

Controllability of systems of Stokes equations with one control force: existence of insensitizing controls

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Abstract

In this paper we establish some exact controllability results for systems of two parabolic equations of the Stokes kind. In a first part, we prove the existence of insensitizing controls for the L^2 norm of the solutions and the *curl* of solutions of linear Stokes equations. Then, in the limit case where one can expect null controllability to hold for a system of two Stokes equations (namely, when the coupling terms concern first and second order derivatives, respectively), we prove this result for some general couplings. © 2006 Elsevier Masson SAS. All rights reserved.

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1. Introduction

Let $\Omega \subset \mathbf{R}^N$ ($N = 2$ or 3) be a bounded simply-connected open set whose boundary $\partial\Omega$ is regular enough. Let $T > 0$ and let $\omega \subset \Omega$ be a (small) nonempty open subset which will usually be referred as *control domain*. We will use the notation $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$.

On the other hand, we will design by C (resp. K) a generic positive constant which depends on Ω , ω and T (resp. Ω and ω). Anyway, it will be made precise each time a constant appear.

Let us recall the definition of some usual spaces in the context of Stokes equations:

$$V = \{y \in H_0^1(\Omega)^N : \nabla \cdot y = 0 \text{ in } \Omega\}$$

and

$$H = \{y \in L^2(\Omega)^N : \nabla \cdot y = 0 \text{ in } \Omega, y \cdot n = 0 \text{ on } \partial\Omega\}.$$

The main objective of this paper is to establish some new controllability results for a system of two strongly coupled Stokes equations.

- The two first main results of this paper concerns insensitizing controls. More precisely, we are interested in insensitizing two different functionals associated to a state system, which is a linear Stokes equation with Dirichlet

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boundary conditions. Let us introduce an open set $\mathcal{O} \subset \Omega$ such that $\mathcal{O} \cap \omega \neq \emptyset$, which is called the *observatory* (or observation open set).

In order to precisely state our problem, we introduce the state system:

$$\begin{cases} y_t - \Delta y + ay + B \cdot \nabla y + \nabla p = v1_\omega + f, & \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & & \text{on } \Sigma, \\ y|_{t=0} = y^0 + \tau \hat{y}^0 & & \text{in } \Omega. \end{cases} \tag{1}$$

Here, v is the control, $y^0 \in L^2(\Omega)^N$ and $a \in \mathbf{R}$ and $B \in \mathbf{R}^N$ are constants. Furthermore, we suppose that \hat{y}^0 is unknown with $\|\hat{y}^0\|_{L^2(\Omega)^N} = 1$ and that τ is a small unknown real number. Then, the interpretation of system (1) is that y is the velocity of the particles of an incompressible fluid, v is a localized source (where we have access to the fluid) to be chosen, f is another source and the initial state of the fluid is partially unknown.

In general, our task is to insensitize a functional J_τ (which is called *sentinel*) by means of the control v . That is to say, we have to find a control v such that the influence of the unknown data $\tau \hat{y}^0$ is not perceptible for J_τ (see (4) below).

In the literature, the usual functional is given by the L^2 norm of the state (see [15,1,3] or [16], for instance). Here, we are not only interested in insensitizing the L^2 norm of the state (solution of (1)) but also the L^2 norm of its curl($\nabla \times y$). Thus, let us introduce the functionals

$$J_{1,\tau}(y) = \iint_{\mathcal{O} \times (0,T)} |y|^2 \, dx \, dt \tag{2}$$

and

$$J_{2,\tau}(y) = \iint_{\mathcal{O} \times (0,T)} |\nabla \times y|^2 \, dx \, dt, \tag{3}$$

where y is the solution of (1).

This kind of problems was first considered by J.-L. Lions in [15], where a lot of another interesting questions concerning insensitizing controls are posed.

Our objective is to find a control v_i such that the presence of the unknown data is imperceptible for $J_{i,\tau}$, that is to say, such that

$$\frac{\partial J_{i,\tau}(y_i)}{\partial \tau} \Big|_{\tau=0} = 0 \quad \text{for all } \hat{y}^0 \in L^2(\Omega)^N \quad \text{such that} \quad \|\hat{y}^0\|_{L^2(\Omega)^N} = 1, \tag{4}$$

for $i = 1, 2$, where y_i is the solution of (1) associated to v_i . If this holds, we will say that the control v_i insensitizes the functional $J_{i,\tau}$ ($i = 1, 2$).

Usually, insensitizing problems are formulated in an equivalent way as a controllability problem of a cascade system (see, for instance, [14] for a rigorous deduction of this fact). Indeed, if we consider the adjoint state of (1) (or apply Lagrange principle), one can see that condition (4) is equivalent to $z|_{t=0} \equiv 0$ in Ω , where z together with w fulfills

$$\begin{cases} w_t - \Delta w + aw + B \cdot \nabla w + \nabla p^0 = v1_\omega + f, & \nabla \cdot w = 0 & \text{in } Q, \\ -z_t - \Delta z + az - B \cdot \nabla z + \nabla q = w1_\mathcal{O}, & \nabla \cdot z = 0 & \text{in } Q, \\ w = 0, \quad z = 0 & & \text{on } \Sigma, \\ w|_{t=0} = y^0, \quad z|_{t=T} = 0 & & \text{in } \Omega \end{cases} \tag{5}$$

for $i = 1$ and

$$\begin{cases} w_t - \Delta w + aw + B \cdot \nabla w + \nabla p^0 = v1_\omega + f, & \nabla \cdot w = 0 & \text{in } Q, \\ -z_t - \Delta z + az - B \cdot \nabla z + \nabla q = \nabla \times ((\nabla \times w)1_\mathcal{O}), & \nabla \cdot z = 0 & \text{in } Q, \\ w = 0, \quad z = 0 & & \text{on } \Sigma, \\ w|_{t=0} = y^0, \quad z|_{t=T} = 0 & & \text{in } \Omega \end{cases} \tag{6}$$

for $i = 2$. Here, we have denoted $w \equiv y|_{\tau=0}$ and $p^0 \equiv p|_{\tau=0}$.

As we said above, all known results around this subject concern parabolic systems of the heat kind. In [1], the authors prove the existence of ε -insensitizing controls (i.e., such that $|\partial_\tau J_{1,\tau}(y)|_{\tau=0}| \leq \varepsilon$) for solutions of a semilinear

heat system with C^1 and globally Lipschitz nonlinearities. In [3], the author proved the existence of insensitizing controls for the same system. For an extension of this results to more general nonlinearities, see [2] and the references therein. Recently, it has been proved in [10] the existence of insensitizing controls for the functional

$$\tilde{J}_\tau(y) = \iint_{\mathcal{O} \times (0, T)} |\nabla y|^2 \, dx \, dt$$

where y is the solution of a heat equation with potentials. Also in this reference, some controllability results for systems of two parabolic equations were considered when a double coupling occurs (the solution of each equation appears in the right-hand side of the other) by means of some combinations of derivatives; first order space derivatives in one equation and second order space derivatives in the other one. See [10] for the details.

All along this paper we will suppose that $\omega \cap \mathcal{O} \neq \emptyset$. This is a condition that has always been imposed in the literature as long as insensitizing controls are concerned. Recently, for the (simpler) situation where we look for a ε -insensitizing control and the functional $J_{1,\tau}$, it has been demonstrated that condition $\omega \cap \mathcal{O} \neq \emptyset$ is not necessary for solutions of linear heat equations (see [4]).

The controllability result for system (5) is given in the following theorem:

Theorem 1. *Let $m > 3$ be a real number and $y^0 \equiv 0$. Then, there exists a constant $\bar{C} > 0$ depending on $\Omega, \omega, \mathcal{O}, T, a$ and B such that for any $f \in L^2(Q)^N$ satisfying $\|e^{\bar{C}/t^m} f\|_{L^2(Q)^N} < +\infty$, there exists a control v_1 such that the corresponding solution (w, p^0, z, q) of (5) satisfies $z|_{t=0} \equiv 0$ in Ω .*

Corollary 2. *There exists insensitizing controls v_1 of the functional $J_{1,\tau}$ given by (2).*

Next, we state the controllability result for system (6):

Theorem 3. *Under the same assumptions of Theorem 1, there exists a control v_2 such that the corresponding solution (w, p^0, z, q) of (6) satisfies $z|_{t=0} \equiv 0$ in Ω .*

Corollary 4. *There exists insensitizing controls v_2 of the functional $J_{2,\tau}$ given by (3).*

Remark 1. The same results stated in Theorems 1 and 3 hold when a and B are functions which depends only on the time variable t and are in $L^\infty(0, T)$. The proof of this fact is direct from that of Theorems 1 and 3.

The proofs of Theorems 1 and 3 will be given in Section 3 but separately, even if the method to solve both problems is similar in some sense. The reason is very simple; there is no reason to think that, once Theorem 1 has been established, Theorem 3 also holds. Actually, the proof of Theorem 3 is much more intrinsic. We will try to make this precise in the sequel.

Let us briefly explain the difficulties a controllability result for systems (5) and (6) possesses. For this, we introduce the associated adjoint systems:

$$\begin{cases} -\varphi_t - \Delta\varphi + a\varphi - B \cdot \nabla\varphi + \nabla\pi = \psi 1_{\mathcal{O}}, & \nabla \cdot \varphi = 0 & \text{in } Q, \\ \psi_t - \Delta\psi + a\psi + B \cdot \nabla\psi + \nabla h = 0, & \nabla \cdot \psi = 0 & \text{in } Q, \\ \varphi = 0, \quad \psi = 0 & & \text{on } \Sigma, \\ \varphi|_{t=T} = 0, \quad \psi|_{t=0} = \psi^0 & & \text{in } \Omega \end{cases} \tag{7}$$

and

$$\begin{cases} -\varphi_t - \Delta\varphi + a\varphi - B \cdot \nabla\varphi + \nabla\pi = \nabla \times ((\nabla \times \psi) 1_{\mathcal{O}}), & \nabla \cdot \varphi = 0 & \text{in } Q, \\ \psi_t - \Delta\psi + a\psi + B \cdot \nabla\psi + \nabla h = 0, & \nabla \cdot \psi = 0 & \text{in } Q, \\ \varphi = 0, \quad \psi = 0 & & \text{on } \Sigma, \\ \varphi|_{t=T} = 0, \quad \psi|_{t=0} = \psi^0 & & \text{in } \Omega, \end{cases} \tag{8}$$

respectively.

It is by now classical to prove that the null controllability results we want to prove for systems (5) and (6) are equivalent to the following *observability inequality*:

$$\iint_{\mathcal{Q}} e^{-C_0/t^m} |\varphi|^2 dx dt \leq C \iint_{\omega \times (0, T)} |\varphi|^2 dx dt, \quad (9)$$

for the solutions of (7) and (8), where m is some positive number and C and C_0 are two positive constants depending on Ω , ω , \mathcal{O} , T , a and B but independent of ψ^0 (see, for instance, [11] or [9]).

The main idea one usually follows in order to prove (9) is a combination of observability inequalities for φ and ψ (as solutions of Stokes equations) and try to eliminate the local term (concentrated in $\omega \times (0, T)$) concerning ψ , resulting from the application of an observability inequality for ψ . Thus, one has to establish a local estimate of the kind

$$\iint_{\tilde{\omega} \times (0, T)} |\psi|^2 dx dt \leq C \iint_{\omega \times (0, T)} |\varphi|^2 dx dt, \quad \tilde{\omega} \subset \omega$$

for both the solutions of (7) and (8). When one tries to prove this inequality (using, of course, the equation satisfied by φ), one finds that the pressure term π is always involved. Indeed, for instance in the simpler situation of system (7), we find

$$\iint_{\tilde{\omega} \times (0, T)} |\psi|^2 dx dt = \iint_{\tilde{\omega} \times (0, T)} \psi (-\varphi_t - \Delta \varphi + a\varphi - B \cdot \nabla \varphi + \nabla \pi) dx dt,$$

provided that $\tilde{\omega} \subset \mathcal{O}$. Hence, one of the terms we have to estimate is

$$\iint_{\tilde{\omega} \times (0, T)} \psi \nabla \pi dx dt,$$

which can never be estimated just in terms of a local integral (in an open set which does not ‘touch’ the boundary) of $|\varphi|^2$.

This means that we have to avoid having a term like

$$\iint_{\tilde{\omega} \times (0, T)} |\psi|^2 dx dt$$

in the right-hand side of an observability inequality for ψ . This is exactly what we do. In fact, for the solutions of system (7) (resp. (8)), we obtain an observability inequality of the following kind:

$$\iint_{\mathcal{Q}} e^{-C_0/t^m} |\psi|^2 dx dt \leq C \iint_{B_0 \times (0, T)} |\nabla \times \psi|^2 dx dt$$

(resp.

$$\iint_{\mathcal{Q}} e^{-C_1/t^m} |\psi|^2 dx dt \leq C \iint_{B_0 \times (0, T)} |\nabla \times \Delta \psi|^2 dx dt),$$

with $B_0 \subset \omega \cap \mathcal{O}$. More comments about the obtention of these inequalities will be given in Section 3.

Once these estimates have been obtained, one can use the equations satisfied by $\nabla \times \varphi$ (resp. $\nabla \times \Delta \varphi$) in $B_0 \times (0, T)$ (which do not contain any pressure term!) and so expect that the previous integrals are bounded just in terms of the local L^2 norm of φ .

Remark 2. Is this result true when the coefficients a and B depend on the space variable? We observe here that, in this situation, not even the following unique continuation property is known:

$$\varphi = 0 \quad \text{in } \omega \times (0, T) \Rightarrow \psi, \varphi \equiv 0 \quad \text{in } \Omega \times (0, T).$$

• The second objective of this paper concerns the coupling of two Stokes systems. We will prove that we can drive both velocity vector fields to zero at any time $T > 0$ when controlling just one of the two solutions. We will distinguish two situations, according to the coupling terms.

On the one hand, we consider the following controllability system:

$$\begin{cases} y_t - \Delta y + P_1(x, t; D)y + \nabla p = v1_\omega + P_2(x, t; D)z & \text{in } Q, \\ z_t - \Delta z + e_0z - (E_0 \cdot \nabla)z + \nabla q = m_0 \nabla \times y & \text{in } Q, \\ \nabla \cdot y = 0, \quad \nabla \cdot z = 0 & \text{in } Q, \\ y = 0, \quad z = 0 & \text{on } \Sigma, \\ y|_{t=0} = y^0, \quad z|_{t=0} = z^0 & \text{in } \Omega, \end{cases} \tag{10}$$

where $y^0, z^0 \in H$, $e_0, m_0 \in \mathbf{R}$, $E_0 \in \mathbf{R}^N$ and $P_j(x, t; D)$ are general partial differential operators of order $j = 1, 2$ with Lipschitz coefficients:

$$\begin{cases} (P_j(x, t; D)w)_i = \sum_{k=1}^N (A_{ik}^j(x, t)w_k - \partial_k(B_k^j(x, t)w_i) - \partial_i(B_k^j(x, t)w_k)) \\ \quad + (j - 1) \sum_{k,l=1}^N (\partial_{kl}(M_{kl}(x, t)w_i) + \partial_{ki}(M_{kl}(x, t)w_l)) \\ \quad i = 1, \dots, N, \quad A^j, M \in L^\infty(0, T; W^{2,\infty}(\Omega)^{N \times N}), \quad B^j \in L^\infty(0, T; W^{2,\infty}(\Omega)^N). \end{cases} \tag{11}$$

Observe that the last term (differential operator of order 2) just concerns the definition of $P_2(x, t; D)$.

On the other hand, we design by $Q_1(t, x; D)$ again a general partial differential operator of first order with coefficients in $L^\infty(0, T; W^{3,\infty}(\Omega)^N)$ and we consider this other system:

$$\begin{cases} y_t - \Delta y + P_1(t, x; D)y + \nabla p = v1_\omega + Q_1(t, x; D)z & \text{in } Q, \\ z_t - \Delta z + e_1z - (E_1 \cdot \nabla)z + \nabla q = m_1 \Delta y & \text{in } Q, \\ \nabla \cdot y = 0, \quad \nabla \cdot z = 0 & \text{in } Q, \\ y = 0, \quad z = 0 & \text{on } \Sigma, \\ y|_{t=0} = y^0, \quad z|_{t=0} = z^0 & \text{in } \Omega, \end{cases} \tag{12}$$

where $y^0, z^0 \in H$ and $e_1, m_1 \in \mathbf{R}$ and $E_1 \in \mathbf{R}^N$.

As explained above, in both situations our objective is to find a control v such that

$$y|_{t=T} = 0, \quad z|_{t=T} = 0 \quad \text{in } \Omega. \tag{13}$$

As long as Stokes systems are concerned, the exact null controllability was established in [11] as a previous result for proving the local exact controllability of the Navier–Stokes system. An improvement of this result was later presented in [6]. On the other hand, we do not know any result concerning coupled Stokes systems.

The reason why we consider a coupling with first and second order derivative terms is explained with detail in [10] in the context of the null controllability of systems of two heat equations. In fact, these are the greatest orders of derivatives one can set in the coupling terms in order to control both variables with just one (N scalar) control force. This fact was checked for the case of heat equations and so we will prove here that the same occurs for the Stokes system.

Our precise results are the following:

Theorem 5. *Let $y^0, z^0 \in H$ and assume that the coefficients of the differential operators P_1 and P_2 satisfy*

$$A^j, M \in L^\infty(0, T; W^{2,\infty}(\Omega)^{N \times N}), \quad B^j \in L^\infty(0, T; W^{2,\infty}(\Omega)^N).$$

Then, there exists a control $v \in L^2(Q)^N$ such that the solution (y, p, z, q) of system (10) satisfies (13).

Theorem 6. *Let $y^0, z^0 \in H$ and assume that the coefficients of P_1 and P_2 satisfy the hypotheses in Theorem 5 and the coefficients of Q_1 belong to $L^\infty(0, T; W^{3,\infty}(\Omega)^N)$. Then, there exists a control $v \in L^2(Q)^N$ such that the solution (y, p, z, q) of system (12) satisfies (13).*

Remark 3. The same results stated in Theorems 5 and 6 hold when e_i, E_i and m_i ($i = 1, 2$) are functions of the time variable belonging to the space $L^\infty(0, T)$.

Remark 4. The same results stated in Theorems 5 and 6 hold when the coupling term $\nabla \times y$ is substituted by any first order operator whose L^2 norm constitutes a norm in $H^1(\Omega)^N$ for the functions $y \in V$ (case of Theorem 5) and when Δy is substituted by any second order operator whose L^2 norm constitutes a norm of y in $H^2(\Omega)^N$ for the functions $y \in H^2(\Omega)^N \cap V$ (case of Theorem 6).

The proofs of Theorems 5 and 6 will again rely on suitable observability estimates for the adjoint systems associated to (10) and (12). All the details of the proofs will be provided in Section 4.

This paper is organized as follows. In Section 2, we present some technical results, most of them known, which will be constantly used in the proofs of Theorems 1, 3, 5 and 6. In Section 3, we provide the proof of Theorems 1 and 3. Next, in Section 4, we prove Theorems 5 and 6 and we end with the proof of a technical result (stated in Section 2) in Appendix A.

2. Some previous results

For the proof of the observability inequalities needed to establish the controllability results stated in the previous Theorems, we will follow a classical approach, consisting of obtaining a suitable weighted-like estimate (so-called *Carleman estimate*) for the associated adjoint systems. For a systematic use of this kind of estimates see, for instance, [11] or [9].

In order to establish these Carleman inequalities, we need to define some weight functions:

$$\alpha(x, t) = \frac{\exp\{\frac{k(m+1)}{m}\lambda\|\eta^0\|_\infty\} - \exp\{\lambda(k\|\eta^0\|_\infty + \eta^0(x))\}}{t^m(T-t)^m},$$

$$\alpha^*(t) = \max_{x \in \bar{\Omega}} \alpha(x, t) = \alpha|_{\partial\Omega}(x, t), \quad \xi(x, t) = \frac{e^{\lambda(k\|\eta^0\|_\infty + \eta^0(x))}}{t^m(T-t)^m}, \tag{14}$$

$$\xi^*(t) = \min_{x \in \bar{\Omega}} \xi(x, t), \quad \hat{\alpha}(t) = \min_{x \in \bar{\Omega}} \alpha(x, t), \quad \hat{\xi}(t) = \max_{x \in \bar{\Omega}} \xi(x, t),$$

where $m > 3$ and $k > m$ are fixed. Here, $\eta^0 \in C^2(\bar{\Omega})$ satisfies

$$|\nabla\eta^0| \geq K > 0 \text{ in } \Omega \setminus \bar{\omega_0}, \quad \eta^0 > 0 \text{ in } \Omega \text{ and } \eta^0 \equiv 0 \text{ on } \partial\Omega, \tag{15}$$

with $\emptyset \neq \omega_0 \subset \omega \cap \mathcal{O}$ an open set. The proof of the existence of such a function η^0 is given in [9]. Weights of the kind (14) were first considered in [9]. In its present form, these weights have already been used in [7] in order to obtain Carleman estimates for the three-dimensional micropolar fluid model and later in [10] for the controllability of strongly coupled parabolic equations.

Accordingly, we define $I_0(s, \lambda; \cdot)$ as follows:

$$I_0(s, \lambda, g) := s\lambda^2 \iint_Q e^{-2s\alpha} \xi |\nabla g|^2 \, dx \, dt + s^3\lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |g|^2 \, dx \, dt.$$

From this expression, we also introduce

$$I(s, \lambda; g) := s^{-1} \iint_Q e^{-2s\alpha} \xi^{-1} (|g_t|^2 + |\Delta g|^2) \, dx \, dt + I_0(s, \lambda, g). \tag{16}$$

Now, we state all technical results. The first one concerns the Laplace operator:

Lemma 1. *Let $\gamma(x) = \exp\{\lambda\eta^0(x)\}$ for $x \in \Omega$ and let $u \in H_0^1(\Omega)$. Then, there exists a positive constant $K(\Omega, \omega_0)$ such that*

$$\tau^3\lambda^4 \int_\Omega e^{2\tau\gamma} \gamma^3 |u|^2 \, dx + \tau\lambda^2 \int_\Omega e^{2\tau\gamma} \gamma |\nabla u|^2 \, dx \leq K \int_{\omega_0} e^{2\tau\gamma} |\Delta u|^2 \, dx, \tag{17}$$

for any $\lambda, \tau \geq K$.

The proof of this lemma can be readily deduced from the corresponding result for parabolic equations included in [9].

The second estimate holds for energy solutions of heat equations with nonhomogeneous Neumann boundary conditions:

Lemma 2. *Let $u^0 \in L^2(\Omega)$, $f_1 \in L^2(Q)$, $f_2 \in L^2(Q)^N$ and $f_3 \in L^2(\Sigma)$. Then, there exists a constant $K(\Omega, \omega_0) > 0$ such that the weak solution u of*

$$\begin{cases} u_t - \Delta u = f_1 + \nabla \cdot f_2 & \text{in } Q, \\ \frac{\partial u}{\partial n} + f_2 \cdot n = f_3 & \text{on } \Sigma, \\ u|_{t=0} = u^0 & \text{in } \Omega \end{cases} \tag{18}$$

satisfies

$$\begin{aligned} I_0(s, \lambda; u) \leq & K \left(s^3 \lambda^4 \iint_{\omega_0 \times (0, T)} e^{-2s\alpha} \xi^3 |u|^2 \, dx \, dt + \iint_Q e^{-2s\alpha} |f_1|^2 \, dx \, dt \right. \\ & \left. + s^2 \lambda^2 \iint_Q e^{-2s\alpha} \xi^2 |f_2|^2 \, dx \, dt + s \lambda \iint_\Sigma e^{-2s\alpha^*} \xi^* |f_3|^2 \, d\sigma \, dt \right) \end{aligned} \tag{19}$$

for any $\lambda \geq K$ and $s \geq K(T^{2m} + T^{2m-1})$.

Let us recall the definition of a weak solution: we say that u is a *weak solution* to (18) if it satisfies

$$\begin{cases} u \in L^2(0, T; H^1(\Omega)) \cap C^0([0, T]; L^2(\Omega)), \\ \langle u_t, v \rangle_{(H^1(\Omega))', H^1(\Omega)} + \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f_1(x, t) v \, dx \\ - \int_\Omega f_2(x, t) \cdot \nabla v \, dx + \int_{\partial\Omega} f_3(x, t) v \, d\sigma \quad \text{a.e. in } (0, T), \forall v \in H^1(\Omega), \\ u(x, 0) = u^0(x) \quad \text{in } \Omega. \end{cases} \tag{20}$$

It is well known that, for $f_1 \in L^2(Q)$, $f_2 \in L^2(Q)^N$, $f_3 \in L^2(\Sigma)$ and $u^0 \in L^2(\Omega)$, (18) possesses exactly one weak solution u .

Lemma 2 was essentially proved in [5]. In fact, the inequality proved there concerns the same weight functions as in (19) but with $m = 1$. Then, one can follow the steps of the proof in [5] (see Theorem 1 in that reference) and adapt the arguments just taking into account that

$$\partial_t \alpha := \alpha_t \leq K T \xi^{(m+1)/m} \quad \text{and} \quad \partial_{tt} \alpha := \alpha_{tt} \leq C T^2 \xi^{(m+2)/m}, \tag{21}$$

with $K > 0$ independent of s , λ and T .

The third estimate that we recall here concerns the solutions of Stokes systems.

Lemma 3. *Let $u^0 \in V$ and $f_4 \in L^2(Q)^N$. Then, there exists a constant $C(\Omega, \omega_0, T) > 0$ such that the solution $u \in L^2(0, T; H^2(\Omega)^N \cap V) \cap L^\infty(0, T; V)$ of*

$$\begin{cases} u_t - \Delta u + \nabla p = f_4, \quad \nabla \cdot u = 0 & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u|_{t=0} = u^0 & \text{in } \Omega \end{cases}$$

satisfies

$$I(s, \lambda; u) \leq C \left(s^{16} \lambda^{40} \iint_{\omega_0 \times (0, T)} e^{-8s\hat{\alpha} + 6s\alpha^*} (\hat{\xi})^{16} |u|^2 \, dx \, dt + s^{15/2} \lambda^{20} \iint_Q e^{-4s\hat{\alpha} + 2s\alpha^*} (\hat{\xi})^{15/2} |f_4|^2 \, dx \, dt \right) \tag{22}$$

for any $s, \lambda \geq C$.

In a similar form, this lemma was proved in [6]. In fact, the weight functions considered in [6] were the ones in (14) but with $m = 4$. Again, one can follow the steps of the proof in [6] (Theorem 1) and, taking into account (21), (22) is readily deduced.

The fourth and last estimate we will need is a new Carleman inequality for the Stokes system which concerns the energy solutions of Stokes systems:

Lemma 4. *Let $u^0 \in H$, $f_5 \in L^2(Q)^N$ and $f_6 \in L^2(Q)^{N \times N}$. Then, there exists a constant $C(\Omega, \omega_0, T) > 0$ such that the weak solution $u \in L^2(0, T; V) \cap L^\infty(0, T; L^2(\Omega)^N)$ of*

$$\begin{cases} u_t - \Delta u + \nabla h = f_5 + \nabla \cdot f_6, & \nabla \cdot u = 0 & \text{in } Q, \\ u = 0 & & \text{on } \Sigma, \\ u|_{t=0} = u^0 & & \text{in } \Omega \end{cases} \tag{23}$$

satisfies

$$\begin{aligned} & s^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |u|^2 \, dx \, dt + s^2 \lambda^4 \iint_Q e^{-2s\alpha^*} (\xi^*)^{2-1/m} |\nabla u|^2 \, dx \, dt \\ & \leq C \left(s^{16} \lambda^{40} \iint_{\omega_0 \times (0, T)} e^{-8s\hat{\alpha} + 6s\alpha^*} (\hat{\xi})^{16} |u|^2 \, dx \, dt + s^{15/2} \lambda^{20} \iint_Q e^{-4s\hat{\alpha} + 2s\alpha^*} (\hat{\xi})^{15/2} |f_5|^2 \, dx \, dt \right. \\ & \quad \left. + s^{47/4} \lambda^{30} \iint_Q e^{-6s\hat{\alpha} + 4s\alpha^*} (\hat{\xi})^{47/4} |f_6|^2 \, dx \, dt \right) \end{aligned} \tag{24}$$

for any $s, \lambda \geq C$.

We will prove this lemma in Appendix A, at the end of this paper.

3. Insensitizing controls for the Stokes system

In this section, we will prove the existence of insensitizing controls for the functionals $J_{1,\tau}$ and $J_{2,\tau}$ (given by (2) and (3), respectively) associated to the Stokes system (1). As we saw in the introduction, we can restrict ourselves to prove Theorems 1 and 3 respectively.

Accordingly, we concentrate in the corresponding adjoint systems

$$\begin{cases} -\varphi_t - \Delta \varphi + a\varphi - B \cdot \nabla \varphi + \nabla \pi = \psi 1_Q, & \nabla \cdot \varphi = 0 & \text{in } Q, \\ \psi_t - \Delta \psi + a\psi + B \cdot \nabla \psi + \nabla h = 0, & \nabla \cdot \psi = 0 & \text{in } Q, \\ \varphi = 0, \quad \psi = 0 & & \text{on } \Sigma, \\ \varphi|_{t=T} = 0, \quad \psi|_{t=0} = \psi^0 & & \text{in } \Omega \end{cases} \tag{25}$$

and

$$\begin{cases} -\varphi_t - \Delta \varphi + a\varphi - B \cdot \nabla \varphi + \nabla \pi = \nabla \times ((\nabla \times \psi) 1_Q), & \nabla \cdot \varphi = 0 & \text{in } Q, \\ \psi_t - \Delta \psi + a\psi + B \cdot \nabla \psi + \nabla h = 0, & \nabla \cdot \psi = 0 & \text{in } Q, \\ \varphi = 0, \quad \psi = 0 & & \text{on } \Sigma, \\ \varphi|_{t=T} = 0, \quad \psi|_{t=0} = \psi^0 & & \text{in } \Omega, \end{cases} \tag{26}$$

where $\psi^0 \in L^2(\Omega)^N$. As explained in the introduction, in the framework of controllability it is classical to see that the null controllability property for system (5) (resp. (6)) is equivalent to the following observability inequality for the solutions of (25) (resp. (26)):

$$\iint_Q e^{-C_2/t^m} |\varphi|^2 \, dx \, dt \leq C \iint_{\omega \times (0, T)} |\varphi|^2 \, dx \, dt, \tag{27}$$

for certain positive constants C_2, C and some positive m independent of ψ^0 .

With this notation, we can prove the following result:

Proposition 1. *There exists a positive constant C which depends on Ω , ω and T such that*

$$\tilde{I}(s, \lambda; \psi) + s^2 \lambda^4 \iint_Q (s e^{-3s\alpha} \xi^3 |\varphi|^2 + e^{-3s\alpha^*} (\xi^*)^{2-1/m} |\nabla \varphi|^2) dx dt \leq C E(s, \lambda; \varphi) \tag{28}$$

for any $s, \lambda \geq C$, where $\tilde{I}(s, \lambda; \psi) = I_0(s, \lambda; \nabla \times \psi)$ and

$$E(s, \lambda; \varphi) = s^{15} \lambda^{16} \iint_{\omega \times (0, T)} e^{-s\alpha} \xi^{15} |\varphi|^2 dx dt$$

for the solutions of system (25) and $\tilde{I}(s, \lambda; \psi) = I_0(s, \lambda; \nabla \times \Delta \psi)$ and

$$E(s, \lambda; \varphi) = s^{6+1/m} \lambda^4 \iint_{\omega \times (0, T)} e^{-4s\hat{\alpha} + 2s\alpha^*} (\hat{\xi})^{6+1/m} |\varphi|^2 dx dt$$

for the solutions of (26).

Remark 5. From the Carleman inequality (28), one can readily deduce the observability inequality (27). Indeed, it suffices to combine an energy type estimate for both $e^{-1/t^m} \varphi$ and $e^{-1/t^m} \psi$. As a consequence, the proofs of Theorems 1 and 3 are achieved.

In the next two paragraphs, we will prove Proposition 1, distinguishing if we deal with system (25) or (26). In both situations, the proof of (28) is divided in two steps. The first and more important one deal with the equation satisfied by ψ (which is independent of φ). In the second one, we combine estimates for both equations.

3.1. Case of system (25)

3.1.1. New Carleman Estimate for ψ

In this paragraph, we deal with the problem

$$\begin{cases} \psi_t - \Delta \psi + a\psi + B \cdot \nabla \psi + \nabla h = 0, & \nabla \cdot \psi = 0 & \text{in } Q, \\ \psi = 0 & & \text{on } \Sigma, \\ \psi|_{t=0} = \psi^0 & & \text{in } \Omega. \end{cases} \tag{29}$$

Recall that $a \in \mathbf{R}$ and $B \in \mathbf{R}^N$ are constants. One may also suppose that a, B depend on the time variable (see Remark 1 for more details).

For this system, we prove the following estimate:

Lemma 5. *There exists a positive constant K depending on Ω and ω_0 such that*

$$I_0(s, \lambda; \nabla \times \psi) \leq K s^3 \lambda^4 \iint_{\omega_0 \times (0, T)} e^{-2s\alpha} \xi^3 |\nabla \times \psi|^2 dx dt, \tag{30}$$

for any $\lambda \geq K$ and $s \geq K(T^{2m} + T^m)$.

Remark 6. Observe that, in particular, we deduce from this inequality the following well-known unique continuation property:

$$\nabla \times \psi = 0 \quad \text{in } \omega_0 \times (0, T) \Rightarrow \psi \equiv 0 \quad \text{in } \Omega \times (0, T). \tag{31}$$

As far as we know, it is new the fact that (31) can be quantified in terms of an inequality like (30) for the solutions of (29). On the other hand, we do not know if (31) holds when a and B are not constant with respect to the space variable.

Proof of Lemma 5. We first look at the equation satisfied by $\nabla \times \psi$:

$$(\nabla \times \psi)_t - \Delta(\nabla \times \psi) + a(\nabla \times \psi) + B \cdot \nabla(\nabla \times \psi) = 0 \quad \text{in } Q.$$

Observe that no boundary conditions are prescribed for $\nabla \times \psi$. At this point, we can apply Lemma 2 and deduce the existence of a constant $K = K(\Omega, \omega_0) > 0$ such that

$$I_0(s, \lambda; \nabla \times \psi) \leq K \left(s^3 \lambda^4 \iint_{\omega_0 \times (0, T)} e^{-2s\alpha} \xi^3 |\nabla \times \psi|^2 dx dt + s \lambda \iint_{\Sigma} e^{-2s\alpha^*} \xi^* \left| \frac{\partial(\nabla \times \psi)}{\partial n} \right|^2 \right) dx dt, \tag{32}$$

for any $\lambda \geq K$ and $s \geq K(T^{2m} + T^{2m-1})$. Recall that $e^{-2s\alpha^*} = \min_{x \in \bar{\Omega}} e^{-2s\alpha}$ (see (14)).

The next step will be to eliminate the last term in the right-hand side of (32). In order to do this, we introduce the function $(\psi^*, h^*) := (\eta(t)\psi, \eta(t)h)$, where

$$\eta(t) = s^{(1/2)-(1/m)} \lambda^2 e^{-s\alpha^*(t)} (\xi^*)^{(1/2)-(1/m)}(t) \tag{33}$$

is a function of $t \in (0, T)$. In view of (29), (ψ^*, h^*) fulfills the following Stokes system:

$$\begin{cases} \psi_t^* - \Delta \psi^* + a\psi^* + B \cdot \nabla \psi^* + \nabla h^* = \eta_t \psi, & \nabla \cdot \psi^* = 0 & \text{in } Q, \\ \psi^* = 0 & & \text{on } \Sigma, \\ \psi^*|_{t=0} = 0 & & \text{in } \Omega. \end{cases} \tag{34}$$

Thanks to the terms appearing in the left-hand side of (32), we are going to deduce that ψ^* is a very regular function. In fact, we have that $\eta_t \psi \in L^2(0, T; H_0^1(\Omega))$, since $\eta_t \nabla \times \psi \in L^2(\Omega)$ (recall that ψ is a divergence-free function) and

$$\begin{aligned} \|\eta_t \nabla \times \psi\|_{L^2(Q)^N} &\leq K T s^{(3/2)-(1/m)} \lambda^2 \|e^{-s\alpha^*} (\xi^*)^{3/2} \nabla \times \psi\|_{L^2(Q)^N} \\ &\leq K s^{3/2} \lambda^2 \|e^{-s\alpha^*} (\xi^*)^{3/2} \nabla \times \psi\|_{L^2(Q)^N}, \end{aligned}$$

for $s \geq C(T^m + T^{2m})$. The square of this last quantity is bounded by the left-hand side of (32), by definition of $I_0(s, \lambda; \cdot)$ (see (16)).

Using Lemma 6 below, we deduce that the solution of (34) satisfies $\psi^* \in L^2(0, T; (H^3 \cap H_0^1)(\Omega))$ and

$$\|\psi^*\|_{L^2(0, T; H^3(\Omega))}^2 = s^{1-2/m} \lambda^4 \int_0^T e^{-2s\alpha^*} (\xi^*)^{1-2/m} \|\psi\|_{H^3(\Omega)}^2 dt \leq K I_0(s, \lambda; \nabla \times \psi). \tag{35}$$

Taking this into account, by a simple integration by parts we deduce that

$$s^{2-1/m} \lambda^4 \int_0^T e^{-2s\alpha^*} (\xi^*)^{2-1/m} \|\psi\|_{H^2(\Omega)^N}^2 dt \leq K I_0(s, \lambda; \nabla \times \psi). \tag{36}$$

From (35) and (36), we obtain in particular that

$$s^{(3/2)-3/(2m)} \lambda^4 \int_0^T e^{-2s\alpha^*} (\xi^*)^{(3/2)-3/(2m)} \left\| \frac{\partial(\nabla \times \psi)}{\partial n} \right\|_{L^2(\partial\Omega)^N}^2 dt \leq K I_0(s, \lambda; \nabla \times \psi).$$

Since $m > 3$, this justifies that the last term in the right-hand side of (32) is absorbed by the left-hand side. As a conclusion, we obtain the desired inequality (30). \square

Lemma 6. Let $a \in \mathbf{R}$ and $B \in \mathbf{R}^N$ be constant and let $f \in L^2(0, T; V)$. Then, the unique solution (u, p) of the Stokes system

$$\begin{cases} u_t - \Delta u + au + B \cdot \nabla u + \nabla p = f & \text{in } Q, \\ \nabla \cdot u = 0 & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u|_{t=0} = 0 & \text{in } \Omega \end{cases} \tag{37}$$

satisfies $u \in L^2(0, T; H^3(\Omega)^N \cap V) \cap H^1(0, T; V)$ and there exists a constant $C > 0$ such that

$$\|u\|_{L^2(0,T;H^3(\Omega)^N)} + \|u\|_{H^1(0,T;H^1(\Omega)^N)} \leq C \|f\|_{L^2(0,T;H^1(\Omega)^N)}. \tag{38}$$

Let us give a sketch of the proof of this lemma. First, we recall that we already have $(u, p) \in L^2(0, T; H^2(\Omega)^N) \times L^2(0, T; H^1(\Omega))$ and the estimate

$$\|u\|_{L^2(0,T;H^2(\Omega)^N)} + \|u\|_{H^1(0,T;L^2(\Omega)^N)} + \|p\|_{L^2(0,T;H^1(\Omega))} \leq C \|f\|_{L^2(0,T;L^2(\Omega)^N)}. \tag{39}$$

Let us first suppose that $f \in C^\infty([0, T]; V)$. Then, if we prove the estimate (38) in this situation, the general case follows from a density argument. In order to simplify the notations, let us denote

$$A(u, p) = -\Delta u + au + B \cdot \nabla u + \nabla p.$$

Let us multiply the equation in (37) by $A(u_t, p_t) \in L^2(Q)$, integrate in Ω and integrate by parts. This yields

$$\int_{\Omega} |\nabla u_t|^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega} |Au|^2 dx = - \int_{\Omega} (au_t + B \cdot \nabla u_t) \cdot u_t dx + \int_{\Omega} \nabla u_t \cdot \nabla f dx + \int_{\Omega} (au_t + B \cdot \nabla u_t) \cdot f dx.$$

Here, we have used that f and u_t are elements of H . From this identity, using Young’s inequality and thanks to (39), we have that $u \in H^1(0, T; V)$ and

$$\|u\|_{H^1(0,T;H^1(\Omega)^N)} \leq C \|f\|_{L^2(0,T;H^1(\Omega)^N)}. \tag{40}$$

Now, regarding system (37) as a stationary Stokes system with right-hand side in V (see, for instance, [17]), one deduces $u \in L^2(0, T; H^3(\Omega)^N)$ and (38).

3.1.2. Carleman estimate for φ and conclusion

First, assuming ψ is given, we look at φ as the solution of

$$\begin{cases} -\varphi_t - \Delta \varphi + a\varphi - B \cdot \nabla \varphi + \nabla \pi = \psi 1_{\mathcal{O}}, & \nabla \cdot \varphi = 0 & \text{in } Q, \\ \varphi = 0 & & \text{on } \Sigma, \\ \varphi|_{t=T} = 0 & & \text{in } \Omega. \end{cases}$$

Here, we apply the Carleman estimate for the Stokes system proved in [6], which was presented in Lemma 3 above:

$$\begin{aligned} L(s, \lambda; \varphi) \leq C & \left(s^{16} \lambda^{40} \iint_{\omega_0 \times (0,T)} e^{-12s\hat{\alpha} + 9s\alpha^*} (\hat{\xi})^{16} |\varphi|^2 dx dt \right. \\ & \left. + s^{15/2} \lambda^{20} \iint_{\mathcal{O} \times (0,T)} e^{-6s\hat{\alpha} + 3s\alpha^*} (\hat{\xi})^{15/2} |\psi|^2 dx dt \right) \end{aligned} \tag{41}$$

for $s, \lambda \geq C$, where $L(s, \lambda; \cdot)$ is given by

$$\begin{aligned} L(s, \lambda; g) := & s^{-1} \iint_Q e^{-3s\alpha} \xi^{-1} (|g_t|^2 + |\Delta g|^2) dx dt + s\lambda^2 \iint_Q e^{-3s\alpha} \xi |\nabla g|^2 dx dt \\ & + s^3 \lambda^4 \iint_Q e^{-3s\alpha} \xi^3 |g|^2 dx dt. \end{aligned}$$

Observe that we have applied this result for smaller exponentials, that is to say, for $e^{-3s\alpha}$ instead of $e^{-2s\alpha}$.

Then, we easily see that the last integral in the right-hand side of (41) is bounded by $I_0(s, \lambda; \nabla \times \psi)$, as long as λ is large enough. In fact, if we denote $\hat{\alpha}(t) = \min_{x \in \bar{\Omega}} \alpha(x, t)$ and $\hat{\xi}(t) = \max_{x \in \bar{\Omega}} \xi(x, t)$ (see (14)), we have

$$\begin{aligned} s^{15/2} \lambda^{20} \iint_{\mathcal{O} \times (0,T)} e^{-6s\hat{\alpha} + 3s\alpha^*} (\hat{\xi})^{20} |\psi|^2 dx dt & \leq s^{15/2} \lambda^{20} \iint_Q e^{-6s\hat{\alpha} + 3s\alpha^*} (\hat{\xi})^{20} |\nabla \times \psi|^2 dx dt \\ & \leq Cs^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |\nabla \times \psi|^2 dx dt, \end{aligned}$$

for a suitable choice of $\lambda \geq C$.

Combining this with (30) and (41), we obtain

$$L(s, \lambda; \varphi) + I_0(s, \lambda; \nabla \times \psi) \leq C \left(s^{16} \lambda^{40} \iint_{\omega_0 \times (0, T)} e^{-12s\hat{\alpha} + 9s\alpha^*} (\hat{\xi})^{16} |\varphi|^2 dx dt + s^3 \lambda^4 \iint_{\omega_0 \times (0, T)} \xi^3 e^{-2s\alpha} |\nabla \times \psi|^2 dx dt \right), \quad (42)$$

for any $s, \lambda \geq C$. Now, since $\omega_0 \subset \mathcal{O}$, from the equation satisfied by φ , we find

$$\nabla \times \psi = -(\nabla \times \varphi)_t - \Delta(\nabla \times \varphi) + a\nabla \times \varphi - B \cdot \nabla(\nabla \times \varphi) \quad \text{in } \omega_0 \times (0, T).$$

Then, we plug this into the last integral in (42) and we obtain:

$$\begin{aligned} & s^3 \lambda^4 \iint_{\omega_0 \times (0, T)} e^{-2s\alpha} \xi^3 |\nabla \times \psi|^2 dx dt \\ &= s^3 \lambda^4 \iint_{\omega_0 \times (0, T)} e^{-2s\alpha} \xi^3 (\nabla \times \psi) (-(\nabla \times \varphi)_t - \Delta(\nabla \times \varphi)) dx dt \\ & \quad + s^3 \lambda^4 \iint_{\omega_0 \times (0, T)} e^{-2s\alpha} \xi^3 (\nabla \times \psi) (a\nabla \times \varphi - B \cdot \nabla(\nabla \times \varphi)) dx dt. \end{aligned}$$

We define a positive function $\theta \in C_c^2(\omega)$ such that $\theta \equiv 1$ in ω_0 . Then, the task turns to estimate

$$\begin{aligned} & s^3 \lambda^4 \iint_{\omega \times (0, T)} \theta e^{-2s\alpha} \xi^3 (\nabla \times \psi) (-(\nabla \times \varphi)_t - \Delta(\nabla \times \varphi)) dx dt \\ & \quad + s^3 \lambda^4 \iint_{\omega \times (0, T)} \theta e^{-2s\alpha} \xi^3 (\nabla \times \psi) (a\nabla \times \varphi - B \cdot \nabla(\nabla \times \varphi)) dx dt. \end{aligned}$$

After several integration by parts (getting all derivatives out of φ) with respect to both space and time, we get:

$$\begin{aligned} s^3 \lambda^4 \iint_{\omega \times (0, T)} \theta e^{-2s\alpha} \xi^3 |\nabla \times \psi|^2 dx dt &= s^3 \lambda^4 \iint_{\omega \times (0, T)} \theta (e^{-2s\alpha} \xi^3)_t (\nabla \times \psi) (\nabla \times \varphi) dx dt \\ & \quad - s^3 \lambda^4 \iint_{\omega \times (0, T)} \Delta(\theta e^{-2s\alpha} \xi^3) (\nabla \times \psi) (\nabla \times \varphi) dx dt \\ & \quad - 2s^3 \lambda^4 \iint_{\omega \times (0, T)} \nabla(\theta e^{-2s\alpha} \xi^3) \cdot \nabla(\nabla \times \psi) (\nabla \times \varphi) dx dt \\ & \quad + s^3 \lambda^4 \iint_{\omega \times (0, T)} B \cdot \nabla(\theta e^{-2s\alpha} \xi^3) (\nabla \times \psi) (\nabla \times \varphi) dx dt. \end{aligned}$$

Here, we have used the equation satisfied by ψ (see (29)) and the fact that θ has compact support in ω . As we have already pointed out, we have the following estimates for the weight functions:

$$(e^{-2s\alpha} \xi^3)_t \leq K T s e^{-2s\alpha} (\xi)^{4+1/m}, \quad \text{and} \quad \Delta(e^{-2s\alpha} \xi^3) \leq K s^2 \lambda^2 e^{-2s\alpha} \xi^5,$$

for $s \geq K T^{2m}$. With this, we obtain

$$s^3 \lambda^4 \iint_{\omega_0 \times (0, T)} e^{-2s\alpha} \xi^3 |\nabla \times \psi|^2 dx dt \leq \varepsilon I_0(s, \lambda; \nabla \times \psi) + C s^7 \lambda^8 \iint_{\omega \times (0, T)} e^{-2s\alpha} \xi^7 |\nabla \times \varphi|^2 dx dt. \quad (43)$$

Similar computations lead to

$$s^7 \lambda^8 \iint_{\omega \times (0, T)} e^{-2s\alpha} \xi^7 |\nabla \times \varphi|^2 \, dx \, dt \leq \varepsilon s^{-1} \iint_Q e^{-3s\alpha} \xi^{-1} |\Delta \varphi|^2 \, dx \, dt + s^{15} \lambda^{16} \iint_{\omega \times (0, T)} e^{-s\alpha} \xi^{15} |\varphi|^2 \, dx \, dt$$

which, combined with (42) and (43), gives the desired inequality (28).

3.2. Case of system (26)

As we have already mentioned at the beginning of this section, the strategy we follow in order to prove estimate (28) in this situation is similar to the previous one, so we will skip repeated arguments.

Precisely, the most important part of the proof concerns a new Carleman inequality for ψ , solution of the Stokes system (29). The inequality we have to prove for this system is much more intrinsic than the one stated in Lemma 5 (see (30)). We present it in the following lemma:

Lemma 7. *There exists a positive constant K depending on Ω and ω_0 such that*

$$\tilde{I}_j(s, \lambda; \psi) + I_0(s, \lambda; \nabla \times \Delta \psi) \leq K s^3 \lambda^4 \iint_{\omega_0 \times (0, T)} e^{-2s\alpha} \xi^3 |\nabla \times \Delta \psi|^2 \, dx \, dt, \tag{44}$$

for any $\lambda \geq K$ and $s \geq K(T^{2m} + T^m)$, where

$$\tilde{I}_j(s, \lambda; \psi) = s^{(6-j)+(3-j)/m} \lambda^4 \int_0^T e^{-2s\alpha^*} (\xi^*)^{(6-j)+(3-j)/m} \|\psi\|_{H^j(\Omega)^N}^2, \quad j \geq 3.$$

Analogously to the previous paragraph, the same comment corresponding to Remark 6 holds here.

Proof of Lemma 7. Again, we can find a complete heat equation satisfied by $\nabla \times \Delta \psi$:

$$(\nabla \times \Delta \psi)_t - \Delta(\nabla \times \Delta \psi) + a(\nabla \times \Delta \psi) + B \cdot \nabla(\nabla \times \Delta \psi) = 0 \quad \text{in } Q.$$

Then, we apply Lemma 2 to $\nabla \times \Delta \psi$ as solution of the previous heat equation and so there exists $K = K(\Omega, \omega_0) > 0$ such that

$$I_0(s, \lambda; \nabla \times \Delta \psi) \leq K \left(s^3 \lambda^4 \iint_{\omega_0 \times (0, T)} e^{-2s\alpha} \xi^3 |\nabla \times \Delta \psi|^2 \, dx \, dt + s \lambda \iint_{\Sigma} e^{-2s\alpha^*} \xi^* \left| \frac{\partial(\nabla \times \Delta \psi)}{\partial n} \right|^2 \right) \, dx \, dt, \tag{45}$$

for any $\lambda \geq K$ and $s \geq K(T^{2m} + T^{2m-1})$.

We observe that the lower order term appearing in $I_0(s, \lambda; \nabla \times \Delta \psi)$ and so in the left-hand side of (45) is

$$s^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |\nabla \times \Delta \psi|^2 \, dx \, dt \geq s^3 \lambda^4 \iint_Q e^{-2s\alpha^*} (\xi^*)^3 |\nabla \times \Delta \psi|^2 \, dx \, dt.$$

Regularity Result: The norm $\|\nabla \times \Delta \psi\|_{L^2(Q)^N}$ constitutes a norm of ψ in $L^2(0, T; H^3(\Omega)^N)$ and there exists a positive constant $K = K(\Omega)$ such that

$$\|\psi\|_{L^2(0, T; H^3(\Omega)^N)} \leq K \|\nabla \times \Delta \psi\|_{L^2(Q)^N}. \tag{46}$$

This is a fact that, in general, is not true for the set of functions ψ having null trace, but the fact that ψ solves a Stokes problem will help us to establish this property.

First, introducing the stream function ζ associated to the velocity vector field ψ ($\nabla \times \zeta = \psi$ and $\zeta \times n = 0$ on $\partial\Omega$, since Ω is a simply-connected set), we realize that on the one hand

$$-\iint_Q \zeta \cdot (\nabla \times \Delta \psi) \, dx \, dt = -\iint_Q (\nabla \times \zeta) \cdot \Delta \psi \, dx \, dt = \iint_Q |\nabla \psi|^2 \, dx \, dt \tag{47}$$

and, on the other hand

$$-\iint_Q \zeta \cdot (\nabla \times \Delta \psi) \, dx \, dt \leq \varepsilon \iint_Q |\zeta|^2 \, dx \, dt + K(\varepsilon) \iint_Q |\nabla \times \Delta \psi|^2 \, dx \, dt. \tag{48}$$

Using the continuous dependence of ζ with respect to ψ , we deduce from a combination of (47) and (48) that

$$\iint_Q |\nabla \psi|^2 \, dx \, dt \leq K \iint_Q |\nabla \times \Delta \psi|^2 \, dx \, dt. \tag{49}$$

Since $\Delta \psi = -\nabla \times (\nabla \times \psi)$ (recall that $\nabla \cdot \psi = 0$), using this last inequality, we also have

$$\iint_Q |\Delta \psi|^2 \, dx \, dt \leq K \iint_Q |\nabla \times \Delta \psi|^2 \, dx \, dt. \tag{50}$$

Taking into account (49) and (50), we realize that, since

$$\nabla \times \psi_t = \nabla \times \Delta \psi - a \nabla \times \psi - B \cdot \nabla \times \nabla \psi,$$

$\nabla \cdot \psi_t = 0$ and $\psi_t|_{\partial\Omega} = 0$, we have $\psi_t \in L^2(0, T; H^1(\Omega)^N)$ and

$$\|\psi_t\|_{L^2(0,T;H^1(\Omega)^N)}^2 \leq K \iint_Q |\nabla \times \Delta \psi|^2 \, dx \, dt. \tag{51}$$

Finally, we see $\psi(t)$ as a solution of the following stationary Stokes problem:

$$\begin{cases} -\Delta \psi + a \psi + B \cdot \nabla \psi + \nabla h = -\psi_t, & \nabla \cdot \psi = 0 & \text{in } \Omega, \\ \psi = 0 & & \text{on } \partial\Omega. \end{cases} \tag{52}$$

Since $\psi_t(t) \in H^1(\Omega)^N$ (almost everywhere $t \in (0, T)$), the solution of (52) verifies $\psi(t) \in H^3(\Omega)^N \cap H_0^1(\Omega)^N$ and

$$\|\psi(t)\|_{H^3(\Omega)^N} \leq K \|\psi_t(t)\|_{H^1(\Omega)^N} \quad \text{a.e. } t \in (0, T).$$

For the proof of this, see for instance, [13] or [17]. Now, from the fact that we actually have $\psi_t \in L^2(0, T; H^1(\Omega)^N)$ and estimate (51) holds, we deduce that $\psi \in L^2(0, T; H^3(\Omega)^N \cap H_0^1(\Omega)^N)$ and that (46) holds. The proof of the technical result is achieved.

Of course, if we consider the function $\eta_0(t) = s^{3/2} \lambda^2 e^{-s\alpha^*(t)} (\xi^*)^{3/2}(t)$ (which only depends on the time variable), the same analysis made above yields

$$\|\eta_0 \psi\|_{L^2(0,T;H^3(\Omega)^N)}^2 \leq K \|\eta_0 \nabla \times \Delta \psi\|_{L^2(Q)^N}^2 \leq K I_0(s, \lambda; \nabla \times \Delta \psi) \tag{53}$$

and so we can add the term $\|\eta_0 \psi\|_{L^2(0,T;H^3(\Omega)^N)}^2$ to the left-hand side of (45).

Next, we consider the couple $(\psi^*, h^*) = (\eta(t)\psi, \eta(t)h)$ where η was defined in (33). Consequently, (ψ^*, h^*) fulfill system (34). Observe that the weight function appearing in the right-hand side in the equation of (34) satisfies

$$|\eta_t| \leq K \eta_0 \quad \text{in } (0, T),$$

for $s \geq K(T^m + T^{2m})$, with $K = K(\Omega) > 0$ independent of T, s and λ . Then, (53) yields $\eta_t \psi \in L^2(0, T; H^3(\Omega)^N \cap H_0^1(\Omega)^N)$ with the same estimate as in (53) and so $\psi^* \in L^2(0, T; H^5(\Omega)^N \cap H_0^1(\Omega)^N)$ (one can follow the lines of the proof of Lemma 6 below) and

$$\|\psi^*\|_{L^2(0,T;H^5(\Omega)^N)}^2 = \lambda^4 \int_0^T e^{-2s\alpha^*} (s\xi^*)^{1-2/m} \|\psi(t)\|_{H^3(\Omega)^N}^2 \, dt \leq K I_0(s, \lambda; \nabla \times \Delta \psi). \tag{54}$$

A simple interpolation argument, yields

$$s^{2-1/m} \lambda^4 \int_0^T e^{-2s\alpha^*} (\xi^*)^{2-1/m} \|\psi(t)\|_{H^4(\Omega)^N}^2 \, dt \leq K I_0(s, \lambda; \nabla \times \Delta \psi). \tag{55}$$

From these two last inequalities, we get

$$s^{(3/2)-3/(2m)}\lambda^4 \int_0^T e^{-2s\alpha^*} (\xi^*)^{(3/2)-3/(2m)} \left\| \frac{\partial(\nabla \times \Delta\psi)(t)}{\partial n} \right\|_{L^2(\partial\Omega)^N}^2 dt \leq K I_0(s, \lambda; \nabla \times \Delta\psi).$$

Again, since $m > 3$, we can absorb the second term in the right-hand side of (45) and so we deduce (44).

The extra terms appearing in the left-hand side of (44) can be obtained successively applying classical regularity results for the Stokes system. Precisely, if

$$s^{(3-j/2)+(3-j)/(2m)}\lambda^2 e^{-s\alpha^*} (\xi^*)^{(3-j/2)+(3-j)/(2m)} \psi := \psi_j \in L^2(0, T; H^j(\Omega)^N),$$

then

$$s^{(2-j/2)+(2-j)/(2m)}\lambda^2 e^{-s\alpha^*} (\xi^*)^{(2-j/2)+(2-j)/(2m)} \psi := \psi_{j+2} \in L^2(0, T; H^{j+2}(\Omega)^N)$$

and there exists a constant $K = K(\Omega) > 0$ such that

$$\|\psi_{j+2}\|_{L^2(0,T;H^{j+2}(\Omega)^N)} \leq K \|\psi_j\|_{L^2(0,T;H^j(\Omega)^N)},$$

for $s \geq K(T^{2m} + T^m)$.

In order to conclude inequality (28) in this situation, we deal with the system fulfilled by φ :

$$\begin{cases} -\varphi_t - \Delta\varphi + a\varphi - B \cdot \nabla\varphi + \nabla\pi = \nabla \times ((\nabla \times \psi)1_{\mathcal{O}}), & \nabla \cdot \varphi = 0 & \text{in } \mathcal{Q}, \\ \varphi = 0 & & \text{on } \Sigma, \\ \varphi|_{t=T} = 0 & & \text{in } \Omega. \end{cases}$$

Here, we apply the Carleman estimate presented in Lemma 4 (and proved in the appendix):

$$\begin{aligned} & s^3 \lambda^4 \iint_{\mathcal{Q}} e^{-3s\alpha} \xi^3 |\varphi|^2 dx dt + s^2 \lambda^4 \iint_{\mathcal{Q}} e^{-3s\alpha^*} (\xi^*)^{2-1/m} |\nabla\varphi|^2 dx dt \\ & \leq C \left(s^{16} \lambda^{40} \iint_{\omega_0 \times (0,T)} e^{-12s\hat{\alpha}+9s\alpha^*} (\hat{\xi})^{16} |\varphi|^2 dx dt + s^{47/4} \lambda^{30} \iint_{\mathcal{O} \times (0,T)} e^{-9s\hat{\alpha}+6s\alpha^*} (\hat{\xi})^{47/4} |\nabla \times \psi|^2 dx dt \right), \end{aligned} \tag{56}$$

for $s, \lambda \geq C$. We have to mention the fact that in Lemma 4 we did not consider terms of order 0 and 1 in the Stokes operator, but obviously, the same result holds when lower order terms appear. Again, we remark here that we have applied the result in Lemma 4 for smaller exponentials $e^{-3s\alpha}$ instead of $e^{-2s\alpha}$.

Analogously as we proved in the previous paragraph, one can see that the last term in the right-hand side of (56) is bounded by $I_0(s, \lambda; \nabla \times \Delta\psi)$, as long as λ is large enough. Combining this with (44) and (56), we obtain

$$\begin{aligned} & s^2 \lambda^4 \iint_{\mathcal{Q}} (s e^{-3s\alpha} \xi^3 |\varphi|^2 + e^{-3s\alpha^*} (\xi^*)^{2-1/m} |\nabla\varphi|^2) dx dt + I_0(s, \lambda; \nabla \times \Delta\psi) \\ & \leq C \left(s^{16} \lambda^{40} \iint_{\omega_0 \times (0,T)} e^{-12s\hat{\alpha}+9s\alpha^*} (\hat{\xi})^{16} |\varphi|^2 dx dt + s^3 \lambda^4 \iint_{\omega_0 \times (0,T)} \xi^3 e^{-2s\alpha} |\nabla \times \Delta\psi|^2 dx dt \right), \end{aligned} \tag{57}$$

for any $s, \lambda \geq C$.

The last step is to eliminate the last term in (57). For this, it suffices to estimate

$$s^3 \lambda^4 \iint_{\omega_0 \times (0,T)} (\hat{\xi})^3 e^{-2s\hat{\alpha}} |\nabla \times \Delta\psi|^2 dx dt.$$

Using the fact that $\omega_0 \subset \mathcal{O}$ and the equation satisfied by φ , we find that

$$\nabla \times \Delta\psi = -(\nabla \times \varphi)_t - \nabla \times \Delta\varphi + a\nabla \times \varphi - B \cdot \nabla(\nabla \times \varphi) \quad \text{in } \omega_0 \times (0, T)$$

and so the integral term equals:

$$\iint_{\omega_0 \times (0,T)} \eta_1(t) (\nabla \times \Delta\psi) (-(\nabla \times \varphi)_t - \Delta(\nabla \times \varphi) + a\nabla \times \varphi - B \cdot \nabla(\nabla \times \varphi)) dx dt \tag{58}$$

with $\eta_1(t) = s^3 \lambda^4 e^{-2s\hat{\alpha}(t)} (\hat{\xi})^3(t)$, $t \in (0, T)$.

We consider again the function $\theta \in C_c^2(\omega)$ such that $\theta \equiv 1$ in ω_0 . We have

$$\begin{aligned} & \iint_{\omega \times (0, T)} \theta \eta_1(t) (\nabla \times \Delta \psi) (-(\nabla \times \varphi)_t - \Delta(\nabla \times \varphi) + a \nabla \times \varphi - B \cdot \nabla(\nabla \times \varphi)) \, dx \, dt \\ &= - \iint_{\omega \times (0, T)} \eta_{1,t} \nabla \times (\theta \nabla \times \Delta \psi) \cdot \varphi \, dx \, dt - \iint_{\omega \times (0, T)} \eta_1 \nabla \times (\Delta \theta \nabla \times \Delta \psi + 2 \nabla \theta \cdot \nabla(\nabla \times \Delta \psi)) \cdot \varphi \, dx \, dt \\ &+ \iint_{\omega \times (0, T)} \eta_1 \nabla \times ((B \cdot \nabla \theta) \nabla \times \Delta \psi) \cdot \varphi \, dx \, dt. \end{aligned}$$

Since $|(e^{-2s\hat{\alpha}}(\hat{\xi}^*)^3)_t| \leq K s e^{-2s\hat{\alpha}} \hat{\xi}^{4+1/m}$, thanks to (44), we deduce

$$\iint_{\omega_0 \times (0, T)} \eta_1 |\nabla \times \Delta \psi|^2 \, dx \, dt \leq \varepsilon (\tilde{I}_4(s, \lambda; \psi) + \tilde{I}_5(s, \lambda; \psi)) + s^{6+1/m} \lambda^4 \iint_{\omega \times (0, T)} e^{-4s\hat{\alpha} + 2s\alpha^*} (\hat{\xi})^{6+1/m} |\varphi|^2 \, dx \, dt.$$

The proof of Proposition 1 is finished.

4. Null controllability of coupled Stokes systems

Analogously as we did in the previous section, in order to prove the controllability results stated in Theorems 5 and 6, we will state the associated adjoint problems and we will prove suitable observability estimates.

In fact, by a formal integration by parts, we obtain both adjoint systems associated to (10) and (12)

$$\begin{cases} -\varphi_t - \Delta \varphi + P_1^*(x, t; D)\varphi + \nabla \pi = m_0 \nabla \times \psi & \text{in } Q, \\ -\psi_t - \Delta \psi + e_0 \psi + (E_0 \cdot \nabla)\psi + \nabla h = P_2^*(x, t; D)\varphi & \text{in } Q, \\ \nabla \cdot \varphi = 0, \quad \nabla \cdot \psi = 0 & \text{in } Q, \\ \varphi = 0, \quad \psi = 0 & \text{on } \Sigma, \\ \varphi|_{t=T} = \varphi^0, \quad \psi|_{t=T} = \psi^0 & \text{in } \Omega \end{cases} \tag{59}$$

and

$$\begin{cases} -\varphi_t - \Delta \varphi + P_1^*(t, x; D)\varphi + \nabla \pi = m_1 \Delta \psi & \text{in } Q, \\ -\psi_t - \Delta \psi + e_1 \psi + (E_1 \cdot \nabla)\psi + \nabla h = Q_1^*(t, x; D)\varphi & \text{in } Q, \\ \nabla \cdot \varphi = 0, \quad \nabla \cdot \psi = 0 & \text{in } Q, \\ \varphi = 0, \quad \psi = 0 & \text{on } \Sigma, \\ \varphi|_{t=T} = \varphi^0, \quad \psi|_{t=T} = \psi^0 & \text{in } \Omega, \end{cases} \tag{60}$$

respectively, with $\varphi^0, \psi^0 \in H$. Here, $P_j^*(x, t; D)$ (resp. $Q_1^*(t, x; D)$) is the formal adjoint operator of $P_j(x, t; D)$ ($j = 1, 2$) (resp. $Q_1(t, x; D)$) defined in (11):

$$(P_j^*(x, t; D)w)_i = \sum_{k=1}^N (A_{ik}^j(x, t)w_k + B_k^j(x, t)(\partial_k w_i + \partial_i w_k)) + (j-1) \sum_{k,l=1}^N M_{kl}(x, t)(\partial_{kl} w_i + \partial_{ki} w_l).$$

For these two systems, we shall prove the existence of a positive constant $C = C(\Omega, \omega, T)$ such that

$$\|\varphi|_{t=0}\|_{L^2(\Omega)^N}^2 + \|\psi|_{t=0}\|_{L^2(\Omega)^N}^2 \leq C \iint_{\omega \times (0, T)} |\varphi|^2 \, dx \, dt \tag{61}$$

for all $\varphi^0, \psi^0 \in H$. Then, it is readily seen that if (61) is satisfied, both systems (10) and (12) are null controllable and so Theorems 5 and 6 hold.

Remark 7. Let us observe that the same proofs of Theorems 1 and 3 cannot be developed here. In fact, in those situations when no term concerning φ appeared in the equation satisfied by ψ , we deduced Carleman inequalities for the Stokes equation of φ with right-hand sides having weight functions ‘smaller’ than the ones appearing in the left-hand side of the inequality of ψ . Obviously, this would not drive to the desired result in the present situation.

4.1. Carleman inequality for system (59)

Proposition 2. Let $\varphi^0 \psi^0 \in H$. Then, there exists a positive constant $K = K(\Omega, \omega)$ such that

$$\begin{aligned} \bar{I}(s, \lambda; \varphi) + \iint_Q e^{-2s\alpha} (s^2 \lambda^3 \xi^2 |\Delta \psi|^2 + \lambda |\nabla \Delta \psi|^2) \, dx \, dt \\ \leq K s^{15+12/m} \lambda^{16} \iint_{\omega \times (0, T)} e^{-10s\alpha + 8s\alpha^*} \xi^{15+12/m} |\varphi|^2 \, dx \, dt, \end{aligned} \tag{62}$$

for any $\lambda \geq K$ and any $s \geq K(T^m + T^{2m})$, where $\bar{I}(s, \lambda; \varphi)$ is defined in (66).

From the fact that (59) is well posed in the space $L^2(0, T; V) \cap L^\infty(0, T; H)$, we deduce in a classical way that the Carleman inequality (62) suffices to prove (61) and so also to prove Theorem 5.

Proof of Proposition 2. We start by considering the heat equation satisfied by $\Delta \varphi$:

$$-(\Delta \varphi)_t - \Delta(\Delta \varphi) = \nabla \times (\nabla \times (P_1^*(x, t; D)\varphi)) + m_0 \nabla \times \Delta \psi, \quad (t, x) \in (0, T) \times \Omega.$$

For the solution of this equation we apply the Carleman inequality presented in Lemma 2 and we obtain the existence of a constant $K(\Omega, \omega) > 0$ such that

$$\begin{aligned} I_0(s, \lambda; \Delta \varphi) \leq K \left(s^2 \lambda^2 \iint_Q e^{-2s\alpha} \xi^2 |\nabla \times (P_1^*(x, t; D)\varphi)|^2 \, dx \, dt + s \lambda \iint_{\Sigma} e^{-2s\alpha^*} \xi^* \left| \frac{\partial \Delta \varphi}{\partial n} \right|^2 \, d\sigma \, dt \right. \\ \left. + s \lambda \iint_{\Sigma} e^{-2s\alpha^*} \xi^* |(n \times \nabla \times)(P_1^*(x, t; D)\varphi)|^2 \, d\sigma \, dt + s^3 \lambda^4 \iint_{\omega_0 \times (0, T)} e^{-2s\alpha} \xi^3 |\Delta \varphi|^2 \, dx \, dt \right. \\ \left. + \iint_Q e^{-2s\alpha} |\nabla \times \Delta \psi|^2 \, dx \, dt \right), \end{aligned} \tag{63}$$

for any $\lambda \geq K$ and $s \geq K(T^{2m-1} + T^{2m})$.

Next, since $\varphi|_{\partial\Omega} = 0$, we can apply Lemma 1 to $\varphi(t)$ and we deduce that there exists $K(\Omega, \omega) > 0$ such that

$$\begin{aligned} \tau^6 \lambda^8 \int_{\Omega} e^{2\tau\gamma} \gamma^6 |\varphi(t)|^2 \, dx + \tau^4 \lambda^6 \int_{\Omega} e^{2\tau\gamma} \gamma^4 |\nabla \varphi(t)|^2 \, dx \\ \leq K \left(\tau^6 \lambda^8 \int_{\omega_0} e^{2\tau\gamma} \gamma^6 |\varphi(t)|^2 \, dx + \tau^3 \lambda^4 \int_{\Omega} e^{2\tau\gamma} \gamma^3 |\Delta \varphi(t)|^2 \, dx \right) \end{aligned}$$

for any $\lambda, \tau \geq K$. Then, if we multiply this inequality by

$$\exp \left\{ -2s \frac{\exp\{\frac{k(m+1)}{m} \lambda \|\eta^0\|_\infty\}}{t^m (T-t)^m} \right\},$$

we choose

$$\tau = s \frac{\exp\{\lambda k \|\eta^0\|_\infty\}}{t^m (T-t)^m}$$

and we integrate in $(0, T)$, we obtain

$$\begin{aligned} s^6 \lambda^8 \iint_Q e^{-2s\alpha} \xi^6 |\varphi|^2 \, dx \, dt + s^4 \lambda^6 \iint_Q e^{-2s\alpha} \xi^4 |\nabla \varphi|^2 \, dx \, dt \\ \leq K \left(s^6 \lambda^8 \iint_{\omega_0 \times (0, T)} e^{-2s\alpha} \xi^6 |\varphi|^2 \, dx \, dt + s^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |\Delta \varphi|^2 \, dx \, dt \right) \end{aligned} \tag{64}$$

for any $\lambda \geq K$ and any $s \geq KT^{2m}$. Then, if we combine this with (63), we can absorb the first term in the right-hand side of (63) (see (11)) and we obtain

$$\begin{aligned} \bar{I}(s, \lambda; \varphi) \leq & K \left(s\lambda \iint_{\Sigma} e^{-2s\alpha^*} \xi^* \left(\left| \frac{\partial \Delta \varphi}{\partial n} \right|^2 + |(n \times \nabla \times)(P_1^*(x, t; D)\varphi)|^2 \right) d\sigma dt \right. \\ & \left. + \iint_Q e^{-2s\alpha} |\nabla \times \Delta \psi|^2 dx dt + s^3 \lambda^4 \iint_{\omega_0 \times (0, T)} e^{-2s\alpha} \xi^3 (s^3 \lambda^4 \xi^3 |\varphi|^2 + |\Delta \varphi|^2) dx dt \right) \end{aligned} \tag{65}$$

for $\lambda \geq K$ and $s \geq C(T^{2m-1} + T^{2m})$, where

$$\bar{I}(s, \lambda; \varphi) = s^6 \lambda^8 \iint_Q e^{-2s\alpha} \xi^6 |\varphi|^2 dx dt + s^4 \lambda^6 \iint_Q e^{-2s\alpha} \xi^4 |\nabla \varphi|^2 dx dt + I_0(s, \lambda; \Delta \varphi). \tag{66}$$

We come back now to the equation of ψ . Applying again the operator $-\nabla \times (\nabla \times \cdot)$, we have:

$$-(\Delta \psi)_t - \Delta(\Delta \psi) + e_0 \Delta \psi + (E_0 \cdot \nabla) \Delta \psi = -\nabla \times (\nabla \times (P_2^*(x, t; D)\varphi)) \quad \text{in } \Omega \times (0, T).$$

Applying thus Lemma 2 to $\Delta \psi$, we obtain

$$\begin{aligned} & \lambda \iint_Q e^{-2s\alpha} (s^2 \lambda^2 \xi^2 |\Delta \psi|^2 + |\nabla \Delta \psi|^2) dx dt \\ & \leq K \left(\iint_{\Sigma} e^{-2s\alpha^*} \left| \frac{\partial \Delta \psi}{\partial n} \right|^2 d\sigma dt + \iint_{\Sigma} e^{-2s\alpha^*} |(n \times \nabla \times)(P_2^*(x, t; D)\varphi)|^2 d\sigma dt \right. \\ & \quad \left. + s\lambda \iint_Q e^{-2s\alpha} \xi |\nabla \times (P_2^*(x, t; D)\varphi)|^2 dx dt + s^2 \lambda^3 \iint_{\omega_0 \times (0, T)} e^{-2s\alpha} \xi^2 |\Delta \psi|^2 dx dt \right), \end{aligned}$$

for any $\lambda \geq K, s \geq K(T^{2m-1} + T^{2m})$. Taking into account (11) and combining this with (65), we get

$$\begin{aligned} \bar{I}(s, \lambda; \varphi) + & \iint_Q e^{-2s\alpha} (s^2 \lambda^3 \xi^2 |\Delta \psi|^2 + \lambda |\nabla \Delta \psi|^2) dx dt \\ & \leq K \left(\iint_{\Sigma} e^{-2s\alpha^*} \left| \frac{\partial \Delta \psi}{\partial n} \right|^2 d\sigma dt + s\lambda \iint_{\Sigma} e^{-2s\alpha^*} \xi^* |(n \times \nabla \times)(P_1^*(x, t; D)\varphi)|^2 d\sigma dt \right. \\ & \quad + s\lambda \iint_{\Sigma} e^{-2s\alpha^*} \xi^* \left| \frac{\partial \Delta \varphi}{\partial n} \right|^2 d\sigma dt + \iint_{\Sigma} e^{-2s\alpha^*} |(n \times \nabla \times)(P_2^*(x, t; D)\varphi)|^2 d\sigma dt \\ & \quad \left. + s^2 \lambda^3 \iint_{\omega_0 \times (0, T)} e^{-2s\alpha} \xi^2 (s^4 \lambda^5 \xi^4 |\varphi|^2 + s\lambda \xi |\Delta \varphi|^2 + |\Delta \psi|^2) dx dt \right), \end{aligned} \tag{67}$$

for any $\lambda \geq K, s \geq K(T^{2m-1} + T^{2m})$.

In order to estimate the boundary terms, we first regard the equation satisfied by the functions

$$(\hat{\varphi}, \hat{\pi}, \hat{\psi}, \hat{q}) = \lambda^{1/2} e^{-s\alpha^*} (\xi^*)^{-1/m} (\varphi, \pi, \psi, q). \tag{68}$$

That of $\hat{\varphi}$ has a right-hand side given by

$$\hat{f}_1 = -\lambda^{1/2} (e^{-s\alpha^*} (\xi^*)^{-1/m})_t \varphi + m_0 \lambda^{1/2} e^{-s\alpha^*} (\xi^*)^{-1/m} \nabla \times \psi \in L^2(0, T; H^2(\Omega)^N)$$

with

$$\|\hat{f}_1\|_{L^2(0, T; H^2(\Omega)^N)}^2 \leq K \left(I_0(s, \lambda; \Delta \varphi) + \lambda \iint_Q e^{-2s\alpha^*} |\nabla \times \Delta \psi|^2 dx dt \right),$$

so the same estimate holds for $\|\hat{\varphi}\|_{L^2(0,T;H^4(\Omega)^N)}^2$. Then, the right-hand side of the equation of $\hat{\psi}$ satisfies

$$\hat{f}_2 = -\lambda^{1/2}(e^{-s\alpha^*}(\xi^*)^{-1/m})_t \psi + \lambda^{1/2}e^{-s\alpha^*}P_2^*(t, x; D)\varphi \in L^2(0, T; H^2(\Omega)^N)$$

and

$$\|\hat{f}_2\|_{L^2(0,T;H^2(\Omega)^N)}^2 \leq K \left(s^2\lambda^3 \iint_Q e^{-2s\alpha^*}(\xi^*)^2 |\Delta\psi|^2 dx dt + \lambda \int_Q e^{-2s\alpha^*} |\nabla \times \Delta\psi|^2 dx dt + I_0(s, \lambda; \Delta\varphi) \right). \tag{69}$$

Consequently, $\hat{\psi} \in L^2(0, T; H^4(\Omega)^N)$ and $\|\hat{\psi}\|_{L^2(0,T;H^4(\Omega)^N)}^2$ is estimated in the same way as in (69). Then, the same estimate also holds for

$$s\lambda^2 \int_0^T e^{-2s\alpha^*}(\xi^*)^{1-1/m} \|\psi(t)\|_{H^3(\Omega)^N}^2 dt \tag{70}$$

and for

$$s^{1/2}\lambda^{3/2} \int_0^T e^{-2s\alpha^*}(\xi^*)^{(1/2)-(3/2m)} \left\| \frac{\partial \Delta\psi}{\partial n} \right\|_{L^2(\partial\Omega)^N}^2 dt.$$

In order to estimate the other boundary term, we define

$$(\bar{\varphi}, \bar{\pi}) = s^{1/2}\lambda e^{-s\alpha^*}(\xi^*)^{(1/2)-(1/m)}(\varphi, \pi) \tag{71}$$

and we see that its right-hand side is

$$\hat{f}_3 = -s^{1/2}\lambda(e^{-s\alpha^*}(\xi^*)^{(1/2)-(1/m)})_t \varphi + m_0 s^{1/2}\lambda e^{-s\alpha^*}(\xi^*)^{(1/2)-(1/m)} \nabla \times \psi \in L^2(0, T; H^2(\Omega)^N).$$

Moreover, this function satisfies

$$\|\hat{f}_3\|_{L^2(0,T;H^2(\Omega)^N)}^2 \leq K \left(I_0(s, \lambda; \Delta\varphi) + s\lambda^2 \int_0^T e^{-2s\alpha^*}(\xi^*)^{1-2/m} \|\psi(t)\|_{H^3(\Omega)^N}^2 dt \right).$$

Once more, this estimate holds for $\|\bar{\varphi}\|_{L^2(0,T;H^4(\Omega)^N)}^2$ and also for

$$s^{3/2}\lambda^{5/2} \int_0^T e^{-2s\alpha^*}(\xi^*)^{(3/2)-(3/2m)} \left(\left\| \frac{\partial \Delta\varphi}{\partial n} \right\|_{L^2(\partial\Omega)}^2 + \|(n \times \nabla \times)(P_2^*(x, t; D)\varphi)\|_{L^2(\partial\Omega)}^2 \right) dt.$$

Again, since $m > 3$, we can absorb the three boundary integrals in the right-hand side of (67) concerning φ and the one concerning ψ and we obtain:

$$\begin{aligned} & \bar{I}(s, \lambda; \varphi) + \iint_Q e^{-2s\alpha} (s^2\lambda^3 \xi^2 |\Delta\psi|^2 + \lambda |\nabla \Delta\psi|^2) dx dt \\ & \leq K \left(s^2\lambda^3 \iint_{\omega \times (0,T)} e^{-2s\alpha} \xi^2 (s^4\lambda^5 \xi^4 |\varphi|^2 + s\lambda \xi |\Delta\varphi|^2 + |\Delta\psi|^2) dx dt \right), \end{aligned} \tag{72}$$

for any $\lambda \geq K, s \geq K(T^{2m-1} + T^{2m})$.

Let us finally estimate the (local) terms concerning $\Delta\varphi$ and $\Delta\psi$. Let $\theta \in C_0^2(\omega)$ with $\theta = 1$ in ω_0 . As long as the first one is concerned, we have

$$\begin{aligned} & \iint_{\omega \times (0, T)} \theta^2 e^{-2s\alpha} \xi^3 (\Delta\varphi)(\Delta\varphi) \, dx \, dt \\ &= - \iint_{\omega \times (0, T)} \theta^2 e^{-2s\alpha} \xi^3 \nabla\varphi \cdot \nabla\Delta\varphi \, dx \, dt - \iint_{\omega \times (0, T)} \nabla(\theta^2 e^{-2s\alpha} \xi^3) \cdot \nabla\varphi(\Delta\varphi) \, dx \, dt \\ &\leq \varepsilon \left(s^{-2}\lambda^{-2} \iint_{\omega \times (0, T)} e^{-2s\alpha} \xi |\nabla\Delta\varphi|^2 \, dx \, dt + \iint_{\omega \times (0, T)} e^{-2s\alpha} \xi^3 |\Delta\varphi|^2 \, dx \, dt \right) \\ &\quad + K s^2 \lambda^2 \iint_{\omega \times (0, T)} \theta^2 e^{-2s\alpha} \xi^5 |\nabla\varphi|^2 \, dx \, dt \end{aligned}$$

and

$$\begin{aligned} & \iint_{\omega \times (0, T)} \theta^2 e^{-2s\alpha} \xi^5 |\nabla\varphi|^2 \, dx \, dt \\ &= - \iint_{\omega \times (0, T)} \theta^2 e^{-2s\alpha} \xi^5 \Delta\varphi\varphi \, dx \, dt + \frac{1}{2} \iint_{\omega \times (0, T)} \Delta(\theta^2 e^{-2s\alpha} \xi^5) |\varphi|^2 \, dx \, dt \\ &\leq \varepsilon s^{-2} \lambda^{-2} \iint_{\omega \times (0, T)} e^{-2s\alpha} \xi^3 |\Delta\varphi|^2 \, dx \, dt + s^2 \lambda^2 \iint_{\omega \times (0, T)} e^{-2s\alpha} \xi^7 |\varphi|^2 \, dx \, dt. \end{aligned}$$

Thus, we deduce from (72) that

$$\begin{aligned} & \bar{I}(s, \lambda; \varphi) + \iint_Q e^{-2s\alpha} (s^2 \lambda^3 \xi^2 |\Delta\psi|^2 + \lambda |\nabla\Delta\psi|^2) \, dx \, dt \\ &\leq K \left(s^2 \lambda^3 \iint_{\omega \times (0, T)} e^{-2s\alpha} \xi^2 (s^5 \lambda^5 \xi^5 |\varphi|^2 + |\Delta\psi|^2) \, dx \, dt \right), \end{aligned} \tag{73}$$

for any $\lambda \geq K, s \geq K(T^{2m-1} + T^{2m})$.

In order to estimate the second one, we observe that

$$m_0 \Delta\psi = (\nabla \times \varphi)_t + \Delta(\nabla \times \varphi) - \nabla \times (P_1^*(t, x; D)\varphi) \quad \text{in } \Omega \times (0, T).$$

Then,

$$\begin{aligned} & \iint_{\omega \times (0, T)} \theta^2 e^{-2s\alpha} \xi^2 (\Delta\psi)((\nabla \times \varphi)_t + \Delta(\nabla \times \varphi) - \nabla \times (P_1^*(t, x; D)\varphi)) \, dx \, dt \\ &\leq \varepsilon \left(s^{-1-1/m} \lambda^{-1} \iint_{\omega \times (0, T)} e^{-2s\alpha^*} (\xi^*)^{1-1/m} (|\nabla \times \psi_t|^2 + |\nabla \times \Delta\psi|^2) \, dx \, dt + \iint_{\omega \times (0, T)} e^{-2s\alpha} \xi^2 |\Delta\psi|^2 \, dx \, dt \right) \\ &\quad + K s^{3+3/m} \lambda^3 \iint_{\omega \times (0, T)} \theta^2 e^{-4s\alpha+2s\alpha^*} \xi^{5+3/m} (|\nabla\varphi|^2 + |D^2\varphi|^2) \, dx \, dt. \end{aligned}$$

We first realize that the first three terms are already bounded like in (73) (see (70)), since

$$\nabla \times \psi_t = -\nabla \times \Delta\psi + e_0 \nabla \times \psi + (E_0 \cdot \nabla)(\nabla \times \psi) - \nabla \times (P_2^*(t, x; D)\varphi) \quad \text{in } \Omega \times (0, T).$$

Finally, the two terms concerning φ are estimated exactly in the same way as we did above for the local integrals of (72). This yields

$$\begin{aligned}
 & s^{5+3/m} \lambda^6 \iint_{\omega \times (0, T)} \theta^2 e^{-4s\alpha + 2s\alpha^*} \xi^{5+3/m} (|\nabla \varphi|^2 + |D^2 \varphi|^2) \, dx \, dt \\
 & \leq K s^{15+12/m} \lambda^{16} \iint_{\omega \times (0, T)} e^{-10s\alpha + 8s\alpha^*} \xi^{15+12/m} |\varphi|^2 \, dx \, dt \\
 & \quad + \varepsilon \left(s \lambda^2 \iint_{\omega \times (0, T)} e^{-2s\alpha} \xi |\nabla \Delta \varphi|^2 \, dx \, dt + s^3 \lambda^4 \iint_{\omega \times (0, T)} e^{-2s\alpha} \xi^3 |\Delta \varphi|^2 \, dx \, dt \right).
 \end{aligned}$$

If we combine this with (73), we get the estimate (62). \square

4.2. Carleman inequality for system (60)

Proposition 3. *Let $\varphi^0 \psi^0 \in H$. Then, there exists a positive constant $K = K(\Omega, \omega)$ such that*

$$\begin{aligned}
 & \bar{I}(s, \lambda; \varphi) + \iint_Q e^{-2s\alpha} (s^2 \lambda^3 \xi^2 |\nabla \times \Delta \psi|^2 + \lambda |\Delta^2 \psi|^2) \, dx \, dt \\
 & \leq K s^{15+12/m} \lambda^{18} \iint_{\omega \times (0, T)} e^{-10s\alpha + 8s\alpha^*} \xi^{15+12/m} |\varphi|^2 \, dx \, dt,
 \end{aligned} \tag{74}$$

for any $\lambda \geq K$ and any $s \geq K(T^{2m-1} + T^{2m})$, where $\bar{I}(s, \lambda; \varphi)$ was defined in (66).

Proof of Proposition 3. Since most of the arguments are the same as in the proof of Proposition 2, we will just provide a sketch of the proof. In this situation, the equation satisfied by $\Delta \varphi$ is

$$-(\Delta \varphi)_t - \Delta(\Delta \varphi) = \nabla \times (\nabla \times (P_1^*(x, t; D)\varphi)) + m_1 \Delta^2 \psi \quad (t, x) \text{ in } (0, T) \times \Omega.$$

We apply again Lemma 2 and we get

$$\begin{aligned}
 I_0(s, \lambda; \Delta \varphi) & \leq K \left(s^2 \lambda^2 \iint_Q e^{-2s\alpha} \xi^2 |\nabla \times (P_1^*(x, t; D)\varphi)|^2 \, dx \, dt \right. \\
 & \quad + s \lambda \iint_{\Sigma} e^{-2s\alpha^*} \xi^* \left(\left| \frac{\partial \Delta \varphi}{\partial n} \right|^2 + |(n \times \nabla \times)(P_1^*(x, t; D)\varphi)|^2 \right) \, d\sigma \, dt \\
 & \quad \left. + \iint_Q e^{-2s\alpha} |\Delta^2 \psi|^2 \, dx \, dt + s^3 \lambda^4 \iint_{\omega_0 \times (0, T)} e^{-2s\alpha} \xi^3 |\Delta \varphi|^2 \, dx \, dt \right),
 \end{aligned} \tag{75}$$

for any $\lambda \geq K$ and $s \geq K(T^{2m-1} + T^{2m})$. Since estimate (64) also holds now ($\varphi|_{\partial\Omega} = 0$), the first term in the right-hand side is absorbed and we analogously obtain

$$\begin{aligned}
 \bar{I}(s, \lambda; \varphi) & \leq K \left(s \lambda \iint_{\Sigma} e^{-2s\alpha^*} \xi^* \left(\left| \frac{\partial \Delta \varphi}{\partial n} \right|^2 + |(n \times \nabla \times)(P_1^*(x, t; D)\varphi)|^2 \right) \, d\sigma \, dt \right. \\
 & \quad \left. + \iint_Q e^{-2s\alpha} |\Delta^2 \psi|^2 \, dx \, dt + s^3 \lambda^4 \iint_{\omega_0 \times (0, T)} e^{-2s\alpha} \xi^3 (s^3 \lambda^4 \xi^3 |\varphi|^2 + |\Delta \varphi|^2) \, dx \, dt \right)
 \end{aligned} \tag{76}$$

for $\lambda \geq K$ and $s \geq C(T^{2m-1} + T^{2m})$, where $\bar{I}(s, \lambda; \varphi)$ was defined in (66).

We are now interested in the equation satisfied by $\nabla \times \Delta \psi$:

$$-(\nabla \times \Delta \psi)_t - \Delta(\nabla \times \Delta \psi) + e_1 \nabla \times \Delta \psi + (E_0 \cdot \nabla)(\nabla \times \Delta \psi) = \nabla \times (\Delta(Q_1^*(x, t; D)\varphi))$$

in $\Omega \times (0, T)$. Applying Lemma 2 to $\nabla \times \Delta \psi$, we deduce

$$\begin{aligned} & \lambda \iint_Q e^{-2s\alpha} (s^2 \lambda^2 \xi^2 |\nabla \times \Delta \psi|^2 + |\nabla(\nabla \times \Delta \psi)|^2) \, dx \, dt \\ & \leq K \left(\iint_{\Sigma} e^{-2s\alpha^*} \left(\left| \frac{\partial \nabla \times \Delta \psi}{\partial n} \right|^2 + |n \times (\Delta(Q_1^*(x, t; D)\varphi))|^2 \right) \, d\sigma \, dt \right. \\ & \quad \left. + s\lambda \iint_Q e^{-2s\alpha} \xi |\Delta(Q_1^*(x, t; D)\varphi)|^2 \, dx \, dt + s^2 \lambda^3 \iint_{\omega_0 \times (0, T)} e^{-2s\alpha} \xi^2 |\nabla \times \Delta \psi|^2 \, dx \, dt \right), \end{aligned}$$

for any $\lambda \geq K, s \geq K(T^{2m-1} + T^{2m})$. Since the coefficients of $Q_1^*(t, x; D)$ are in $L^\infty(0, T; W^{2,\infty}(\Omega)^N)$, we combine the previous inequality with (76) and we get

$$\begin{aligned} & \bar{I}(s, \lambda; \varphi) + \iint_Q e^{-2s\alpha} (s^2 \lambda^3 \xi^2 |\nabla \times \Delta \psi|^2 + \lambda |\nabla(\nabla \times \Delta \psi)|^2) \, dx \, dt \\ & \leq K \left(\iint_{\Sigma} e^{-2s\alpha^*} \left| \frac{\partial \nabla \times \Delta \psi}{\partial n} \right|^2 \, d\sigma \, dt + s\lambda \iint_{\Sigma} e^{-2s\alpha^*} \xi^* \left| \frac{\partial \Delta \varphi}{\partial n} \right|^2 \, d\sigma \, dt \right. \\ & \quad \left. + \iint_{\Sigma} e^{-2s\alpha^*} (s\lambda \xi^* |(n \times \nabla \times)(P_1^*(x, t; D)\varphi)|^2 + |n \times (\Delta(Q_1^*(x, t; D)\varphi))|^2) \, d\sigma \, dt \right. \\ & \quad \left. + s^2 \lambda^3 \iint_{\omega_0 \times (0, T)} e^{-2s\alpha} \xi^2 (s^4 \lambda^5 \xi^4 |\varphi|^2 + s\lambda \xi |\Delta \varphi|^2 + |\nabla \times \Delta \psi|^2) \, dx \, dt \right), \end{aligned} \tag{77}$$

for any $\lambda \geq K, s \geq K(T^{2m-1} + T^{2m})$.

Next, we realize that $s\lambda^{3/2} e^{-s\alpha^*} \xi^* \varphi \in L^2(0, T; H^2(\Omega)^N)$ and

$$\|s\lambda^{3/2} e^{-s\alpha^*} \xi^* \varphi\|_{L^2(0, T; H^2(\Omega)^N)}^2 \leq K \bar{I}(s, \lambda; \varphi),$$

for $s \geq CT^{2m}$. Taking this into account, exactly the same analysis developed between (46) and (53) can be done here and we deduce that

$$\|s\lambda^{3/2} e^{-s\alpha^*} \xi^* \psi\|_{L^2(0, T; H^3(\Omega)^N)}^2 \leq K(\bar{I}(s, \lambda; \varphi) + \|s\lambda^{3/2} e^{-s\alpha^*} \xi^* \nabla \times \Delta \psi\|_{L^2(Q)^N}^2). \tag{78}$$

At this point, we can carry out the same estimates for the functions $(\hat{\varphi}, \hat{\psi})$ defined in (68) and $(\bar{\varphi}, \bar{\pi})$ defined in (71) with the only difference that we have a one degree improvement (in the space variable) in the Sobolev space where $\hat{\psi}$ is bounded. That's the reason why we need the coefficients of the operator Q_1^* to be in $L^\infty(0, T; W^{3,\infty}(\Omega)^N)$. This yields

$$\begin{aligned} & s^{1/2} \lambda^{3/2} \int_0^T e^{-2s\alpha^*} (\xi^*)^{(1/2)-(3/2m)} \left\| \frac{\partial(\nabla \times \Delta \psi)}{\partial n} \right\|_{L^2(\partial\Omega)^N}^2 \, dt \\ & \quad + s^{3/2} \lambda^{5/2} \int_0^T e^{-2s\alpha^*} (\xi^*)^{(3/2)-(3/2m)} \left\| \frac{\partial(\Delta \varphi)}{\partial n} \right\|_{L^2(\partial\Omega)^N}^2 \, dt \, dt \\ & \quad + s^{3/2} \lambda^{5/2} \iint_{\Sigma} e^{-2s\alpha^*} (\xi^*)^{(3/2)-(3/2m)} |(n \times \nabla \times)(P_1^*(x, t; D)\varphi)|^2 \, d\sigma \, dt \\ & \quad + s^{3/2} \lambda^{5/2} \int_0^T e^{-2s\alpha^*} (\xi^*)^{(3/2)-(3/2m)} \|n \times (\Delta(Q_1^*(x, t; D)\varphi))\|_{L^2(\partial\Omega)^N}^2 \, dt \\ & \leq K(\bar{I}(s, \lambda; \varphi) + \|s\lambda^{3/2} e^{-s\alpha^*} \xi^* \nabla \times \Delta \psi\|_{L^2(Q)^N}^2). \end{aligned}$$

Observe that the powers of s , λ and ξ^* in the boundary integral of $(n \times \nabla \times)(P_1^*(x, t; D)\varphi)$ are clearly not optimal, but this will suffice for our purposes, so we do not worry about its optimality.

This justifies that the four boundary integrals in (77) are absorbed. This provides

$$\begin{aligned} \bar{I}(s, \lambda; \varphi) &+ \iint_Q e^{-2s\alpha} (s^2 \lambda^3 \xi^2 |\nabla \times \Delta \psi|^2 + \lambda |\nabla \times \nabla \Delta \psi|^2) \, dx \, dt \\ &\leq K \left(s^2 \lambda^3 \iint_{\omega_0 \times (0, T)} e^{-2s\alpha} \xi^2 (s^4 \lambda^5 \xi^4 |\varphi|^2 + s \lambda \xi |\Delta \varphi|^2 + |\nabla \times \Delta \psi|^2) \, dx \, dt \right), \end{aligned} \tag{79}$$

for any $\lambda \geq K$, $s \geq K(T^{2m-1} + T^{2m})$.

Then, one can develop exactly the same computations as in the previous paragraph and deduce the desired estimate (74). We do not include this local estimate here since we think it is evident from the one proved above. \square

Appendix A. Proof of Lemma 4

We will develop here the *duality method* introduced in [12] in the context of the heat equation. The same argument has already been used in the context of the heat equation with nonhomogeneous Robin boundary conditions in [5] and in the context of the heat equation with right-hand sides belonging to $L^2(0, T; H^{-2}(\Omega)) \cap H^{-1}(0, T; L^2(\Omega))$, which only permits to talk about solutions in $L^2(Q)$; this is explained with detail in [8].

First, we view u as a solution by transposition of (23). This means that u is the unique function in $L^2(Q)^N$ satisfying

$$\iint_Q u \cdot f \, dx \, dt = \iint_Q f_5 \cdot z \, dx \, dt - \iint_Q f_6 : \nabla z \, dx \, dt + \int_{\Omega} u^0 z|_{t=0} \, dx \quad \forall f \in L^2(Q)^N, \tag{80}$$

where we have denoted by $z \in L^2(0, T; H^2(\Omega)^N \cap V) \cap H^1(0, T; L^2(\Omega)^N)$, together with q , the (strong) solution of the following problem:

$$\begin{cases} -z_t - \Delta z + \nabla q = f, & \nabla \cdot z = 0 & \text{in } Q, \\ z = 0 & & \text{on } \Sigma, \\ z|_{t=T} = 0 & & \text{in } \Omega. \end{cases}$$

Let us first get an estimate of the lower order term in the left-hand side of (24), i.e.

$$s^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |u|^2 \, dx \, dt. \tag{81}$$

In order to do this, let us consider the following constrained extremal problem:

$$\begin{cases} \text{Minimize } \frac{1}{2} I^*(s, \lambda; z, v) \\ \text{subject to } v \in L^2(Q)^N \text{ and} \\ \begin{cases} -z_t - \Delta z + \nabla q = s^3 \lambda^4 e^{-2s\alpha} \xi^3 u + v 1_{\omega_0}, & \nabla \cdot z = 0 & \text{in } Q, \\ z = 0 & & \text{on } \Sigma, \\ z|_{t=0} = 0, \quad z|_{t=T} = 0 & & \text{in } \Omega, \end{cases} \end{cases} \tag{82}$$

where

$$I^*(s, \lambda; z, v) = \left(s^{-15/2} \lambda^{-20} \iint_Q e^{4s\hat{\alpha} - 2s\alpha^*} (\hat{\xi})^{-15/2} |z|^2 \, dx \, dt + s^{-16} \lambda^{-40} \iint_{\omega_0 \times (0, T)} e^{8s\hat{\alpha} - 6s\alpha^*} (\hat{\xi})^{-16} |v|^2 \, dx \, dt \right).$$

Here, s and λ are chosen like in Lemma 3.

By virtue of Lagrange’s principle, there exist p, p_0 and q_0 such that the following optimality system is satisfied:

$$\begin{cases} s^{15/2}\lambda^{20}\mathcal{L}^*(e^{-4s\hat{\alpha}+2s\alpha^*}(\hat{\xi})^{15/2}(\mathcal{L}(p, p_0), q_0)) \\ \quad + s^{16}\lambda^{40}e^{-8s\hat{\alpha}+6s\alpha^*}(\hat{\xi})^{16}p1_{\omega_0} = s^3\lambda^4e^{-2s\alpha}\xi^3u & \text{in } Q, \\ \nabla \cdot p = 0, \quad \nabla \cdot \mathcal{L}(p, p_0) = 0 & \text{in } Q, \\ p = 0, \quad \mathcal{L}(p, p_0) = 0 & \text{on } \Sigma, \\ e^{-4s\hat{\alpha}+2s\alpha^*}(\hat{\xi})^{15/2}(\mathcal{L}(p, p_0)|_{t=0}, \mathcal{L}(p, p_0)|_{t=T}) = (0, 0) & \text{in } \Omega. \end{cases} \tag{83}$$

Here, $\mathcal{L}(a_1, a_2) = \partial_t a_1 - \Delta a_1 + \nabla a_2$ is the Stokes operator and $\mathcal{L}^*(a_1, a_2) = -\partial_t a_1 - \Delta a_1 + \nabla a_2$ is its formal adjoint. If p (together with some (p_0, q_0)) is a solution to (83) (in an appropriate sense), then

$$\hat{v} = -s^{16}\lambda^{40}e^{-8s\hat{\alpha}+6s\alpha^*}(\hat{\xi})^{16}p1_{\omega_0} \quad \text{and} \quad \hat{z} = s^{15/2}\lambda^{20}e^{-4s\hat{\alpha}+2s\alpha^*}(\hat{\xi})^{15/2}\mathcal{L}(p, p_0) \tag{84}$$

solve (82).

Let us show that (83) has a unique *weak* solution. In order to do this, we shall rewrite this problem as a Lax–Milgram variational equation. Let us introduce the space

$$Z_0 = \{(z, z_0) \in C^2(\bar{Q}) \times C^1(\bar{Q}): z = 0 \text{ on } \Sigma \text{ and } \nabla \cdot z = 0 \text{ in } \Omega\}$$

and the norm $\|\cdot\|_Z$, with

$$\|(w, w_0)\|_Z^2 = s^{15/2}\lambda^{20} \iint_Q e^{-4s\hat{\alpha}+2s\alpha^*}(\hat{\xi})^{15/2}|\mathcal{L}(w, w_0)|^2 dx dt + s^{16}\lambda^{40} \iint_{\omega_0 \times (0,T)} e^{-8s\hat{\alpha}+6s\alpha^*}(\hat{\xi})^{16}|w|^2 dx dt$$

for all $(w, w_0) \in Z_0$. Due to Lemma 3, $\|\cdot\|_Z$ is indeed a norm in Z_0 . Let Z be the completion of Z_0 for the norm $\|\cdot\|_Z$. Then Z is a Hilbert space for the scalar product $(\cdot, \cdot)_Z$, with

$$\begin{aligned} ((p_1, p_{1,0}), (p_2, p_{2,0}))_Z &= s^{15/2}\lambda^{20} \iint_Q e^{-4s\hat{\alpha}+2s\alpha^*}(\hat{\xi})^{15/2}(\mathcal{L}(p_1, p_{1,0}))(\mathcal{L}(p_2, p_{2,0})) dx dt \\ &\quad + s^{16}\lambda^{40} \iint_{\omega_0 \times (0,T)} e^{-8s\hat{\alpha}+6s\alpha^*}(\hat{\xi})^{16}p_1 p_2 dx dt. \end{aligned}$$

With this notation, system (83) is equivalent to find a function $p \in Z$ such that

$$((p, p_0), (\tilde{p}, \tilde{p}_0))_Z = \ell(\tilde{p}, \tilde{p}_0) \quad \forall (\tilde{p}, \tilde{p}_0) \in Z, \tag{85}$$

where

$$\ell(\tilde{p}, \tilde{p}_0) = s^3\lambda^4 \iint_Q e^{-2s\alpha}\xi^3u\tilde{p} dx dt \quad \forall (\tilde{p}, \tilde{p}_0) \in Z.$$

By virtue of Lemma 3, one can easily check that $\ell \in Z'$. Consequently, one can apply Lax–Milgram lemma and deduce that there exists a unique solution to (83).

Let us now take

$$f = s^3\lambda^4e^{-2s\alpha}\xi^3u + \hat{v}1_{\omega}$$

in (80). This gives

$$s^3\lambda^4 \iint_Q e^{-2s\alpha}\xi^3|u|^2 dx dt = \iint_Q f_5 \cdot \hat{z} dx dt - \iint_Q f_6 : \nabla \hat{z} dx dt - \iint_{\omega_0 \times (0,T)} u\hat{v} dx dt \tag{86}$$

(recall that \hat{v} and \hat{z} are given by (84)).

Let us multiply the equation in (83) by p and integrate in Q , which gives

$$\|(p, p_0)\|_Z^2 \leq \|\ell\|_{Z'} \|(p, p_0)\|_Z.$$

Consequently,

$$\begin{aligned} \|(P, p_0)\|_Z^2 &= s^{-15/2}\lambda^{-20} \iint_Q e^{4s\hat{\alpha}-2s\alpha^*} (\hat{\xi})^{-15/2} |\hat{z}|^2 \, dx \, dt + s^{-16}\lambda^{-40} \iint_{\omega_0 \times (0, T)} e^{8s\hat{\alpha}-6s\alpha^*} (\hat{\xi})^{-16} |\hat{v}|^2 \, dx \, dt \\ &\leq C s^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |u|^2 \, dx \, dt, \end{aligned} \tag{87}$$

for $\lambda, s \geq C = C(\Omega, \omega, T) > 0$, since

$$\|\ell\|_{Z'} \leq s^{3/2} \lambda^2 \left(\iint_Q e^{-2s\alpha} \xi^3 |u|^2 \, dx \, dt \right)^{1/2}.$$

Now, we multiply the equation satisfied by \hat{z} by $s^{-47/4}\lambda^{-30} e^{6s\hat{\alpha}-4s\alpha^*} (\hat{\xi})^{-47/4} \hat{z}$ and we integrate in Q . After integration by parts, we obtain:

$$\begin{aligned} &s^{-47/4}\lambda^{-30} \iint_Q e^{6s\hat{\alpha}-4s\alpha^*} (\hat{\xi})^{-47/4} |\nabla \hat{z}|^2 \, dx \, dt \\ &= -\frac{1}{2} s^{-47/4}\lambda^{-30} \iint_Q \frac{\partial}{\partial t} (e^{6s\hat{\alpha}-4s\alpha^*} (\hat{\xi})^{-47/4}) |\hat{z}|^2 \, dx \, dt + s^{-35/4}\lambda^{-26} \iint_Q e^{-2s\alpha+6s\hat{\alpha}-4s\alpha^*} (\hat{\xi})^{-47/4} \xi^3 u \hat{z} \, dx \, dt \\ &\quad + s^{-47/4}\lambda^{-30} \iint_{\omega_0 \times (0, T)} e^{6s\hat{\alpha}-4s\alpha^*} (\hat{\xi})^{-47/4} \hat{v} \hat{z} \, dx \, dt. \end{aligned} \tag{88}$$

Using Hölder’s inequality in the last two terms and taking into account that $e^{2s\hat{\alpha}} = \min_{x \in \bar{\Omega}} e^{2s\alpha}$, we obtain

$$\begin{aligned} &s^{-47/4}\lambda^{-30} \iint_Q e^{6s\hat{\alpha}-4s\alpha^*} (\hat{\xi})^{-47/4} |\nabla \hat{z}|^2 \, dx \, dt \\ &\leq C \left(s^{-15/2}\lambda^{-20} \iint_Q e^{4s\hat{\alpha}-2s\alpha^*} (\hat{\xi})^{-15/2} |\hat{z}|^2 \, dx \, dt + s^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |u|^2 \, dx \, dt \right. \\ &\quad \left. + s^{-16}\lambda^{-40} \iint_{\omega_0 \times (0, T)} e^{8s\hat{\alpha}-6s\alpha^*} (\hat{\xi})^{-16} |\hat{v}|^2 \, dx \, dt \right), \end{aligned}$$

where we have taken $s \geq C$. This, together with (87), provides

$$\begin{aligned} &s^{-15/2}\lambda^{-20} \iint_Q e^{4s\hat{\alpha}-2s\alpha^*} (\hat{\xi})^{-15/2} |\hat{z}|^2 \, dx \, dt + s^{-47/4}\lambda^{-30} \iint_Q e^{6s\hat{\alpha}-4s\alpha^*} (\hat{\xi})^{-47/4} |\nabla \hat{z}|^2 \, dx \, dt \\ &\quad + s^{-16}\lambda^{-40} \iint_{\omega_0 \times (0, T)} e^{8s\hat{\alpha}-6s\alpha^*} (\hat{\xi})^{-16} |\hat{v}|^2 \, dx \, dt \leq C s^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |u|^2 \, dx \, dt. \end{aligned} \tag{89}$$

A combination of this inequality with (86) yields the estimate of the zero order term (81) in terms of the right-hand side of (24).

Let us now show that the first order term ∇u can also be bounded in the same way. To this end, we multiply the equation of u by

$$s^2 \lambda^4 e^{-2s\alpha^*} (\xi^*)^{2-1/m} u$$

and we obtain

$$\begin{aligned} &\frac{1}{2} s^2 \lambda^4 \iint_Q e^{-2s\alpha^*} (\xi^*)^{2-1/m} \frac{\partial}{\partial t} |u|^2 \, dx \, dt + s^2 \lambda^4 \iint_Q e^{-2s\alpha^*} (\xi^*)^{2-1/m} |\nabla u|^2 \, dx \, dt \\ &= s^2 \lambda^4 \iint_Q e^{-2s\alpha^*} (\xi^*)^{2-1/m} f_5 \cdot u \, dx \, dt - s^2 \lambda^4 \iint_Q e^{-2s\alpha^*} (\xi^*)^{2-1/m} f_6 : \nabla u \, dx \, dt. \end{aligned}$$

Finally, integrating by parts with respect to t in the first integral and using that

$$\left(e^{-2s\alpha^*} (\xi^*)^{2-1/m}\right)_t \leq C s e^{-2s\alpha^*} (\xi^*)^3, \quad s \geq C,$$

we conclude that the first order term is also bounded by the right-hand side of (24).

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