Available online at www.sciencedirect.com
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Ann. I. H. Poincaré – AN 24 (2007) 341–359
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

An evolutionary double-well problem

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Received 24 February 2002; accepted 24 April 2002

Available online 20 December 2006

Abstract

We establish the existence theorem and study the long time behaviour of the following PDE problem:

$$\begin{cases} u_t - \operatorname{div} \nabla W(\nabla u) - f(x) = 0 \quad \text{in } \Omega \times (0, -\infty), \\ \nabla W(\nabla u) \cdot \mathbf{n}|_{\partial\Omega \times (0,\infty)} = 0, \\ u(x,0) = u_0(x) \quad \text{in } \Omega \end{cases} \tag{0.1}$$

where W is a specially given quasiconvex double-well function and $f \in L^2(\Omega)$ is a given function independent of time t. In particular, the existence theorem is established for general given source term f, the long time behaviour is analyzed under the assumption that $\int_\Omega f(x)\,\mathrm{d}x = 0$.

The system is an evolutionary quasimonotone system. We believe that the existence of solutions established here is stronger than the usual Young Measure solution and is the first of its kind. The existence of a compact ω-limit set as $t \to \infty$ is also established under some non-restrictive conditions.

MSC: 35K55; 49J45

Keywords: Quasiconvexity; Long time behaviour; Semi-flow

1. Introduction

In [8], it was established that for a general parabolic system, generated as the gradient flow of a rank-one convex energy functional, the energy at any time is less than or equal to the energy 'smoothly sampled' at earlier times. In this paper, we give a concrete example to this result. Furthermore, for the given example, despite lack of higher regularity, we obtain the long time convergence of solutions. Before we start our discussion, we mention the recent interesting papers of [18] and [7]. In [18,3], the existence of Young measure solutions for non-convex elasto-dynamics was discussed. In [7], the existence of weak but non-Young measure solutions for the non-linear flow equation of the type we dealt with was discussed. The setting of [7] is abstract with strong assumptions on the quasimonotonicity of the energy potential. In our setting, we look at a concrete quasiconvex function that has double-well potential and not necessarily strongly quasimonotone. Related problems have been discussed in [13], [14] and [17]. The distinct feature

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doi:10.1016/j.anihpc.2006.11.002

of our paper is that our potential energy function is specifically given, with clear double-well structure [5,12] and is not necessarily strongly quasimonotone, we can obtain the existence of strong solution that has long time convergence properties. We believe that this is the first result of its kind.

It is worth pointing out that if we replace the Neumann boundary condition by Dirichlet boundary condition in (0.1), then the corresponding problem becomes strongly monotone and the analysis can be carried out along the lines of standard non-linear parabolic theory.

We conclude that the evolution of micro-structure in the particular setting of our problem settles down to the steady state over the long time period. However, it is not known if the evolutionary solution will settle down to the energy minimizers. In a separate paper (cf. [21]), we establish some examples illustrating that under some special boundary conditions, the solution of the heat-flow problem could converge to solutions at different energy levels and provide a general picture for analyzing these problems.

In [24], the steady state problem of (0.1) has been studied

$$\begin{cases} \operatorname{div} \nabla W_\lambda(\nabla u) + f(x) = 0 & \text{in } \Omega, \\ \nabla W_\lambda(\nabla u) \cdot \mathbf{n}|_{\partial\Omega} = 0 \end{cases} \tag{1.1}$$

where f is some given smooth function on Ω, Ω is a bounded open smooth subset of $\mathbb{R}^n$, u is a mapping from Ω into $\mathbb{R}^N$, N and n are any positive integers > 1, $W_\lambda : M^{N\times n} \to \mathbb{R}$ is a non-negative, quasiconvex double-well function vanishing at two matrix points. In this paper, we consider the corresponding evolutionary system (0.1).

In Section 2, we give some preliminary results and introduce our double-well model and give the detailed properties established in [24]. In Section 3, we present preliminary results and introduce the weak formulation using finite difference. In Section 4, we establish a priori estimates independent of discretization. In Section 5, we discuss how to obtain convergence using insufficient a priori estimates, the existence of solutions is established using the given $f \in L^2(\Omega)$. In Section 6, we use Galerkin method to briefly discuss the existence of solutions using f with constraint $\int_\Omega f(x)\,\mathrm{d}x = 0$, subsequently, we establish the foundation for discussing the long time behaviour of solutions [11,4] and established the convergence results. In Section 7, we look at a precise result on how the flow evolves as $t \to \infty$ in some specific circumstances.

2. Preliminaries and the model

In this section, we describe the energy density W_λ in (1.1) which we will use throughout this paper.

Definition 2.1. Let $W : M^{N\times n} \to \mathbb{R}$ be a continuous function. The following are some conditions related to weak lower semicontinuity of the integral $I(u) = \int_\Omega W(Du(x))\,\mathrm{d}x$:

(i) W is *rank-one convex* if for any $N \times n$ matrix P, any rank one matrix $B = a \otimes b = (a_i b_j)$ and any real number $s \in [0, 1]$, we have

$$W(P + sB) \leqslant sW(P + a \otimes b) + (1 - s)W(P).$$

(ii) W is *quasiconvex* at the constant matrix $P \in M^{N\times n}$ if for a given, non-empty bounded open set $\Omega \subset R^n$ and every $\phi \in W_0^{1,\infty}(\Omega, R^n)$, $\int_\Omega W(P + \nabla\phi)\,\mathrm{d}x \geqslant W(P)\,\mathrm{meas}(\Omega)$. W is quasiconvex if it is quasiconvex at every $A \in M^{N\times n}$. The class of quasiconvex functions is independent of the choice of Ω.

It is well known that (ii) implies (i), while (i) does not imply (ii) (cf. [2,16,6,19]).

For quadratic forms on $M^{N\times n}$, the two definitions (i) and (ii) in Definition 2.1 are the same.

For a continuous function $W : M^{N\times n} \to \mathbb{R}$ that is bounded from below, we may define its quasiconvex envelop (cf. [6]) as the largest quasiconvex function less than or equal to W. More precisely, $QW = \sup\{G \leqslant W,\ G \text{ quasiconvex}\}$. Similarly, the convex envelop of f is $CW = \sup\{G \leqslant W,\ G \text{ convex}\}$.

The following is the ellipticity condition which was first introduced in [22] to prove the existence of weak solution for elliptic systems [10], and in [9] for the partial regularity property for weak solutions. From now on, we use the summation convention for repeated indices with I from 1 to N and μ from 1 to n.

Definition 2.2. A continuous mapping $B : M^{N\times n} \to M^{N\times n}$ is strongly quasimonotone if for every constant matrix $P \in M^{N\times n}$, every bounded open set $\Omega \subset \mathbb{R}^n$ and every $\phi \in W_0^{1,\infty}(\Omega, \mathbb{R}^N)$,

$$\int_\Omega B^i_\alpha(P + \nabla\phi) D_\alpha\phi^i(x)\,\mathrm{d}x \geqslant c_0 \int_\Omega |\nabla\phi|^2\,\mathrm{d}x$$

where $c_0 > 0$ is a constant independent of P, Ω and ϕ.

If under above notation assumptions, we have only

$$\int_\Omega B(P + \nabla\phi) : \nabla\phi\,\mathrm{d}x \geqslant 0$$

then the mapping B is called quasimonotone.

When $B(P) = \nabla W(P)$ for some scalar valued W, if $B(P)$ is strongly quasimonotone (respectively, quasimonotone), we call ∇W a strongly quasimonotone (respectively, quasimonotone) gradient mapping.

It is easy to check that convex functions and rank-one convex quadratic forms give gradient quasimonotone mappings.

Now we introduce our model double-well integrands W_λ and show that $W_\lambda = G_\lambda + H_\lambda$ where G_λ is convex and H_λ is a rank-one convex quadratic form, which implies that ∇W_λ is quasimonotone.

Let $A \in M^{N\times n}$ be a given matrix with $\operatorname{rank}(A) > 1$, $|A| = 1$. Let $E = \operatorname{span}[A]$ and $K = \{-A, A\}$, C_E denote the convexification operation on the one-dimensional space E, P_E be the standard Euclidean projection operator onto E such that for any given arbitrary $N \times n$ matrix X: $P_E(X) = (X \cdot A)A = (\sum_{i,j} X_{ij}A_{ij})A$. Sometimes, where the circumstance is clear, we may also use the notation $P_E(X)$ for the scalar quantity $X \cdot A$. Subsequently, we have $P_{E^\perp}(X) = X - P_E(X)$. Let λ_0 be the largest eigenvalue of $A^{\mathrm{T}}A$. From $|A| = 1$, we know that $0 < \lambda_0 \leqslant 1$. We define

$$\begin{aligned} &W_\lambda(X) = C_E\bigl(\operatorname{dist}^2\bigl(P_E(X), K\bigr) + \lambda\bigl|P_E(X)\bigr|^2\bigr) + H_\lambda(X) = G_\lambda(X) + H_\lambda(X), \\ &H_\lambda(X) = \bigl|P_{E^\perp}(X)\bigr|^2 - \lambda\bigl|P_E(X)\bigr|^2 \end{aligned} \tag{2.1}$$

where

$$0 \leqslant \lambda \leqslant \lambda^* = \frac{1 - \lambda_0}{\lambda_0}. \tag{2.2}$$

It is easy to check that under our assumptions, for each fixed λ satisfying $0 \leqslant \lambda \leqslant \lambda^*$, the function $H_\lambda(X)$ defined on $M^{N\times n}$ is a rank-one convex quadratic form, so is quasiconvex. Furthermore, $\nabla H_\lambda(X)$ is strongly quasimonotone when $0 \leqslant \lambda < \lambda^*$.

If $f(t)$ is a continuous function on the real line, we denote by $Cf(t)$ the convexification of f. In the case of $G(P_E(X))$, it is a continuous function defined on E. Let $g(t) = G(tA)$, $t \in \mathbb{R}$. We define the convexification of $G(P_E(X))$ on E as

$$C_E G\bigl(P_E(X)\bigr) = Cg(X \cdot A).$$

It can be verified that

$$G_\lambda(X) = G_\lambda\bigl(P_E(X)\bigr) = f_\lambda(X \cdot A)$$

where

$$f_\lambda(t) = \begin{cases} (t+1)^2 + \lambda t^2, & t \leqslant -\dfrac{1}{1+\lambda}, \\ \dfrac{\lambda}{1+\lambda}, & |t| \leqslant \dfrac{1}{1+\lambda}, \\ (t-1)^2 + \lambda t^2, & t \geqslant \dfrac{1}{1+\lambda} \end{cases}$$

for $0 < \lambda \leqslant \lambda^*$. Since f is an explicit function, we can easily verify that there is a constant $C > 0$ such that for all $0 < \lambda \leqslant \lambda^*$ and $t, s \in \mathbb{R}$,

$$\bigl(f'_\lambda(t) - f'_\lambda(s)\bigr)^2 \leqslant C\bigl(f'_\lambda(t) - f'_\lambda(s)\bigr)(t - s). \tag{2.3}$$

The function W_λ defined below are our model integrands (cf. [24]).

Remark 2.1. Our discussion also works for matrices A that $|A| \neq 1$. In this case, we need to introduce $A_0 = A/|A|$ and the corresponding projection operators will need to be modified. To avoid unnecessary complication of notation, we use A with unit modulus.

Proposition 2.1. *Let W_λ be as in* (2.1) *with $0 \leqslant \lambda \leqslant \lambda^*$. Then*

(1) *$W_\lambda(X)$ is quasiconvex. Furthermore, when $\lambda = \lambda^*$, $W_\lambda(X) = Q\operatorname{dist}^2(X, K)$ and when $\lambda = 0$, $W_\lambda(X) = C\operatorname{dist}^2(X, K)$ for all $X \in M^{N\times n}$.*
(2) *For $0 \leqslant \lambda \leqslant \tau \leqslant \lambda^*$, $W_\lambda(X) \leqslant W_\tau(X)$.*
(3) *For $0 < \lambda \leqslant \lambda^*$,*
$$\frac{\lambda}{1+\lambda}\operatorname{dist}^2(X, K) \leqslant W_\lambda(X) \leqslant \operatorname{dist}^2(X, K)$$
for all $X \in M^{N\times n}$.
(4) *For $0 < \lambda < \lambda^*$, $\nabla W_\lambda(X)$ is strongly quasimonotone, while $\nabla W_{\lambda^*}(X)$ is quasimonotone.*
(5) *For $0 < \lambda \leqslant \lambda^*$, we have $W_\lambda(X) = 0$ if and only if $X = A$ or $X = -A$. Furthermore, when $\operatorname{dist}(X, K) < \lambda/(1+\lambda)$, $W_\lambda(X) = \operatorname{dist}^2(X, K)$.*
(6) *There are positive constants c_0, C_0 and c_1 depending on $|A|$ and λ such that*
$$c_0\big(|X|^2 - 1\big) \leqslant W_\lambda(X) \leqslant C_0\big(|X|^2 + 1\big), \qquad \nabla W_\lambda(X) : X \geqslant c_1\big(|X|^2 - 1\big)$$
for $0 \leqslant \lambda \leqslant \lambda^*$.
(7) *$\nabla W_\lambda(X)$ is a global Lipschitz function*:
$$\big|\nabla W_\lambda(X) - \nabla W_\lambda(Y)\big| \leqslant C_0|X - Y|$$
for all $X, Y \in M^{N\times n}$.

Let $I \in M^{2\times 2}$ be the identity matrix,
$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \in M^{2\times 2} \text{ be an arbitrarily given matrix.}$$
An example of W_λ for $K = \{-\frac{1}{\sqrt{2}}I, \frac{1}{\sqrt{2}}I\}$ is
$$W_\lambda(X) = f_\lambda\left(\frac{1}{\sqrt{2}}(X_{11} + X_{22})\right) + H_\lambda(X)$$
with $0 < \lambda \leqslant \sqrt{2} - 1 = \lambda^*$. Note that in this special case, $\lambda_0 = 1/\sqrt{2}$. If we restrict W_λ to the one-dimensional subspace $E = \operatorname{span}[A]$, then
$$W_\lambda(tA) = f_\lambda(t) - \lambda t^2 = \begin{cases} (t+1)^2, & t \leqslant \dfrac{1}{1+\lambda}, \\ \dfrac{\lambda}{1+\lambda} - \lambda t^2, & |t| \leqslant \dfrac{1}{1+\lambda}, \\ (t-1)^2, & t \geqslant \dfrac{1}{1+\lambda}. \end{cases}$$
Therefore, along E, W_λ has a double well structure.

Finally, we state the following result that is a special case of a general theorem due to J. Krestensen [15].

Proposition 2.2. *Suppose $\Omega \subset \mathbb{R}^n$ is bounded and smooth, and $u_j \to u$ in $W^{1,p}(\Omega)$ where $1 < p < \infty$. Then there are two bounded sequences (v_j) in $W^{1,p}(\Omega)$ and (w_j) in $W_0^{1,p}(\Omega)$ such that $u_j - u_0 = v_j + w_j$, and up to a subsequence*

(i) *$\nabla v_j \to 0$ almost everywhere in Ω,*
(ii) *$w_j \rightharpoonup 0$ in $W_0^{1,p}(\Omega)$ and $|\nabla w_j|^p$ is equi-integrable on Ω.*

3. The discretized problem

We discretize the problem and try to solve, for any $\phi \in H^1(\Omega)$ and $m = 1, 2, \ldots,$

$$\begin{aligned} &\int_\Omega \frac{u_m - u_{m-1}}{\Delta t} \cdot \phi \,\mathrm{d}x + \int_\Omega \nabla W_\lambda(\nabla u_m) : \nabla\phi \,\mathrm{d}x - \int_\Omega f(x) \cdot \phi \,\mathrm{d}x = 0, \\ &u_0 = u^0. \end{aligned} \tag{3.1}$$

Theorem 3.1. *Let W_λ be given as in* (2.1)*, f be given in $L^2(\Omega)$, then the following minimization problem*

$$\inf_{u_m \in H^1(\Omega)} \int_\Omega \left(\frac{1}{2\Delta t}\bigl(|u_m|^2 - 2u_m \cdot u_{m-1}\bigr) + W_\lambda(\nabla u_m) - f \cdot u_m \right) \mathrm{d}x \tag{3.2}$$

admits at least one solution in $H^1(\Omega)$.

Proof. Because of the estimate in Proposition 2.1(6), the growth condition required by [1] is satisfied, we also know that by Proposition 2.1(1), W_λ is quasiconvex, hence we can use the main result of [1] to conclude the existence of an energy minimizer.

Due to the growth condition satisfied by W_λ, the energy minimizers lie in the space $H^1(\Omega)$. □

Corollary 3.1. *The energy minimizers of* (3.2) *solves the variational equality* (3.1).

Proof. (3.1) is the Euler–Lagrange equation of (3.2). □

4. A priori estimates independent of discretization

For technicalities, we should restrict to a bounded time interval $(0, T)$ in the first place and then extend the result to $(0, \infty)$. To simplify the argument, we only deal with the solutions $\{u_m\}$ obtained by minimization method.

Proposition 4.1. *Let $\{u_m\}$ be the sequence of solutions obtained by minimizing energies defined in* (3.2)*, we have, for some constant C depending only on the initial data,*

$$\sup_m \int_\Omega \left(\frac{1}{2}\frac{|u_m - u_{m-1}|^2}{\Delta t} + W_\lambda(\nabla u_m) + f \cdot u_m \right) \mathrm{d}x \leqslant C \tag{4.1}$$

and

$$\sum_m \int_\Omega \frac{|u_m - u_{m-1}|^2}{(\Delta t)^2} \Delta t \leqslant C(T+1). \tag{4.2}$$

Proof. First, from the definition of the energy minimizers, we obtain, for any m,

$$\begin{aligned} \int_\Omega \left(\frac{1}{2}\frac{|u_m - u_{m-1}|^2}{\Delta t} + W_\lambda(\nabla u_m) + f \cdot u_m \right) \mathrm{d}x &= \inf_{u \in H^1(\Omega)} \int_\Omega \left(\frac{1}{2}\frac{|u - u_{m-1}|^2}{\Delta t} + W_\lambda(\nabla u) + f \cdot u \right) \mathrm{d}x \\ &\leqslant \int_\Omega \left(\frac{1}{2}\frac{|u - u_{m-1}|^2}{\Delta t} + W_\lambda(\nabla u) + f \cdot u \right) \mathrm{d}x \Big|_{u=u_{m-1}} \\ &= \int_\Omega \bigl(W_\lambda(\nabla u_{m-1}) + f \cdot u_{m-1} \bigr) \mathrm{d}x \\ &\leqslant \int_\Omega \left(\frac{1}{2}\frac{|u_{m-1} - u_{m-2}|^2}{\Delta t} + W_\lambda(\nabla u) + f \cdot u \right) \mathrm{d}x \end{aligned}$$

$$\vdots$$
$$\leqslant \int_\Omega \big(W_\lambda(\nabla u_0) + f\cdot u_0\big)\,\mathrm{d}x.$$

This implies (4.1). Furthermore, since the energy minimizers satisfy the Euler–Lagrange equation (3.1), by choosing $\phi = (u_m - u_{m-1})/\Delta t$, we have, for some constant C

$$\begin{aligned}0 &= \sum_m \Delta t \int_\Omega \left(\left|\frac{u_m - u_{m-1}}{\Delta t}\right|^2 + \nabla W_\lambda(\nabla u_m) : \nabla \frac{u_m - u_{m-1}}{\Delta t} + f\cdot \frac{u_m - u_{m-1}}{\Delta t}\right)\\ &\geqslant \sum_m \Delta t \int_\Omega \left(\nabla W_\lambda(\nabla u_m) : \nabla \frac{u_m - u_{m-1}}{\Delta t} + \frac{1}{2}\left|\frac{u_m - u_{m-1}}{\Delta t}\right|^2\right)\mathrm{d}x - CT.\end{aligned}$$

To study the term

$$\sum_m \Delta t \int_\Omega \nabla W_\lambda(\nabla u_m) : \nabla \frac{u_m - u_{m-1}}{\Delta t}\,\mathrm{d}x,$$

we introduce the following functions

$$U_N(x,t) = u_m(x) \quad \text{when } t \in (t_{m-1}, t_m],\ m = 0, \dots, N, \tag{4.3}$$

$$Y_N(x,t) = \frac{t-(m-1)\Delta t}{\Delta t} u_m(x) + \left(1 - \frac{t-(m-1)\Delta t}{\Delta t}\right) u_{m-1}(x) \quad \text{when } t \in (t_{m-1}, t_m],\ m = 1, \dots, N \tag{4.4}$$

where $N = \frac{T}{\Delta t}$ is assumed to be an integer. Then we have

$$\begin{aligned}&\sum_m \Delta t \int_\Omega \nabla W_\lambda(\nabla u_m) : \nabla \frac{u_m - u_{m-1}}{\Delta t}\,\mathrm{d}x\\ &\quad = \sum_m \Delta t \int_\Omega \nabla W_\lambda(\nabla U_N) : \nabla \frac{u_m - u_{m-1}}{\Delta t}\,\mathrm{d}x\\ &\quad = \sum_m \Delta t \int_\Omega \nabla\big(W_\lambda(\nabla U_N) - W_\lambda(\nabla Y_N) + W_\lambda(\nabla Y_N)\big) : \nabla \frac{u_m - u_{m-1}}{\Delta t}\,\mathrm{d}x\\ &\quad \geqslant -\sum_m \Delta t \int_\Omega \big|\nabla(u_m - u_{m-1})\big|^2\,\mathrm{d}x + \int_0^T\!\!\int_\Omega \nabla W_\lambda(\nabla Y_N) : \nabla \frac{\partial Y_N}{\partial t}\,\mathrm{d}x\\ &\quad \geqslant -CT - 1.\end{aligned}$$

Summarizing, we easily deduce (4.2). The proof of the proposition is finished. □

5. Convergence

We now have, for any $\phi \in L^2(0,T; H^1(\Omega))$, notice that $u_0 = u^0$,

$$\sum_m \int_{t_{m-1}}^{t_m} \mathrm{d}t \int_\Omega \left(\frac{u_m - u_{m-1}}{\Delta t}\cdot \phi + \nabla W_\lambda(\nabla u_m) : \nabla \phi - f(x)\cdot \phi\right)\mathrm{d}x = 0. \tag{5.1}$$

Because of the estimates obtained in the previous section, we conclude that both $\{U_N\}$ and $\{Y_N\}$ admit common subsequences still denoted by $\{U_N\}$ and $\{Y_N\}$ such that (by $N \to \infty$ we mean $\Delta t \to 0$)

$$U_N \to u \quad \text{and} \quad Y_N \to u \quad \text{as } N \to \infty.$$

The convergence is strong in $L^2(\Omega \times (0,t))$, weak-$*$ in $L^\infty(0,T;L^2(\Omega))$ and weak in $L^2(0,T;H^1(\Omega))$. In addition, $\{Y_N\}$ converges weakly in $H^1(\Omega \times (0,T))$.

Moreover, we know that

$$\|U_N - Y_N\|_{L^2(0,T;H^1(\Omega))} \to 0 \quad \text{as } N \to \infty, \tag{5.2}$$

$$\frac{u_m - u_{m-1}}{\Delta t} = \partial_t Y_N \quad \text{on } \big((m-1)\Delta t, m\Delta t\big). \tag{5.3}$$

Using the results from [14,15,23] and the discussions above, we know that

$$U_N = Y_N + (U_N - Y_N) = u + v_N + w_N + (U_N - Y_N)$$

where

(i) v_N is bounded in $H^1(\Omega \times (0,T))$, $\nabla v_N \to 0$ almost everywhere in $\Omega \times (0,T)$.
(ii) w_N is bounded in $H_0^1(\Omega \times (0,T))$, $w_N \rightharpoonup 0$ in $H_0^1(\Omega \times (0,T))$ and $|\nabla w_N|^2 + |\partial_t w_N|^2$ is equi-integrable on $\Omega \times (0,T)$.
(iii) $U_N - V_N$ converges strongly in $L^2(0,T;H^1(\Omega))$.

In order to prove that there is a solution, we need to show that $\nabla v_N + \nabla w_N$ tends to 0 strongly in $L^2(\Omega \times (0,T))$. To achieve this, choose $\phi = v_N + w_N + (U_N - Y_N)$ in (5.1), we obtain

$$\int_0^T \mathrm{d}t \int_\Omega \bigg(\frac{\partial V_N}{\partial t} \cdot \big(v_N + w_N + (U_N - Y_N)\big) + \nabla W_\lambda(\nabla U_N) : \nabla\big(v_N + w_N + (U_N - Y_N)\big) \\ - f(x) \cdot \big(v_N + w_N + (U_N - Y_N)\big)\bigg)\,\mathrm{d}x = 0.$$

Take the limit $N \to \infty$, from what we know, it is easy to conclude that

$$\lim_{N\to\infty} \int_0^T \mathrm{d}t \int_\Omega \nabla W_\lambda\big(\nabla u + \nabla\big(v_N + w_N + (U_N - Y_N)\big)\big) : \nabla(v_N + w_N)\,\mathrm{d}x = 0. \tag{5.4}$$

Using the fact that ∇W_λ is Lipschitz and that $\nabla(U_N - Y_N)$ converges strongly in $L^2(\Omega \times (0,T))$, we have

$$\lim_{N\to\infty} \int_0^T \mathrm{d}t \int_\Omega \nabla W_\lambda\big(\nabla u + \nabla(v_N + w_N)\big) : \nabla(v_N + w_N)\,\mathrm{d}x = 0. \tag{5.5}$$

From here, we can show that $\nabla(v_N + w_N) \to 0$ in $L^2(\Omega \times (0,T))$:

Recall that

$$W_\lambda(X) = C_E\big(\mathrm{dist}^2\big(P_E(X), \{-A, A\}\big) + \lambda\big|P_E(X)\big|^2\big) + H_\lambda(X) \\ = G_\lambda(X) + H_\lambda(X),$$

which is the sum of a convex function of quadratic growth and a rank-one convex quadratic form. Due to convexity, we know that

$$\big(\nabla G_\lambda(X) - \nabla G_\lambda(Y)\big) : (X - Y) \geqslant 0 \quad \text{for all } X, Y \in M^{2\times 2}.$$

From the property of decomposition, we know that $\nabla u \in L^2(\Omega \times (0,T))$ and $|\nabla w_N|^2$ equi-integrable, so for any $\mu > 0$, there is $0 < \delta < \mu$ such that for any measurable set $F \subset \Omega \times (0,T)$ with $\operatorname{meas} F < \delta$, we have

$$\int_F |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant \mu, \qquad \int_F |\nabla w_N|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant \mu.$$

Since $\nabla v_N \to 0$ almost everywhere, using Egorov's Theorem that for every $\delta > 0$, there is a measurable subset $E_1 \subset \Omega \times (0,T)$ with meas $E_1 < \delta$ such that $\nabla v_N \to 0$ uniformly on $(\Omega \times (0,T)) \setminus E_1$ as $j \to \infty$. Therefore, we have (where $Q = \Omega \times (0,T)$)

$$\begin{aligned}
&\int_Q \nabla W_\lambda(\nabla u + \nabla v_N + \nabla w_N) : (\nabla v_N + \nabla w_N)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_{Q\setminus E_1} + \int_{E_1} \big(\nabla W_\lambda(\nabla u + \nabla v_N + \nabla w_N) : (\nabla v_N + \nabla w_N)\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = A_N + B_N.
\end{aligned}$$

We estimate A_N first

$$\begin{aligned}
A_N &= \int_{Q\setminus E_1} \nabla W_\lambda(\nabla u + \nabla v_N + \nabla w_N) : (\nabla v_N + \nabla w_N)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{Q\setminus E_1} \big(\nabla W_\lambda(\nabla u + \nabla v_N + \nabla w_N) - \nabla W_\lambda(\nabla u)\big) : (\nabla v_N + \nabla w_N)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{Q\setminus E_1} \nabla W_\lambda(\nabla u) : (\nabla v_N + \nabla w_N)\,\mathrm{d}x\,\mathrm{d}t \\
&= A_N^{(1)} + A_N^{(2)}.
\end{aligned}$$

It is easy to see that

$$A_N^{(2)} \to 0 \quad \text{as } N \to \infty.$$

Using the facts that ∇H_λ is linear and G_λ is convex, we have

$$\begin{aligned}
A_N^{(1)} &= \int_{Q\setminus E_1} \big(\nabla W_\lambda(\nabla u + \nabla v_N + \nabla w_N) - \nabla W_\lambda(\nabla u)\big) : (\nabla v_N + \nabla w_N)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{Q\setminus E_1} \big(\nabla G_\lambda(\nabla u + \nabla v_N + \nabla w_N) - \nabla G_\lambda(\nabla u)\big) : (\nabla v_N + \nabla w_N)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{Q\setminus E_1} \big(\nabla H_\lambda(\nabla u + \nabla v_N + \nabla w_N) - \nabla H_\lambda(\nabla u)\big) : (\nabla v_N + \nabla w_N)\,\mathrm{d}x\,\mathrm{d}t \\
&\geqslant \int_{Q\setminus E_1} \big(\nabla H_\lambda(\nabla u + \nabla v_N + \nabla w_N) - \nabla H_\lambda(\nabla u)\big) : (\nabla v_N + \nabla w_N)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{Q\setminus E_1} \nabla H_\lambda(\nabla u + \nabla v_N + \nabla w_N) : (\nabla v_N + \nabla w_N)\,\mathrm{d}x\,\mathrm{d}t - \int_{Q\setminus E_1} \nabla H_\lambda(\nabla u) : (\nabla v_N + \nabla w_N)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{Q\setminus E_1} \big[\nabla H_\lambda(\nabla u + \nabla v_N + \nabla w_N) : \nabla v_N + \nabla H_\lambda(\nabla u + \nabla v_N) : \nabla w_N \\
&\quad - \nabla H_\lambda(\nabla u) : \nabla(v_N + w_N)\big]\,\mathrm{d}x\,\mathrm{d}t + \int_{Q\setminus E_1} \nabla H_\lambda(\nabla w_N) : \nabla w_N\,\mathrm{d}x\,\mathrm{d}t \\
&= C_N^{(1)} + C_N^{(2)}
\end{aligned}$$

where

$$\int_{Q\setminus E_1} \nabla H_\lambda(\nabla u + \nabla v_N + \nabla w_N) : \nabla v_N \,\mathrm{d}x\,\mathrm{d}t \to 0, \quad \text{as } N \to \infty$$

because $\nabla v_N \to 0$ uniformly in $\Omega \setminus E_1$ and $|\nabla H_\lambda(\nabla u + \nabla v_N + \nabla w_N)|$ is bounded in $L^2(Q)$;

$$\lim_{N\to\infty} \int_{Q\setminus E_1} \nabla H_\lambda(\nabla u + \nabla v_N) : \nabla w_N \,\mathrm{d}x\,\mathrm{d}t = 0$$

because $\nabla H_\lambda(\nabla u + \nabla v_N) \to \nabla H_\lambda(Du)$ strongly in $L^2(\Omega \setminus E_1)$ as $N \to \infty$ and $\nabla w_N \rightharpoonup 0$ in $L^2(\Omega)$;

$$\lim_{N\to\infty} \int_{Q\setminus E_1} \nabla H_\lambda(\nabla u) : \nabla (v_N + w_N) \,\mathrm{d}x\,\mathrm{d}t = 0$$

because $\nabla(v_N + w_N) \rightharpoonup 0$ in $L^2(Q)$. Hence

$$C_N^{(1)} \to 0$$

as $N \to \infty$. To study the convergence of $C_N^{(2)}$, we study two cases:

Case I: $\lambda < \lambda^*$. Using the fact that $\nabla H_\lambda(X) : X = 2H_\lambda(X)$, we have

$$\begin{aligned}
C_N^{(2)} &= \int_{Q\setminus E_1} \nabla H_\lambda(\nabla w_N) : \nabla w_N \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{Q\setminus E_1} 2H_\lambda(\nabla w_N) \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_Q 2H_\lambda(\nabla w_N) \,\mathrm{d}x\,\mathrm{d}t - \int_{E_1} 2H_\lambda(\nabla w_N) \,\mathrm{d}x\,\mathrm{d}t \\
&\geqslant 2\varepsilon_0 \int_Q |\nabla w_N|^2 \,\mathrm{d}x\,\mathrm{d}t - 2C_0 \int_{E_1} \big(1 + |\nabla w_N|^2\big) \,\mathrm{d}x\,\mathrm{d}t \\
&\geqslant 2\varepsilon_0 \int_Q |\nabla w_N|^2 \,\mathrm{d}x\,\mathrm{d}t - 4C_0\mu.
\end{aligned}$$

Here ε_0 is defined such that $\lambda = (1 - \lambda_0 - \varepsilon_0)/\lambda_0$ as discussed in Proposition 2.1 and C_0 is as defined in Proposition 2.1(6).

Consequently, we obtain

$$A_N \geqslant 2\varepsilon_0 \int_\Omega |\nabla w_N|^2 \,\mathrm{d}x + A_N^{(2)} + C_N^{(1)} - 4C_0\mu \tag{5.6}$$

where $A_N^{(2)} + C_N^{(1)} \to 0$ as $N \to \infty$.

Now we estimate B_N.

$$\begin{aligned}
B_N &= \int_{E_1} \nabla W_\lambda(\nabla u + \nabla v_N + \nabla w_N) : (\nabla v_N + \nabla w_N) \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{E_1} \nabla W_\lambda(\nabla u + \nabla v_N + \nabla w_N) : (\nabla u + \nabla v_N + \nabla w_N) \,\mathrm{d}x\,\mathrm{d}t - \int_{E_1} \nabla W_\lambda(\nabla u + \nabla v_N + \nabla w_N) : \nabla u \,\mathrm{d}x\,\mathrm{d}t \\
&\geqslant c_1 \int_{E_1} \big(\big|\nabla(u + v_N + w_N)\big|^2 - 1\big) - C_1 \int_{E_1} \big(1 + \big|\nabla(u + v_N + w_N)\big|\big)|\nabla u| \,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&\geqslant \frac{c_1}{2}\int_{E_1}|\nabla v_N|^2\,\mathrm{d}x\,\mathrm{d}t - c_1\int_{E_1}\big(|\nabla u|+|\nabla w_N|\big)^2\,\mathrm{d}x\,\mathrm{d}t-\mu\\
&\quad - C_1\sqrt{\int_{\Omega}\big(1+|\nabla u+\nabla v_N+\nabla w_N|\big)^2\,\mathrm{d}x\,\mathrm{d}t}\sqrt{\int_{E_1}|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t}\\
&\geqslant \frac{c_1}{2}\int_{E_1}|\nabla v_N|^2\,\mathrm{d}x\,\mathrm{d}t - C_4\sqrt{\mu}.
\end{aligned}
$$

This leads to

$$
\begin{aligned}
&\int_Q \nabla W_\lambda\big(\nabla u+\nabla(v_N+w_N)\big):\nabla(v_N+w_N)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\geqslant 2\varepsilon_0\int_Q|\nabla w_N|^2\,\mathrm{d}x\,\mathrm{d}t + D_N - 4C_0\mu + \frac{c_1}{2}\int_{E_1}|\nabla v_N|^2\,\mathrm{d}x\,\mathrm{d}t - C_4\sqrt{\mu},
\end{aligned}
\tag{5.7}
$$

where $\int_Q \nabla W_\lambda(\nabla u+\nabla(v_N+w_N)):\nabla(v_N+w_N)\,\mathrm{d}x\,\mathrm{d}t\to 0$, $D_N=C_N^{(1)}+A_N^{(2)}\to 0$. Consequently, for sufficiently large N, we have

$$
2\varepsilon_0\int_Q|\nabla w_N|^2\,\mathrm{d}x\,\mathrm{d}t+\frac{c_1}{2}\int_{E_1}|\nabla v_N|^2\,\mathrm{d}x\,\mathrm{d}t\leqslant C_5\sqrt{\mu}. \tag{5.8}
$$

This implies that $\nabla w_N\to 0$ strongly in $L^2(Q,M^{N\times n})$ as $N\to\infty$ and $|\nabla v_N|^2$ is equi-integrable on Ω. Therefore $\nabla(v_N+w_N)\to 0$ as $N\to\infty$. So v_N+w_N converges strongly to 0 in $L^2(0,T;H^1(\Omega))$.

Therefore, we can take the limit in (5.1) to get a solution of (0.1).

Case II: We now have $\lambda=\lambda^*$. To simplify notation, we still use λ in the text.

Since $\nabla H_\lambda(X):X=2H_\lambda(X)$ and $\nabla H_\lambda(X)$ is only quasimonotone, parallel to Case I, we get

$$
\begin{aligned}
C_N^{(2)}&=\int_{Q\setminus E_1}\nabla H_\lambda(\nabla w_N):\nabla w_N\,\mathrm{d}x\,\mathrm{d}t\\
&=\int_Q 2H_\lambda(\nabla w_N)\,\mathrm{d}x\,\mathrm{d}t-\int_{E_1}2H_\lambda(\nabla w_N)\,\mathrm{d}x\,\mathrm{d}t\\
&\geqslant -2C_0\int_{E_1}\big(1+|\nabla w_N|^2\big)\,\mathrm{d}x\,\mathrm{d}t\geqslant -4C_0\mu.
\end{aligned}
\tag{5.9}
$$

This is because the constant ε_0 in Case I is now 0, while $C_0>0$ is given by Proposition 2.1(6). Subsequently, (5.6) becomes

$$
A_N\geqslant D_N-4C_0\mu \tag{5.10}
$$

where $D_N=A_N^{(2)}+C_N^{(1)}\to 0$. It is easy to see that estimate of B_N in Case I still holds, we obtain

$$
\begin{aligned}
B_N&=\int_{E_1}\nabla W_\lambda\big(\nabla u+\nabla(v_N+w_N)\big):\nabla(v_N+w_N)\,\mathrm{d}x\,\mathrm{d}t\\
&=\int_{E_1}\nabla W_\lambda\big(\nabla u+\nabla(v_N+w_N)\big):\big(\nabla u+\nabla(v_N+w_N)\big)\,\mathrm{d}x\,\mathrm{d}t-\int_{E_1}\nabla W_\lambda\big(\nabla u+\nabla(v_N+w_N)\big):\nabla u\,\mathrm{d}x\,\mathrm{d}t\\
&\geqslant\frac{c_1}{2}\int_{E_1}|\nabla v_N|^2\,\mathrm{d}x\,\mathrm{d}t-C_4\sqrt{\mu},
\end{aligned}
\tag{5.11}
$$

hence

$$\int_Q \nabla W_\lambda\big(\nabla u+\nabla(v_N+w_N)\big):\nabla(v_N+w_N)\,\mathrm{d}x\,\mathrm{d}t\geqslant D_N-4C_0\mu+\frac{c_1}{2}\int_{E_1}|\nabla v_N|^2\,\mathrm{d}x\,\mathrm{d}t-C_4\sqrt{\mu} \tag{5.12}$$

where $\int_Q \nabla W_\lambda(\nabla u+\nabla(v_N+w_N)):\nabla(v_N+w_N)\,\mathrm{d}x\,\mathrm{d}t\to 0$, $D_N\to 0$. Consequently, for sufficiently large $j>0$, we have

$$\frac{c_1}{2}\int_{E_1}|\nabla v_N|^2\,\mathrm{d}x\,\mathrm{d}t\leqslant C_5\sqrt{\mu}, \tag{5.13}$$

therefore $|\nabla v_N|^2$ is equi-integrable on Ω which implies that $\nabla v_N\to 0$ in $L^2(Q)$ strongly. Therefore $\nabla(v_N+w_N)\rightharpoonup 0$ weakly in $L^2(Q)$ as $N\to\infty$ and is equi-integrable on Q.

From now on, we write $W_{\lambda^*}(X)=G(P_A(X))+H(X)$ with $G_{P_A}(X)=G_{\lambda^*}(X)$ and $H(X)=H_{\lambda^*}(X)$.

Since $\phi_N=v_N+w_N$ as given above, we see that $\nabla v_N\to 0$ in L^2, $|\nabla w_N|^2$ is equi-integrable in Q, $\nabla w_N\rightharpoonup 0$ and $w_N=0$ on ∂Q. We also have

$$\int_Q \nabla G\big(P_A(\nabla u+\nabla\phi_N)\big):P_A(\nabla\phi_N)\,\mathrm{d}x\,\mathrm{d}t+\int_Q \nabla H(\nabla u+\nabla\phi_N):\nabla\phi_N\,\mathrm{d}x\,\mathrm{d}t=t_N\to 0, \tag{5.14}$$

as $N\to\infty$. We have in (5.14) that

$$\begin{aligned}
&\int_Q \nabla H(\nabla u+\nabla\phi_N):\nabla\phi_N\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\int_Q \nabla H(\nabla u+\nabla v_N+\nabla w_N):\nabla(v_N+w_N)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=2\int_Q H(\nabla w_N)\,\mathrm{d}x\,\mathrm{d}t+\int_Q \nabla H(\nabla u+\nabla v_N):\nabla\phi_N\,\mathrm{d}x\,\mathrm{d}t+\int_Q \nabla H(\nabla u+\nabla\phi_N):\nabla v_N\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=2\int_Q H(\nabla w_N)\,\mathrm{d}x\,\mathrm{d}t+r_N\geqslant r_N,
\end{aligned} \tag{5.15}$$

where

$$r_N=\int_Q \nabla H(\nabla u+\nabla v_N):\nabla\phi_N\,\mathrm{d}x\,\mathrm{d}t+\int_Q \nabla H(\nabla u+\nabla\phi_N):\nabla v_N\,\mathrm{d}x\,\mathrm{d}t\to 0$$

and

$$2\int_Q H(\nabla w_N)\,\mathrm{d}x\geqslant 0 \tag{5.16}$$

because $w_N\in H_0^1(Q)$, for a.e. $t\in(0,T)$, $w_N(t,\cdot)\in H_0^1(\Omega)$. Also H is quasiconvex and $H(0)=0$ hence (5.16) holds. Similarly

$$\int_Q H(\nabla u+\nabla\phi_N)\,\mathrm{d}x\,\mathrm{d}t=\int_Q H(\nabla u)\,\mathrm{d}x\,\mathrm{d}t+\int_Q H(\nabla w_N)\,\mathrm{d}x\,\mathrm{d}t+l_N \tag{5.17}$$

where $l_N\to 0$. Substituting (5.15) into (5.14), we have

$$\int_Q \nabla G\big(P_A(\nabla u+\nabla\phi_N)\big):P_A(\nabla\phi_N)\,\mathrm{d}x+2\int_Q H(\nabla w_N)\,\mathrm{d}x\,\mathrm{d}t=t_N-r_N\to 0 \tag{5.18}$$

as $N \to \infty$. Since G is convex,

$$\int_Q \nabla G\big(P_A(\nabla u + \nabla\phi_N)\big) : P_A(\nabla\phi_N)\,\mathrm{d}x\,\mathrm{d}t \geqslant \int_Q \nabla G\big(P_A(\nabla u)\big) : P_A(\nabla\phi_N)\,\mathrm{d}x\,\mathrm{d}t \to 0 \tag{5.19}$$

as $N \to \infty$. Therefore (5.16), (5.18) and (5.19) imply

$$\lim_{N\to\infty}\int_Q \nabla G\big(P_A(\nabla u + \nabla\phi_N) : P_A(\nabla\phi_N)\big)\,\mathrm{d}x\,\mathrm{d}t = 0, \qquad \lim_{N\to\infty}\int_Q H(\nabla w_N)\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{5.20}$$

Thus from (5.17) and (5.20),

$$\lim_{N\to\infty}\int_Q H(\nabla u + \nabla\phi_N)\,\mathrm{d}x\,\mathrm{d}t = \int_Q H(\nabla u)\,\mathrm{d}x\,\mathrm{d}t. \tag{5.21}$$

Since G is convex,

$$\int_Q G\big(P_A(\nabla u + \nabla\phi_N)\big)\,\mathrm{d}x\,\mathrm{d}t - \int_Q G\big(P_A(\nabla u)\big)\,\mathrm{d}x\,\mathrm{d}t \geqslant \int_Q \nabla G\big(P_A(\nabla u)\big) : P_A(\nabla\phi_N)\,\mathrm{d}x\,\mathrm{d}t \to 0,$$

$$\int_Q G\big(P_A(\nabla u)\big)\,\mathrm{d}x\,\mathrm{d}t - \int_Q G\big(P_A(\nabla u_N)\big)\,\mathrm{d}x\,\mathrm{d}t \geqslant \int_Q \nabla G\big(P_A(\nabla u + \nabla\phi_N)\big) : P_A(-\nabla\phi_N)\,\mathrm{d}x \to 0,$$

hence

$$\lim_{N\to\infty}\int_Q G\big(P_A(\nabla u + \nabla\phi_N)\big)\,\mathrm{d}x\,\mathrm{d}t = \int_Q G\big(P_A(\nabla u)\big)\,\mathrm{d}x\,\mathrm{d}t. \tag{5.22}$$

From (5.21) and (5.22), we obtain, as $N \to \infty$, that

$$\int_Q G\big(P_A(\nabla u + \nabla\phi_N)\big)\,\mathrm{d}x\,\mathrm{d}t + \int_Q H(\nabla u_N)\,\mathrm{d}x\,\mathrm{d}t + \int_Q f\cdot(u+\phi_N)\,\mathrm{d}x\,\mathrm{d}t$$
$$\to \int_Q G\big(P_A(\nabla u)\big)\,\mathrm{d}x\,\mathrm{d}t + \int_Q H(\nabla u)\,\mathrm{d}x\,\mathrm{d}t + \int_Q f\cdot u\,\mathrm{d}x\,\mathrm{d}t.$$

Now we prove that u is a solution of (0.1).

Since H is a quadratic form, for any test function ψ, we have

$$\int_Q \nabla H(\nabla u + \nabla\phi_N) : \nabla\psi\,\mathrm{d}x\,\mathrm{d}t \to \int_Q \nabla H(\nabla u) : \nabla\psi\,\mathrm{d}x\,\mathrm{d}t.$$

We only need to prove that

$$\int_Q \nabla G\big(P_A(\nabla u + \nabla\phi_N)\big) : P_A(\nabla\psi)\,\mathrm{d}x\,\mathrm{d}t \to \int_Q \nabla G\big(P_A(\nabla u)\big) : P_A(\nabla\psi)\,\mathrm{d}x\,\mathrm{d}t. \tag{5.23}$$

We also have, from Proposition 2.1 that

$$\int_Q \big|\nabla G\big(P_A(\nabla u + \nabla\phi_N)\big) - \nabla G\big(P_A(\nabla u)\big)\big|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant C\int_Q \big(\nabla G\big(P_A(\nabla u + \nabla\phi_N)\big) - \nabla G\big(P_A(\nabla u)\big)\big)\cdot P_A(\nabla\phi_N)\,\mathrm{d}x\,\mathrm{d}t \to 0$$

as $N \to \infty$ and (5.23) follows. Therefore, u is a solution of (0.1).

Furthermore, as T is arbitrary, we have proved in fact

Theorem 5.1. *The problem* (0.1) *admits at least one weak solution on* $(0,\infty)$ *in the following sense*: *for any* $T>0$, *any* $\phi\in L^2(0,T;H^1(\Omega))$,

$$\begin{cases} \int_0^T\int_\Omega \big(u_t\cdot\phi+\nabla W_\lambda(\nabla u):\nabla\phi+f\cdot\phi\big)\,\mathrm{d}x\,\mathrm{d}t=0,\\ u(x,0)=u_0(x). \end{cases} \tag{5.24}$$

Moreover, the energy functional

$$\int_\Omega \big(W_\lambda(\nabla u)+f(x)\cdot u\big)\,\mathrm{d}x \tag{5.25}$$

is a decreasing function of time.

Proof. The existence proof is obvious. The fact that the energy is decreasing in time can be proved by standard differentiation argument. For the proof under a more general setting, we refer to [8] for details. □

Remark 5.1. The use of Proposition 2.1 is not essential for establishing the weak continuity of the integrals. In fact, it is possible to establish same kind of results in a much more general setting for $W=G+H$ where G is convex and H a quadratic form under various boundedness and coercivity conditions. However, without the special form of $G(X)=f(A\cdot X)$, the proof is more complicated and involves the use of Young measures. We will examine more general cases later.

6. Long time behaviour of solutions

6.1. The case where $\lambda<\lambda^*$

First, we establish a general results on the existence of ω-limit by using similar ideas explored in [4] and [20]. First, we concentrate our discussion to the case when $\lambda<\lambda^*$.

Definition 6.1. A generalized semi-flow G on X is a family of mappings $\phi:[0,\infty)\to X$ (called solutions) satisfying

(H1) (Existence) For each $z\in X$, there exists at least one $\phi\in G$ with $\phi(0)=z$;
(H2) (Translation) If $\phi\in G$ and $\tau\geqslant 0$, then $\phi^\tau(t)=\phi(t+\tau)\in G$;
(H3) (Concatenation) If $\phi,\psi\in G$, $t\geqslant 0$ with $\psi(0)=\phi(t)$ then $\theta\in G$ where

$$\theta(s)=\begin{cases}\phi(s), & s\in[0,t],\\ \psi(s-t), & t<s.\end{cases}$$

(H4) (Upper-semi-continuity) If $\phi_j\in G$ with $\phi_j(0)\to z$, then there exists a subsequence ϕ_μ of ϕ_j and $\phi\in G$ with $\phi(0)=z$ such that

$$\phi_\mu(t)\to\phi(t)\quad\text{for a.e. } t\geqslant 0.$$

The definition of upper-semi continuity is weaker than that found in [4] but it is enough for most application problems.

Proposition 6.1. *The solution set established in Theorem* 5.1 *for all possible initial data in* $H^1(\Omega)$ *defines a generalized semi-flow.*

Proof. The solutions we find in the previous section satisfy obviously (H1)–(H3). As to (H4), let u_j be a sequence of solutions obtained in Section 4, with $u_j(0) \to z$ in $H^1(\Omega)$ as $j \to \infty$, we know then u_j is bounded in $H^1(\Omega \times (0,T))$, up to choosing a subsequence u_μ, we get

$$u_\mu \rightharpoonup u_\infty \quad \text{for some } u_\infty \in H^1\big(\Omega \times (0,T)\big). \tag{6.1}$$

By trace theorem, we have

$$u_\infty(0) = z. \tag{6.2}$$

Using the equation

$$\int_0^T \int_\Omega \big(u_{\mu t} \cdot \phi + \nabla W(\nabla u_\mu) : \nabla \phi + f \cdot \phi\big)\, \mathrm{d}x\, \mathrm{d}t = 0,$$

we obtain

$$\int_0^T \int_\Omega \big(u_{\mu t} \cdot (u_\mu - u_\infty) + \nabla W(\nabla u_\mu) : \nabla (u_\mu - u_\infty) + f \cdot (u_\mu - u_\infty)\big)\, \mathrm{d}x\, \mathrm{d}t = 0.$$

From the boundedness of $u_{\mu t}$ in $L^2(\Omega \times (0,T))$, the strong convergence of u_μ in $L^2(\Omega \times (0,T))$, we derive

$$\lim_{\mu \to \infty} \int_0^T \int_\Omega \nabla W(\nabla u_\mu) : \nabla (u_\mu - u_\infty)\, \mathrm{d}x\, \mathrm{d}t = 0.$$

Using the same proof as in Section 5, we conclude that u_∞ is also a solution and

$$u_\mu \to u_\infty \quad \text{in } L^2\big(0,T; H^1(\Omega)\big) \text{ strongly},$$

we obtain that $u_\mu(t) \to u_\infty(t)$ for a.e. $t > 0$ in $H^1(\Omega)$. $\square$

Remark 6.1. From the weak formulation

$$\int_0^T \int_\Omega \big(u_t \cdot \phi + \nabla W_\lambda(\nabla u) : \nabla \phi + f \cdot \phi\big)\, \mathrm{d}x\, \mathrm{d}t = 0 \tag{6.3}$$

for any $\phi \in L^2(\Omega \times (0,T))$, we know that there is a set $K_1 \subset R_+ = (0,\infty)$ with $\mathrm{meas}(K_1) = 0$ such that

$$\int_\Omega \big(u_t \cdot \phi + \nabla W_\lambda(\nabla u) : \nabla \phi + f \cdot \phi\big)\, \mathrm{d}x = 0 \quad \text{for all } t \in R_+ \setminus K_1. \tag{6.4}$$

To actually discuss the long time behaviour, we have to make the following assumption to be able to improve the T dependent estimates of the solutions obtained in Section 4:

(HH) $\int_\Omega f(x)\, \mathrm{d}x = 0$.

Remark 6.2. (HH) can also be replaced by the assumption that $f(x,t) \in L^2(\Omega \times (0,\infty))$ plus some additional decay conditions on t. However, we are not interested in the case where f depends on t at the moment.

Theorem 6.1. *Under assumption* (HH), *for any given initial data $u^0 \in H^1(\Omega)$, there exists a weak solution of* (0.1) *which satisfies*

$$\int_0^\infty \int_\Omega |u_t|^2\, \mathrm{d}x\, \mathrm{d}t < C \tag{6.5}$$

for some constant C.

Proof. Let $\{\phi_j\}$ be the sequence of eigenfunctions of Neumann Laplace operator on Ω, assume that

$$u_N = \sum_{j=1}^{N} a_j(t)\phi_j(x)$$

solves for $l = 1, \ldots, N$,

$$\dot{a}_l - \int_\Omega \left(\nabla W_\lambda\left(\sum_j a_j \nabla\phi_j\right) : \nabla\phi_l + f \cdot \phi_l\right) \mathrm{d}x = 0. \tag{6.6}$$

This is the Galerkin approximation of the problem and a simple computation leads to

$$\int_0^T \left(\sum_j |\dot{a}_j|^2\right) \mathrm{d}t + \int_\Omega W_\lambda\left(\sum_j a_j(T)\nabla\phi_j\right) \mathrm{d}x + \int_\Omega f \cdot \sum_j a_j(T)\phi_j \,\mathrm{d}x \leqslant C. \tag{6.7}$$

Because $\phi_1 \equiv$ constant and $\int_\Omega f\,\mathrm{d}x = 0$, the W_λ term dominates the f term in (6.7), hence we obtain

$$\int_0^\infty \left(\sum_j |\dot{a}_j|^2\right) \mathrm{d}t \leqslant C$$

for some constant C. Similar to discussions carried out in Section 4, it is not difficult to show that the Galerkin approximation converge to a solution. Subsequently, the theorem holds. We omit the details here. □

Remark 6.3. We also derive, for the solution obtained above, for a.e. $t > t' > 0$,

$$\int_{t'}^{t}\int_\Omega |u_t|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \big(W(\nabla u) + f \cdot u\big)(t)\,\mathrm{d}x - \int_\Omega \big(W(\nabla u) + f \cdot u\big)(t')\,\mathrm{d}x \leqslant 0. \tag{6.8}$$

Consequently, there is a set K_2 of measure 0 such that if $t > t' > 0$, $t, t' \in R_+ \setminus K_2$, then (6.8) holds. (6.5) also implies that $u_t \in L^2(\Omega \times (0, \infty))$. Hence, for any given positive number M and α, there is a positive number L such that

$$\operatorname{meas} K_{\mathrm{LM}} = \operatorname{meas}\big\{t > L,\ \|u_t(t)\|_{L^2(\Omega)} > M,\ u_t(t) \text{ and } \nabla u(t) \text{ are well defined } L^2(\Omega) \text{ functions}\big\} < \alpha.$$

Now we discuss the property of the solutions as $t \to \infty$. Since there is a lack of regularity in general, we have to obtain asymptotic compactness of solutions as $t \to \infty$ by imposing certain conditions.

Proposition 6.2. *Let u be a weak solution of* (0.1) *satisfying* (6.8), *let $\{t_j\} \subset R_+ \setminus (K_1 \cup K_2 \cup K_{\mathrm{LM}})$ for some arbitrary L and M, $t_j \to \infty$, then the sequence $\{u(t_j)\}$ is pre-compact in $H^1(\Omega)$.*

Proof. Since the energy is decreasing, we know that $\{u(t_j)\}$ admits a subsequence, still denoted by $\{u(t_j)\}$ which converges weakly in $H^1(\Omega)$. Denoting the limit by u^*, then we have

$$\int_\Omega \Big(u_t(t_j) \cdot \big(u(t_j) - u^*\big) + \nabla W\big(\nabla u(t_j)\big) : \nabla\big(u(t_j) - u^*\big) + f \cdot \big(u(t_j) - u^*\big)\Big)\,\mathrm{d}x = 0.$$

Since $\{u_t(t_j)\}$ is bounded in $L^2(\Omega)$, we obtain again

$$\lim_{j\to\infty} \int_\Omega \nabla W\big(\nabla u(t_j)\big) : \nabla\big(u(t_j) - u^*\big)\,\mathrm{d}x = 0.$$

The strong convergence follows. □

Now we define the ω-limit set for our problem:

Definition 6.2. For any given initial data $u^0 \in H^1(\Omega)$, we define

$$\omega\big(u^0\big) = \big\{u^*:\ \text{there exists a sequence } t_j \to \infty \text{ such that } u(t_j) \text{ is well-defined,}\\ \text{satisfying (6.4) and } u(t_j) \to u^* \text{ in } H^1(\Omega)\big\}.$$

By our discussion, it is easy to know that $\omega(u^0)$ is non-empty and it is easy to conclude that it is closed. Now we prove the

Theorem 6.2. *If $t_j \to \infty$, $t_j \in R_+ \setminus (K_1 \cup K_2 \cup K_{\mathrm{LM}})$ such that $u(t_j) \to u^*$ in $H^1(\Omega)$, then u^* is a stationary point of the energy functional* (5.25).

Proof. First, if there are two sequences $t_j \in R_+ \setminus (K_1 \cup K_2 \cup K_{\mathrm{LM}})$ and $s_j \in R_+ \setminus (K_1 \cup K_2 \cup K_{\mathrm{LM}})$ such that

$$u(t_j) \to u^*,$$
$$u(s_j) \to u^{**},$$

then because the energy is decreasing, we must have

$$\int_\Omega \big(W(\nabla u^*) + f \cdot u^*\big)\,\mathrm{d}x = \int_\Omega \big(W(\nabla u^{**}) + f \cdot u^{**}\big)\,\mathrm{d}x.$$

As a consequence, since we know that

$$u(t_j) \to u^*,$$

up to choosing a subsequence, let $S(t)v$ be the solution of (0.1) obtained as in Theorem 6.1 with initial data v, we have

$$S(t)u(t_j) \to S(t)u^* \quad \text{in } L^2\big(0, T; H^1(\Omega)\big) \text{ for any } T > 0.$$

But as

$$\int_\Omega \big(W\big(\nabla S(t)u^*\big) + f \cdot S(t)u^*\big)\,\mathrm{d}x = \text{constant}.$$

We have $S(t)u^*$ is independent of t and is a solution of the steady state problem. □

It is known from [24] that the set of steady state solutions is bounded and compact in $H^1(\Omega)$, hence we have:

Corollary 6.1. *The ω-limit set defined in Definition* 6.2 *for our problem is a compact, closed set in $H^1(\Omega)$.*

6.2. The case where $\lambda = \lambda^*$

In this case, the condition (H4) no longer stands and needs to be replaced by

(H4*) (Weak upper-semi-continuity): If $\phi_j \in G$ with $\phi_j(0) \to z$, then there exists a subsequence ϕ_μ of ϕ_j and $\phi \in G$ with $\phi(0) = z$ such that

$$\phi_\mu(t) \rightharpoonup \phi(t) \quad \text{in } L^2\big(0, T; H^1(\Omega)\big).$$

Definition 6.3. The assumptions (H1)–(H3) and (H4*) define a *generalized weak semi-flow*.

In general, generalized weak semi-flow is not useful in discussing long time convergence for non-linear evolutionary solutions because the convergence is weak and limit can therefore not be taken. However, the essence of Section 5 is to say that even with weak convergence, we can take the limit in the energy expression. Hence, we can argue in the same way as in Section 6.1 to establish

Theorem 6.3. *Let $\lambda = \lambda^*$, u be a weak solution of* (0.1) *satisfying* (6.8), *there exists a set $K \subset \mathbb{R}_+$ with measure* 0, *a function $u^* \in H^1(\Omega)$. For any sequence*

$$t_j \in \mathbb{R}_+ \setminus K,$$

there is a subsequence of t_j still denoted by t_j such that

$$u(t_j) \rightharpoonup u^* \quad \text{in } H^1(\Omega).$$

Moreover, under the same assumptions as in Section 6.1, *u^* is a solution of the corresponding steady state problem*

$$\int_\Omega \nabla W(\nabla u^*) : \nabla \phi + f \cdot \phi \, \mathrm{d}x = 0.$$

In fact the set u^*s form a weakly closed 'weak' ω-limit set for the problem.

7. Precise convergence

Generally speaking, the structure of the ω-limit set (i.e., some subset of steady state solutions) of an evolutionary solution is rather complicated. The complication of the set of steady state solutions was partially discussed in [24].

However, as noticed in [24], in the neighborhood of A and $-A$, the system has a simple structure. We will show that the solution converge to Ax and $-Ax$ respectively if the initial data is appropriately located and if the source term f is small.

In the following, for convenience, we assume that ε is a small positive constant, f is a very smooth function with $\|f\|_{C^2(\Omega)} \leqslant \varepsilon$. Let v satisfy

$$-\Delta v = f \quad \text{in } \Omega,$$
$$\frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega,$$

then we have $\|v\|_{C^2(\Omega)} \leqslant C_f$ for some constant C_f.

Using the results of [24], when $|X \pm A| \leqslant \lambda|A|/(1+\lambda)$,

$$W_\lambda(X) = |X \pm A|^2.$$

Let $w = u \pm Ax - \varepsilon v$, we know that when

$$\left|\nabla w(x)\right| \leqslant \frac{\lambda|A|}{1+\lambda} - C_f \varepsilon,$$

w satisfies the partial differential equation

$$w_t - \Delta w = 0 \quad \text{in } \Omega \times (0, T),$$
$$\frac{\partial w}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0, T),$$
$$w(x, 0) = w_0(x) \quad \text{in } \Omega.$$

Proposition 7.1. *Let the spatial domain Ω be smooth and strictly convex, let the initial data u_0 be in $C^2(\overline{\Omega})$ and the right-hand side function f be in $C(\overline{\Omega})$, then there exists a constant $\delta > 0$, such that when $|u_0|_{C^1(\overline{\Omega})} < \delta$, then the solution of* (0.1) *satisfies, for all $t > 0$,*

$$\left|(u \pm Ax)_1\right|_{C^1(\Omega)} < \frac{\lambda|A|}{\sqrt{2}(1+\lambda)}, \qquad \left|(u \pm Ax)_2\right|_{C^1(\Omega)} < \frac{\lambda|A|}{\sqrt{2}(1+\lambda)}.$$

Here $(v)_1$ and $(v)_2$ are the first and second component of the vector function v respectively.

Furthermore, the ω-limit of u as $t \to \infty$ is the only solution (*up to an added constant vector*) *of the steady state problem*

$$-\Delta u = f,$$
$$\frac{\partial u}{\partial n} = A \cdot n.$$

Sketch of proof. We start with the function w. Since w is the solution of a homogeneous second type initial boundary value problem for heat equation, standard PDE theory implies w is in $C^2(\overline{\Omega})$ as long as $|\nabla w|$ remain in the range where the Laplace equation holds.

Now we estimate the $C(\overline{\Omega})$ norm of ∇w. In the following we assume that the boundary (the tangent direction $\boldsymbol{\tau}$) of Ω is oriented in the anticlockwise direction and the normal vector $\boldsymbol{n}$ is the outward normal unit vector. We use κ to denote the curvature. Under this setting, we have

$$
\begin{aligned}
&\kappa > 0,\\
&\frac{\partial \boldsymbol{n}}{\partial \boldsymbol{\tau}} = \kappa \boldsymbol{\tau},\\
&\frac{\partial \boldsymbol{\tau}}{\partial \boldsymbol{\tau}} = -\kappa \boldsymbol{n},\\
&\nabla w_j|_{\partial\Omega\times(0,T)} = (\nabla w_j \cdot \boldsymbol{\tau})\boldsymbol{\tau} \quad \text{for } j = 1, 2 \text{ and for any } T > 0.
\end{aligned}
$$

A simple computation then leads to, for each $j = 1, 2$,

$$
\begin{cases}
\partial_t |\nabla w_j|^2(x, y, t) - \Delta |\nabla w_j|^2(x, y, t) = -2|H w_j|^2(x, y, t) \leqslant 0,\\
\frac{\partial |\nabla w_j|^2}{\partial \boldsymbol{n}} = -2\kappa |\partial_{\boldsymbol{\tau}} w_j|^2.
\end{cases}
\tag{7.1}
$$

The first equation leads to that $|\nabla w|^2$ has no (positive) maximum in the interior of $\Omega \times (0, T)$ for any $T > 0$. The boundary condition leads to that the positive maximum can not appear on $\partial\Omega \times (0, T)$. Hence we have

$$
|\nabla w|^2(x, t) \leqslant \max_x \left|\nabla w_0(x)\right|^2. \tag{7.2}
$$

Using the relation between w and u, we obtain that as long as

$$
|\nabla u_0 \pm A||_{C(\overline{\Omega})} < \frac{\lambda |A|}{(1+\lambda)} - C_f \varepsilon,
$$

the solution stays in the desired domain and we have the heat equation on u.

The conclusion on ω-limit is now standard. □

Acknowledgements

The authors would like to thank their colleague K. Deckelnick for constructive discussions.

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