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On the optimality of the observability inequalities for parabolic and hyperbolic systems with potentials ☆

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Abstract

In this paper we prove the optimality of the observability inequality for parabolic systems with potentials in even space dimensions $n \geqslant 2$. This inequality (derived by E. Fernández-Cara and the third author in the context of the scalar heat equation with potentials in any space dimension) asserts, roughly, that for small time, the total energy of solutions can be estimated from above in terms of the energy localized in a subdomain with an observability constant of the order of $\exp(C\|a\|_\infty^{2/3})$, a being the potential involved in the system. The problem of the optimality of the observability inequality remains open for scalar equations.

The optimality is a consequence of a construction due to V.Z. Meshkov of a complex-valued bounded potential $q = q(x)$ in $\mathbb{R}^2$ and a nontrivial solution u of $\Delta u = q(x)u$ with the decay property $|u(x)| \leqslant \exp(-|x|^{4/3})$. Meshkov's construction may be generalized to any even dimension. We give an extension to odd dimensions, which gives a sharp decay rate up to some logarithmic factor and yields a weaker optimality result in odd space-dimensions.

We address the same problem for the wave equation. In this case it is well known that, in space-dimension $n = 1$, observability holds with a sharp constant of the order of $\exp(C\|a\|_\infty^{1/2})$. For systems in even space dimensions $n \geqslant 2$ we prove that the best constant one can expect is of the order of $\exp(C\|a\|_\infty^{2/3})$ for any $T > 0$ and any observation domain. Based on Carleman inequalities, we show that the positive counterpart is also true when T is large enough and the observation is made in a neighborhood of the boundary. As in the context of the heat equation, the optimality of this estimate is open for scalar equations.

We address similar questions, for both equations, with potentials involving the first order term. We also discuss issues related with the impact of the growth rates of the nonlinearities on the controllability of semilinear equations. Some other open problems are raised.

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1. Introduction and main results

In this paper we discuss the optimality of some observability estimates for the heat and wave equations with potentials that may be obtained by means of Carleman inequalities and that arise naturally in the context of control theory. To better illustrate the problem under consideration, let us first analyze the case of the heat equation.

Let $n \geqslant 1$ and $N \geqslant 1$ be two integers, $T > 0$, Ω be a simply connected, bounded domain of $\mathbb{R}^n$ with smooth boundary Γ. Put $Q = (0,T) \times \Omega$ and $\Sigma = (0,T) \times \Gamma$. Consider the heat equation with a potential $a = a(t,x)$ in $L^\infty(Q;\mathbb{R}^{N\times N})$:

$$\begin{cases} \varphi_t - \Delta\varphi + a\varphi = 0, & \text{in } Q, \\ \varphi = 0, & \text{on } \Sigma, \\ \varphi(0,x) = \varphi^0(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where φ takes values in $\mathbb{R}^N$. Denote by $\|\cdot\|_\infty$ and $|\cdot|$ the (usual) norms on $L^\infty(Q;\mathbb{R}^{N\times N})$ and $\mathbb{R}^N$, respectively. We recall the following known result:

Theorem A. ([11]) *Assume that ω is an open non-empty subset of Ω. Then, there exists a constant $C = C(\Omega,\omega) > 0$, depending on Ω and ω but independent of T, the potential $a = a(t,x)$ and the solution φ of* (1.1)*, such that*

$$\|\varphi(T)\|^2_{(L^2(\Omega))^N} \leqslant \exp\left(C\left(1 + \frac{1}{T} + T\|a\|_\infty + \|a\|_\infty^{2/3}\right)\right) \int_0^T \int_\omega |\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t, \tag{1.2}$$

for every solution φ of (1.1)*, potential $a \in L^\infty(Q;\mathbb{R}^{N\times N})$ and time $T > 0$.*

Inequality (1.2), that we shall refer to as *observability inequality* for system (1.1), plays a key role for solving control problems for linear and nonlinear heat processes (see [11] and [12]). This inequality was proved in [11] using the Carleman inequality approach developed by A. Fursikov and O. Imanuvilov (see, for example, [13]) by paying special attention to the dependence of the observability constants on the size of the potential $a = a(t,x)$ entering in the equation. Indeed, the inequality was only proven in the case $N = 1$, but the method of proof yields the same result in the vectorial case $N \geqslant 2$.

The observability constant in this situation is, by definition, the best constant in the above observability inequality. We shall denote it by $C^*(T, a)$. According to (1.2) we can guarantee that the following upper bound holds:

$$C^*(T,a) \leqslant \exp\left(C\left(1 + \frac{1}{T} + T\|a\|_\infty + \|a\|_\infty^{2/3}\right)\right). \tag{1.3}$$

In this paper we show that this estimate is sharp, in a sense that will be made precise. Similar questions will be addressed for the wave equation as well.

Before going further it is convenient to observe that the estimate in (1.3) includes three different terms. More precisely:

$$\exp\left(C\left(1 + \frac{1}{T} + T\|a\|_\infty + \|a\|_\infty^{2/3}\right)\right) = C_1^*(T,a)C_2^*(T,a)C_3^*(T,a), \tag{1.4}$$

where

$$C_1^*(T,a) = \exp\left(C\left(1 + \frac{1}{T}\right)\right), \quad C_2^*(T,a) = \exp\big(CT\|a\|_\infty\big), \quad C_3^*(T,a) = \exp\big(C\|a\|_\infty^{2/3}\big). \tag{1.5}$$

The role that each of these constants plays in the observability inequality is of different nature:

- When $a \equiv 0$, i.e. in the absence of potential, the observability constant is simply $C_1^*(T,a)$. This constant blows-up exponentially as $T \downarrow 0$. This growth rate is easily seen to be optimal by inspection of the heat kernel and has been analyzed in more detail in [11] and [19] to see what the influence of the geometry of Ω and ω is. We refer also to [22] for a discussion of the optimal growth rate in one space dimension. For large a, the constant C_1^* is the leading one in the region $T \lesssim \|a\|_\infty^{-2/3}$.[1]

- The second constant $C_2^*(T,a)$ is very natural as it arises when applying Gronwall's inequality to analyze the time evolution of the L^2-norm of solutions. For large a, this is the leading constant in the region $T \gtrsim \|a\|_\infty^{-1/3}$.
- The constant $C_3^*(T,a)$, which, actually, only depends on the potential a, is the most intriguing one. Indeed, the $2/3$ exponent does not seem to arise naturally in the context of the heat equation since, taking into account that the heat operator is of order one and two in the time and space variables respectively, one could rather expect terms of the form $\exp(c\|a\|_\infty)$ and $\exp(c\|a\|_\infty^{1/2})$, as a simple ODE argument would indicate.

This paper is devoted to discuss the optimality of this third contribution, $C_3^*(T,a)$, in the observability constant. Note that this contribution is only predominant in the region:

$$\|a\|_\infty^{-2/3} \lesssim T \lesssim \|a\|_\infty^{-1/3}. \tag{1.6}$$

Nevertheless, its study is of great importance in view of applications to nonlinear problems, as it is the largest constant depending on a for small T and large a. In fact, it is the key constant in the main result in [12] that asserts that some weakly blowing-up nonlinear heat processes may be controlled to zero under suitable growth conditions on the nonlinearity. We shall return to this matter bellow.

In this paper we show that the estimate (1.3) is sharp in what concerns the dependence on the potential a, for times of order less than $\|a\|_\infty^{-1/3}$, in space dimension $n \geqslant 2$ for systems with at least two equations. More precisely, in even dimension, the following holds:

Theorem 1.1. *Assume that $n \geqslant 2$ is even and that $N \geqslant 2$. Let ω be any given nonempty open subset of Ω such that $\Omega \setminus \overline{\omega} \neq \emptyset$. Then there exist two constants $c > 0$ and $\mu > 0$, a family of potentials $\{a_R\}_{R>0} \subset L^\infty(Q;\mathbb{R}^{N\times N})$ satisfying*

$$\|a_R\|_\infty \underset{R\to+\infty}{\longrightarrow} +\infty,$$

and a family of initial data $\{\varphi_R^0\}_{R>0}$ in $(L^2(\Omega))^N$ such that the corresponding solution φ_R of (1.1) satisfies

$$\lim_{R\to\infty}\left\{\inf_{T\in I_\mu}\frac{\|\varphi_R(T)\|^2_{(L^2(\Omega))^N}}{\exp(c\|a_R\|_\infty^{2/3})\int_0^T\int_\omega|\varphi_R|^2\,\mathrm{d}x\,\mathrm{d}t}\right\} = +\infty, \tag{1.7}$$

where $I_\mu \triangleq (0, \mu\|a_R\|_\infty^{-1/3}]$.

Remark 1.1. As we shall see, the construction of Theorem 1.1 only requires potentials $a = a(x)$ which are independent of time.

According to (1.7) the inequality (1.2) is sharp in the sense that, for large potentials a, when the observation time is of order smaller than $\|a\|_\infty^{-1/3}$, the observability constant may not grow slower than an exponential factor of the order of $\exp(c\|a\|_\infty^{2/3})$ for some $c > 0$. This sharpness of the constant for small time is particularly important in view of the applications to semilinear equations (see Section 7). In Section 4 (see Theorem 4.1), we give a slightly weaker extension to odd dimensions, for $N \geqslant 8$, and with a logarithmic correction. Getting exactly (1.7) for n odd is an open problem.

Note that Theorems 1.1 and 4.1 do not exclude a possible improvement of the estimate for $N < 8$ in odd space dimension, and for scalar equations in any space dimension. It will be clear from the proof of Theorem 1.1 that this last issue is closely related to that of the optimal decay at infinity for solutions of scalar equations on $\mathbb{R}^n$ of the form $\Delta u = q(x)u$ with bounded potential q, a problem that, to our knowledge, is not completely solved.

Theorem 1.1 is a consequence of the following known result:

[1] This means that $T \leqslant C\|a\|_\infty^{-1/3}$ for some generic constant $C > 0$. We shall use a similar notation later, for example for $T \gtrsim \|a\|_\infty^{-1/3}$.

Theorem B. (Meshkov [18]) *Assume that the space dimension is $n = 2$. Then, there exists a nonzero complex-valued bounded potential $q = q(x)$ and a nontrivial complex valued solution $u = u(x)$ of*

$$\Delta u = qu, \quad \text{in } \mathbb{R}^2, \tag{1.8}$$

with the property that

$$\left|u(x)\right| \leqslant C \exp\left(-|x|^{4/3}\right), \quad \forall x \in \mathbb{R}^2, \tag{1.9}$$

for some positive constant $C > 0$.

Taking into account that the potential q and the solution u are complex valued, Eq. (1.8) can be viewed as an elliptic system with real valued coefficients and two components ($N = 2$).

As we shall see, Theorem 1.1 holds from the construction by Meshkov by scaling and localization arguments. The proof of Theorem 1.1 will be given in Section 4.

By separation of variables, Theorem B holds in any even dimension. The validity of this result in odd dimensions is, to the knowledge of the authors, an open problem. Section 3 is devoted to this issue. We obtain a slightly weaker version of Theorem B in $3-d$ for $\mathbb{C}^4$-valued solutions and with a potential q growing at infinity in a logarithmic way. This construction is the main tool to prove Theorem 4.1 on the optimality of the observability constant in odd space-dimensions.

Let us mention some variants of the preceding results. One may assume that the potential a is in $L^\infty(0,T;L^p(\Omega;\mathbb{R}^{N\times N}))$, where $n \leqslant p < +\infty$, but is not necessarily bounded. For such a, an analogue of the observability inequality (1.2) is shown in Theorem 2.1. In this case, the constant C_3^* has to be replaced by $\exp(C\|a\|^\alpha_{L^\infty(0,T;L^p(\Omega;\mathbb{R}^{N\times N}))})$, for some α depending on p. Surprisingly, Meshkov's construction is not sufficient to show that this constant is sharp. We shall also consider the case of convective potentials, that is first order operators of the form $a_1.\nabla$, where a_1 is bounded. The analogue of the observability inequality (1.2) is also stated in Theorem 2.1. In this case, in space-dimension $n = 2$, we are able to show the optimality of this inequality, using an easy adaptation of Meshkov's construction (see Section 6).

We now consider the wave equation and, more precisely, a system of $N \geqslant 1$ wave equations of the form

$$\begin{cases} w_{tt} - \Delta w + aw = 0, & \text{in } Q, \\ w = 0, & \text{on } \Sigma, \\ w(0,x) = w^0(x), \quad w_t(0,x) = w^1(x), & \text{in } \Omega, \end{cases} \tag{1.10}$$

where the unknown function w takes value in $\mathbb{R}^N$ and $a = a(t,x)$ is a matrix-valued potential as in (1.1).

Let ω be a nonempty open subset of Ω. As above, we shall study the observability constant $D^* = D^*(T,a)$, defined, for fixed $a \in L^\infty(Q;\mathbb{R}^{N\times N})$ and $T > 0$, as the smallest (possibly infinite) positive constant such that the following observability inequality holds:

$$\left\|w^0\right\|^2_{(L^2(\Omega))^N} + \left\|w^1\right\|^2_{(H^{-1}(\Omega))^N} \leqslant D^*(T,a) \int_0^T\!\!\int_\omega |w|^2 \,\mathrm{d}x\,\mathrm{d}t, \tag{1.11}$$

for every solution of (1.10).

Concerning the dependence of $D^*(T,a)$ on the potential a the following can be said:

- Unlike the heat case, the existence of (finite) $D^*(T,a)$ is not guaranteed for all triple (Ω,T,ω). In [5], Bardos, Lebeau and Rauch have established (under strong smoothness assumptions) an essentially necessary and sufficient condition for (1.11) to hold, the *geometric control condition*, which asserts that all rays of geometric optics in Ω enter the subdomain ω in an uniform time $T > 0$. But the micro-local techniques used in their work do not seem to give the explicit dependence of the constant $D^*(T,a)$ on the potential a. In Section 2 we shall introduce a stronger condition on (Ω,T,ω) which yields the existence of $D^*(T,a)$ for the multi-dimensional case. Indeed, by means of Carleman estimates, we show in Theorem 2.2 that under this condition:

$$D^*(T,a) \leqslant \exp\left(C(T)\left(1 + \|a\|^{2/3}_\infty\right)\right), \tag{1.12}$$

  with a positive constant $C(T)$ that depends only on (Ω,T,ω).

- Observability constant (1.12) is similar to that in [11] for the heat equation (see (1.3) above). Note however that there is an important difference between $C^*(T,a)$ for the heat equation and $D^*(T,a)$ for the wave one. Indeed, in (1.3), there is a term $\exp(CT\|a\|_\infty)$, due to the use of Gronwall's inequality. Because of this term, one needs to choose T to be of the order (1.6) (thus small when a is large), to get an observability constant bounded by $\exp(C\|a\|^{2/3})$. The situation is different for the wave equation. Indeed, as noted in [29], a modified energy estimate gives an upper bound for the evolution of the energy of the order of $\exp(CT\|a\|_\infty^{1/2})$. As we shall see, this point is crucial to derive (1.12).
- In one space dimension, in [29], using sidewise energy estimates it was proved that

$$D^*(T,a) \leqslant C \exp\bigl(C\|a\|_\infty^{1/2}\bigr). \tag{1.13}$$

This is true whatever the number N of components of the system is and even for 1-d wave equations with BV-coefficients in the principal part. But it may fail for equations with Hölder continuous coefficients (see [8]). Estimate (1.13) is known to be sharp. It is even sharp for the scalar 1-d wave equations with constant potentials.

However, as we shall see, (1.13) is not satisfied in space dimensions $n \geqslant 2$. Indeed, our next main goal of this paper is to show that estimate (1.12) is sharp in several space dimensions. We state the result in even space dimension. As in the case of parabolic systems a similar, but weaker result holds in odd space-dimensions (Theorem 5.1).

Theorem 1.2. *Assume that $n \geqslant 2$ is even and $N \geqslant 2$. Let ω be any given open nonempty subset of Ω such that $\Omega \setminus \overline{\omega} \neq \emptyset$. Then, for all $T > 0$ there exist a constant $c > 0$, a family of potentials $\{a_R\}_{R>0} \subset L^\infty(Q;\mathbb{R}^{N\times N})$ satisfying*

$$\|a_R\|_\infty \underset{R\to+\infty}{\longrightarrow} +\infty,$$

and a family of initial data $\{(w_R^0, w_R^1)\}_{R>0}$ in $(L^2(\Omega))^N \times (H^{-1}(\Omega))^N$ such that the corresponding solution w_R of (1.10) *satisfies*

$$\lim_{R\to\infty}\left\{\frac{\|w_R^0\|^2_{(L^2(\Omega))^N} + \|w_R^1\|^2_{(H^{-1}(\Omega))^N}}{\exp(c\|a_R\|_\infty^{2/3})\int_0^T\int_\omega |w_R|^2\,\mathrm{d}x\,\mathrm{d}t}\right\} = +\infty. \tag{1.14}$$

Remark 1.2. The potentials a_R in Theorem 1.2 will be chosen to be time-invariant. Furthermore, as one shall see in the proof of Theorem 1.2 in Section 5, we actually choose the initial velocity $w_R^1 = 0$.

As in the context of the heat equation, the proof of this result is based on the construction by Meshkov in Theorem B. Theorem 1.2 shows that the estimate (1.12) is sharp for systems of wave equations in even dimensions. Hence, in this case, when T is fixed, the observability constant $D^*(T,a)$ has to grow, at least, at the order of $\exp(C\|a\|_\infty^{2/3})$ as $\|a\|_\infty \to \infty$. The problem of the optimality of the estimate (1.12) for scalar equations is open. As in the parabolic case, Theorem 3.2 yields a weaker version of Theorem 1.2 when n is odd, that we state in Section 5 (see Theorem 5.1).

The rest of this paper is organized as follows. In Section 2, we shall show sharp observability estimates for parabolic and hyperbolic equations with both zero and, in some cases, first order potentials. We also consider the case of boundary observability for the wave equation. In Section 3, we construct solutions on $\mathbb{R}^n$ of elliptic linear equations with a maximal speed of decay at infinity. These constructions generalize that of Meshkov, as it includes first order potentials, and odd n. They are the main tool of the three next sections, which are devoted to prove negative results. In Sections 4 and 5 we prove respectively the optimality of the observability estimates for heat and wave equations with zero order potentials, i.e., Theorems 1.1 and 1.2. In Section 6 we address the case of heat and wave equations with convective potentials, and show the optimality of the observability estimates which depend exponentially on the square of the L^∞-norm of the convective potential. It is well-known that, in the context of both heat and wave equations, the observability inequalities with explicit bounds in terms of the potentials are intimately related with the optimal growth rates for the control of semilinear equations. This issue will be briefly discussed in Section 7. Finally, in Section 8 we comment some closely related issues and open problems.

2. Sharp observability estimates for hyperbolic and parabolic equations with potentials

This section is devoted to show sharp observability estimates for hyperbolic and parabolic equations with zero and first order potentials. We shall consider elliptic operators with variable coefficients, which does not change the proof with respect to the simpler case of Laplace's operator, and gives the same dependence of the observability constant on the potentials. As it was mentioned in the introduction, we also consider the case of zero order potentials which are in L^p-spaces in the space variable, with $n \leqslant p \leqslant \infty$.

2.1. Statement of the results

In the sequel, we fix real valued functions $b^{ij} \in C^1(\overline{\Omega})$ satisfying:

$$b^{ij}(x) = b^{ji}(x), \quad \forall x \in \overline{\Omega},\ i, j = 1, 2, \ldots, n, \tag{2.1}$$

and

$$\sum_{i,j=1}^{n} b^{ij}(x)\xi_i\xi_j \geqslant \beta|\xi|^2, \quad \forall (x, \xi) \equiv (x, \xi_1, \ldots, \xi_n) \in \overline{\Omega} \times \mathbb{R}^n, \tag{2.2}$$

for some constant $\beta > 0$.

We also consider $\mathbb{R}^{N\times N}$-valued functions a, a_1^k $(k = 1, \ldots, n)$ and a_2 on Q satisfying:

(H1) $a \in L^\infty(0, T; L^p(\Omega; \mathbb{R}^{N\times N}))$ *for some* $p \in [n, \infty]$, *and* $a_1^1, \ldots, a_1^n, a_2 \in L^\infty(Q; \mathbb{R}^{N\times N})$.

Put

$$r_0 \triangleq \|a\|_{L^\infty(0,T;L^p(\Omega;\mathbb{R}^{N\times N}))}, \quad r_1 \triangleq \sum_{k=1}^{n} \|a_1^k\|_\infty, \quad r_2 \triangleq r_1 + \|a_2\|_\infty. \tag{2.3}$$

2.1.1. The parabolic system

We consider first the following parabolic system:

$$\begin{cases} \varphi_t - \displaystyle\sum_{i,j=1}^{n} \big(b^{ij}(x)\varphi_{x_i}\big)_{x_j} = a\varphi + \sum_{k=1}^{n} a_1^k \varphi_{x_k}, & \text{in } Q, \\ \varphi = 0, & \text{on } \Sigma, \\ \varphi(0) = \varphi^0, & \text{in } \Omega. \end{cases} \tag{2.4}$$

We have the following observability estimate for system (2.4).

Theorem 2.1. *Let ω be an open nonempty subset of Ω and $b^{ij}(\cdot) \in C^1(\overline{\Omega})$ satisfy* (2.1)–(2.2). *Then, there exists a constant $C = C(\Omega, \omega) > 0$ depending only on Ω and ω such that for any time $T > 0$, potentials a and a_1^k $(k = 1, \ldots, n)$ satisfying* (H1)*, and initial data $\varphi^0 \in (L^2(\Omega))^N$, the corresponding solution $\varphi \in C([0, T]; (L^2(\Omega))^N)$ of* (2.4) *satisfies*

$$\|\varphi(T)\|^2_{(L^2(\Omega))^N} \leqslant \exp\left\{C\left[1 + \frac{1}{T} + Tr_0 + r_0^{\frac{1}{3/2-n/p}} + (1+T)r_1^2\right]\right\} \int_0^T\!\!\int_\omega |\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{2.5}$$

By adapting an argument in Step 5 in the proof of Theorem 2.4 in Subsection 2.2 (for the hyperbolic system), the proof of Theorem 2.1 is almost the same as that in [9, Theorem 2.3] and [11, Theorem 1.2]. Hence we omit the details. Theorem 1.1 shows that, when $(b^{ij})_{n\times n} = I$ (the identity matrix), the exponent $2/3$ in $r_0^{2/3}$ (for the special case $p = \infty$ in the estimate (2.5)) is sharp. As we shall see in Section 6, the quadratic dependence on r_1 is also sharp. The problem of the optimality of the term in $r_0^{\frac{1}{3/2-n/p}}$ is completely open when $p < \infty$.

2.1.2. The hyperbolic system

Let us now consider the following hyperbolic system:

$$\begin{cases} w_{tt} - \displaystyle\sum_{i,j=1}^{n} \big(b^{ij}(x) w_{x_i}\big)_{x_j} = aw + \sum_{k=1}^{n} a_1^k w_{x_k} + a_2 w_t, & \text{in } Q, \\ w = 0, & \text{on } \Sigma, \\ w(0) = w^0, \ w_t(0) = w^1, & \text{in } \Omega, \end{cases} \tag{2.6}$$

where $w = (w_1, \ldots, w_N)^\top$ is a $\mathbb{R}^N$-valued unknown. Under some assumption on the observation domain, we will show boundary and interior observability inequalities for system (2.6), with an explicit dependence on the constants r_0 (and, in the case of boundary observability also r_2). For this, we introduce the following condition.

(H2) *There exists a function $d(\cdot) \in C^2(\overline{\Omega})$ satisfying the following*:

(i) *For some constant $\mu_0 > 0$,*

$$\sum_{i,j=1}^{n} \left\{ \sum_{i',j'=1}^{n} \left[2b^{ij'}\big(b^{i'j} d_{x_{i'}}\big)_{x_{j'}} - b^{ij}_{x_{j'}} b^{i'j'} d_{x_{i'}}\right] \right\} \xi_i \xi_j \geqslant \mu_0 \sum_{i,j=1}^{n} b^{ij} \xi_i \xi_j, \quad \forall (x, \xi) \in \overline{\Omega} \times \mathbb{R}^n; \tag{2.7}$$

(ii) *The function $d(\cdot)$ does not have any critical point in $\overline{\Omega}$, i.e.*,

$$\min_{x \in \overline{\Omega}} \big|\nabla d(x)\big| > 0. \tag{2.8}$$

Denote by $\nu = \nu(x) = (\nu_1, \nu_2, \ldots, \nu_n)$ the unit outward normal vector of Ω at $x \in \Gamma$. For the function $d(\cdot)$ satisfying Condition (H2), we introduce the following set:

$$\Gamma_0 \triangleq \left\{ x \in \Gamma \;\middle|\; \sum_{i,j=1}^{n} b^{ij} \nu_i d_{x_j} > 0 \right\}. \tag{2.9}$$

Note that for the case of $(b^{ij})_{n \times n} = I$, by choosing $d(x) = |x - x_0|^2$ with any fixed $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$, (H2) is satisfied with $\mu_0 = 4$ and (2.7) holds with an equality. In this case, Γ_0 in (2.9) is given by

$$\big\{x \in \Gamma \mid (x - x_0) \cdot \nu(x) > 0\big\},$$

which coincides with the subset of the boundary arising usually in the context of the multiplier method [17].

On the other hand, it is easy to check that, if $d(\cdot) \in C^2(\overline{\Omega})$ satisfies (2.7), then for any given constants $\alpha \geqslant 1$ and $\beta \in \mathbb{R}$, the function

$$\hat{d} = \hat{d}(x) \triangleq \alpha d(x) + \beta \tag{2.10}$$

still satisfies Condition (H2) with μ_0 replaced by $\alpha\mu_0$, the set Γ_0 remaining unchanged. Hence, without loss of generality, we may assume that

$$\begin{cases} \text{(2.7) holds with } \mu_0 \geqslant 4, \\ \dfrac{1}{4} \displaystyle\sum_{i,j=1}^{n} b^{ij}(x) d_{x_i}(x) d_{x_j}(x) \geqslant \max_{x \in \overline{\Omega}} d(x) \geqslant \min_{x \in \overline{\Omega}} d(x) > 0, \quad \forall x \in \overline{\Omega}. \end{cases} \tag{2.11}$$

In what follows, put

$$R_1 \triangleq \max_{x \in \overline{\Omega}} \sqrt{d(x)}, \qquad T_0 \triangleq 2 \inf\big\{R_1 \mid d(\cdot) \text{ satisfies (2.11)}\big\}. \tag{2.12}$$

For the interior observation, we introduce the following assumption:

(H3) *There is a constant $\delta > 0$ such that*

$$\omega = \mathcal{O}_\delta(\Gamma_0) \cap \Omega, \tag{2.13}$$

where $\mathcal{O}_\delta(\Gamma_0) = \{x \in \mathbb{R}^n \mid |x - x'| < \delta$ for some $x' \in \Gamma_0\}$.

In other words, the observation subdomain ω is assumed to be a neighborhood of a subset of the boundary satisfying the condition above for boundary observability.

In the rest of this section, we will use $C = C(T, \Omega)$ to denote a generic positive constant which may vary from line to line. Our observability estimates for system (2.6) are stated as follows.

Theorem 2.2. *Let $b^{ij} \in C^1(\overline{\Omega})$ satisfy* (2.1)–(2.2). *Assume that conditions* (H2)–(H3) *hold. Let $T > T_0$, where T_0 is defined in* (2.12). *Then the following two assertions hold*:

(i) boundary observability: *There is a constant $C > 0$ such that for any potentials $a, a_1^1, \ldots, a_1^n$ and a_2 satisfying* (H1) *for some $p \in [n, +\infty]$, and initial data $(w^0, w^1) \in (H_0^1(\Omega))^N \times (L^2(\Omega))^N$, the corresponding weak solution*:

$$w \in C\big([0,T]; \big(H_0^1(\Omega)\big)^N\big) \cap C^1\big([0,T]; \big(L^2(\Omega)\big)^N\big)$$

of (2.6) *satisfies*

$$\|w^0\|_{(H_0^1(\Omega))^N} + \|w^1\|_{(L^2(\Omega))^N} \leqslant \exp\big(C\big(1 + r_0^{\frac{1}{3/2-n/p}} + r_2^2\big)\big) \left\| \frac{\partial w}{\partial \nu} \right\|_{(L^2((0,T)\times\Gamma_0))^N}. \tag{2.14}$$

(ii) interior observability: *If $a_1^k \equiv 0$ $(k = 1, \ldots, n)$ and $a_2 \equiv 0$, then there is a constant $C > 0$ such that for any potential a satisfying* (H1) *and initial data $(w^0, w^1) \in (L^2(\Omega))^N \times (H^{-1}(\Omega))^N$, the corresponding weak solution*:

$$w \in C\big([0,T]; \big(L^2(\Omega)\big)^N\big) \cap C^1\big([0,T]; \big(H^{-1}(\Omega)\big)^N\big)$$

of (2.6) *satisfies*

$$\|w^0\|_{(L^2(\Omega))^N} + \|w^1\|_{(H^{-1}(\Omega))^N} \leqslant \exp\big(C\big(1 + r_0^{\frac{1}{3/2-n/p}}\big)\big) \|w\|_{(L^2((0,T)\times\omega))^N}. \tag{2.15}$$

Part (ii) of Theorem 2.2 is the internal observability inequality for the wave equation announced in (1.11). Note that unlike the case of the heat equation (Theorem 2.1) and the boundary observability for the wave equation (part (i) of Theorem 2.2), we do not give a generalization of this inequality to first order potentials. The proof of such a generalization would yield new technical difficulties, essentially due to the fact that in view of control-theoretic applications, we treat the case of the adjoint system (with initial data in $(L^2(\Omega) \times H^{-1}(\Omega))^N$), instead of the usual energy space (with initial data in $(H_0^1(\Omega) \times L^2(\Omega))^N$).

As in the parabolic case, Theorem 1.2 shows that the exponent 2/3 in the estimate $r_0^{2/3}$ (in (2.15) for the special case $p = \infty$) is sharp. As we shall see in Section 6, the exponent 2 in the estimate r_2^2 (in (2.14)) is sharp, too.

In the rest of this section, to simplify the presentation, we only consider the case $N = 1$. The proof in the case $N \geqslant 2$ is exactly the same. The proof of Theorem 2.2 will be given in Subsection 2.3. For this, we need to show a global Carleman estimate for hyperbolic operators, which is the object of the next subsection.

2.2. Global Carleman estimate for hyperbolic operators

Recall (2.12) for the definitions of R_1 and T_0. Let $T > T_0$ be given. We may choose d such that

$$T > 2R_1. \tag{2.16}$$

By (2.16), one may choose a constant $c \in (0,1)$ so that

$$\left(\frac{2R_1}{T}\right)^2 < c < \frac{2R_1}{T}. \tag{2.17}$$

Put

$$\phi = \phi(t,x) \triangleq d(x) - c\left(\frac{t-T}{2}\right)^2, \tag{2.18}$$

with T and c satisfying respectively (2.16) and (2.17).

Define a formal differential operator $\mathcal{P}$ by

$$\mathcal{P}u \triangleq u_{tt} - \sum_{i,j=1}^{n} \big(b^{ij}(x)u_{x_i}\big)_{x_j}. \tag{2.19}$$

By [14, (5.15) in the proof of Theorem 5.1], we have the following Carleman estimate.

Theorem 2.3. *Let $b^{ij} \in C^1(\overline{\Omega})$ satisfy* (2.1)–(2.2). *Assume that condition* (H2) *holds and Γ_0 is given by* (2.9). *Then there exists a $\lambda_0 > 1$ such that for all $\lambda \geqslant \lambda_0$ and all $u \in H_0^1(Q)$ with $\mathcal{P}u \in L^2(Q)$, it holds*:

$$\lambda \int_Q e^{2\lambda\phi}\big(\lambda^2 u^2 + u_t^2 + |\nabla u|^2\big)\,dx\,dt \leqslant C\Bigg(\int_Q e^{2\lambda\phi}|\mathcal{P}u|^2\,dx\,dt + \lambda \int_0^T\!\!\int_{\Gamma_0} e^{2\lambda\phi}\left|\frac{\partial u}{\partial \nu}\right|^2 dx\,dt\Bigg). \tag{2.20}$$

In order to prove the interior observability result in Theorem 2.2, we also need the following:

Theorem 2.4. *Let $b^{ij} \in C^1(\overline{\Omega})$ satisfy* (2.1)–(2.2), *and $a \in L^\infty(0,T;L^p(\Omega))$ with $p \in [n,\infty]$. Assume that conditions* (H2)–(H3) *hold. Then there exists a $\lambda_0 > 1$ such that for all $\lambda \geqslant \lambda_0$, any $u \in C([0,T];L^2(\Omega))$ satisfying $u(0,x) = u(T,x) = 0$ for $x \in \Omega$, $\mathcal{P}u \in H^{-1}(Q)$ and*

$$(u, \mathcal{P}\eta)_{L^2(Q)} = \langle \mathcal{P}u, \eta\rangle_{H^{-1}(Q),H_0^1(Q)}, \quad \forall \eta \in H_0^1(Q) \text{ with } \mathcal{P}\eta \in L^2(Q), \tag{2.21}$$

it holds

$$\begin{aligned}\lambda\big\|e^{\lambda\phi}u\big\|_{L^2(Q)}^2 \leqslant C\bigg(&\big\|e^{\lambda\phi}(\mathcal{P}u - au)\big\|_{H^{-1}(Q)}^2 + \frac{1}{\lambda^{2(1-n/p)}}\big\|e^{\lambda\phi}au\big\|_{L^2(0,T;H^{-n/p}(\Omega))}^2 \\ &+ \lambda^2\big\|e^{\lambda\phi}u\big\|_{L^2((0,T)\times\omega)}^2\bigg).\end{aligned} \tag{2.22}$$

Proof. We shall borrow some idea from [16], which consists in applying (2.21) to some special choice of η with $\mathcal{P}\eta = \cdots + \lambda e^{2\lambda\phi}u$, which yields the desired term $\lambda\|e^{\lambda\phi}u\|_{L^2(Q)}^2$ and reduces the estimate (2.22) to an estimate on $\|\eta\|_{H_0^1(Q)}$.

In [14, (7.20) and (7.21) in the proof of Theorem 7.1], it is shown, using an optimization argument, that there is a λ_0 greater than 1 so that for any λ greater than λ_0 and any u in $C([0,T];L^2(\Omega))$ vanishing at $t = 0$ and $t = T$, there exist $(\check{z}, \check{r})$ in $H_0^1(Q) \times L^2((0,T)\times\Omega)$, such that:

$$\begin{cases} \operatorname{supp}\check{r} \subset \overline{(0,T)\times\omega}, \\ \mathcal{P}\check{z} = \check{r} + \lambda e^{2\lambda\phi}u, & \text{in } Q, \\ \check{z} = 0, & \text{on } \partial Q, \end{cases} \tag{2.23}$$

and for some constant $C > 0$, independent of λ, it holds

$$\int_Q e^{-2\lambda\phi}\big(|\nabla\check{z}|^2 + \check{z}_t^2 + \lambda^2\check{z}^2\big)\,dx\,dt + \frac{1}{\lambda^2}\int_0^T\!\!\int_\omega e^{-2\lambda\phi}\check{r}^2\,dx\,dt \leqslant C\lambda\int_Q e^{2\lambda\phi}u^2\,dx\,dt. \tag{2.24}$$

Now, by (2.21) with η replaced by $\check{z}$ above, one gets

$$\big(u, \check{r} + \lambda\,e^{2\lambda\phi}u\big)_{L^2(Q)} = \langle \mathcal{P}u, \check{z}\rangle_{H^{-1}(Q),H_0^1(Q)}.$$

Hence, noting $\operatorname{supp}\check{r} \subset \overline{(0,T)\times\omega}$:

$$\begin{aligned}\lambda\big\|e^{\lambda\phi}u\big\|_{L^2(Q)}^2 &= \langle \mathcal{P}u, \check{z}\rangle_{H^{-1}(Q),H_0^1(Q)} - (u,\check{r})_{L^2((0,T)\times\omega)} \\ &= \langle \mathcal{P}u - au, \check{z}\rangle_{H^{-1}(Q),H_0^1(Q)} + (au,\check{z})_{L^2(Q)} - (u,\check{r})_{L^2((0,T)\times\omega)} \\ &\leqslant \big\|e^{\lambda\phi}(\mathcal{P}u - au)\big\|_{H^{-1}(Q)}\big\|e^{-\lambda\phi}\check{z}\big\|_{H_0^1(Q)} + \big\|e^{\lambda\phi}au\big\|_{L^2(0,T;H^{-n/p}(\Omega))}\big\|e^{-\lambda\phi}\check{z}\big\|_{L^2(0,T;H_0^{n/p}(\Omega))}\end{aligned}$$

$$
\begin{aligned}
&+ \|e^{\lambda\phi}u\|_{L^2((0,T)\times\omega)}\|e^{-\lambda\phi}\check{r}\|_{L^2((0,T)\times\omega)} \\
&\leqslant C\sqrt{\mathcal{Q}}\Big[\|e^{-\lambda\phi}\check{z}\|^2_{H_0^1(Q)} + \lambda^{2(1-n/p)}\|e^{-\lambda\phi}\check{z}\|^2_{L^2(0,T;H_0^{n/p}(\Omega))} \\
&\quad + \frac{1}{\lambda^2}\|e^{-\lambda\phi}\check{r}\|^2_{L^2((0,T)\times\omega)}\Big]^{1/2},
\end{aligned}
\tag{2.25}
$$

where

$$
\mathcal{Q} \triangleq \|e^{\lambda\phi}(\mathcal{P}u - au)\|^2_{H^{-1}(Q)} + \frac{1}{\lambda^{2(1-n/p)}}\|e^{\lambda\phi}au\|^2_{L^2(0,T;H^{-n/p}(\Omega))} + \lambda^2\|e^{\lambda\phi}u\|^2_{L^2((0,T)\times\omega)}
$$

is the right-hand term of (2.22). On the other hand, for any $f \in H^1(\mathbb{R}^n)$, by Hölder's inequality, one has

$$
\|f\|^2_{H^{n/p}(\mathbb{R}^n)} = \int_{\mathbb{R}^n}\big(1+|\xi|^2\big)^{n/p}\big|\hat{f}(\xi)\big|^{2n/p}\big|\hat{f}(\xi)\big|^{2(1-n/p)}\,d\xi \leqslant \|f\|^{n/p}_{H^1(\mathbb{R}^n)}\|f\|^{1-n/p}_{L^2(\mathbb{R}^n)}.
$$

This yields immediately:

$$
\|g\|^2_{H_0^{n/p}(\Omega)} \leqslant C\|g\|^{n/p}_{H_0^1(\Omega)}\|g\|^{1-n/p}_{L^2(\Omega)}, \quad \forall g \in H_0^1(\Omega),
\tag{2.26}
$$

for some constant $C > 0$, independent of g. Hence, for any $h \in L^2(0,T;H_0^1(\Omega))$:

$$
\|h\|_{L^2(0,T;H_0^{n/p}(\Omega))} \leqslant C\|h\|^{n/p}_{L^2(0,T;H_0^1(\Omega))}\|h\|^{1-n/p}_{L^2(Q)}.
\tag{2.27}
$$

Now, by (2.27) and using Young's inequality, it follows

$$
\begin{aligned}
\lambda^{2(1-n/p)}\|e^{-\lambda\phi}\check{z}\|^2_{L^2(0,T;H_0^{n/p}(\Omega))} &\leqslant C\lambda^{2(1-n/p)}\|e^{-\lambda\phi}\check{z}\|^{2n/p}_{L^2(0,T;H_0^1(\Omega))}\|e^{-\lambda\phi}\check{z}\|^{2(1-n/p)}_{L^2(Q)} \\
&\leqslant C\big[\|e^{-\lambda\phi}\check{z}\|^2_{L^2(0,T;H_0^1(\Omega))} + \lambda^2\|e^{-\lambda\phi}\check{z}\|^2_{L^2(Q)}\big].
\end{aligned}
\tag{2.28}
$$

Finally, using (2.24), (2.25) and (2.28), we arrive at the desired estimate (2.22). This completes the proof of Theorem 2.4. □

2.3. *Proof of sharp observability estimates for hyperbolic equations with potentials*

We now prove Theorem 2.2. The main idea is to use the Carleman estimate in Theorems 2.3–2.4. The proof is divided into several steps.

Step 1: Choice of a cutoff function. Note that our w satisfying (2.6) does not necessarily vanish at $t = 0, T$. Therefore we need to introduce a suitable cutoff function. To this end, set (recall (2.11))

$$
T_i \triangleq T/2 - \varepsilon_i T, \quad T_i' \triangleq T/2 + \varepsilon_i T, \quad R_0 \triangleq \min_{x\in\overline{\Omega}}\sqrt{d(x)}\ (>0),
\tag{2.29}
$$

where $i = 0, 1$; $0 < \varepsilon_0 < \varepsilon_1 < 1/2$ will be given below.

From (2.17), (2.18) and the definition (2.12) of R_1, it is easy to see that

$$
\phi(0,x) = \phi(T,x) \leqslant R_1^2 - cT^2/4 < 0, \quad \forall x \in \Omega.
\tag{2.30}
$$

Therefore there exists an $\varepsilon_1 \in (0, 1/2)$, close to $1/2$, such that

$$
\phi(t,x) \leqslant R_1^2/2 - cT^2/8 < 0, \quad \forall (t,x) \in \big((0,T_1)\cup(T_1',T)\big)\times\Omega,
\tag{2.31}
$$

with T_1 and T_1' given by (2.29). Further, by (2.18), we see that

$$
\phi(T/2,x) = d(x) \geqslant R_0^2, \quad \forall x \in \Omega.
$$

Hence, one can find an $\varepsilon_0 \in (0, 1/2)$, close to 0, such that

$$
\phi(t,x) \geqslant R_0^2/2, \quad \forall (t,x) \in (T_0,T_0')\times\Omega,
\tag{2.32}
$$

with T_0 and T_0' given by (2.29). We now choose a nonnegative function $\xi \in C_0^\infty(0,T)$ so that

$$\xi(t) \equiv 1 \quad \text{in } (T_1, T_1'). \tag{2.33}$$

We start to show the second assertion of Theorem 2.2, whose proof is more technical.

Step 2: An intermediate inequality. Assume that the assumptions of the second part of Theorem 2.2 hold. We first shall apply Carleman inequality (2.22) to prove the following observability inequality:

$$\exists \lambda_1 > 0,\ \forall \lambda \geqslant \left(1 + r_0^{\frac{1}{3/2-n/p}}\right)\lambda_1, \quad \lambda \int_Q e^{2\lambda\phi} w^2 \, dx\, dt \leqslant C\left(\lambda^2 \int_0^T\!\!\int_\omega e^{2\lambda\phi} w^2 \, dx\, dt + \|w\|^2_{L^2(J\times\Omega)}\right), \tag{2.34}$$

where $J \triangleq (0, T_1) \cup (T_1', T)$. Note that in the right-hand side of (2.34) the square of the L^2 norm of w on all Ω appears, but only for time smaller than T_1 or greater than T_1'. To get read of these term, we will need to use, in the next step, a modified energy method.

Recall that ξw vanishes at $t = 0, T$. Hence, by Theorem 2.4, for any $\lambda \geqslant \lambda_0$, we have

$$\lambda \int_Q e^{2\lambda\phi} (\xi w)^2 \, dx\, dt \leqslant C\Bigg[\left\|e^{\lambda\phi}\left(\mathcal{P}(\xi w) - a\xi w\right)\right\|^2_{H^{-1}(Q)} + \frac{1}{\lambda^{2(1-n/p)}} \left\|e^{\lambda\phi} a \xi w\right\|^2_{L^2(0,T;H^{-n/p}(\Omega))} + \lambda^2 \int_0^T\!\!\int_\omega e^{2\lambda\phi} w^2 \, dx\, dt\Bigg]. \tag{2.35}$$

By Eq. (2.6), recalling $a_1^k \equiv 0$ ($k = 1, \dots, n$) and $a_2 \equiv 0$, and noting (2.31) and (2.33), we have

$$\begin{aligned}
\left\|e^{\lambda\phi}\left(\mathcal{P}(\xi w) - a\xi w\right)\right\|_{H^{-1}(Q)} &= \left\|e^{\lambda\phi}(2\xi_t w_t + w\xi_{tt})\right\|_{H^{-1}(Q)} \\
&= \sup_{|f|_{H_0^1(Q)}=1} \int_Q e^{\lambda\phi}(2\xi_t w_t + w\xi_{tt}) f \, dx\, dt \\
&= \sup_{|f|_{H_0^1(Q)}=1} \int_Q e^{\lambda\phi} w(-\xi_{tt} f - 2\xi_t f_t - 2\lambda\phi_t\xi_t f) \, dx\, dt \\
&\leqslant C e^{(R_1^2/2 - cT^2/8)\lambda}(1+\lambda)\|w\|_{L^2(J\times\Omega)}.
\end{aligned} \tag{2.36}$$

Recalling the definition of r_0 in (2.3) and noting that the Sobolev embedding $H_0^{n/p}(\Omega) \hookrightarrow L^{\frac{2p}{p-2}}(\Omega)$, implies by duality the embedding $L^{\frac{2p}{p+2}}(\Omega) \hookrightarrow H^{-n/p}(\Omega)$, we get:

$$\begin{aligned}
\left\|e^{\lambda\phi} a\xi w\right\|_{L^2(0,T;H^{-n/p}(\Omega))} &\leqslant \left\|e^{\lambda\phi} a\xi w\right\|_{L^2(0,T;L^{2p/(p+2)}(\Omega))} \\
&\leqslant C r_0 \left\|e^{\lambda\phi} w\right\|_{L^2(Q)},
\end{aligned} \tag{2.37}$$

where at the second line we simply used Hölder's inequality. Further, by (2.31) and (2.33), we have

$$\begin{aligned}
\int_Q e^{2\lambda\phi}(\xi w)^2 \, dx\, dt &= \int_Q e^{2\lambda\phi} w^2 \, dx\, dt - \int_Q e^{2\lambda\phi}\left(1-\xi^2\right) w^2 \, dx\, dt \\
&\geqslant \int_Q e^{2\lambda\phi} w^2 \, dx\, dt - C e^{(R_1^2 - cT^2/4)\lambda}\|w\|^2_{L^2(J\times\Omega)}.
\end{aligned} \tag{2.38}$$

Combining (2.35)–(2.38), we conclude that there is a constant $C_1 = C_1(T, \Omega)$, independent of λ and r_0, such that:

$$\lambda \int_Q e^{2\lambda\phi} w^2 \, dx\, dt \leqslant C_1\Bigg[\frac{r_0^2}{\lambda^{2(1-n/p)}} \int_Q e^{2\lambda\phi} w^2 \, dx\, dt + \lambda^2 \int_0^T\!\!\int_\omega e^{2\lambda\phi} w^2 \, dx\, dt$$

$$+\, \mathrm{e}^{(R_1^2 - cT^2/4)\lambda}\big(1+\lambda^2\big)\|w\|^2_{L^2(J\times\Omega)}\Bigg]. \tag{2.39}$$

Since $R_1^2 - cT^2/4 < 0$, one may find a $\lambda_1' \geqslant \lambda_0$ such that $\mathrm{e}^{(R_1^2-cT^2/4)\lambda}(1+\lambda^2) < 1$ for all $\lambda \geqslant \lambda_1'$. Take a $\lambda_1 \geqslant \lambda_1'$ and such that:

$$\lambda \geqslant r_0^{\frac{1}{3/2-n/p}}\lambda_1 \Longrightarrow \lambda - C_1\frac{r_0^2}{\lambda^{2(1-n/p)}} \geqslant \frac{\lambda}{2}. \tag{2.40}$$

For such a choice of λ_1, the desired inequality (2.34) follows from (2.39).

Step 3: A modified energy method. From (2.32), we see that

$$\int_Q \mathrm{e}^{2\lambda\phi} w^2\,\mathrm{d}x\,\mathrm{d}t \geqslant \mathrm{e}^{R_0^2\lambda}\int_{T_0}^{T_0'}\int_\Omega w^2\,\mathrm{d}x\,\mathrm{d}t. \tag{2.41}$$

Put

$$E(t) \triangleq \frac{1}{2}\big[\big\|w(t,\cdot)\big\|^2_{L^2(\Omega)} + \big\|w_t(t,\cdot)\big\|^2_{H^{-1}(\Omega)}\big]. \tag{2.42}$$

For any $S_0 \in (T_0, T/2)$ and $S_0' \in (T/2, T_0')$, by means of the classical energy estimate, one has

$$\int_{S_0}^{S_0'} E(t)\,\mathrm{d}t \leqslant C(1+r_0)\int_{T_0}^{T_0'}\int_\Omega w^2\,\mathrm{d}x\,\mathrm{d}t. \tag{2.43}$$

We claim that, there is a constant $C > 0$ such that

$$E(t) \leqslant C\mathrm{e}^{Cr_0^{\frac{1}{2-n/p}}}E(s), \quad \forall t, s \in [0, T]. \tag{2.44}$$

Note however that this does not follow from the usual energy method. Instead, we need to use the duality argument and adopt a modified energy estimate introduced in [29]. For this, for any $(z^0, z^1) \in H_0^1(\Omega) \times L^2(\Omega)$, we introduce the following system

$$\begin{cases} z_{tt} - \displaystyle\sum_{i,j=1}^n \big(b^{ij}(x)z_{x_i}\big)_{x_j} = az, & \text{in } Q,\\ z = 0, & \text{on } \Sigma,\\ z(T) = z^0, \quad z_t(T) = z^1, & \text{in } \Omega. \end{cases} \tag{2.45}$$

Denote a (modified) energy of system (2.45) by

$$\mathcal{E}(t) = \frac{1}{2}\int_\Omega\Bigg[\big|z_t(t,x)\big|^2 + \sum_{i,j=1}^n b^{ij}(x)z_{x_i}(t,x)z_{x_j}(t,x) + r_0^{\frac{2}{2-n/p}}\big|z(t,x)\big|^2\Bigg]\mathrm{d}x. \tag{2.46}$$

Then, by (2.45) and recalling the definition of r_0 in (2.3), it follows

$$\frac{\mathrm{d}\mathcal{E}(t)}{\mathrm{d}t} = \int_\Omega azz_t\,\mathrm{d}x + r_0^{\frac{2}{2-n/p}}\int_\Omega zz_t\,\mathrm{d}x. \tag{2.47}$$

Put $p_1 = \frac{2p}{n-2}$ and $p_2 = \frac{2p}{p-n}$. Noting that $\frac{1}{p} + \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{2} = 1$ and $\frac{1}{2(n/p)^{-1}} + \frac{1}{2(1-n/p)^{-1}} + \frac{1}{2} = 1$, by Hölder's inequality and Sobolev's embedding theorem, and recalling (2.46), we get

$$\begin{aligned}\int_\Omega azz_t\,\mathrm{d}x &\leqslant \int_\Omega |a||z|^{\frac{n}{p}}|z|^{1-\frac{n}{p}}|z_t|\,\mathrm{d}x\\ &\leqslant r_0\big\|\big|z(t,\cdot)\big|^{\frac{n}{p}}\big\|_{L^{p_1}(\Omega)}\big\|\big|z(t,\cdot)\big|^{1-\frac{n}{p}}\big\|_{L^{p_2}(\Omega)}\big\|z_t(t,\cdot)\big\|_{L^2(\Omega)}\end{aligned}$$

$$
\begin{aligned}
&= r_0 \|z(t,\cdot)\|^{\frac{n}{p}}_{L^{\frac{2n}{n-2}}(\Omega)} \|z(t,\cdot)\|^{1-\frac{n}{p}}_{L^2(\Omega)} \|z_t(t,\cdot)\|_{L^2(\Omega)} \\
&= r_0^{\frac{1}{2-n/p}} \underbrace{\|z(t,\cdot)\|^{\frac{n}{p}}_{L^{\frac{2n}{n-2}}(\Omega)}}_{\leqslant \mathcal{E}(t)^{\frac{n}{2p}}} \underbrace{\big(r_0^{\frac{1-n/p}{2-n/p}} \|z(t,\cdot)\|^{1-\frac{n}{p}}_{L^2(\Omega)}\big)}_{\leqslant \mathcal{E}(t)^{\frac{1}{2}-\frac{n}{2p}}} \underbrace{\|z_t(t,\cdot)\|_{L^2(\Omega)}}_{\leqslant \mathcal{E}(t)^{1/2}} \\
&\leqslant C r_0^{\frac{1}{2-n/p}} \mathcal{E}(t).
\end{aligned}
\tag{2.48}
$$

Similarly,

$$
r_0^{\frac{2}{2-n/p}} \int_\Omega z z_t \,\mathrm{d}x \leqslant \frac{r_0^{\frac{1}{2-n/p}}}{2} \int_\Omega \big(r_0^{\frac{2}{2-n/p}} z^2 + z_t^2\big)\,\mathrm{d}x \leqslant C r_0^{\frac{1}{2-n/p}} \mathcal{E}(t). \tag{2.49}
$$

Hence, combining (2.47)–(2.49), we conclude that

$$
\frac{\mathrm{d}\mathcal{E}(t)}{\mathrm{d}t} \leqslant C r_0^{\frac{1}{2-n/p}} \mathcal{E}(t).
$$

By this and noting the time reversibility of system (2.45), we get

$$
\mathcal{E}(t) \leqslant C \mathrm{e}^{C r_0^{\frac{1}{2-n/p}}} \mathcal{E}(s), \quad \forall t, s \in [0,T].
$$

Hence,

$$
\big\|\big(z(t), z_t(t)\big)\big\|_{H_0^1(\Omega)\times L^2(\Omega)} \leqslant C \mathrm{e}^{C r_0^{\frac{1}{2-n/p}}} \big\|\big(z(s), z_t(s)\big)\big\|_{H_0^1(\Omega)\times L^2(\Omega)}, \quad \forall t, s \in [0,T]. \tag{2.50}
$$

Now, taking the scalar product of the first equation of (2.6) by z, integrating it in $(t,s)\times\Omega$, recalling that by assumption $a_1^k \equiv 0$ ($k = 1, \dots, n$) and $a_2 \equiv 0$, and using (2.45) and integrations by parts, we get

$$
\begin{aligned}
&\big(w(s), z_t(s)\big)_{L^2(\Omega)} + \big\langle w_t(s), -z(s)\big\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \\
&\quad = \big(w(t), z_t(t)\big)_{L^2(\Omega)} + \big\langle w_t(t), -z(t)\big\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad \forall t, s \in [0,T].
\end{aligned}
\tag{2.51}
$$

Hence, by (2.42) and (2.51), and noting the last equation in (2.45), and using (2.50), we get (denoting by S the unit sphere of the space $H_0^1(\Omega)\times L^2(\Omega)$)

$$
\begin{aligned}
\sqrt{2E(T)} &= \sup_{(z^0,z^1)\in S} \big[\big(w(T), z_1\big)_{L^2(\Omega)} + \big\langle w_t(T), -z_0\big\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}\big] \\
&= \sup_{(z^0,z^1)\in S} \big[\big(w(t), z_t(t)\big)_{L^2(\Omega)} + \big\langle w_t(t), -z(t)\big\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}\big] \\
&\leqslant C\sqrt{E(t)} \sup_{(z^0,z^1)\in S} \big\|\big(z(t), z_t(t)\big)\big\|_{H_0^1(\Omega)\times L^2(\Omega)} \\
&\leqslant C \mathrm{e}^{C r_0^{\frac{1}{2-n/p}}} \sqrt{E(t)} \sup_{(z^0,z^1)\in S} \big\|\big(z(T), z_t(T)\big)\big\|_{H_0^1(\Omega)\times L^2(\Omega)} \\
&= C \mathrm{e}^{C r_0^{\frac{1}{2-n/p}}} \sqrt{E(t)}.
\end{aligned}
$$

This fact, combined with the time reversibility of system (2.6), yields the desired estimate (2.44).

Step 4: We now return to the proof of the second assertion. By (2.44), we get

$$
\|w\|^2_{L^2(J\times\Omega)} \leqslant C E(0)\, \mathrm{e}^{C r_0^{\frac{1}{2-n/p}}}, \tag{2.52}
$$

and

$$\int_{S_0}^{S_0'} E(t)\,\mathrm{d}t \geqslant \frac{1}{C} E(0)\,\mathrm{e}^{-Cr_0^{\frac{1}{2-n/p}}}. \tag{2.53}$$

Combining (2.53) with (2.41) and (2.43), we get:

$$\lambda \int_Q w^2\,\mathrm{d}x\,\mathrm{d}t \geqslant \frac{\lambda}{C(1+r_0)} \mathrm{e}^{R_0^2\lambda - Cr_0^{\frac{1}{2-n/p}}}\, E(0). \tag{2.54}$$

Inequality (2.34) together with (2.52) and (2.54) yields a constant C_2 such that

$$\lambda \geqslant \big(1 + r_0^{\frac{1}{3/2-n/p}}\big)\lambda_1$$
$$\Longrightarrow \underbrace{\big[\lambda\,\mathrm{e}^{R_0^2\lambda - C_2 r_0^{\frac{1}{2-n/p}}} - C_2(1+r_0)\mathrm{e}^{C_2 r_0^{\frac{1}{2-n/p}}}\big]}_{\alpha(\lambda, r_0)} E(0) \leqslant C_2\lambda^2(1+r_0)\,\mathrm{e}^{C_2\lambda} \int_0^T\!\!\int_\omega w^2\,\mathrm{d}x\,\mathrm{d}t. \tag{2.55}$$

Assume that $\lambda \geqslant (1 + r_0^{\frac{1}{3/2-n/p}})\lambda_1$. Taking, if necessary, a greater λ_1 we have:

$$\lambda\,\mathrm{e}^{\frac{R_0^2\lambda}{2}} \geqslant 1 + C_2(1+r_0), \qquad \frac{R_0^2\lambda}{2} \geqslant 2C_2 r_0^{\frac{1}{2-n/p}}. \tag{2.56}$$

(To obtain the second inequality we used that $\frac{1}{2-n/p} < \frac{1}{3/2-n/p}$.) Thus:

$$\alpha(\lambda, r_0) \geqslant \mathrm{e}^{C_2 r_0^{\frac{1}{2-n/p}}} \geqslant 1. \tag{2.57}$$

Then, from (2.55) and (2.57), we obtain:

$$\exists\lambda_1, \quad \lambda \geqslant \big(1 + r_0^{\frac{1}{3/2-n/p}}\big)\lambda_1 \Longrightarrow E(0) \leqslant C_2\lambda^2(1+r_0)\,\mathrm{e}^{C_2\lambda}\|w\|^2_{L^2((0,T)\times\omega)}, \tag{2.58}$$

taking the preceding inequality at $\lambda = (1 + r_0^{\frac{1}{3/2-n/p}})\lambda_1$ gives the desired observability inequality (2.15).

Step 5: Let us now show the first assertion. Again we will show an intermediate inequality:

$$\exists\lambda_1 > 0\colon \forall\lambda > \lambda_1\big(1 + r_0^{\frac{1}{3/2-n/p}} + r_2^2\big),$$
$$\lambda \int_Q \mathrm{e}^{2\lambda\phi}\big(w_t^2 + |\nabla w|^2\big)\,\mathrm{d}x\,\mathrm{d}t \leqslant C\left[\|w\|^2_{H^1(J\times\Omega)} + \lambda \int_0^T\!\!\int_{\Gamma_0} \mathrm{e}^{2\lambda\phi}\left|\frac{\partial w}{\partial\nu}\right|^2\mathrm{d}x\,\mathrm{d}t\right]. \tag{2.59}$$

Clearly, ξw vanishes at $t = 0, T$. Therefore, by Theorem 2.3, we get

$$\lambda \int_Q \mathrm{e}^{2\lambda\phi}\big[\lambda^2(\xi w)^2 + \big|(\xi w)_t\big|^2 + \big|\nabla(\xi w)\big|^2\big]\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant C\left(\int_Q \mathrm{e}^{2\lambda\phi}\big|\mathcal{P}(\xi w)\big|^2\,\mathrm{d}x\,\mathrm{d}t + \lambda\int_0^T\!\!\int_{\Gamma_0} \mathrm{e}^{2\lambda\phi}\left|\frac{\partial w}{\partial\nu}\right|^2\mathrm{d}x\,\mathrm{d}t\right), \quad \forall\lambda \geqslant \lambda_0. \tag{2.60}$$

By Eq. (2.6), we get:

$$\int_Q \mathrm{e}^{2\lambda\phi}\big|\mathcal{P}(\xi w)\big|^2\,\mathrm{d}x\,\mathrm{d}t = \int_Q \mathrm{e}^{2\lambda\phi}\left|\xi_{tt}w + 2\xi_t w_t + \xi\left(aw + \sum_{k=1}^n a_1^k w_{x_k} + a_2 w_t\right)\right|^2\mathrm{d}x\,\mathrm{d}t. \tag{2.61}$$

Furthermore, recalling the definition of r_0 in (2.3), and using successively Hölder's and Sobolev's inequalities, then inequality (2.27):

$$\begin{aligned}\left\|a\mathrm{e}^{\lambda\phi}\xi w\right\|_{L^2(Q)} &\leqslant r_0\left\|\mathrm{e}^{\lambda\phi}\xi w\right\|_{L^2(0,T;L^s(\Omega))}, \quad 1/s+1/p=1/2\\ &\leqslant r_0\left\|\mathrm{e}^{\lambda\phi}\xi w\right\|_{L^2(0,T;H_0^{n/p}(\Omega))}\\ &\leqslant r_0\left\|\mathrm{e}^{\lambda\phi}\xi w\right\|^{n/p}_{L^2(0,T;H_0^1(\Omega))}\left\|\mathrm{e}^{\lambda\phi}\xi w\right\|^{1-n/p}_{L^2(Q)}.\end{aligned}$$

Hence, using Young's inequality we get, for any $\varepsilon>0$:

$$\left\|a\mathrm{e}^{\lambda\phi}\xi w\right\|^2_{L^2(Q)} \leqslant \varepsilon\lambda\left\|\mathrm{e}^{\lambda\phi}\xi w\right\|^2_{L^2(0,T;H_0^1(\Omega))} + C_\varepsilon r_0^{2p/(p-n)}\lambda^{-n/(p-n)}\left\|\mathrm{e}^{\lambda\phi}\xi w\right\|^2_{L^2(Q)}, \tag{2.62}$$

where C_ε is a positive constant depending on ε. Further, by (2.31) and (2.33), we have

$$\begin{aligned}&\int_Q \mathrm{e}^{2\lambda\phi}\left[\left|(\xi w)_t\right|^2+\left|\nabla(\xi w)\right|^2\right]\mathrm{d}x\,\mathrm{d}t\\ &\quad\geqslant \int_Q \mathrm{e}^{2\lambda\phi}\left[w_t^2+|\nabla w|^2\right]\mathrm{d}x\,\mathrm{d}t - C\mathrm{e}^{(R_1^2-cT^2/4)\lambda}\left(\|w\|^2_{H^1((0,T_1)\times\Omega)}+\|w\|^2_{H^1((T_1',T)\times\Omega)}\right).\end{aligned} \tag{2.63}$$

Combining (2.60)–(2.63), and taking $\varepsilon>0$ small enough in (2.62) we get, for some large constant $C_3>0$:

$$\begin{aligned}\lambda\int_Q \mathrm{e}^{2\lambda\phi}\left[\lambda^2(\xi w)^2+w_t^2+|\nabla w|^2\right]\mathrm{d}x\,\mathrm{d}t \leqslant C_3\Bigg[&\mathrm{e}^{(R_1^2-cT^2/4)\lambda}\|w\|^2_{H^1(J\times\Omega)} + r_0^{2p/(p-n)}\lambda^{-n/(p-n)}\left\|\mathrm{e}^{\lambda\phi}\xi w\right\|^2_{L^2(Q)}\\ &+ r_2^2\int_Q \mathrm{e}^{2\lambda\phi}\left(w_t^2+|\nabla w|^2\right)\mathrm{d}x\,\mathrm{d}t + \lambda\int_0^T\!\!\int_{\Gamma_0}\mathrm{e}^{2\lambda\phi}\left|\frac{\partial w}{\partial\nu}\right|^2\mathrm{d}x\,\mathrm{d}t\Bigg].\end{aligned} \tag{2.64}$$

Now, choosing $\lambda_1>0$ large enough such that:

$$\lambda>\lambda_1\left(1+r_0^{\frac{1}{3/2-n/p}}+r_2^2\right) \Longrightarrow C_3\left(r_0^{\frac{2p}{p-n}}+r_2^2\right)\leqslant\frac{\lambda}{2},$$

and noting that by (2.17), $R_1^2-cT^2/4<0$, we get (2.59).

As in Step 3, (2.59) and the modified energy method lead to the first assertion in Theorem 2.2. Since the preceding case is more complex we omit the details. This completes the proof of Theorem 2.2. □

3. Super-exponentially decaying solutions to elliptic equations

In this section, we construct solutions u of equations on $\mathbb{R}^n$, $n\geqslant 2$, of one of the following forms:

$$\Delta u+qu=0, \tag{3.1}$$

or:

$$\Delta u+q_1\cdot\nabla u=0, \tag{3.2}$$

where q and q_1 admit some bounds at infinity, and such that u decays at infinity as fast as possible. When $n=1$, if q is bounded, the solution u of (3.1) may not decay faster than exponentially. In higher dimensions, an elementary but optimal Carleman estimate due to Meshkov [18], shows that one cannot hope a decay at infinity faster than $\mathrm{e}^{-C|x|^{4/3}}$ for all $C>0$. In the same work is also given a surprising example on $\mathbb{R}^2$ of complex-valued functions q and u satisfying (3.1), with q bounded and u decaying like $\mathrm{e}^{-|x|^{4/3}}$ (see Theorem B). The example of Meshkov gives by separation of variables examples of solutions with the optimal rate of decay in any even dimension. In Theorem 3.1 we state this result and its counterpart in dimension 2 in the case of Eq. (3.2), where the convective potential q_1 decays like $|x|^{-1/3}$ (which is sharp according to the Carleman estimate in [18]). For the sake of completeness we give a proof of both results.

The case of odd dimension is more complex and to our knowledge no results exist in this direction. In Theorem 3.2, we give, for odd n, an example of a solution u of (3.1) on $\mathbb{R}^n$ decaying at the same super-exponential speed, but which is $\mathbb{C}^4$-valued, and with some q growing logarithmically at infinity. Once again, according to the Carleman inequality, this construction is quasi-optimal. The existence of a solution u of (3.1) taking values in $\mathbb{C}$ and/or with a bounded q remains open, as are similar questions concerning real-valued functions in all dimensions $n \geqslant 2$.

For x in $\mathbb{R}^n$, we shall write $r = |x|$.

Theorem 3.1. *Let $n \geqslant 2$ be even and $c_* > 0$. There exists nontrivial functions*:

$$u \in C^\infty(\mathbb{R}^n; \mathbb{C}), \qquad q \in C^\infty(\mathbb{R}^n; \mathbb{C}) \cap L^\infty(\mathbb{R}^n; \mathbb{C})$$

such that (3.1) *is satisfied on $\mathbb{R}^n$, and, for some constant C*:

$$\left|u(x)\right| \leqslant C\,\mathrm{e}^{-c_* r^{4/3}}. \tag{3.3}$$

Furthermore, when $n = 2$, and for the same function u, there exists:

$$q_1 \in C^\infty(\mathbb{R}^2; \mathbb{C}^2), \quad \text{with } (r+1)^{1/3} q_1 \in L^\infty(\mathbb{R}^2; \mathbb{C}^2), \tag{3.4}$$

such that Eq. (3.2) *holds.*

Theorem 3.2. *Let $n \geqslant 3$ be odd and $c_* > 0$. There exist nontrivial functions*:

$$u \in C^\infty(\mathbb{R}^n; \mathbb{C}^4), \qquad q \in C^\infty(\mathbb{R}^n; \mathbb{C}^{4\times 4}),$$

fulfilling (3.1) *and such that, for a constant $C > 0$*:

$$\left(\log(2+r)\right)^{-3} q \in L^\infty(\mathbb{R}^n), \tag{3.5}$$

$$\left|u(x)\right| \leqslant C\,\mathrm{e}^{-c_* r^{4/3}}. \tag{3.6}$$

Remark 3.1. Theorem 3.1 is optimal in the following sense. Fix a bounded function q on $\mathbb{R}^n$. Then the only solution of Eq. (3.1) on $\mathbb{R}^n$ satisfying:

$$\forall c > 0, \quad \int_{\mathbb{R}^n} \left|u(x)\right| \mathrm{e}^{c|x|^{4/3}}\,\mathrm{d}x < +\infty, \tag{3.7}$$

is $u = 0$. Indeed this is a consequence of the following Carleman inequality by Meshkov [18, Lemma 1], which holds for large τ, in any space dimension n:

$$\forall v \in C_0^\infty(\{r > 1\}),$$

$$\tau^3 \int |v|^2 \exp(2\tau r^{4/3}) r^{1-n}\,\mathrm{d}x + \tau \int |\nabla v|^2 \exp(2\tau r^{4/3}) r^{1/3-n}\,\mathrm{d}x \leqslant C \int |\Delta v|^2 \exp(2\tau r^{4/3}) r^{1-n}\,\mathrm{d}x. \tag{3.8}$$

The same argument shows that if q_1 satisfies the decay property (3.4), then the only solution u of (3.2) on $\mathbb{R}^n$ satisfying (3.7) is $u = 0$. These results remain valid with vector-valued functions.

If q (scalar or matrix-valued) satisfies a logarithmic bound such as:

$$\exists N > 0, \quad \frac{q}{\log^N(2+r)} \in L^\infty(\mathbb{R}^n),$$

one may write weaker uniqueness results for (3.2). For example:

$$\int_{\mathbb{R}^n} \left|u(x)\right| \mathrm{e}^{\varepsilon |x|^{\varepsilon + 4/3}}\,\mathrm{d}x < +\infty \quad \text{for some } \varepsilon > 0 \Longrightarrow u = 0.$$

In this sense Theorem 3.2 is quasi-optimal.

Remark 3.2. The construction of Theorem 3.1 may be adapted to potentials with polynomial bounds at infinity. Precisely, if $2/3 < \alpha \leqslant 2$ and $\gamma \triangleq (4-3\alpha)/2$, there exist a potential q_α bounded by $C|x|^{-\gamma}$ and a solution u decaying like $\mathrm{e}^{-|x|^\alpha}$ such that Eq. (3.1) holds. By a variant of Carleman inequality (3.8), this decay is also optimal.

One may think this result would be of some help to test the optimality of the constant of the observability inequality for potentials $a \in L^\infty(0,T;L^p(\Omega))$, $n \leqslant p < +\infty$ (see Theorems 2.1 and 2.2). Unfortunately this is not the case. Indeed the arguments of Sections 4 and 5 applied on the potentials q_α would only show that the constant of observation may not be better than $C\mathrm{e}^{C\|a\|^{2/3}_{L^\infty(0,T;L^p(\Omega))}}$, which is only interesting in the case $p=\infty$, that is in the case of Theorem 3.1.

All the polynomially decaying potentials q_α we are referring to are locally bounded. It seems likely that, to prove the optimality of the observability constants given by Theorems 2.1 and 2.2 for potentials a in L^p spaces, one will have to construct variants of the Meshkov function u with potentials q that are in L^p, but not locally bounded (see also open problem 8.4).

Remark 3.3. The uniqueness result of Meshkov, and the constructions of this section are closely related to the issue of uniqueness of solutions of equations such as (3.1) or (3.2) vanishing to some specified (finite or infinite) order at a point or at a general submanifold of $\mathbb{R}^n$. See the book of Zuily [31] and the constructions of counterexamples in [2,3,15,26].

3.1. Construction in even dimension

In this part we prove Theorem 3.1. We first remark that in the case of Eq. (3.1), we may assume that $n=2$. Indeed, if there exists a function u and a potential q fulfilling the first part of Theorem 3.1 for $n=2$, and if $n=2m$ is even, we can define

$$\tilde{u}(x_1,x_2,\ldots,x_{2m}) \triangleq u(x_1,x_2)u(x_3,x_4)\cdots u(x_{2m-1},x_{2m}).$$

It then follows that

$$\Delta\tilde{u} + \underbrace{\big(q(x_1,x_2)+q(x_3,x_4)+\cdots+q(x_{2m-1},x_{2m})\big)}_{\tilde{q}}\,\tilde{u} = 0.$$

The potential $\tilde{q}$ is bounded on $\mathbb{R}^n$ and $\tilde{u}$ satisfies the decay property:

$$\big|\tilde{u}(x)\big| \leqslant C\exp\big(-c_* r^{4/3}\big), \quad x \in \mathbb{R}^n.$$

Note that this argument does not work in the case of convective equation (3.2), as it gives only a bounded q_1.

We now assume that $n=2$. Denote by $\theta = \frac{x}{|x|}$ the polar variable. We will construct a function u which is harmonic in the neighborhood of any of its zero. Thus the existence of a function q, bounded and C^∞ on $\mathbb{R}^2$, and such that Eq. (3.1) holds is equivalent to the existence of a constant C_0 such that, on $\mathbb{R}^2$:

$$|\Delta u| \leqslant C_0|u|. \tag{3.9}$$

Indeed, if u satisfies (3.9), it suffices to take q to be $-\Delta u/u$ where u does not vanish, and 0 elsewhere, which implies trivially equation (3.1). Likewise, the existence of a function w satisfying (3.4) and such that Eq. (3.2) holds is equivalent to the existence of a constant C_0' such that, on $\mathbb{R}^2$:

$$|\Delta u| \leqslant C_0'(r+1)^{-1/3}|\nabla u|. \tag{3.10}$$

Furthermore, it suffices to show Theorem 3.1 for some c_*. One can then obtain the general case by dilatation of u.

3.1.1. Construction for large r

We first construct u on $\{r \geqslant \rho\}$, where ρ is large. As in the article of Meshkov [18], we shall construct u on well-chosen rings $\{\rho_k \leqslant r \leqslant \rho_{k+1}\}$, $k \in \mathbb{N}$.

Notation 3.1. In all the following, we shall write, for sequences of real numbers (A_k) and (B_k):

$$A_k = \mathrm{O}(B_k),$$

when there exist constants C and k_0 such that:

$$\forall k \geqslant k_0, \quad |A_k| \leqslant C|B_k|.$$

When A_k and B_k also depend on $r \in I$, I being an interval, the estimate is also assumed to be uniform with respect to r in I.

We will also use the notation $\approx$ in the following sense:

$$A_k \approx B_k \iff \big(A_k = \mathrm{O}(B_k) \text{ and } B_k = \mathrm{O}(A_k)\big).$$

Let ρ_0 be a large enough real number, and define the sequence ρ_k by:

$$\rho_{k+1} \triangleq \rho_k + 6\rho_k^{1/3}. \tag{3.11}$$

Denote by $\rho_{kl} \triangleq \rho_k + l\rho_k^{1/3}$, for $l = 0, \dots, 6$. This divides the interval $[\rho_k, \rho_{k+1}]$ in 6 sub-intervals. Consider the harmonic function:

$$u_k \triangleq a_k r^{-n_k} \mathrm{e}^{in_k\theta}, \quad a_k > 0,\ n_k \in \mathbb{N}. \tag{3.12}$$

The crucial point of the proof is a lemma which allows to pass from u_k to $-\bar{u}_{k+1}$ within the interval (ρ_k, ρ_{k+1}) with a function solving (3.9). We first choose the values of n_k and a_k. Let:

$$\begin{aligned} &n_k \triangleq 2\left[\frac{\rho_k^{4/3}}{2}\right], \qquad d_k \triangleq \frac{n_{k+1} - n_k}{2} \in \mathbb{N}, \\ &a_0 \triangleq 1, \qquad a_{k+1} \triangleq a_k \rho_{k3}^{2d_k}, \end{aligned} \tag{3.13}$$

where $[y]$ stands for the integer part of y. The a_k's have been chosen so that $|u_k|$ and $|u_{k+1}|$ coincide when $r = \rho_{k3}$. An easy calculation shows that for some positive constant δ, independent of k:

$$d_k = \delta\rho_k^{2/3} + \mathrm{O}(1). \tag{3.14}$$

Lemma 3.1. *There exist a large integer k_0, a constant C_0 independent of $k \geqslant k_0$, and*:

$$u \in C^\infty\big(\big\{x \in \mathbb{R}^2 \mid \rho_k \leqslant |x| \leqslant \rho_{k+1}\big\}\big)$$

such that:

$$\begin{aligned} &u(r,\theta) = u_k(r,\theta), \quad \rho_k = \rho_{k0} \leqslant r \leqslant \rho_{k1}, \\ &u(r,\theta) = -\bar{u}_{k+1}(r,\theta), \quad \rho_{k5} \leqslant r \leqslant \rho_{k6} = \rho_{k+1}, \end{aligned} \tag{3.15}$$

$$\big|u(r,\theta)\big| = \mathrm{O}\big(a_k r^{-n_k}\big), \quad \rho_k \leqslant r \leqslant \rho_{k+1}, \tag{3.16}$$

and, for $\rho_k \leqslant r \leqslant \rho_{k+1}$:

$$|\Delta u| \leqslant C_0|u|, \tag{3.17}$$

$$|\Delta u| \leqslant C_0 r^{-1/3}|\nabla u|. \tag{3.18}$$

The result remains valid when replacing (3.15) *by*:

$$\begin{aligned} &u(r,\theta) = -\bar{u}_k(r,\theta), \quad \rho_{k0} \leqslant r \leqslant \rho_{k1}, \\ &u(r,\theta) = u_{k+1}(r,\theta), \quad \rho_{k5} \leqslant r \leqslant \rho_{k6}. \end{aligned} \tag{3.15$'$}$$

Proof. We will just do the proof in the first case (when (3.15) holds), the proof of the other case being the same almost word by word.

A simple idea would be to let $u = \xi_k u_k - \tilde{\xi}_k \bar{u}_{k+1}$, where ξ_k and $\tilde{\xi}_k$ are smooth functions of r, ξ_k (resp. $\tilde{\xi}_k$) is equal to 1 (resp. 0) near ρ_k, 0 (resp. 1) near ρ_{k+1}. Indeed, with a suitable choice of the functions ξ_k and $\tilde{\xi}_k$, such a u solves (3.17) except in the neighborhood of its zeros. This is a nonnegligible difficulty, taking into account that, for homotopy reasons, a continuous function u satisfying (3.15) has to vanish: when r increases, u passes continuously from a continuous function of θ in S^1 winding clockwise n_k times around the origin to one winding counterclockwise

n_{k+1} times, which is impossible without vanishing. Note that $n_k \neq n_{k+1}$, so that the same argument would hold with u_{k+1} (which winds clockwise n_{k+1} times around the origin) instead of $-\bar{u}_{k+1}$ in (3.15).

One way to avoid this problem is to consider a $\mathbb{C}^2$-valued function (see Remark 3.4). To treat the harder case of a complex-valued function, we need to use a trick due to Meshkov [18] consisting in introducing an intermediate function v_k, close to $-\bar{u}_{k+1}$, which is nonharmonic, but nevertheless solution of an inequality of the form (3.17) on $\rho_k \leqslant r \leqslant \rho_{k+1}$.

Choice of an intermediate state between u_k and $-\bar{u}_{k+1}$. Consider a 2π-periodic function $\varphi_k(\theta)$ (which we shall make explicit later) such that:

$$\left|\varphi_k(\theta)\right| = \mathrm{O}\bigl(\rho_k^{-2/3}\bigr), \quad \left|\varphi_k'(\theta)\right| = \mathrm{O}\bigl(\rho_k^{2/3}\bigr), \quad \left|\varphi_k''(\theta)\right| = \mathrm{O}\bigl(\rho_k^{2}\bigr). \tag{3.19}$$

Let:

$$v_k(r,\theta) \triangleq -r^{4d_k}\rho_{k3}^{-4d_k}\mathrm{e}^{\mathrm{i}\varphi_k(\theta)}\bar{u}_{k+1}(r,\theta)\mathrm{e}^{\mathrm{i}\varphi_k}\bar{u}_{k+1}. \tag{3.20}$$

As $-\bar{u}_{k+1}(r,\cdot)$, the function $v_k(r,\cdot)$, when r is fixed, winds n_{k+1} times, counterclockwise, around the origin. But unlike $-\bar{u}_{k+1}$, it decreases slower than u_k, so that it is more natural to replace u_k by v_k than by $-\bar{u}_{k+1}$ as r is increasing. For these reasons, v_k is an appropriate intermediate state between u_k and $-\bar{u}_{k+1}$. The constant in (3.20) has been chosen so that:

$$\forall\theta, \quad \left|u_k(\rho_{k3},\theta)\right| = \left|u_{k+1}(\rho_{k3},\theta)\right| = \left|v_k(\rho_{k3},\theta)\right|. \tag{3.21}$$

Let $g(r) \triangleq \log(r^{d_k}/\rho_k^{d_k})$. By (3.14), we have:

$$g'(r) = \frac{d_k}{r} = \mathrm{O}\bigl(\rho_k^{-1/3}\bigr).$$

Noting that $g(\rho_k) = 0$ and $\rho_{k+1} - \rho_k = \mathrm{O}(\rho_k^{1/3})$, we get that $g(r) = \mathrm{O}(1)$ for r in the interval $[\rho_k, \rho_{k+1}]$. Thus:

$$r^{d_k} \approx \rho_k^{d_k}, \quad \rho_k \leqslant r \leqslant \rho_{k+1}. \tag{3.22}$$

The following lemma gathers some estimates on u_k, v_k and u_{k+1}.

Lemma 3.2. *We have*:

$$u_k \approx v_k \approx u_{k+1}, \quad \rho_k \leqslant r \leqslant \rho_{k+1}. \tag{3.23}$$

Furthermore, any of the three sequences $w_k = u_k$, v_k or u_{k+1}, satisfies, for $\rho_k \leqslant r \leqslant \rho_{k+1}$:

$$|\partial_r w_k| = \bigl(r^{1/3} + \mathrm{O}\bigl(r^{-1/3}\bigr)\bigr)|w_k|, \qquad \left|\frac{1}{r}\partial_\theta w_k\right| = \bigl(r^{1/3} + \mathrm{O}\bigl(r^{-1/3}\bigr)\bigr)|w_k|. \tag{3.24}$$

Finally there is a constant C_1 such that:

$$|\Delta v_k| \leqslant C_1|v_k|, \quad \rho_k \leqslant r \leqslant \rho_{k+1}, \tag{3.25}$$

$$|\Delta v_k| \leqslant C_1 r^{-1/3}|\nabla v_k|, \quad \rho_k \leqslant r \leqslant \rho_{k+1}. \tag{3.26}$$

Proof. The estimates (3.23) are a direct consequence of (3.22) and the definitions of u_k and v_k. By simple computations:

$$\partial_r u_k = -\frac{n_k}{r}u_k, \qquad \partial_r v_k = \frac{2d_k - n_k}{r}v_k,$$
$$\frac{1}{r}\partial_\theta u_k = \frac{\mathrm{i}n_k}{r}u_k, \qquad \frac{1}{r}\partial_\theta v_k = \frac{-\mathrm{i}n_{k+1} + \mathrm{i}\varphi_k'(\theta)}{r}v_k.$$

With the definition (3.13) of n_k, and the bounds (3.19) on φ_k, we easily get (3.24). Furthermore:

$$\Delta v_k = -2\rho_{k3}^{-4d_k}\nabla\bigl(r^{4d_k}\mathrm{e}^{\mathrm{i}\varphi_k}\bigr)\cdot\nabla\bar{u}_{k+1} - \rho_{k3}^{-4d_k}\Delta\bigl(r^{4d_k}\mathrm{e}^{\mathrm{i}\varphi_k}\bigr)\bar{u}_{k+1}. \tag{3.27}$$

Keeping in mind the estimates (3.14) and (3.22), we get:

$$\nabla\bigl(r^{4d_k}\bigr) \approx r^{4d_k}\rho_k^{-1/3}, \qquad \Delta\bigl(r^{4d_k}\bigr) = \mathrm{O}\bigl(\rho_k^{-2/3}r^{4d_k}\bigr).$$

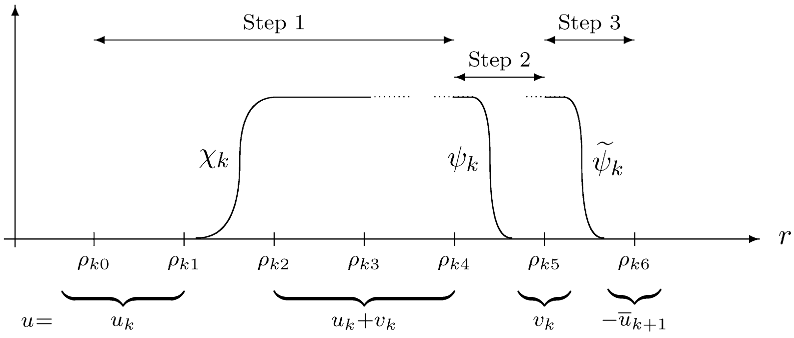

Fig. 1. Steps of the construction.

The bounds (3.19) on φ_k and its derivatives imply easily:

$$\nabla \mathrm{e}^{\mathrm{i}\varphi_k} = \mathrm{O}\big(\rho_k^{-1/3}\big), \qquad \Delta\big(\mathrm{e}^{\mathrm{i}\varphi_k}\big) = \mathrm{O}(1).$$

Combining these estimates, one gets:

$$\big|\nabla\big(r^{4d_k}\mathrm{e}^{\mathrm{i}\varphi_k}\big)\big| = \mathrm{O}\big(\rho_k^{-1/3} r^{4d_k}\big), \qquad \Delta\big(r^{4d_k}\mathrm{e}^{\mathrm{i}\varphi_k}\big) = \mathrm{O}\big(r^{4d_k}\big).$$

Together with the estimates (3.23), (3.24), Eq. (3.27) yields (3.25) and (3.26). □

The construction of u takes three steps (see Fig. 1).

Step 1. Construction of u on $[\rho_{k0}, \rho_{k4}]$.

Lemma 3.3. *There exists u, C^∞ on $\{\rho_{k0} \leqslant r \leqslant \rho_{k4}\}$, satisfying* (3.17) *and* (3.18) *and such that*:

$$u(r,\theta) = u_k(r,\theta), \quad \rho_{k0} \leqslant r \leqslant \rho_{k1}, \tag{3.28}$$

$$u(r,\theta) = u_k(r,\theta) + v_k(r,\theta), \quad \rho_{k2} \leqslant r \leqslant \rho_{k4}. \tag{3.29}$$

Proof. Let χ be a nondecreasing, C^∞ function such that:

$$\chi(s) = 0 \quad \text{if } s \leqslant 1 \quad \text{and} \quad \chi(s) = 1 \quad \text{if } s \geqslant 2. \tag{3.30}$$

Let:

$$\chi_k \triangleq \chi\left(\frac{r-\rho_k}{\rho_k^{1/3}}\right),$$

so that $\chi_k(r)$ is 0 if $r \leqslant \rho_{k1}$ and 1 if $r \geqslant \rho_{k2}$, and that the derivatives of χ_k satisfy the estimates:

$$\big|\chi_k^{(p)}\big| = \mathrm{O}\big(\rho_k^{-p/3}\big). \tag{3.31}$$

Let:

$$u \triangleq u_k + \chi_k v_k \quad \text{for } \rho_{k0} \leqslant r \leqslant \rho_{k4}. \tag{3.32}$$

Obviously (3.28) and (3.29) are satisfied. To show that u satisfies (3.17) and (3.18) with C_0 independent of k, we divide $[\rho_{k0}, \rho_{k4}]$ into two subintervals. When $\rho_{k0} \leqslant r \leqslant \rho_{k2}$, we will simply use that u_k satisfy (3.17) and (3.18), that v_k satisfy (3.25) and (3.26), and that $|u_k|$ is larger than $c|v_k|$, $c > 1$, so that the sum u of the two is of the order of u_k. When $\rho_{k2} \leqslant r \leqslant \rho_{k4}$, the two absolute values may coincide, and we will have to build an adequate phase φ_k so that the function u is harmonic around its zeros. This is the most tricky and nontrivial part of Meshkov's construction.

The region $\rho_{k0} \leqslant r \leqslant \rho_{k2}$. Let:

$$\tilde{g}(r) \triangleq \log\frac{|v_k|}{|u_k|} = \log\frac{a_{k+1} r^{-n_{k+1}+4d_k}\rho_{k3}^{-4d_k}}{a_k r^{-n_k}}.$$

Then:

$$\tilde{g}(r) = 2d_k \log r + C(k),$$

where $C(k)$ is a constant which depends only on k. By the choice of the constants a_k and a_{k+1} (see (3.21)), $\tilde{g}(\rho_{k3}) = 0$. Furthermore $\tilde{g}'(r) = \frac{2d_k}{r}$. Using (3.14) we get that if k is large enough, $\tilde{g}'(r)$ is greater than $\delta\rho_k^{-1/3}$, from which we deduce the two following crucial comparison estimates:

$$|u_k| \geqslant \mathrm{e}^{\delta}|v_k|, \quad r \leqslant \rho_{k2}, \tag{3.33}$$

$$|v_k| \geqslant \mathrm{e}^{\delta}|u_k|, \quad r \geqslant \rho_{k4}. \tag{3.34}$$

The inequality (3.33) implies that when $\rho_{k0} \leqslant r \leqslant \rho_{k2}$:

$$2|u_k| \geqslant |u| \geqslant \left(1 - \mathrm{e}^{-\delta}\right)|u_k| \geqslant c|u_k| \geqslant c|v_k|, \quad \text{where } c > 0. \tag{3.35}$$

Furthermore:

$$\Delta u = \chi_k \Delta v_k + 2\nabla\chi_k \cdot \nabla v_k + (\Delta\chi_k)v_k. \tag{3.36}$$

According to (3.31), (3.35), (3.36), and the estimates of Lemma 3.2, the function u satisfies (3.17) when $\rho_{k0} \leqslant r \leqslant \rho_{k2}$.

Furthermore, using again (3.35), and Lemma 3.2, we get:

$$\begin{aligned}
&\partial_r u = \partial_r u_k + \chi_k' v_k + \chi_k \partial_r v_k,\\
&|\partial_r u| \geqslant r^{1/3}|u_k| + \mathrm{O}\left(r^{-1/3}\right)\left(|u_k| + |v_k|\right),\\
&|\partial_r u| \geqslant \left\{\frac{1}{2}r^{1/3} + \mathrm{O}\left(r^{-1/3}\right)\right\}|u|, \quad \rho_{k0} \leqslant r \leqslant \rho_{k2},
\end{aligned}$$

which, with inequality (3.17), yields inequality (3.18).

The region $\rho_{k2} \leqslant r \leqslant \rho_{k4}$. Notice that in this region, χ_k is equal to 1, so that:

$$u = u_k + v_k = a_k r^{-n_k}\,\mathrm{e}^{\mathrm{i}n_k\theta}\Big\{1 - \underbrace{\rho_{k3}^{-2d_k} r^{2d_k}\,\mathrm{e}^{-\mathrm{i}(2n_k+2d_k)\theta + \mathrm{i}\varphi_k(\theta)}}_{w_k}\Big\}. \tag{3.37}$$

Let:

$$T_k \triangleq \frac{\pi}{n_k + d_k}, \quad \theta_{jk} \triangleq jT_k, \quad 0 \leqslant j \leqslant 2n_k + 2d_k - 1. \tag{3.38}$$

The θ_{jk}'s are the solutions of the equation $\mathrm{e}^{-\mathrm{i}(2n_k+2d_k)\theta} = 1$, so that according to (3.37) (φ_k being small), the function u vanishes near each θ_{jk}. In order to satisfy (3.17) and (3.18), u (thus v_k) has to be harmonic near each θ_{jk}. The function v_k being equal, up to a multiplicative constant, to

$$v_k = -\rho_{k3}^{-4d_k} r^{-n_k+2d_k}\,\mathrm{e}^{-\mathrm{i}n_k\theta - 2\mathrm{i}d_k\theta + \mathrm{i}\varphi_k(\theta)},$$

it suffices to choose φ_k satisfying the following lemma:

Lemma 3.4. *There exists a real-valued $\varphi_k \in C^{\infty}(\mathbb{R})$, 2π-periodic and satisfying* (3.19)*, such that for all j, there is a constant c_{jk} with*:

$$\varphi_k(\theta) = 4d_k\theta + c_{jk}, \quad \theta_{jk} - \frac{T_k}{4} \leqslant \theta \leqslant \theta_{jk} + \frac{T_k}{4}. \tag{3.39}$$

Proof. Consider a function f_k on $[0, T_k]$ (see Fig. 2) so that:

$$\int_0^{T_k} f_k(s)\,\mathrm{d}s = 0, \tag{3.40}$$

$$f_k(s) = 4d_k, \quad s \in [0, T_k/4] \cup [3T_k/4, T_k], \tag{3.41}$$

$$\left|f_k(s)\right| \leqslant C\rho_k^{2/3}, \quad |f_k'| \leqslant C\rho_k^2, \quad C \text{ independent of } k. \tag{3.42}$$

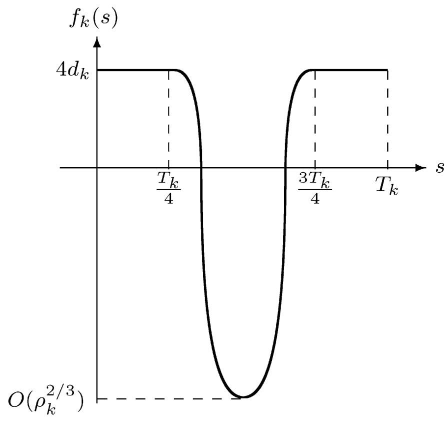

Fig. 2. The function f_k.

Noting that d_k is of the order $\rho_k^{2/3}$, and $1/T_k$ of the order $\rho_k^{4/3}$, such a function exists. We extend f_k to $\mathbb{R}$ into a T_k-periodic function, still denoted by f_k. Let:

$$\varphi_k(\theta) \triangleq \int_0^\theta f_k(s)\,\mathrm{d}s, \tag{3.43}$$

which defines, taking into account (3.40), (3.42), and the fact that T_k is of the order $\rho_k^{-4/3}$, a T_k-periodic function which satisfies the desired bounds (3.19). In particular φ_k is 2π-periodic. Furthermore:

$$\theta_{jk} - \frac{T_k}{4} \leqslant \theta \leqslant \theta_{jk} + \frac{T_k}{4} \Longrightarrow \varphi_k(\theta) = \int_0^{\theta_{jk}} f_k(s)\,\mathrm{d}s + \int_{\theta_{jk}}^{\theta} f_k(s)\,\mathrm{d}s,$$

so that according to (3.40) and (3.41), equality (3.39) holds. $\square$

We go back to the proof of Lemma 3.3. To show (3.17), we distinguish two cases. Let θ be in $[0, 2\pi)$, and choose j so that $\theta = \theta_{jk} + \tau$, $|\tau| \leqslant T_k/2$.

- First assume that $|\tau| \geqslant T_k/4$. Note that $(2n_k + 2d_k)\theta_{jk} \in 2\pi\mathbb{Z}$. Thus the phase of the second term w_k in (3.37) is, by estimates (3.19):

  $$\underbrace{-(2n_k + 2d_k)\theta + \varphi_k(\theta)}_{\tilde{\varphi}_k(\theta)} \equiv -(2n_k + 2d_k)\tau + \mathrm{O}\big(\rho_k^{-2/3}\big) + 2\pi l_{jk}, \quad l_{jk} \in \mathbb{Z}.$$

  Furthermore, depending on the sign of τ:

  $$-\pi \leqslant -(2n_k + 2d_k)\tau \leqslant -\frac{\pi}{2} \quad \text{or} \quad \frac{\pi}{2} \leqslant -(2n_k + 2d_k)\tau \leqslant \pi.$$

  This implies, for some constant C independent of k and θ:

  $$\mathrm{Re}\big(\mathrm{e}^{\mathrm{i}\tilde{\varphi}_k}\big) \leqslant C\rho_k^{-2/3}.$$

  Thus, for large k:

  $$\mathrm{Re}\big(1 - \rho_{k3}^{-2d_k} r^{2d_k} \mathrm{e}^{\mathrm{i}\tilde{\varphi}_k}\big) \geqslant \frac{1}{2}.$$

  Using formula (3.37), we get that if k is large enough:

  $$2|u| \geqslant a_k r^{-n_k} = |u_k|. \tag{3.44}$$

With Lemma 3.2 and the fact that $\Delta u = \Delta v_k$, we get inequality (3.17) with a C_0 independent of k.
By a simple calculation:

$$\partial_r u = -\frac{n_k}{r}u + \frac{2d_k}{r}v_k,$$

so that, using successively (3.44) and (3.23):

$$|\partial_r u| \geqslant \frac{n_k}{r}|u| + \mathrm{O}\big(r^{-1/3}\big)|v_k| \geqslant \big\{r^{1/3} + \mathrm{O}\big(r^{-1/3}\big)\big\}|u|,$$

which yields, together with (3.17), inequality (3.18).

- Now assume that $|\tau| \leqslant T_k/4$. With (3.39), we have:

$$v_k = -\rho_{k3}^{-4d_k} a_k r^{-n_k+2d_k} \mathrm{e}^{-\mathrm{i}(n_k-2d_k)\theta} \mathrm{e}^{\mathrm{i}c_{jk}},$$

so that u is harmonic and inequalities (3.17) and (3.18) are trivially satisfied. □

Step 2. Construction of u on $[\rho_{k4}, \rho_{k5}]$.

Lemma 3.5. *There exists u in $C^\infty(\{\rho_{k4} \leqslant r \leqslant \rho_{k5}\})$ satisfying* (3.17) *and* (3.18)*, and so that*:

$$u(r,\theta) = u_k(r,\theta) + v_k(r,\theta), \quad \text{near } \rho_{k4}, \tag{3.45}$$

$$u(r,\theta) = v_k(r,\theta), \quad \text{near } \rho_{k5}. \tag{3.46}$$

Proof. Let:

$$\psi_k \triangleq 1 - \chi\big(2\rho_k^{-1}(r-\rho_{k4})\big),$$

where χ is the function defined is Step 1, and satisfying (3.30). We have:

$$\begin{gathered}\psi_k(r) = 1 \quad \text{near } \rho_{k4}, \qquad \psi_k(r) = 0 \quad \text{near } \rho_{k5},\\ \big|\psi_k^{(p)}\big| \lesssim \rho_k^{-p/3}.\end{gathered} \tag{3.47}$$

Let:

$$u \triangleq \psi_k u_k + v_k,$$

so that (3.45) and (3.46) are satisfied. Note also that the comparison estimate (3.34) implies that for some $c > 0$ independent of k:

$$2|v_k| \geqslant |u| \geqslant c|v_k| \geqslant c|u_k|, \qquad \rho_{k4} \leqslant r \leqslant \rho_{k5}. \tag{3.48}$$

We have:

$$\Delta u = (\Delta\psi_k)u_k + 2\nabla\psi_k \cdot \nabla u_k + \Delta v_k. \tag{3.49}$$

Using the estimates of Lemma 3.2 together with estimates (3.47), (3.48) and Eq. (3.49) one gets (3.17).

Inequality (3.18), as in the first case of Step 1, comes easily from the explicit computation of $\partial_r u$, inequality (3.17) and estimates (3.24), (3.47) and (3.48). □

Step 3. Construction of u on $[\rho_{k5}, \rho_{k6}]$.

Lemma 3.6. *There exists $u \in C^\infty(\{\rho_{k5} \leqslant r \leqslant \rho_{k6}\})$, satisfying* (3.17) *and* (3.18) *and so that*:

$$u(r,\theta) = v_k(r,\theta), \quad \text{near } \rho_{k5}, \tag{3.50}$$

$$u(r,\theta) = -\bar{u}_{k+1}(r,\theta), \quad \text{near } \rho_{k6}. \tag{3.51}$$

Proof. Consider the function $\tilde{\psi}_k$ defined by:

$$\tilde{\psi}_k(r) \triangleq \psi_k\big(r - \rho_k^{1/3}\big),$$

where ψ_k is the function of Lemma 3.5. The function $\tilde{\psi}_k$ is 1 near ρ_{k5} and 0 near ρ_{k6}, and satisfies estimates (3.47). Recall the definition (3.20) of v_k and let:

$$u \triangleq \tilde{\psi}_k v_k - (1-\tilde{\psi}_k)\bar{u}_{k+1} = -\bar{u}_{k+1}\big\{1 - \tilde{\psi}_k + \tilde{\psi}_k r^{4d_k} \rho_{k3}^{-4d_k} \mathrm{e}^{\mathrm{i}\varphi_k(\theta)}\big\}.$$

According to (3.19), there is a constant $C > 0$ independent of k and θ such that:

$$\operatorname{Re} \mathrm{e}^{\mathrm{i}\varphi_k(\theta)} \geqslant 1 - C\rho_k^{-4/3},$$

so that for k large enough, and using (3.22):

$$\operatorname{Re}\big(1 - \tilde{\psi}_k + \tilde{\psi}_k r^{4d_k} \rho_{k3}^{-4d_k} \mathrm{e}^{\mathrm{i}\tilde{\varphi}_k(\theta)}\big) \geqslant 1 - \tilde{\psi}_k + c_1\tilde{\psi}_k \geqslant c_1,$$

for some positive constant c_1. Thus:

$$|u| \geqslant c_1|u_{k+1}|, \qquad \rho_{k5} \leqslant r \leqslant \rho_{k6}. \tag{3.52}$$

Furthermore:

$$\Delta u = \tilde{\psi}_k \Delta v_k + 2\nabla\tilde{\psi}_k \cdot \nabla v_k + (\Delta\tilde{\psi}_k) v_k - 2\nabla\tilde{\psi}_k \cdot \nabla u_{k+1} - (\Delta\tilde{\psi}_k) u_{k+1}. \tag{3.53}$$

Using Lemma 3.2 and the estimates (3.47) on $\tilde{\psi}_k$ together with Eq. (3.53), we get:

$$\Delta u = \mathrm{O}(u_{k+1}), \qquad \rho_{k5} \leqslant r \leqslant \rho_{k6}.$$

We conclude with (3.52) that u satisfies (3.17) on $[\rho_{k5}, \rho_{k6}]$.

To finish Step 3, we have to show inequality (3.18). As in the preceding steps, it comes easily from inequality (3.17), estimates (3.24) and the explicit computation of $\partial_r u$. □

The construction of u on $[\rho_k, \rho_{k+1}]$ is complete. According to Lemmas 3.3, 3.5 and 3.6, u satisfies (3.17) and (3.18).

It remains to check that on $[\rho_k, \rho_{k+1}]$, u satisfies the bound (3.16). Indeed, by the definition of u at each step, it is easy to deduce (3.16) from the same bound on functions u_k, v_k and u_{k+1}. This bound is trivial for u_k. But u_k, v_k and u_{k+1} are of the same order (see (3.23)), hence (3.16). This concludes the proof of Lemma 3.1. □

If k is large enough for the preceding lemma to hold and $r \in [\rho_k, \rho_{k+1}]$, we take $u(r)$ to be the function constructed in the lemma, satisfying (3.15) if k is odd and (3.15′) is k is even. In this way, the pieces of u stick up well together at each ρ_k, and this defines a C^∞ function u for $r \geqslant \rho$, where $\rho = \rho_K$ is a large positive real number. According to the uniform inequalities (3.17) and (3.18) satisfied by u on each $[\rho_k, \rho_{k+1}]$, the function u is solution of (3.9) and (3.10). It remains to check the decay of u at infinity, and to extend u to all $\mathbb{R}^2$.

3.1.2. Decay of u at infinity

Take a point of $\mathbb{R}^2$ with coordinates (r, θ) such that:

$$\rho_k \leqslant r \leqslant \rho_{k+1}. \tag{3.54}$$

Let:

$$h \triangleq \frac{r - \rho_k}{\rho_k} = \mathrm{O}\big(\rho_k^{-2/3}\big).$$

Estimate (3.16) yields a constant C, independent of k and r satisfying (3.54), such that:

$$\big|u(r,\theta)\big| \leqslant C a_k r^{-n_k}.$$

Thus:

$$\begin{aligned}\log\big|u(r,\theta)\big| - \log\big|u(\rho_k,\theta)\big| &\leqslant -n_k \log r + n_k \log\rho_k + \mathrm{O}(1)\\ &\leqslant -n_k \log(1+h) + \mathrm{O}(1) \leqslant -n_k h + \mathrm{O}(1),\end{aligned}$$

using the fact that $n_k h^2$ is bounded independently of r and k. On the other hand, if $m(r) \triangleq \mathrm{e}^{-\frac{3}{4}r^{4/3}}$:

$$\begin{aligned}\log m(r) - \log m(\rho_k) &= -\frac{3}{4}r^{4/3} + \frac{3}{4}\rho_k^{4/3}\\ &= -\frac{3}{4}\bigl\{\rho_k^{4/3}\bigl((1+h)^{4/3}-1\bigr)\bigr\} = -\rho_k^{4/3}h + \mathrm{O}(1).\end{aligned}$$

Thus, recalling that $n_k = 2[\rho_k^{4/3}/2]$:

$$\log\bigl|u(r,\theta)\bigr| - \log\bigl|u(\rho_k,\theta)\bigr| \leqslant \log m(r) - \log m(\rho_k) + \mathrm{O}(1). \tag{3.55}$$

The same argument yields, if $K \leqslant j \leqslant k$:

$$\log\bigl|u(\rho_j,\theta)\bigr| - \log\bigl|u(\rho_{j-1},\theta)\bigr| \leqslant \log m(\rho_j) - \log m(\rho_{j-1}) + \mathrm{O}(1). \tag{3.56}$$

Adding inequality (3.55) and all inequalities (3.56), $K \leqslant j \leqslant k$, we get:

$$\log\bigl|u(r,\theta)\bigr| \leqslant \log m(r) + \mathrm{O}(k).$$

It is classical that a sequence ρ_k defined by the induction relation (3.11) is of order $k^{3/2}$. Hence:

$$\bigl|u(r,\theta)\bigr| \leqslant \mathrm{e}^{-3/4r^{4/3}+Cr^{2/3}}.$$

Which gives (3.3) for any $c_* < 3/4$.

3.1.3. Extension of u to all $\mathbb{R}^2$

So far, we have constructed u on $r \geqslant \rho$, equal to $ar^{-n}\mathrm{e}^{\mathrm{i}n\theta}$ near ρ for some integer n and real a. Let ψ be a smooth, nondecreasing, function equal to 1 for $r \geqslant 2\rho/3$ and 0 for $r \leqslant \rho/3$. Let:

$$u(r,\theta) \triangleq \bigl(\psi(r)r^{-n} + \bigl(1-\psi(r)\bigr)r^n\bigr)a\mathrm{e}^{\mathrm{i}n\theta}, \quad r \leqslant \rho.$$

This extends u to a C^∞ function on $\mathbb{R}^2$, harmonic in a neighborhood of 0, and who does not vanish, which implies trivially (3.9) for $r \leqslant \rho$. Similarly, ∇u does not vanish for $r > 0$ (because $\partial_\theta u$ does not) which gives (3.10) for $r \leqslant \rho$. The construction is complete. □

3.2. Construction in odd dimension

In this part we prove Theorem 3.2. We first remark that we only need to do the construction for $n = 3$. Indeed if Theorem 3.2 holds for $n = 3$, and $n = m + 3$ is an odd number larger than 3, one can define the function:

$$\tilde{u}(x_1, x_2, \dots, x_{m+3}) \triangleq v(x_1, \dots, x_m)u(x_{m+1}, x_{m+2}, x_{m+3}),$$

where v is the complex-valued function u defined on $\mathbb{R}^m$ given by Theorem 3.1, and u is the $\mathbb{C}^4$-valued function defined on $\mathbb{R}^3$ given by Theorem 3.2. Note that $\tilde{u}$ takes values in $\mathbb{C}^4$. A straightforward computation shows that the function $\tilde{u}$ and potential $\tilde{q}$ are solutions of Eq. (3.1), where $\tilde{q}$ satisfies the bound (3.5) and $\tilde{u}$ decays at the desired speed (3.6).

We now turn to the proof of the case $n = 3$. One of the main ingredients of the preceding construction was the sequence of eigenfunctions $(\mathrm{e}^{\mathrm{i}n_k\theta})_k$ of the Laplace operator on S^1, which trivially satisfies the estimate (in the sense given by Notation 3.1):

$$\mathrm{e}^{\mathrm{i}n_k\theta} \approx \mathrm{e}^{\mathrm{i}n_{k+1}\theta}. \tag{3.57}$$

The construction is difficult to adapt in dimension 3, since there is no sequence of spherical harmonics on S^2 satisfying (3.57). To show Theorem 3.2, we write an abstract theorem showing that an estimate of the form (3.57), but with polynomial loss in n_k, is sufficient to construct a vector-valued, superexponentially decaying solution of an equation of the form (3.1), with a potential q which only grows logarithmically.

Consider a smooth manifold M without boundary, and an operator:

$$R : C^\infty(M) \longrightarrow C^\infty(M).$$

We define, for $\rho \geqslant 0$:

$$\widetilde{M}_\rho \triangleq (\rho, +\infty) \times M, \qquad P \triangleq \frac{\partial^2}{\partial_r^2} + \frac{1}{r^2} R.$$

The operator P acts on $C^\infty(\widetilde{M}_0)$. Up to the conjugation by a power of r, and the addition of a zero-order potential, this framework includes the Laplace operator on $\mathbb{R}^n$, $n \geqslant 2$. Let $p \geqslant 1$. Assume that R admits a sequence of bounded eigenfunctions $(\Phi_k)_{k\geqslant 0}$:

$$\Phi_k : M \longrightarrow \mathbb{R}^p, \quad R\Phi_k \triangleq -\lambda_k \Phi_k, \quad \lambda_k > 0, \tag{3.58}$$

$$\|\Phi_k\|_{L^\infty(M)} = 1, \tag{3.59}$$

where the sequence $(\lambda_k)_k$ is increasing and tends to infinity. Define n_k and ρ_k by:

$$\begin{aligned} &n_k(n_k+1) = \lambda_k, \quad n_k \geqslant 0, \\ &\rho_k \triangleq n_k^{3/4}, \qquad d_k \triangleq \frac{n_{k+1} - n_k}{2}. \end{aligned} \tag{3.60}$$

Denote by $|\cdot|$ the Euclidean norm on $\mathbb{R}^p$. Then the following holds:

Theorem 3.3. *Assume* (3.58), (3.59) *and that there exist positive constants δ, C, N such that*:

$$\forall \omega \in M, \quad \frac{1}{C n_k^N} \leqslant \frac{|\Phi_k(\omega)|}{|\Phi_{k+1}(\omega)|} \leqslant C n_k^N, \tag{3.61}$$

$$d_k = \delta n_k^{1/2} + \mathrm{O}(1). \tag{3.62}$$

Let $c_* > 0$. Then, if ρ is large enough:

$$\exists u \in C^\infty\big(\widetilde{M}_\rho; \mathbb{R}^{2p}\big), \ \exists C > 0, \quad Pu = qu, \tag{3.63}$$

$$q \in C^\infty\big(\widetilde{M}_\rho; \mathbb{R}^{2p\times 2p}\big), \qquad \big(\log(r+2)\big)^{-3} q \in L^\infty, \tag{3.64}$$

$$\big|u(r,\omega)\big| \leqslant C\,\mathrm{e}^{-c_* r^{4/3}}. \tag{3.65}$$

Remark 3.4. As will appear clearly in the proof, when the power N of n_k is 0 in (3.61), the same result remains valid with a bounded q. This would yield Theorem 3.1, with an easier proof, but a $\mathbb{C}^2$-valued solution u.

Proof of Theorem 3.3. This construction is very similar, although much simpler because of the vectorial setting, than the preceding one. Denote by

$$\rho_{kj} \triangleq \rho_k + j\,\frac{\rho_{k+1} - \rho_k}{4},$$

which divides (ρ_k, ρ_{k+1}) in 4 subintervals. Note that according to (3.62):

$$\rho_{k+1} - \rho_k = \frac{3}{2}\delta \rho_k^{1/3} + \mathrm{O}\big(\rho_k^{-1/3}\big). \tag{3.66}$$

Consider the following solutions of the equation $PE = 0$:

$$E_k(r,\omega) \triangleq a_k r^{-n_k} \Phi_k(\omega), \tag{3.67}$$

where the sequence a_k is defined by:

$$a_0 \triangleq 1, \qquad a_{k+1} \triangleq \rho_k^{2d_k} a_k,$$

so that $a_k r^{-n_k}$ and $a_{k+1} r^{-n_{k+1}}$ coincide when $r = \rho_k$. Consider the $\mathbb{R}^{2p}$-valued functions $\mathbf{E}_k$:

$$\mathbf{E}_k(r,\omega) \triangleq \begin{pmatrix} E_k(r,\omega) \\ 0 \end{pmatrix} \quad \text{if } k \text{ is even}, \qquad \mathbf{E}_k(r,\omega) \triangleq \begin{pmatrix} 0 \\ E_k(r,\omega) \end{pmatrix} \quad \text{if } k \text{ is odd}.$$

Then we have the following lemma, analogous to Lemma 3.1:

Lemma 3.7. *Let k be a large enough integer. There exist a constant C_0 independent of k, and*:

$$u \in C^\infty\big(\{\rho_k \leqslant r \leqslant \rho_{k+1}\}; \mathbb{R}^{2p}\big),$$

such that:

$$\begin{aligned} &u(r,\omega) = \mathbf{E}_k(r,\omega), \quad \rho_{k0} \leqslant r \leqslant \rho_{k1},\\ &u(r,\omega) = \mathbf{E}_{k+1}(r,\omega), \quad \rho_{k3} \leqslant r \leqslant \rho_{k4}, \end{aligned} \tag{3.68}$$

$$\big|u(r,\omega)\big| = \mathrm{O}\big(a_k r^{-n_k}\big), \quad \rho_k \leqslant r \leqslant \rho_{k+1}, \tag{3.69}$$

and satisfying the inequality:

$$|Pu| \leqslant C_0(\log r)^3|u|. \tag{3.70}$$

Proof. Using that the logarithmic derivative of $r^{d_k}/\rho_k^{d_k}$ is bounded by $\rho_k^{-1/3}$, one gets, as in the proof of Theorem 3.1, that for $\rho_k \leqslant r \leqslant \rho_{k+1}$:

$$r^{d_k} \approx \rho_k^{d_k}, \tag{3.71}$$

$$a_k r^{n_k} \approx a_{k+1} r^{n_{k+1}}. \tag{3.72}$$

We divide the construction into several steps.

Step 1: Definition of u. Let χ be a C^∞ nonincreasing function on $\mathbb{R}$ such that:

$$\begin{aligned} &s \leqslant 0 \Longrightarrow \chi(s) = 0, \qquad s \geqslant 1 \Longrightarrow \chi(s) = 1,\\ &0 < s \leqslant 1/2 \Longrightarrow \chi(s) = \mathrm{e}^{-1/s}. \end{aligned} \tag{3.73}$$

Near $s = 0$, χ, χ' and χ'' are increasing functions of s. Let (see Fig. 3):

$$\tilde{\chi}_k(r) \triangleq \chi\big(\rho_k^{-1/3}(\rho_{k3} - r)\big), \qquad \chi_k(r) \triangleq \chi\big(\rho_k^{-1/3}(r - \rho_{k1})\big).$$

We have:

$$\big|\chi_k^{(p)}\big| = \mathrm{O}\big(\rho_k^{-p/3}\big), \qquad \big|\tilde{\chi}_k^{(p)}\big| = \mathrm{O}\big(\rho_k^{-p/3}\big). \tag{3.74}$$

Assume for example that k is odd. Let:

$$u(r,\omega) \triangleq \begin{pmatrix} \chi_k(r)E_k(r,\omega) \\ \tilde{\chi}_k(r)E_{k+1}(r,\omega) \end{pmatrix},$$

so that (3.68) holds. The bound (3.69) is immediate from (3.59), (3.67) and (3.72). We have:

$$Pu = \begin{pmatrix} (\chi_k'' + 2\frac{n_k}{r}\chi_k')E_k \\ (\tilde{\chi}_k + 2\frac{n_{k+1}}{r}\tilde{\chi}_k')E_{k+1} \end{pmatrix}. \tag{3.75}$$

Consider the first p components of Pu:

$$v \triangleq \left(\chi_k'' + 2\frac{n_k}{r}\chi_k'\right)E_k. \tag{3.76}$$

We will show that if k is large enough:

$$\big|v(r,\omega)\big| = \mathrm{O}\big((\log r)^3\big|u(r,\omega)\big|\big), \quad \rho_k \leqslant r \leqslant \rho_{k+1},\ \omega \in M. \tag{3.77}$$

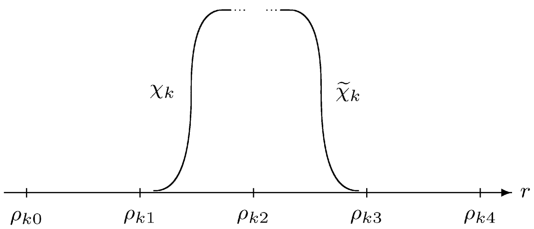

Fig. 3. Step functions for Theorem 3.3.

Let $s \triangleq \rho_k^{-1/3}(r-\rho_{k1})$. We distinguish two regions.

Step 2: Pointwise bound on v for $s \notin (0, (\log\rho_k)^{-3/2})$. Using the explicit form (3.73) of χ near 0, a straightforward computation shows that, for $0 < s \leqslant 1/2$:

$$\chi_k'(r) = \rho_k^{-1/3}\chi'(s) = \rho_k^{-1/3}s^{-2}\chi_k(r), \tag{3.78}$$

$$\chi_k''(r) = \rho_k^{-2/3}\chi''(s) = \rho_k^{-2/3}\bigl(-2s^{-3}+s^{-4}\bigr)\chi_k(r). \tag{3.79}$$

This shows that if $(\log\rho_k)^{-3/2} \leqslant s \leqslant \frac{1}{2}$:

$$\frac{n_k}{r}\bigl|\chi_k'(r)\bigr| = \mathrm{O}\bigl((\log\rho_k)^3\chi_k(r)\bigr), \quad \bigl|\chi_k''(r)\bigr| = \mathrm{O}\bigl((\log\rho_k)^3\chi_k(r)\bigr).$$

Furthermore, these inequalities are trivial for $s<0$ (where χ_k is identically). When $s \geqslant 1/2$ they are a direct consequence of the estimates (3.74) on the derivatives of χ_k. Going back to the definition of v, we have:

$$\bigl|v(r,\omega)\bigr| \leqslant (\log r)^3\chi_k(r)\bigl|E_k(r,\omega)\bigr|, \quad s \leqslant 0 \text{ or } s \geqslant (\log\rho_k)^{-3/2}.$$

This shows inequality (3.77), outside of the region $\{s \in (0, (\log\rho_k)^{-3/2})\}$.

Step 3: Pointwise bound on v for $s \in (0, (\log\rho_k)^{-3/2})$. We now assume that $s \in (0, \log(\rho_k)^{-3/2})$. Then, using formulas (3.78), (3.79) and the fact that χ' is increasing we get, if k is large:

$$\begin{aligned}&\frac{n_k}{r}\bigl|\chi_k'(s)\bigr| = \frac{n_k}{r}\rho_k^{-1/3}\bigl|\chi'(s)\bigr| \leqslant C\bigl|\chi'\bigl(\log^{-3/2}\rho_k\bigr)\bigr|,\\ &\frac{n_k}{r}\bigl|\chi_k'(s)\bigr| \leqslant C(\log\rho_k)^3\mathrm{e}^{-(\log\rho_k)^{3/2}}.\end{aligned} \tag{3.80}$$

Similarly:

$$\begin{aligned}\bigl|\chi_k''(s)\bigr| &= \rho_k^{-2/3}\bigl|\chi''(s)\bigr| \leqslant \rho_k^{-2/3}\bigl|\chi''\bigl(\log^{-3/2}\rho_k\bigr)\bigr|\\ &\leqslant \rho_k^{-2/3}(\log\rho_k)^6\mathrm{e}^{-(\log\rho_k)^{3/2}}.\end{aligned} \tag{3.81}$$

By the definition (3.76) of v, together with (3.80), (3.81), we get:

$$\bigl|v(r,\omega)\bigr| \leqslant (\log\rho_k)^3\mathrm{e}^{-(\log\rho_k)^{3/2}}\bigl|E_k(\omega)\bigr| \leqslant C(\log\rho_k)^3\mathrm{e}^{-(\log\rho_k)^{3/2}}n_k^N\bigl|E_{k+1}(\omega)\bigr|.$$

For the second inequality, we used the assumption (3.61) on the sequence (Φ_k) together with (3.72). Noting that $\tilde\chi_k$ takes value 1 near ρ_{k1}, we get that (3.77) holds for large k.

End of the proof. By the same argument, one may show the property analogous to (3.77) for the last p components of Pu, namely:

$$\left(\tilde\chi_k + 2\frac{n_{k+1}}{r}\tilde\chi_k'\right)E_{k+1} = \mathrm{O}\bigl((\log r)^3u(r,\omega)\bigr), \quad \rho_k \leqslant r \leqslant \rho_{k+1},\ \omega \in M.$$

Thus:

$$Pu = \mathrm{O}\bigl((\log u)^3u\bigr), \quad \rho_k \leqslant r \leqslant \rho_{k+1}, \omega \in M,$$

which completes the proof of the lemma. □

The end of the proof of Theorem 3.3, which consists in sticking up the pieces of u defined by Lemma 3.7, and checking the decay of u at infinity, is exactly the same as the one of Theorem 3.1, and therefore we omit it. □

Proof of Theorem 3.2. We shall use Theorem 3.3 with $M = S^2$. For this we need to choose suitable spherical harmonics. Let θ and ϕ be the spherical coordinates on the sphere S^2, $\theta \in [0,\pi]$ being the polar coordinate and $\phi \in [0,2\pi)$ the azimuthal one. Let $l = 2j$ be an even integer and F_l be the $\mathbb{C}^2$-valued spherical harmonic:

$$F_l(\phi,\theta) \triangleq \begin{pmatrix} P_l^0(\cos\theta)\\ \mathrm{e}^{\mathrm{i}\phi}P_l^1(\cos\theta)\end{pmatrix}. \tag{3.82}$$

Here and in the sequel, the P_l^m are the associated Legendre polynomials:

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}\left(1-x^2\right)^{m/2}\frac{\mathrm{d}^{l+m}}{\mathrm{d}x^{l+m}}\left(x^2-1\right)^l. \tag{3.83}$$

The functions P_l^m are solutions to the equation:

$$\left(1-x^2\right)\frac{\mathrm{d}^2 P}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}P}{\mathrm{d}x} + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P = 0, \quad x \in (-1,1). \tag{3.84}$$

It is standard (cf. [25,23]), that $\Delta_{S^2} F_l = -l(l+1)F_l$. To use Theorem 3.3 we need to give a pointwise estimate on F_l:

Lemma 3.8. *The eigenfunctions F_l satisfy, for large $l \in 2\mathbb{N}$:*

$$\frac{1}{Cl^{7/4}} \leqslant \left|F_l(\phi,\theta)\right| \leqslant Cl. \tag{3.85}$$

Proof. Let:

$$g(x) \triangleq l(l+1)\left|P_l^0(x)\right|^2 + \left|P_l^1(x)\right|^2.$$

Note that according to (3.83),

$$g(x) = l(l+1)\left|P_l^0(x)\right|^2 + \left(1-x^2\right)\left|\frac{\mathrm{d}P_l^0}{\mathrm{d}x}(x)\right|^2.$$

Using Eq. (3.84) with $m=0$ we get:

$$g'(x) = 2x\left|\frac{\mathrm{d}P_l^0}{\mathrm{d}x}(x)\right|^2,$$

so that the minimum of g is in 0, and its maxima are in 1 and -1. Note that l being even, P_l^0 is even and P_l^1 is odd. In particular:

$$g(0) = l(l+1)\left|P_l^0(0)\right|^2. \tag{3.86}$$

Let $\alpha_j \triangleq |P_{2j}^0(0)| = |P_l^0(0)|$. According to formula (3.83),

$$P_{2j}^0(x) = \frac{1}{2^{2j}(2j)!}\frac{\mathrm{d}^{2j}}{\mathrm{d}x^{2j}}\left(x^2-1\right)^{2j}.$$

The coefficient of x^{2j} in $(x^2-1)^{2j}$ is $(-1)^j\binom{2j}{j}$, so that we have:

$$\alpha_j = \frac{1}{2^{2j}(2j)!}(2j)!\binom{2j}{j} = \frac{(2j)!}{2^{2j}(j!)^2}.$$

Stirling formula, yields:

$$\left|P_l^0(0)\right| = \left|P_{2j}^0(0)\right| = \alpha_j \sim \frac{1}{2\sqrt{\pi j}}, \quad \text{as } j \to +\infty. \tag{3.87}$$

Furthermore, going back to the definition (3.83) of P_m^l, we get $|P_l^0(1)| = |P_l^0(-1)| = 1$, so that:

$$g(1) = g(-1) = l(l+1). \tag{3.88}$$

Using (3.86), (3.87) and (3.88) we get:

$$\frac{1}{Cl^{3/2}} \leqslant l(l+1)\left|P_l^0(x)\right|^2 + \left|P_l^1(x)\right|^2 \leqslant Cl(l+1),$$

which shows (3.85) according to the definition (3.82) of F_l. $\square$

Choose a large odd number n_0 and define the sequence $(n_k)_k$ of odd integers, and the sequence $(\Phi_k)_k$ of eigenfunctions of Δ_{S^2} by:

$$n_{k+1} \triangleq n_k + 2\big[n_k^{1/2}\big], \qquad \Phi_k(\phi,\theta) \triangleq c_k F_{n_k-1}(\phi,\theta),$$

where c_k is a normalizing constant such that (3.59) holds. Noting that (3.62) is fulfilled by the choice of $(n_k)_k$, and that (3.85) implies (3.61), one can use Theorem 3.3 with $R = \Delta_{S^2}$. This yields a function $\tilde{u} \in C^\infty(\{x \in \mathbb{R}^3 \mid |x| \geqslant \rho\}; \mathbb{C}^4)$, solution of:

$$P\tilde{u} = \left(\frac{\partial^2}{\partial_r^2} + \frac{1}{r^2}\Delta_{S^2}\right)\tilde{u} = \tilde{q}\tilde{u},$$

and so that $\tilde{q}$ and $\tilde{u}$ decrease at the desired speed at infinity. Taking into account that $\Delta_{\mathbb{R}^3} f = r^{-1}P(rf)$, the function $u \triangleq r^{-1}\tilde{u}$ and potential $q \triangleq \tilde{q}$ satisfy all the assertions of Theorem 3.2. It remains to extend u to $r \leqslant \rho$, which is left to the reader. $\square$

4. Optimality of the observability constant for the heat equation with zero order potential

This section is addressed to the proof of Theorem 1.1 and its weaker counterpart in odd space dimensions:

Theorem 4.1. *Assume that $n \geqslant 3$ is odd and that $N \geqslant 8$. Let ω be a nonempty open subset of Ω such that $\Omega \setminus \overline{\omega} \neq \emptyset$. Then there exist two constants $c > 0$ and $\mu > 0$, a family of potentials $\{a_R\}_{R>0} \subset L^\infty(Q; \mathbb{R}^{N\times N})$ satisfying*

$$\|a_R\|_\infty \underset{R\to+\infty}{\longrightarrow} +\infty,$$

and a family of initial data $\{\varphi_R^0\}_{R>0}$ in $(L^2(\Omega))^N$ such that the corresponding solution φ_R of (1.1) satisfies

$$\lim_{R\to\infty}\left\{\inf_{T\in J_\mu}\frac{\|\varphi_R(T)\|^2_{(L^2(\Omega))^N}}{\exp(c(\log\|a_R\|_\infty)^{-2}\|a_R\|_\infty^{2/3})\int_0^T\int_\omega|\varphi_R|^2\,\mathrm{d}x\,\mathrm{d}t}\right\} = +\infty, \tag{4.1}$$

where $J_\mu \triangleq (0, \mu(\log\|a_R\|_\infty)^{-2}\|a_R\|_\infty^{-1/3}]$.

Theorem 4.1 would show the optimality of an observability constant a little smaller than $C_3^*(T,a)$ defined in (1.5). This is due to the logarithmic loss in the construction of Theorem 3.2.

The proofs of Theorems 1.1 and 4.1 are very similar so that we do not need to distinguish between the two cases in the main part of this section. First observe that we only need to show Theorem 1.1 in the case $N = 2$ and Theorem 4.1 in the case $N = 8$. Indeed, to get the other cases, it suffices to consider the solutions φ_R in which the first 2 components (respectively 8 components in the case of odd dimensions) are as in the case $N = 2$ (respectively $N = 8$), the other being identically zero. The matrix a_R of the corresponding system can be built in a similar way by adding zero entries to the 2×2 (respectively 8×8) matrix a_R.

The proof is divided into several steps.

Step 1: Construction on $\mathbb{R}^n$.

Consider the solution u and potential q given by Theorem 3.1 if n is even, and by Theorem 3.2 if n is odd. Recalling that both u and q are complex-valued, by setting

$$u_R(x) = \begin{pmatrix}\operatorname{Re} u(Rx)\\ \operatorname{Im} u(Rx)\end{pmatrix}, \qquad a_R(x) = -R^2\begin{pmatrix}\operatorname{Re} q(Rx) & -\operatorname{Im} q(Rx)\\ \operatorname{Im} q(Rx) & \operatorname{Re} q(Rx)\end{pmatrix}, \tag{4.2}$$

we obtain a one-parameter family of potentials $\{a_R\}_{R>0}$ and solutions $\{u_R\}_{R>0}$ satisfying

$$\Delta u_R = a_R(x)u_R, \quad \text{in } \mathbb{R}^n, \tag{4.3}$$

and (using Theorems 3.1–3.2 with $c_* = 1$)

$$\big|u_R(x)\big| \leqslant C\exp\big(-R^{4/3}|x|^{4/3}\big), \quad \text{in } \mathbb{R}^n. \tag{4.4}$$

Furthermore, for some constant $C > 0$, the potential a_R is such that

$$C^{-1}R^2 \leqslant \|a_R\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^{N\times N})} \leqslant CR^2, \quad \text{if } n \text{ is even}, \tag{4.5}$$

$$C^{-1}R^2 \leqslant \|a_R\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^{N\times N})} \leqslant CR^2(\log R)^3, \quad \text{if } n \text{ is odd}. \tag{4.6}$$

The functions $\{u_R\}_{R>0}$ may also be viewed as stationary solutions of the corresponding parabolic systems. Indeed, set

$$\psi_R(t,x) = u_R(x), \quad x \in \mathbb{R}^n,\ t > 0. \tag{4.7}$$

Then, it satisfies

$$\psi_{R,t} - \Delta\psi_R + a_R\psi_R = 0, \quad x \in \mathbb{R}^n,\ t > 0, \tag{4.8}$$

and

$$\left|\psi_R(x,t)\right| \leqslant C\exp\left(-R^{4/3}|x|^{4/3}\right), \quad x \in \mathbb{R}^n,\ t > 0. \tag{4.9}$$

Assume now that ω is an open bounded subset of $\mathbb{R}^n \setminus B$, B being the unit ball in $\mathbb{R}^n$. Then

$$\int_0^T\!\!\int_\omega |\psi_R|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant TC\int_\omega \exp\left(-2R^{4/3}|x|^{4/3}\right)\mathrm{d}x, \tag{4.10}$$

and, taking into account that $|x| \geqslant 1$ on ω,

$$\int_\omega \exp\left(-2R^{4/3}|x|^{4/3}\right)\mathrm{d}x = \mathrm{O}\left(\exp\left(-2R^{4/3}\right)\right). \tag{4.11}$$

On the other hand, for some constant $c > 0$,

$$\left\|\psi_R(T)\right\|^2_{(L^2(\mathbb{R}^n))^N} = \|u_R\|^2_{(L^2(\mathbb{R}^n))^N} = \frac{1}{R^n}\|u\|^2_{(L^2(\mathbb{R}^n))^N} = \frac{c}{R^n}. \tag{4.12}$$

In view of (4.5) and (4.10)–(4.12) it is easy to see that, if n is even, an estimate of the form (1.2) would be sharp in the whole $\mathbb{R}^n$ in what concerns the dependence of the observability constant C_3^* on the potential. The same conclusion would hold for odd n, up to a logarithmic factor.

Step 2: Restriction to Ω.

Let us now consider the case of a bounded domain Ω and ω to be a nonempty open subset Ω such that $\Omega \setminus \overline{\omega} \neq \emptyset$. Without loss of generality (by translation and scaling) we can assume that $B \subset \Omega\setminus\overline{\omega}$.

We can then view the functions $\{\psi_R\}_{R>0}$ above as a family of solutions of the Dirichlet problem in Ω with non-homogeneous Dirichlet boundary conditions:

$$\begin{cases} \psi_{R,t} - \Delta\psi_R + a_R\psi_R = 0, & \text{in } Q, \\ \psi_R = \varepsilon_R, & \text{on } \Sigma, \end{cases} \tag{4.13}$$

where

$$\varepsilon_R = \psi_R|_\Gamma = u_R|_\Gamma. \tag{4.14}$$

Taking into account that both ω and Γ are contained in the complement of B, we deduce that, for a suitable C:

$$\left|\psi_R(t,x)\right| \leqslant C\exp\left(-R^{4/3}\right), \quad x \in \omega,\ 0 < t < T, \tag{4.15}$$

$$\left|\varepsilon_R(t,x)\right| \leqslant C\exp\left(-R^{4/3}\right), \quad x \in \Gamma,\ 0 < t < T. \tag{4.16}$$

We can then correct these solutions to fulfill the Dirichlet homogeneous boundary condition. For this purpose, we introduce the correcting terms

$$\begin{cases} \rho_{R,t} - \Delta\rho_R + a_R\rho_R = 0, & \text{in } Q, \\ \rho_R = \varepsilon_R, & \text{on } \Sigma, \\ \rho_R(0,x) = 0, & \text{in } \Omega, \end{cases} \tag{4.17}$$

and then set

$$\varphi_R = \psi_R - \rho_R. \tag{4.18}$$

Clearly $\{\varphi_R\}_{R>0}$ is a family of solutions to parabolic systems of the form (1.1) with potentials $a = a_R(x)$.

Let us show that φ_R is the family of solutions that fulfills (1.7). By the bounds (4.5) and (4.6) on a_R, we have, for some $C > 0$:

$$C^{-1}R^2 \leqslant \|a_R\|_\infty \leqslant CR^2, \quad \text{if } n \text{ is even,} \tag{4.19}$$

$$C^{-1}R^2 \leqslant \|a_R\|_\infty \leqslant CR^2(\log R)^3, \quad \text{if } n \text{ is odd.} \tag{4.20}$$

Furthermore, according to (4.10)–(4.11) and (4.12):

$$\int_0^T\!\!\int_\omega |\psi_R|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant T\int_\omega |u_R|^2\,\mathrm{d}x \leqslant CT\exp\bigl(-2R^{4/3}\bigr), \quad \text{as } R \to \infty, \tag{4.21}$$

$$\bigl\|\psi_R(T)\bigr\|^2_{(L^2(\Omega))^N} = \frac{1}{R^n}\|u\|^2_{(L^2(R\Omega))^N} \geqslant \frac{c_2}{R^n}, \quad \text{for some constant } c_2 > 0. \tag{4.22}$$

Let us now analyze the corrector term ρ_R. We decompose it as

$$\rho_R = \sigma_R + \xi_R, \tag{4.23}$$

where

$$\begin{cases} \sigma_{R,t} - \Delta\sigma_R = 0, & \text{in } Q,\\ \sigma_R = \varepsilon_R, & \text{on } \Sigma,\\ \sigma_R(0) = 0, & \text{in } \Omega. \end{cases} \tag{4.24}$$

By the maximum principle and (4.16) we know that

$$\|\sigma_R\|_{(L^\infty(Q))^N} \leqslant \|\varepsilon_R\|_{(L^\infty(\Sigma))^N} \leqslant C\exp\bigl(-R^{4/3}\bigr). \tag{4.25}$$

On the other hand, the reminder ξ_R fulfills

$$\begin{cases} \xi_{R,t} - \Delta\xi_R + a_R\xi_R = -a_R\sigma_R, & \text{in } Q,\\ \xi_R = 0, & \text{on } \Sigma,\\ \xi_R(0) = 0, & \text{in } \Omega. \end{cases} \tag{4.26}$$

A standard energy estimate shows, together with (4.25), that

$$\bigl\|\xi_R(t)\bigr\|^2_{(L^2(\Omega))^N} \leqslant C\exp\bigl(-2R^{4/3}\bigr)\exp\bigl(2t\|a_R\|_\infty\bigr), \quad \forall t \geqslant 0. \tag{4.27}$$

We now distinguish between the two cases: n even or n odd.

End of the proof when n is even.

Let $T \leqslant \mu\|a_R\|_\infty^{-1/3}$. According to (4.27), we have, for time $t \leqslant T$:

$$\bigl\|\xi_R(t)\bigr\|^2_{(L^2(\Omega))^N} \leqslant C\exp\bigl(-2R^{4/3} + 2\mu\|a_R\|_\infty^{2/3}\bigr). \tag{4.28}$$

Therefore, by choosing $\mu > 0$ small enough we deduce, using estimate (4.19) on a_R:

$$\bigl\|\xi_R(t)\bigr\|_{(L^2(\Omega))^N} \leqslant C\exp\biggl(-\frac{1}{2}R^{4/3}\biggr). \tag{4.29}$$

Combining (4.25) and (4.29) we deduce that

$$\bigl\|\rho_R(T)\bigr\|_{(L^2(\Omega))^N} \leqslant C\exp\biggl(-\frac{1}{2}R^{4/3}\biggr). \tag{4.30}$$

In view of (4.22) and (4.30) we conclude that

$$\bigl\|\varphi_R(T)\bigr\|^2_{(L^2(\Omega))^N} \geqslant \frac{c_2}{2R^n} \tag{4.31}$$

as $R \to \infty$.

On the other hand, integrating (4.30) with respect to time we deduce that

$$\int_0^T\!\!\int_\Omega |\rho_R|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant CT\exp\biggl(-\frac{1}{2}R^{4/3}\biggr).$$

Obviously this implies, in particular, that

$$\int_0^T\int_\omega |\rho_R|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C\exp\left(-\frac{1}{2}R^{4/3}\right). \tag{4.32}$$

Combining (4.21) and (4.32) we get

$$\int_0^T\int_\omega |\varphi_R|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C\exp\left(-\frac{1}{2}R^{4/3}\right). \tag{4.33}$$

Estimate (4.31) together with (4.33) and the bound (4.19) on a_R, guarantees (1.7) for small $c > 0$, which concludes the proof of Theorem 1.1.

End of the proof when n is odd.

Let T be such that:

$$T \leqslant \mu\big(\log\|a_R\|_\infty\big)^{-2}\|a_R\|_\infty^{-1/3}.$$

Note that this implies, with (4.6):

$$T \leqslant C\mu(\log R)^{-2}\|a_R\|_\infty^{-1/3}. \tag{4.34}$$

Consider a time $t < T$. By estimates (4.27) and (4.34) we have:

$$\big\|\xi_R(t)\big\|^2_{(L^2(\Omega))^N} \leqslant C\exp\big(-2R^{4/3} + 2C\mu(\log R)^{-2}\|a_R\|_\infty^{2/3}\big). \tag{4.35}$$

Therefore, by choosing $\mu > 0$ small enough we deduce, using estimate (4.20) on a_R:

$$\big\|\xi_R(t)\big\|_{(L^2(\Omega))^N} \leqslant C\exp\left(-\frac{1}{2}R^{4/3}\right). \tag{4.36}$$

Combining (4.25) and (4.36) we deduce that

$$\big\|\rho_R(T)\big\|_{(L^2(\Omega))^N} \leqslant C\exp\left(-\frac{1}{2}R^{4/3}\right). \tag{4.37}$$

As in the even-dimensional case we have, with (4.22) and (4.37):

$$\big\|\varphi_R(T)\big\|^2_{(L^2(\Omega))^N} \geqslant \frac{c_2}{2R^n}, \tag{4.38}$$

as $R \to \infty$.

On the other hand, integrating (4.37) with respect to time we get:

$$\int_0^T\int_\omega |\rho_R|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C\exp\left(-\frac{1}{2}R^{4/3}\right). \tag{4.39}$$

Combining with (4.21) we get

$$\int_0^T\int_\omega |\varphi_R|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C\exp\left(-\frac{1}{2}R^{4/3}\right). \tag{4.40}$$

Note that the bounds (4.20) on a_R imply, for large R:

$$\big(\log\|a_R\|_\infty\big)^{-2}\|a_R\|^{2/3} \leqslant CR^{4/3}. \tag{4.41}$$

Estimate (4.31) together with (4.40) and on (4.41), guarantees (4.1) for small $c > 0$. The proof of Theorem 4.1 is completed. □

5. Optimality of the observability constant for the wave equation with zero order potential

This section is devoted to prove Theorem 1.2. We start to write a slightly weaker counterpart of Theorem 1.2 in odd dimension.

Theorem 5.1. *Assume that $n \geqslant 3$ is odd and $N \geqslant 8$. Let ω be a given open nonempty subset of Ω such that $\Omega \setminus \overline{\omega} \neq \emptyset$. Then, for all $T > 0$ there exist a constant $c > 0$, a family of potentials $\{a_R\}_{R>0} \subset L^\infty(Q; \mathbb{R}^{N\times N})$ satisfying*

$$\|a_R\|_\infty \underset{R\to+\infty}{\longrightarrow} +\infty,$$

a family of initial data $\{(w_R^0, w_R^1)\}_{R>0} \subset (L^2(\Omega))^N \times (H^{-1}(\Omega))^N$ such that the corresponding solution w_R of (1.10) *satisfies*

$$\lim_{R\to\infty}\left\{\frac{\|w_R^0\|^2_{(L^2(\Omega))^N} + \|w_R^1\|^2_{(H^{-1}(\Omega))^N}}{\exp(c(\log\|a_R\|_\infty)^{-2}\|a_R\|_\infty^{2/3})\int_0^T\int_\omega |w_R|^2\,\mathrm{d}x\,\mathrm{d}t}\right\} = +\infty. \tag{5.1}$$

The proof of Theorem 5.1 is very similar to the one of Theorem 1.2 and we thus leave it to the reader. To prove Theorem 1.2, we argue as in the previous section. As it was observed in that section, one may assume that $N = 2$. Consider the solution u and potential q on $\mathbb{R}^n$ given by Theorem 3.1. The family $\{\psi_R\}_{R>0}$ as in (4.7) and (4.2) can be viewed as a family of stationary solutions of the Cauchy problem

$$\begin{cases} \psi_{R,tt} - \Delta\psi_R + a_R\psi_R = 0, & \text{in } \mathbb{R}^n \times (0,T),\\ \psi_R(0) = u_R, \quad \psi_{R,t}(0) = 0, & \text{in } \mathbb{R}^n, \end{cases} \tag{5.2}$$

with potentials $a_R = a_R(x)$ as in (4.2). They can also be viewed as solutions of the wave system in the domain Ω with nonhomogeneous boundary conditions:

$$\begin{cases} \psi_{R,tt} - \Delta\psi_R + a_R\psi_R = 0, & \text{in } Q,\\ \psi_R = \varepsilon_R, & \text{on } \Sigma,\\ \psi_R(0) = u_R, \quad \psi_{R,t}(0) = 0, & \text{in } \Omega, \end{cases} \tag{5.3}$$

where

$$\varepsilon_R(x) = u_R(x), \quad \forall x \in \Gamma. \tag{5.4}$$

We can assume, without loss of generality, that both ω and Γ are contained in the exterior of the unit ball. Then, (4.15) and (4.16) hold.

We correct the boundary conditions by introducing the weak solutions $\rho_R = \rho_R(t,x)$ of

$$\begin{cases} \rho_{R,tt} - \Delta\rho_R + a_R\rho_R = 0, & \text{in } Q,\\ \rho_R = \varepsilon_R, & \text{on } \Sigma,\\ \rho_R(0) = \rho_{R,t}(0) = 0, & \text{in } \Omega, \end{cases} \tag{5.5}$$

and then setting

$$w_R = \psi_R - \rho_R. \tag{5.6}$$

Clearly $\{w_R\}_{R>0}$ is a family of solutions to the hyperbolic systems of the form (1.10) with potentials $a = a_R$ of size $\frac{1}{c_1}R^2 \leqslant \|a_R\|_\infty \leqslant c_1R^2$ for some constant $c_1 > 0$.

Let us show that w_R satisfies (1.14). We have

$$w_R(0) = u_R, \quad w_{R,t}(0) = 0, \quad \text{in } \Omega. \tag{5.7}$$

Hence, for some constant $c_2 > 0$

$$\|w_R(0)\|^2_{(L^2(\Omega))^N} + \|w_{R,t}(0)\|^2_{(H^{-1}(\Omega))^N} = \|u_R\|^2_{(L^2(\Omega))^N} = \frac{1}{R^n}\|u\|^2_{(L^2(R\Omega))^N} \geqslant \frac{c_2}{R^n}, \tag{5.8}$$

as $R \to \infty$.

On the other hand, taking into account that $\omega \subset \mathbb{R}^n \backslash B$ and by (4.21), we deduce that

$$\int_0^T\int_\omega |w_R|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant 2\int_0^T\int_\omega |\psi_R|^2\,\mathrm{d}x\,\mathrm{d}t + 2\int_0^T\int_\omega |\rho_R|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant 2C\exp\bigl(-2R^{4/3}\bigr) + 2\int_0^T\int_\omega |\rho_R|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{5.9}$$

Consequently, in order to conclude that (1.14) holds, it is sufficient to get an upper bound of the form

$$\int_0^T\int_\omega |\rho_R|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C\exp\bigl(-c_0R^{4/3}\bigr) \tag{5.10}$$

for suitable constants $C>0$ and $c_0>0$.

The rest of the proof is devoted to proving (5.10).

The solution ρ_R of (5.5) is defined by transposition. More precisely, consider the adjoint problem

$$\begin{cases}\theta_{tt}-\Delta\theta+a_R\theta=f, & \text{in } Q,\\ \theta=0, & \text{on } \Sigma,\\ \theta(T)=\theta_t(T)=0, & \text{in } \Omega.\end{cases} \tag{5.11}$$

Then, multiplying the first equation in (5.5) by θ, integrating it in Q and using formal integration by parts we get

$$\int_Q f\cdot\rho_R\,\mathrm{d}x\,\mathrm{d}t = -\int_\Sigma \frac{\partial\theta}{\partial\nu}\cdot\varepsilon_R\,\mathrm{d}\sigma\,\mathrm{d}t. \tag{5.12}$$

Here, $\cdot$ means the usual scalar product in $\mathbb{R}^N$. We adopt (5.12) as definition of solution ρ_R of (5.5) in the sense of transposition.

It is well known that there exists $C=C(R)$ such that

$$\|\rho_R\|_{(L^\infty(0,T;L^2(\Omega)))^N} \leqslant C(R)\|\varepsilon_R\|_{(L^2(\Sigma))^N}. \tag{5.13}$$

This is a consequence, by duality, of the following hidden regularity property for solutions of (5.11):

$$\left\|\frac{\partial\theta}{\partial\nu}\right\|_{(L^2(\Sigma))^N} \leqslant C(R)\|f\|_{(L^1(0,T;L^2(\Omega)))^N}. \tag{5.14}$$

It remains to show that (5.14) holds with a constant $C(R)$ such that

$$C(R)\leqslant C\exp\bigl(CR^{4/3}\bigr). \tag{5.15}$$

First of all, by an energy estimate, we observe that

$$\|\theta\|_{(L^\infty(0,T;H_0^1(\Omega)))^N} + \|\theta_t\|_{(L^\infty(0,T;L^2(\Omega)))^N} \leqslant C\exp(CR)\|f\|_{(L^1(0,T;L^2(\Omega)))^N}. \tag{5.16}$$

However, to prove this, we cannot simply apply the Gronwall inequality to the energy

$$E(t)=\frac12\bigl[\bigl\|\theta(t)\bigr\|^2_{(H_0^1(\Omega))^N} + \bigl\|\theta_t(t)\bigr\|^2_{(L^2(\Omega))^N}\bigr]. \tag{5.17}$$

This would yield a constant in the inequality of the order of $\exp(CR^2)$ since the norm $\|a_R\|_\infty$ of the potential is of the order of R^2. To improve this estimate and get a constant of the order of $\exp(CR)$ we have to work, rather, with the modified energy

$$E_R(t)=\frac12\bigl[\bigl\|\theta(t)\bigr\|^2_{(H_0^1(\Omega))^N} + \bigl\|\theta_t(t)\bigr\|^2_{(L^2(\Omega))^N} + R^2\bigl\|\theta(t)\bigr\|^2_{(L^2(\Omega))^N}\bigr], \tag{5.18}$$

as in [29]. We have:

$$\left|\frac{\mathrm{d}E_R(t)}{\mathrm{d}t}\right| = \left|\int\limits_\Omega f\cdot\theta_t\,\mathrm{d}x - \int\limits_\Omega (a_R\theta)\cdot\theta_t\,\mathrm{d}x\right|$$
$$\leqslant \|f(t)\|_{(L^2(\Omega))^N}\|\theta_t(t)\|_{(L^2(\Omega))^N} + CR^2\|\theta(t)\|_{(L^2(\Omega))^N}\|\theta_t(t)\|_{(L^2(\Omega))^N}$$
$$\leqslant CRE_R(t) + \|f(t)\|_{(L^2(\Omega))^N}\sqrt{E_R(t)}. \tag{5.19}$$

Solving this differential inequality and taking into account $E_R(T)=0$ (because the data of θ at $t=T$ vanish) we easily get (5.16).

Once (5.16) is proved we can show (5.14) using the Rellich multiplier as in [17]. It follows that (5.14) holds with a constant $C(R)$ of the order of $C\exp(CR)$ which, clearly, fulfills (5.15).

This concludes the proof of Theorem 1.2. □

6. Equations with convective terms

In Section 4 and 5 we have discussed equations with potentials entering in the zero order term. In Theorem 3.1 we have also considered an equation with a convective potential, of the form:

$$\Delta u + q_1\cdot\nabla u = 0, \quad x\in\mathbb{R}^2,$$

with:

$$|u|\leqslant C\exp\bigl(-|x|^{4/3}\bigr), \qquad |q_1|\leqslant C\bigl(1+|x|\bigr)^{-1/3}.$$

This allows extending the optimality construction of the previous sections to heat equations or systems with convective potentials of the form:

$$\begin{cases}\varphi_t - \Delta\varphi + a_1\cdot\nabla\varphi = 0, & \text{in } Q,\\ \varphi = 0, & \text{on } \Sigma.\end{cases} \tag{6.1}$$

In this case the observability inequality is known to hold with an observability constant of the order of $\exp(C\|a_1\|_\infty^2)$ (see Theorem 2.1 of this paper, or Theorem 2.3 in [9]).

The construction in Theorem 3.1 allows arguing as above, by scaling and cut-off, and to show that this observability estimate is sharp in what concerns the exponentially quadratic growth of the observability constant on the potential.

Let us briefly check why this is the case: We set,

$$u_R(x) = \begin{pmatrix}\operatorname{Re}u(Rx)\\ \operatorname{Im}u(Rx)\end{pmatrix}, \qquad b_R(x) = -R\begin{pmatrix}\operatorname{Re}q_1(Rx) & -\operatorname{Im}q_1(Rx)\\ \operatorname{Im}q_1(Rx) & \operatorname{Re}q_1(Rx)\end{pmatrix}. \tag{6.2}$$

Then,

$$\Delta u_R = b_R(x)\cdot\nabla u_R, \quad \text{in } \mathbb{R}^2, \tag{6.3}$$

and

$$\bigl|u_R(x)\bigr|\leqslant C\exp\bigl(-c_*R^{4/3}|x|^{4/3}\bigr), \quad \text{in } \mathbb{R}^2. \tag{6.4}$$

Moreover,

$$\bigl|b_R(x)\bigr|\leqslant CR\bigl(1+R|x|\bigr)^{-1/3}\leqslant CR^{2/3}, \quad \forall x\colon |x|\leqslant 1. \tag{6.5}$$

Combining (6.4) and (6.5) one easily observes that the observability constant must necessarily grow exponentially on the square of the L^∞-norm of the potential. The functions u_R are defined on all $\mathbb{R}^2$. By restriction arguments, as in Step 2 of Section 4, one can prove the optimality of the known observability inequality for heat equation with convective potentials on a bounded open subset Ω of $\mathbb{R}^2$.

Note however that, in this case, due to the fact that the construction only holds in dimension 2, very little is known about the sharpness of the estimates in other dimensions. Unfortunately, the argument of separation of variables that allows to deduce, from any even n, n-dimensional examples from 2-dimensional examples does not work in the convective case. Indeed, in this argument, we would lose the decay at infinity of q_1 at the speed $|x|^{-1/3}$ that is necessary to show the sharpness of the estimate. Thus we do not know anything about the optimality in even dimension $n \geqslant 4$. As for odd space-dimension, adapting the proof of Theorem 3.2 to a convective potential one may give a weaker, vectorial form of the optimality result in dimension $n = 3$. Again, we do not know anything about the case of superior odd dimensions.

7. Connections with the controllability of semilinear equations

As we said in the introduction the problem of the explicit dependence of the observability constant on the size of the potentials entering in the system is also closely related with the problem of controllability of semilinear equations. Indeed, the controllability of semilinear equations is usually obtained by a fixed point argument. The growth condition one needs to impose at infinity to the nonlinearity depends very much on the cost of controlling the linearized equation perturbed by a potential, which is precisely given by the observability constant. Consequently the growth of the observability constant on the potential and the growth condition on the nonlinearity for controllability to hold are closely connected. In this section we briefly describe the problem of controllability of semilinear equations, and some open problems related to the optimality results of the previous sections. We first address the semilinear heat equation to later consider the wave equation.

7.1. The semilinear heat equation

Consider the semilinear heat equation

$$\begin{cases} y_t - \Delta y + f(y) = v\chi_\omega, & \text{in } Q, \\ y = 0, & \text{on } \Sigma, \\ y(0,x) = y_0(x), & \text{in } \Omega. \end{cases} \tag{7.1}$$

Here ω is an open non-empty subset ω of Ω, and we denote by χ_ω the characteristic function of ω. To make the problem nontrivial we also assume that $\Omega \setminus \overline{\omega} \neq \emptyset$ throughout this section.

We discuss the problem of null-controllability and more precisely whether for all $T > 0$ and $y_0 \in L^2(\Omega)$ there exists a control $v \in L^2(Q)$ such that the solution $y = y(t,x)$ of (7.1) satisfies

$$y(x,T) \equiv 0, \quad \text{in } \Omega. \tag{7.2}$$

In [12] the following result was proved:

Theorem C. ([12]) *Assume that the nonlinearity $f \in C^1(\mathbb{R})$ is such that $f(0) = 0$ and*

$$\limsup_{|s|\to\infty} \frac{|f'(s)|}{\log^{3/2}(s)} = 0. \tag{7.3}$$

Then, whatever the open nonempty subset ω of Ω is, and for all $T > 0$, system (7.2) *is null-controllable.*

In particular, Theorem C guarantees the possibility of controlling some blowing-up equations. Indeed, when f is of the form

$$f(s) = -s\log^r\big(1 + |s|\big) \tag{7.4}$$

with $r > 1$ solutions of (7.1), in the absence of control, i.e. with $v \equiv 0$, blow-up in finite time occurs. According to Theorem C the process can be controlled, and, in particular, the blow-up be avoided when $1 < r \leqslant 3/2$.

The growth condition in (7.3) is intimately related with the observability inequality (1.2). Indeed, the logarithmic function in (7.3) is precisely the inverse of the exponential one in (1.2).

The proof in [12] relies on the by now classical argument of fixed point consisting on linearizing the equation and estimating the cost of the control in terms of the size of the potential entering in the system.

Theorem C was proved for scalar semilinear heat equations, but the same technique applies with the same conclusions for semilinear systems.

According to the results in Section 4 the estimate (1.2) cannot be improved, and the fixed point argument in [12] may not lead to any improvement of the growth condition (7.3) for controllability.

By the contrary, in [12] it was proved that there are some nonlinearities f satisfying

$$\limsup_{|s|\to\infty} \frac{|f(s)|}{s\log^r(1+|s|)} = 0 \tag{7.5}$$

with a growth rate of the order for $r > 2$ for which controllability fails because the control may not avoid blow-up to occur.

Whether controllability occurs for nonlinearities with a growth rate of the order of (7.5) with $3/2 \leqslant r \leqslant 2$ is an open problem. However, in the light of the optimality of the explicit observability estimate we can guarantee that the method in [12] does not allow to cover the case $r > 3/2$.

Similar questions arise for nonlinearities depending on the gradient of the state considered in [9].

7.2. *The semilinear wave equation*

The problem of controllability of the semilinear wave equation and the existing results were recently discussed in [28]. Let us recall the main known results.

Consider the semilinear wave equation:

$$\begin{cases} y_{tt} - \Delta y + f(y) = v\chi_\omega, & \text{in } Q, \\ y = 0, & \text{on } \Sigma, \\ y(0,x) = y_0(x), \quad y_t(0,x) = y_0(x), & \text{in } \Omega. \end{cases} \tag{7.6}$$

In one space dimension (i.e., $n = 1$), roughly speaking, controllability is known to hold provided the nonlinearity grows as

$$\left|f(s)\right| \leqslant C|s|\log^2|s|, \quad \text{as } |s| \to \infty \tag{7.7}$$

(see [28,29,7]). This result is sharp in the sense that blow-up is known to occur for some nonlinearities growing as (7.4) with $r > 2$ and, due to the finite speed of propagation, controllability may not hold if blow-up occurs. According to this, to some extent, the picture is complete in one space dimension.

In the multidimensional case, i.e., $n > 1$, less is known. Once again, due to blow-up, one cannot expect controllability to hold for nonlinearities of the form (7.4) with $r > 2$. But, in principle, one could expect the system to be controllable for $r \leqslant 2$. However, in view of the optimality result of Theorem 1.2 on the growth of the observability constant, one may not expect the fixed point methods in [29] to apply for $r > 3/2$. Thus the problem is completely open for $3/2 < r \leqslant 2$.

As we mentioned above, Theorem 2.2 suggests that controllability should hold for nonlinearities satisfying the growth condition (7.3). However, even this is an open problem. Indeed, controllability is only known under the more restrictive condition (see [14])

$$\lim_{|s|\to\infty} \frac{|f(s)|}{s\log^{1/2}|s|} = 0. \tag{7.8}$$

This is so because of the lack of smoothing effect of the wave equation. Finite energy solutions belong to $y \in C([0,T]; H_0^1(\Omega))$ but fail to be bounded in dimensions $n \geqslant 2$ and this an obstacle to apply the fixed point argument for nonlinearities (7.5) with $r = 3/2$. Note also that, by using instead the sharp observability internal observability in part (ii) of Theorem 2.2, similar to [29] and [14], it is easy to show the exact controllability of system (7.6) with nonlinearities (7.5) up to the exponent $r < 3/2$.

In view of this, in the application of the fixed point argument for control, one is not allowed to assume the potential to be bounded and consequently one has to work with potentials in some $L^\infty(0,T; L^s(\Omega))$ with $s < \infty$ in which case the observability constant has a faster growth, as we have seen in Theorem 2.2. Consequently, the controllability of the semilinear hyperbolic equations requires a stronger growth condition (7.5) (with $r < 3/2$, compared with the parabolic one in which $r = 3/2$ is allowed). Accordingly, the controllability for nonlinearities of the form (7.4) with $3/2 \leqslant r \leqslant 2$ is an open problem in the multidimensional hyperbolic case.

8. Open problems

A lot of problems remain open in this field. Some of them could need important new ideas and developments.

8.1. *Meshkov's construction for scalar equations*

One of the very first open problems in this frame is whether for scalar multidimensional equations one may construct solutions of (1.8) decaying super-exponentially. The results in [19], based on Carleman inequalities, show that the decay rate $\exp(-C|x|^{4/3})$ is critical in the sense that all solutions that decay faster as $|x| \to \infty$ do necessarily vanish. But the construction we have recalled in Theorem A showing that there is a nontrivial solution decaying as $\exp(-C|x|^{4/3})$ is only available for a system of two real equations. Whether such a construction can be adapted to scalar equations is an interesting and very likely difficult open problem.

Obviously this is connected with the possibility of extending the optimality results in Theorems 1.1 and 1.2 to scalar equations.

The construction in [26] of a nonzero real function vanishing to infinite order at 0 and satisfying a critical differential inequality may be a first step in this direction, at least in the case of convective potentials.

8.2. *Meshkov's construction in dimension $n = 3$*

As we mentioned above the construction of Meshkov is valid for space dimension $n = 2$ and, by separation of variables, can be easily extended to any even dimension. In Section 3, we gave a weaker version of this construction in dimension 3, with a $\mathbb{C}^4$-valued solution u and a potential q that is not bounded, but grows only logarithmically. This could be adapted to get a bounded potential q on $\mathbb{R}^3$, and a solution u decaying almost like $\mathrm{e}^{-|x|^{4/3}}$ (that is, faster than any $\mathrm{e}^{-|x|^{\alpha}}$ with $\alpha < 4/3$). The questions whether in odd dimensions, one can exactly get the optimal decay $\mathrm{e}^{-|x|^{4/3}}$, or replace the $\mathbb{C}^4$-valued solution u by a complex-valued function (as in the original construction of Meshkov) remains open. Indeed it is not clear whether the distinction between even and odd space dimensions is purely technical or not. Note that there is no distinction in what concerns the observability inequality.

The general question of the fastest speed of decay at infinity for eigenfunctions of the Laplace operator with a potential, which includes open problems 8.1 and 8.2, does not seem to have been studied intensively. In [24], the author uses Carleman-type inequalities to give some lower bound on the eigenfunctions. In [4] and [6], some lower bounds are given, but with a strong positivity assumption on the potential. We refer to the book by Agmon [1] for a study of the exponential decay of the eigenfunctions.

8.3. *Sharp observability and time-dependent potentials*

The constructions of Sections 4 and 5 are based on time-independent functions, so that both the heat and wave equation are dealt with as perturbations of an elliptic equation. One possible way to improve the optimality results would be to consider time-dependent variants of the Meshkov's construction. Note that in [18], similar constructions are done for evolution equations, but with the purpose of getting fast-decaying solutions of these equations for large times, which is not what we need in our context.

8.4. *L^r-potentials*

In Section 2 an observability inequality is shown for potentials that are in $L^{\infty}(0,T;L^p(\Omega))$, $n \leqslant p \leqslant \infty$. In Sections 4 and 5, we have shown that these estimates are sharp only in the case $p = \infty$. The question remains open for finite p. As it was already noticed (see Remark 3.2), it is not sufficient to construct optimally decaying solutions u of the elliptic equation (3.1) on $\mathbb{R}^n$ with a potential q in $L^p(\mathbb{R}^n)$ to solve this problem.

8.5. 1-d *problems*

The construction by Meshkov is impossible in one space dimension. In that case solutions of equations of the form (1.8) may not decay in a super-exponential way.

On the other hand, for heat equations, the observability estimates one obtains in 1-d and several space dimensions are the same. Accordingly whether the observability estimate (1.2) is sharp is an open problem.

As we mentioned in Section 7, in the context of the wave equation, using sidewise energy estimates one can obtain a much better estimate with an observability constant of the order of $\exp(C\|a\|_{\infty}^{1/2})$. This estimate is sharp even for constant potentials as can be easily seen from the Fourier representation of solutions and the analysis of how the spectral gap behaves as the constant potential tends to infinity (see [10]).

When the potentials under consideration only depend on x one can use well known transformations (see [19] and [21]) from wave into heat processes to obtain estimates on the observability constant of the order of $\exp(C\|a\|_\infty^{1/2})$.

Whether the estimate (1.2) is sharp for 1-d heat equations with bounded potentials depending both on x and t is an open problem.

8.6. Other equations

In this article we have addressed the heat and wave equations. But the same problems arise for other equations like Schrödinger, plate and KdV equations. We refer to [30,27] and [20] for a discussion of observability estimates for these models.

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