_{\varepsilon}(y)-V\left(y /|y|^{2}\right)|y|^{2-N}\right]^{q+1}|y|^{-\sigma(q)} \mathrm{d} y \rightarrow 0 \quad \text { as } \varepsilon \rightarrow 0 .
$$

Therefore for any $\lambda$, we can take $r$ small such that

$$
\int_{\Omega_{\varepsilon}^{r}}\left[J w_{\varepsilon}\right]^{(q+1) \frac{q_{\varepsilon}-1}{q-1}}(y)|y|^{-\sigma(q)} \mathrm{d} y \leqslant \int_{\Omega_{\varepsilon}^{r}}\left[J w_{\varepsilon}\right]^{(q+1)}(y)|y|^{-\sigma(q)} \mathrm{d} y \leqslant \frac{\lambda}{2 C(\delta)}
$$

and $M$ large such that for all $\varepsilon \leqslant \varepsilon_{0}$ we have

$$
\begin{align*}
\int_{\Omega_{\varepsilon}^{R}}\left[Q(s)|y|^{-\gamma}\right]^{\frac{q+1}{q-1}} \mathrm{~d} y & \leqslant C(\delta) \int_{\Omega_{\varepsilon}^{r}}\left[J w_{\varepsilon}\right]^{(q+1) \frac{q_{\varepsilon}-1}{q-1}}|y|^{-\sigma(q)} \mathrm{d} y+\frac{C(\delta)}{M^{\frac{q+1}{q-1}} \int_{B_{R} \backslash B_{r}}\left[J w_{\varepsilon}\right]^{(q+1)^{\frac{q_{\varepsilon}-1}{q-1}}|y|^{-\sigma(q)} \mathrm{d} y}} \underset{ }{ } \leqslant \lambda,
\end{align*}
$$

where we have used $b(y) \leqslant C(\delta)\left[J w_{\varepsilon}\right]^{q_{\varepsilon}-1}$ with $\delta$ given by Lemma 2.2.
Now we show that $K$ is bounded from $L^{m}\left(\Omega_{\varepsilon}^{R}\right)$ to $L^{m}\left(\Omega_{\varepsilon}^{R}\right)$.

$$
\begin{aligned}
\|K J \tilde{w}\|_{L^{m}\left(\Omega_{\varepsilon}^{R}\right)} & \leqslant C_{1}\left\|\mathcal{P}(-L)^{-1} \mathcal{Q} J \tilde{w}\right\|_{L^{r}\left(\Omega_{\varepsilon}^{R}\right)} \\
& \leqslant C_{1}\left\||y|^{\gamma} 2 \eta_{1} P\right\|_{L^{\frac{p+1}{p-1}\left(\Omega_{\varepsilon}^{R}\right)}}\left\|(-L)^{-1} \mathcal{Q} J \tilde{w}\right\|_{L^{r^{\prime}}\left(\Omega_{\varepsilon}^{R}\right)} \\
& \leqslant C_{1}\left\||y|^{\gamma} 2 \eta_{1} P\right\|_{L^{\frac{p+1}{p-1}\left(\Omega_{\varepsilon}^{R}\right)}} C_{2}\|\mathcal{Q} J \tilde{w}\|_{L^{s^{\prime}}\left(\Omega_{\varepsilon}^{R}\right)} \\
& \leqslant C_{1} C_{2}\left\||y|^{\gamma} 2 \eta_{1} P\right\|_{L^{\frac{p+1}{p+1}\left(\Omega_{\varepsilon}^{R}\right)}}\left\||y|^{-\gamma} 2 \eta_{2} Q\right\|_{L^{\frac{q+1}{q-1}\left(\Omega_{\varepsilon}^{R}\right)}}\|J \tilde{w}\|_{L^{m^{\prime}}\left(\Omega_{\varepsilon}^{R}\right)} \\
& \leqslant \bar{C}\left\||y|^{\gamma} P\right\|_{L^{\frac{p+1}{p-1}\left(\Omega_{\varepsilon}^{R}\right)}}\left\||y|^{-\gamma} Q\right\|_{L^{q+1}\left(\Omega_{\varepsilon}^{R}\right)}\|J \tilde{w}\|_{L^{m^{\prime}}\left(\Omega_{\varepsilon}^{R}\right)}
\end{aligned}
$$

with $1 / r=1 / m+2 / N$, so $r^{\prime}>1$ implies $m>N /(N-2) .1 / r=(p-1) /(p+1)+1 / r^{\prime}$ and $1 / s^{\prime}=1 / r^{\prime}+2 / N$, so condition (b) in (5.1) implies $N-2+N / m>2 N /(p+1)$ and $s^{\prime}>1$ implies $m>(q+1) / 2$ so (a) in (5.1) holds since $\gamma>0$ and $1 / s^{\prime}=(q-1) /(q+1)+1 / m^{\prime}$. Since

$$
\frac{q-1}{q+1}+\frac{p-1}{p+1}=\frac{4}{N}, \quad \text { we have } m^{\prime}=m .
$$

By

$$
\int_{\Omega_{\varepsilon}^{*}}\left[z_{\varepsilon}(y)-U\left(y /|y|^{2}\right)|y|^{2-N}\right]^{p+1}|y|^{-\sigma(p)} \mathrm{d} y \rightarrow 0 \quad \text { as } \varepsilon \rightarrow 0,
$$

we deduce that $\left\||y|^{\gamma-\sigma(p)} z_{\varepsilon}^{p-1}\right\|_{L^{\frac{p+1}{p-1}\left(\Omega_{\varepsilon}^{R}\right)}}=\left\||y|^{\gamma} P\right\|_{L^{p+1}\left(\Omega_{\varepsilon}^{R}\right)} \leqslant C\left(\varepsilon_{0}\right)$ with $C\left(\varepsilon_{0}\right)>0$ and for all $\varepsilon \in\left(0, \varepsilon_{0}\right)$. Since $\lambda$ in (4.13) can be arbitrarily small then the norm of $K$ is small and so $I-K: L^{m}\left(\Omega_{\varepsilon}^{R}\right) \rightarrow L^{m}\left(\Omega_{\varepsilon}^{R}\right)$ invertible for $m$ large. We have that

$$
\left\||y|^{-\gamma} f_{2}\right\|_{L^{m}\left(\Omega_{\varepsilon}^{R}\right)} \leqslant r^{-\gamma}\left\|f_{2}\right\|_{L^{\infty}\left(\Omega_{r}^{R}\right)}\left(\operatorname{meas}\left(\Omega_{r}^{R}\right)\right)^{1 / m}
$$

is bounded, since $f_{2}$ is zero outside $\Omega_{r}^{R}$ and

$$
\left\|\Delta^{-1} f_{1}\right\|_{L^{m}\left(\Omega_{\varepsilon}^{R}\right)} \leqslant C_{1}\left\|f_{2}\right\|_{L^{r}\left(\Omega_{\varepsilon}^{R}\right)} \leqslant\left\|f_{1}\right\|_{L^{\infty}\left(\Omega_{r}^{R}\right)}\left(\operatorname{meas}\left(\Omega_{\mu}^{R}\right)\right)^{1 / r} \leqslant C\left(z_{0}\right)\left(\operatorname{meas}\left(B_{R}\right)\right)^{1 / r} .
$$

This implies $\|J w\|_{L^{m}\left(\Omega_{\varepsilon}^{R}\right)} \leqslant M$ for every $m$ large, and consequently for every $m \geqslant 1$. (Use the $w_{0}$ to get that $\left\|J w_{\varepsilon}\right\|_{L^{m}\left(\Omega_{\varepsilon}^{R}\right)} \leqslant M$.) Now we have that

$$
-\Delta \tilde{z}=b(y) J w_{\varepsilon}=|y|^{-\sigma(q)+\left(q-q_{\varepsilon}\right)(N-2)}\left[J w_{\varepsilon}\right]^{q_{\varepsilon}} .
$$

Since $\sigma(q) \leqslant 0$, if we take $m$ large such that $m q_{\varepsilon}>N / 2$ then

$$
\begin{equation*}
\|\tilde{z}\|_{L^{\infty}\left(\Omega_{\varepsilon}^{R}\right)} \leqslant \tilde{M} \quad \text { and therefore } \quad\left\|z_{\varepsilon}\right\|_{L^{\infty}\left(\Omega_{\varepsilon}^{R}\right)} \leqslant M \tag{4.14}
\end{equation*}
$$

for some $M$ independent of $\varepsilon$. We study now each case of $J$ separately. We have

$$
\begin{equation*}
-\Delta J w_{\varepsilon}=|y|^{-\sigma(p)} z_{\varepsilon}^{p} \quad \text { in } \Omega_{\varepsilon}^{*} . \tag{4.15}
\end{equation*}
$$

(a) In the case $J=1$, since $\sigma(p)<2$, using (4.14), we have $-\Delta \tilde{w}_{\varepsilon} \in L^{q}(\Omega)$ for any $q \in(N / 2, N / \sigma(p))$. By regularity, we get

$$
\left\|w_{\varepsilon}\right\|_{L^{\infty}\left(\Omega_{\varepsilon}^{R}\right)} \leqslant M
$$

(b) For $J(|y|)=-\log (|y| / \bar{R})>\log (\bar{R} / R)$, we have

$$
-\Delta \tilde{w}-\frac{\nabla J}{J} \nabla \tilde{w}-\frac{\Delta J}{J} \tilde{w}=\frac{1}{J|y|^{2}} z_{\varepsilon}^{p} \quad \text { in } \Omega_{\varepsilon}^{R}
$$

or equivalently

$$
-\Delta \tilde{w}+\frac{1}{J|y|^{2}}(y, \nabla \tilde{w})+\frac{1}{J|y|^{2}}(N-2) \tilde{w}=\frac{1}{J|y|^{2}} z_{\varepsilon}^{p} \quad \text { in } \Omega_{\varepsilon}^{R} \text {. }
$$

Using (4.14), we can take $u=\tilde{w}-M$ with $M=\sup _{\varepsilon>0} \sup _{y \in \Omega_{\varepsilon}^{R}} z_{\varepsilon}^{p}(y) /(N-2)$, and we get

$$
-J|y|^{2} \Delta u+(y, \nabla u)+(N-2) u \leqslant 0 \quad \text { in } \Omega_{\varepsilon}^{R} .
$$

Since $u=-M<0$ on the boundary, $u \leqslant 0$ in $\Omega_{\varepsilon}^{R}$. This gives $w_{\varepsilon} \leqslant M$ in $\Omega_{\varepsilon}^{R}$.
For the remaining case $p<N /(N-2)$, we have

$$
-\Delta \tilde{w}-\frac{\nabla J}{J} \nabla \tilde{w}-\frac{\Delta J}{J} \tilde{w}=\frac{1}{|y|^{2}} z_{\varepsilon}^{p} \quad \text { in } \Omega_{\varepsilon}^{R}
$$

As before, defining $u=\tilde{w}-M$ with $M=\sup _{\varepsilon>0} \sup _{y \in \Omega_{\varepsilon}^{R}} z_{\varepsilon}^{p} /[(\sigma(p)-2)(N-\sigma(p))]$ then

$$
-|y|^{2} \Delta u-(2-\sigma(p))(y, \nabla u)-(2-\sigma(p))(N-\sigma(p)) u \leqslant 0 \quad \text { in } \Omega_{\varepsilon}^{R} .
$$

Since $u=-M<0$ on the boundary, $u \leqslant 0$ in $\Omega_{\varepsilon}^{R}$. This implies $w_{\varepsilon} \leqslant M$ in $\Omega_{\varepsilon}^{R}$.

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## Appendix A

Let $N>2$. Let $h$ and $v$ be functions in $L^{s^{\prime}}\left(\Omega_{\varepsilon}^{R}\right)$. Given the Green's function $G$ solution of $-\Delta G(x, \cdot)=\delta_{x}$ in $\Omega_{\varepsilon}^{R}$, $G(x, \cdot)=0$ on $\partial \Omega_{\varepsilon}^{R}$, we define

$$
(-\Delta)^{-1} h(\xi)=\int_{\Omega_{\varepsilon}^{R}} G(x, \xi) h(x) \mathrm{d} x, \quad \xi \in \Omega_{\varepsilon}^{R}
$$

and

$$
(-L)^{-1} v(\xi)=|\xi|^{-\gamma} \int_{\Omega_{\varepsilon}^{R}} G(x, \xi)|x|^{\gamma} v(x) \mathrm{d} x, \quad \xi \in \Omega_{\varepsilon}^{R}
$$

Note that $G$ is positive, so both operators are positive. We know that $(-\Delta)^{-1}$ is bounded, independently of $\varepsilon$, from $L^{s^{\prime}}\left(\Omega_{\varepsilon}^{R}\right)$ to $L^{r^{\prime}}\left(\Omega_{\varepsilon}^{R}\right)$ with $1 / r^{\prime}=1 / s^{\prime}-2 / N$. Next we prove the same result for $(-L)^{-1}$. By the weighted Hardy-Littlewood-Sobolev inequality [5,18], for $|\xi|^{-\gamma} f \in L^{s^{\prime}}\left(\Omega_{\varepsilon}^{R}\right)$, we have that

$$
\left\|\xi^{-\gamma}(-\Delta)^{-1} f\right\|_{L^{r^{\prime}\left(\Omega_{\varepsilon}^{R}\right)}} \leqslant 2\left\||\xi|^{-\gamma} \int_{\Omega_{\varepsilon}^{R}} \frac{C}{|x-\xi|^{N-2}} f(x) \mathrm{d} x\right\|_{L^{r^{\prime}\left(\Omega_{\varepsilon}^{R}\right)}} \leqslant C\left\||\xi|^{-\gamma} f\right\|_{L^{s^{\prime}}\left(\Omega_{\varepsilon}^{R}\right)}
$$

for $1<s^{\prime}<r^{\prime}<\infty$, with $1 / r^{\prime}=1 / s^{\prime}-2 / N$ and

$$
\begin{equation*}
\text { (a) }-\gamma<N\left(1-1 / s^{\prime}\right)=N-2-N / r^{\prime} \quad \text { and } \quad \text { (b) } \gamma<N / r^{\prime} \tag{5.1}
\end{equation*}
$$

In other words, for any $v \in L^{s^{\prime}}\left(\Omega_{\varepsilon}^{R}\right)$, we have

$$
\begin{align*}
\left\|(-L)^{-1} v\right\|_{L^{r^{\prime}\left(\Omega_{\varepsilon}^{R}\right)}} & =\left\||\xi|^{-\gamma}(-\Delta)^{-1}|x|^{\gamma} v\right\|_{L^{r^{\prime}}\left(\Omega_{\varepsilon}^{R}\right)} \\
& \leqslant 2\left\||\xi|^{-\gamma} \int_{\Omega_{\varepsilon}^{R}} \frac{C}{|x-\xi|^{N-2}}|x|^{\gamma} v(x) \mathrm{d} x\right\|_{L^{r^{\prime}}\left(\Omega_{\varepsilon}^{R}\right)} \\
& \leqslant C\|v\|_{L^{s^{\prime}}\left(\Omega_{\varepsilon}^{R}\right)} \tag{5.2}
\end{align*}
$$

## Lemma 5.1. Let u solve

$$
\begin{cases}-\Delta u=f & \text { in } \Omega \subset \mathbb{R}^{N} \\ u=0 & \text { on } \partial \Omega\end{cases}
$$

## Let $\omega$ be a neighborhood of $\partial \Omega$. Then

$$
\|u\|_{W^{1, q}(\Omega)}+\|\nabla u\|_{C^{0, \alpha}\left(\omega^{\prime}\right)} \leqslant C\left(\|f\|_{L^{1}(\Omega)}+\|f\|_{L^{\infty}(\omega)}\right)
$$

for $q<N /(N-1), \alpha \in(0,1)$ and $\omega^{\prime} \subset \omega$ is a strict subdomain of $\omega$.

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