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Partial hyperbolicity for symplectic diffeomorphisms ☆

Vanderlei Horita [a,*], Ali Tahzibi [b]

[a] *UNESP – Universidade Estadual Paulista, Departamento de Matemática, IBILCE, Rua Cristóvão Colombo 2265, 15054-000 S. J. Rio Preto, SP, Brazil*
[b] *Departamento de Matemática e de Computação, ICMC/USP, Av. Trabalhador São Carlense, 400-Cx. Postal 668, 13560-320 São Carlos, SP, Brazil*

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Abstract

We prove that every robustly transitive and every stably ergodic symplectic diffeomorphism on a compact manifold admits a dominated splitting. In fact, these diffeomorphisms are partially hyperbolic.

Keywords: Partial hyperbolicity; Dominated splitting; Symplectic diffeomorphisms; Robust transitivity; Stable ergodicity

1. Introduction

In the 1960's, one of the main goals in the study of dynamical systems was to characterize the structurally stable systems and to verify their genericity.

The theory of uniformly hyperbolic systems has introduced a foundation to approach these subjects. Indeed, uniform hyperbolicity proved to be the key ingredient to characterize structurally stable systems.

We have a good description of uniformly hyperbolic dynamics. The Smale spectral Theorem asserts that the non-wandering set of a hyperbolic diffeomorphism splits into basic pieces. The dynamic restricted to each piece is topologically transitive and the transitivity property persists after small perturbations.

Recall a diffeomorphism is *transitive* if there exists a point x such that its forward orbit is dense. Transitive systems do not have neither sinks nor source. We say a diffeomorphism is C^r-*robustly transitive* if it belongs to the C^r-interior of the set of transitive diffeomorphisms.

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* Corresponding author.
E-mail addresses: vhorita@ibilce.unesp.br (V. Horita), tahzibi@icmc.usp.br (A. Tahzibi).
URL: http://www.icmc.usp.br/~tahzibi.

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An interesting question is whether robust transitivity implies hyperbolicity in uniform fashion or not.

It is well known that every C^1-robustly transitive diffeomorphism on a closed surface is an Anosov diffeomorphism. This is a direct consequence of a C^1-generic dichotomy between hyperbolicity and C^1-Newhouse phenomenon (the coexistence of infinitely many sinks and sources in a locally residual subset) proved by Mañé in [11].

In compact manifolds of dimension greater than two, robust transitivity does not imply uniform hyperbolicity, see [14,10,4,6,15]. However, all these examples present a weak form of hyperbolicity called *dominated splitting*. Let M be a compact manifold and $f : M \to M$ a diffeomorphism. A Df-invariant decomposition $TM = E \oplus F$ of the tangent bundle of M, is dominated if for every positive integer ℓ and any x in M,

$$\|Df^\ell|E(x)\| \cdot \|Df^{-\ell}|F(f^\ell(x))\| < C\lambda^\ell,$$

for some constants $C > 0$ and $0 < \lambda < 1$.

If for the dominated splitting $TM = E \oplus F$ at least one of the subbundles is uniformly hyperbolic then f is called *partially hyperbolic*.

Díaz, Pujals and Ures in [9] show that every C^1-robustly transitive diffeomorphism in a 3-dimensional compact manifold should be partially hyperbolic. This result cannot be generalized for higher dimension: Bonatti and Viana present in [6] an example in 4-dimensional compact manifold of a non-partially hyperbolic C^1-robustly transitive diffeomorphism. This example can be extended in any dimension, see [15].

Bonatti, Díaz and Pujals in [5] proved that every C^1-robustly transitive diffeomorphism on a n-dimensional compact manifold, $n \geqslant 1$ admits a dominated splitting. Recently Vivier proved similar results for flows in [16]. She proved that robustly transitive C^1-vector fields on a compact manifold do not admit singularity and that robust transitivity implies the existence of dominated structure.

In this paper we address the problem of the existence of dominated splitting for C^1-robustly transitive symplectic diffeomorphisms. In the symplectic setting dominated splitting implies strong partial hyperbolicity. This fact was first observed by Mañé, see [12]. A proof is given in [2] for $\dim M = 4$ and in [3] for the general case.

Let (M, ω) be a $2N$-dimensional symplectic manifold, where ω is a non-degenerated symplectic form. We denote $\mathrm{Diff}^r_\omega(M)$, $r \geqslant 1$, the set of C^r-symplectic diffeomorphisms. We say $f \in \mathrm{Diff}^r_\omega(M)$ is *C^r-robustly transitive symplectic diffeomorphism* if there exists a neighborhood $\mathcal{U} \subset \mathrm{Diff}^r_\omega(M)$ of f such that every $g \in \mathcal{U}$ is also transitive.

Theorem 1. *Every C^1-robustly transitive symplectic diffeomorphism on a $2N$-dimensional, $N \geqslant 1$, compact manifold is partially hyperbolic.*

We emphasize that if $f \in \mathrm{Diff}^1_\omega$ and is C^1-robustly transitive symplectic diffeomorphism then just symplectic nearby diffeomorphism are transitive. So, our theorem is not a consequence of results in [5]. In 4-dimensional setting, $N = 2$, Theorem 1 is a consequence of [2].

In the context of volume preserving diffeomorphisms, ergodicity of the Lebesgue measure is a basic feature. Recall that a diffeomorphism in $\mathrm{Diff}^1_\omega(M)$ preserves the 2-form ω and consequently the volume form $\omega \wedge \cdots \wedge \omega$ is also preserved. This volume form induces in a natural way a Lebesgue measure defined on M.

The stable ergodicity of symplectic diffeomorphism is defined as follows. A symplectic diffeomorphism $f \in \mathrm{Diff}^2_\omega(M)$ is *C^1-stably ergodic* if there exists a neighborhood $\mathcal{U} \subset \mathrm{Diff}^1_\omega(M)$ of f such that any $g \in \mathcal{U} \cap \mathrm{Diff}^2_\omega(M)$ is ergodic.

The theories of stable ergodicity and robust transitivity had very parallel development in last few years. Like as in the robust transitivity case we know examples of stably ergodic diffeomorphisms with weak form of hyperbolicity. We propose the reader to see [7] for approaches to prove stable ergodicity in the partially hyperbolic case. However, there are stably ergodic diffeomorphisms which are not partially hyperbolic, see [15].

A natural question is whether stable ergodicity of volume preserving or even symplectic diffeomorphisms implies the existence of a dominated splitting of the tangent bundle. For the symplectic case we give an affirmative answer.

Theorem 2. *Every C^1-stably ergodic symplectic diffeomorphism on a $2N$-dimensional, $d \geqslant 1$, compact manifold is partially hyperbolic.*

To prove our theorems we follow the arguments in [5] where they obtain a dichotomy between dominated splitting and the Newhouse phenomenon. This is done by showing that the lack of dominated splitting leads to creation of sinks or sources by a convenient perturbation. Of course a symplectic diffeomorphism does not admit sink or source. The idea in the symplectic case is to make a perturbation and create a totally elliptic periodic point.

In Section 2 we state this dichotomy for symplectic diffeomorphisms in Theorem 2.1.

We use this dichotomy to finish the proof of our main results in Section 3. The idea is to eliminate the possibility of creation of totally elliptic periodic points for robustly transitive or stably ergodic symplectic diffeomorphisms. For this purpose in Section 3 we use generating functions for symplectic diffeomorphisms to prove that "stably ergodic and robustly transitive symplectic diffeomorphisms are C^1 far from having totally elliptic points".

To prove the dichotomy for symplectic diffeomorphisms we prove a similar result for linear symplectic systems. Symplectic linear systems are introduced in Section 4 and some perturbation results are proved there which will be used in the rest of the paper.

The difficulty to get the dichotomy in the symplectic case is that we have much less space to perform symplectic perturbations. The perturbations have to preserve the symplectic form ω.

In Section 5 we state the precise result of dichotomy for linear symplectic systems and in Sections 6 and 7 we prove this statements. In these two final sections we prove new results for symplectic linear systems which enable us to adapt the approach of [5] for symplectic linear systems.

We mention that for robustly transitive volume preserving diffeomorphisms the dichotomy as mentioned above is straightforward from the result in [5]. Arbieto and Matheus [1] use this dichotomy and prove that robustly transitive conservative diffeomorphisms have dominated splitting. So, the main difficulty in the proof of similar to our results in the volume preserving case is to show that the robustly transitive conservative diffeomorphisms cannot have totally elliptic points. They overcome this difficulty with a new "Pasting Lemma" which uses a theorem of Dacorogna and Moser [8].

2. A dichotomy for symplectic diffeomorphisms

An invariant decomposition $TM = E \oplus F$ is called *ℓ-dominated* if for any $n \geqslant \ell$ and every x in M,

$$\left\|Df^n|E(x)\right\| \cdot \left\|Df^{-n}|F\left(f^n(x)\right)\right\| < \frac{1}{2}.$$

From now on we use the notation $E \prec_\ell F$ for ℓ-dominated splitting and $E \prec F$ for dominated splitting without specifying the strength of the splitting.

It is easy to verify from definition that ℓ-dominated splitting has the following properties:

1. If $\Lambda \subset M$ is an invariant subset that admits an ℓ-dominated splitting then the same is true for the closure of Λ.
2. If a sequence $(f_n)_n$ of maps admitting an ℓ-dominated splitting converges to f in C^1-topology then f also admits an ℓ-dominated splitting.

Theorem 2.1. *Let $f \in \mathrm{Diff}^1_\omega(M)$ be a symplectic diffeomorphism of a $2N$-dimensional manifold M. Then there is $\ell \in \mathbb{N}$ such that,*

(a) *either there is a symplectic ε-C^1-perturbation g of f having a periodic point x of period $n \in \mathbb{N}$ such that $Dg^n(x) = \mathrm{Id}$,*
(b) *or for any symplectic diffeomorphism g ε-C^1-close to f and every periodic saddle x of g the homoclinic class $H(x, g)$ admits an ℓ-dominated splitting.*

To prove this theorem we introduce the concept of symplectic linear systems in Sections 4. Then, in Section 5, we reduce the proof of Theorem 2.1 to proof a similar result for that symplectic linear systems.

By the following lemma we are able to extend dominated splittings over homoclinic classes to the whole manifold for a generic symplectic diffeomorphism. Before stating this result let us recall a symplectic version of Connecting lemma due to Xia in [17].

Theorem 2.2 *((Xia)). Let M be a compact n-dimensional manifold with a symplectic or volume form ω. Then there is a residual subset $\mathcal{R}_2 \subset \mathrm{Diff}^1_\omega(M)$ such that if $g \in \mathcal{R}_2$ and $p \in M$ are such that p is a hyperbolic periodic point of g, then $W^s(p) \cap W^u(p)$ is dense in both $W^s(p)$ and $W^u(p)$.*

Using this result we are able to prove the following result.

Lemma 2.3. *There is a residual subset $\mathcal{R} \subset \mathrm{Diff}^1_\omega(M)$ of diffeomorphisms f such that the non-trivial homoclinic classes of hyperbolic periodic points of f are dense in M.*

Proof. The version of this lemma for conservative diffeomorphisms is given in [5, Lemma 7.8]. We can use the same arguments adapted for the symplectic case by means of a result of Newhouse and connecting lemma of Xia.

First of all, we recall that for a symplectic diffeomorphism the set of recurrent points is dense in M. Indeed a symplectic diffeomorphism preserves a Lebesgue measure. Moreover, using C^1-Closing lemma of Pugh and Robinson and the Birkhoff fixed point theorem, Newhouse [13, Corollary 3.2] proved that there is a C^1-residual subset $\mathcal{R}_1 \in \mathrm{Diff}^1_\omega(M)$ such that for any $g \in \mathcal{R}_1$ the set of hyperbolic periodic points of g is dense in M.

By using Theorem 2.2 we take $\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2$. Thus, for any $g \in \mathcal{R}$ the union of non-trivial homoclinic classes is dense. This completes the proof of lemma. □

Remark 2.4. Using the above lemma and the continuity of dominated splitting the item (b) of Theorem 2.1 can be rewritten as follows.

(b_1) the manifold M is the union of finitely many invariant (by f) compact sets having a dominated splitting.

We remark that the invariant compact sets mentioned above are Λ_i, $i < 2N$, where Λ_i is closure of the union of non-trivial homoclinic classes with an ℓ-dominated splitting $E \oplus F$ with dimension i (i.e. $\dim(E) = i$). Of course for a transitive diffeomorphism the above item is equivalent to have a dominated splitting on the whole manifold.

3. Proof of Theorems 1 and 2

In this section we prove Theorems 1 and 2 using Theorem 2.1.

3.1. Elliptic points vs. robust transitivity

Let us prove that the item (a) of the dichotomy given in Theorem 2.1 does not occur for C^1-robustly transitive diffeomorphisms.

Lemma 3.1. *If $f \in \mathrm{Diff}^1_\omega(M)$ has a totally elliptic periodic point p of period n ($Df^n(p) = \mathrm{Id}$) then there exists $g \in \mathrm{Diff}^1_\omega(M)$ and C^1-close to f such that g is not transitive.*

Proof. In order to simplify our arguments, let us suppose that p is a fixed point. We use generating functions to construct g in such a way that g coincide to the identity map in a small neighborhood of p.

Let us introduce generating function as in [17]. We fix a local coordinated system $(x_1, \dots, x_d, y_1, \dots, y_d)$ in a neighborhood U of p such that p correspond to $(0, 0)$ in this coordinate system. Suppose $f(x, y) = (\xi(x, y), \eta(x, y))$. The fact f is symplectic implies

$$\sum_{i=1}^{d} \mathrm{d}x_i \wedge \mathrm{d}y_i = \sum_{i=1}^{d} \mathrm{d}\xi_i \wedge \mathrm{d}\eta_i.$$

Moreover, we may suppose that partial derivative $\partial\eta/\partial y$ of η with respect to y is non-singular at every point of U. This enable us to solve $\eta = \eta(x, y)$ to obtain $y = y(x, \eta)$. Hence, we can define a new system of coordinates $(x_1, \dots, x_d, \eta_1, \dots, \eta_d)$. Let $\gamma : (x, y) \to (x, \eta(x, y))$ be the map of change of coordinates.

Since the 1-form

$$\alpha := \sum_{i=1}^{d} \xi_i \, \mathrm{d}\eta_i + y_i \, \mathrm{d}x_i$$

is closed, there exists a C^2 function $S_f(x, \eta)$, defined on a neighborhood of $(0, \eta(0, 0))$ such that $\mathrm{d}S_f = \alpha$. The function S_f is unique up to a constant. Moreover,

$$\frac{\partial S_f}{\partial x_i} = y_i \quad \text{and} \quad \frac{\partial S_f}{\partial \eta_i} = \xi_i.$$

Conversely, for a real C^2-function $S(x, \eta)$ defined on a neighborhood of $(0, \eta(0, 0))$ such that the second partial derivative

$$\frac{\partial^2 S}{\partial x \partial \eta}$$

is non-singular in this neighborhood, we define

$$\xi_i(x, \eta) = \frac{\partial S}{\partial \eta_i} \quad \text{and} \quad y_i(x, \eta) = \frac{\partial S}{\partial x_i}.$$

Then, solving η in terms of x, y we find a symplectic diffeomorphism which maps (x, y) to (ξ, η).

Observe that in the above construction the generating function of f is C^{k+1} whenever f is C^k. Moreover, given a C^{k+1} generating function we obtain locally a C^k symplectic diffeomorphism.

We also have that f is C^1-close to g if and only if S_f is C^2-close to S_g, provided they are defined on the same domain.

Let ρ be a C^∞ bump function such that

$$\rho(z) = \begin{cases} 1 & \text{if } z \in \gamma\big(B(\beta/2)\big), \\ 0 & \text{if } z \notin \gamma\big(B(\beta)\big) \end{cases}$$

where $B(r)$ is the ball of radius r centered in $(0, 0)$ in the (x, y)-coordinates.

Define a C^2 function given by

$$S_g := \rho(x, \eta) S_{\mathrm{id}} + \big(1 - \rho(x, \eta)\big) S_f.$$

More important is that, if we take $\beta > 0$ small enough, S_g is C^2-close to S_f. Indeed, f is C^1-close to identity in a neighborhood of the origin and consequently, S_{id} and S_f are C^2-close enough on $\gamma(B(\beta))$.

Finally, it is easy to see that if S_{id} and S_f are ε-close on $\gamma(B(\beta))$ in the C^2-topology then

$$\rho(x,\eta)S_{\mathrm{id}} + \big(1-\rho(x,\eta)\big)S_f$$

is $K\varepsilon$-close to S_f in the C^2-topology, where K depends on the bump function ρ and β.

Let $g \in \mathrm{Diff}^1_\omega(M)$ be the corresponding symplectic diffeomorphism to S_g. As g is C^1-close to f and locally it is equal to the identity, we conclude that g cannot be transitive and this conclude the proof of the lemma. □

The previous lemma proves that a robustly transitive diffeomorphism cannot have a totally elliptic periodic point. So, if f is robustly transitive as in Theorem 1 then by Remark 2.4 we can conclude that f admits a dominated splitting. In this way we prove Theorem 1.

3.2. Elliptic point vs. stable ergodicity

In this subsection we prove that any C^1-stably ergodic symplectic diffeomorphism admits a dominated splitting. Let us recall that by C^1-stably ergodic diffeomorphism we mean a symplectic C^2-diffeomorphism such that all symplectic C^2-diffeomorphisms in a C^1-neighborhood are ergodic.

Let $f \in \mathrm{Diff}^2_\omega(M)$ be a stably ergodic diffeomorphism and $\mathcal{U}$ be a C^1-neighborhood of f such that any $g \in \mathcal{U} \cap \mathrm{Diff}^2_\omega(M)$ is ergodic. We prove that any diffeomorphism inside $\mathcal{R} \cap \mathcal{U}$ admits an ℓ-dominated splitting where $\mathcal{R}$ is given in Lemma 2.3. Since the ℓ-dominated splitting property is a closed property in C^1-topology we obtain a dominated splitting for f. The proof is by contradiction.

Suppose that there exists $f_1 \in \mathcal{R} \cap \mathcal{U}$ close to f. If f_1 does not admit a dominated splitting then, by Theorem 2.1, there exists $g_1 \in \mathrm{Diff}^1_\omega(M)$ with a totally elliptic periodic point.

We claim that there exists $g \in \mathrm{Diff}^2_\omega(M)$ and C^1-close to g_1 such that g is not ergodic. This gives a contradiction with the stable ergodicity of f.

Just to simplify our arguments let us suppose that g_1 has a totally elliptic fixed point. Using the same technics used in the previous subsection we construct $g_2 \in \mathrm{Diff}^1_\omega(M)$ and C^1-close to g_1 in such a way that g_2 coincides with the identity map on $B(\beta)$, for some $\beta > 0$. Note that g_2 is just C^1 and we do not get a contradiction with the stable ergodicity of f.

However, from a result of Zehnder in [18] there exists $g_3 \in \mathrm{Diff}^2_\omega(M)$ and C^1-close to g_2. In order to get a contradiction, similarly to the previous subsection, let $\tilde{\rho}$ be a C^∞ bump function such that

$$\tilde{\rho}(z) = \begin{cases} 1 & \text{if } z \in \gamma\big(B(\beta/3)\big), \\ 0 & \text{if } z \notin \gamma\big(B(\beta/2)\big). \end{cases}$$

We define

$$S_g(x,\eta) := \tilde{\rho}(x,\eta)S_{\mathrm{id}} + \big(1-\tilde{\rho}(x,\eta)\big)S_{g_3}.$$

Since g_3 is a C^2-diffeomorphism we have S_g is a C^3-function. Moreover, S_g is C^2-close to S_{g_3}. Therefore, S_g is a generating function for a symplectic diffeomorphism $g \in \mathrm{Diff}^2_\omega(M)$ such that g is C^1-close to g_3 and consequently C^1-close to f.

By construction, g coincides with the identity map on a neighborhood of zero. Hence g cannot be ergodic. This gives a contradiction, because we have supposed that f is stably ergodic.

The proof of Theorem 2 is complete.

4. Symplectic linear systems

In this section we introduce the key ingredient in the proof of Theorem 2.1. Following the techniques of Mañé, we use symplectic linear systems enriched with transition by Bonatti, Díaz, Pujals (see [5, Section 1]) in order to prove a dichotomy for linear systems in Proposition 5.3. However, we stress that in our case the symplectic perturbations we have to perform is much more difficult to realize.

Let Σ be a topological space and f a homeomorphism defined on Σ. Consider a locally trivial vector bundle $\mathcal{E}$ over Σ such that $\mathcal{E}(x)$ is a symplectic vector space with anti-symmetric non-degenerated 2-form ω. Furthermore, we require the $\dim(\mathcal{E}(x))$ does not depend on x.

We denote by $\mathcal{S}(\Sigma, f, \mathcal{E})$ the set of maps $A:\mathcal{E} \to \mathcal{E}$ such that for every $x \in \Sigma$ the induced map $A(x,\cdot)$ is a linear symplectic isomorphism from $\mathcal{E}(x) \to \mathcal{E}(f(x))$, that is,

$$\omega(u,v) = \omega\big(A(u), A(v)\big).$$

Thus, $A(x,\cdot)$ belongs to $\mathcal{L}_\omega(\mathcal{E}(x), \mathcal{E}(f(x)))$.

For each $x \in \mathcal{E}$ the Euclidean metric on $\mathcal{E}(x)$ and $\mathcal{E}(f(x))$ induces in a natural way a norm on $\mathcal{L}_\omega(\mathcal{E}(x), \mathcal{E}(f(x)))$:

$$\big|B(x,\cdot)\big| = \sup\big\{\big|B(x,v)\big|,\ v \in \mathcal{E}(x),\ |v| = 1\big\}.$$

Furthermore, for $A \in \mathcal{S}(\Sigma, f, \mathcal{E})$ we define $|A| = \sup_{x\in\Sigma} |A(x,\cdot)|$. Note that, if A belongs to $\mathcal{S}(\Sigma, f, \mathcal{E})$ its inverse A^{-1} belongs to $\mathcal{S}(\Sigma, f^{-1}, \mathcal{E})$. The *norm* of $A \in \mathcal{S}(\Sigma, f, \mathcal{E})$ is defined by $\|A\| = \sup\{|A|, |A^{-1}|\}$.

Let $(\Sigma, f, \mathcal{E}, A)$ be a *linear symplectic system* (or *linear symplectic cocycle* over f), that is, a 4-tuple where Σ is a topological space, f is a homeomorphism of Σ, $\mathcal{E}$ is an Euclidean bundle over Σ, A belongs to $\mathcal{S}(\Sigma, f, \mathcal{E})$ and $\|A\| < \infty$. We say that $(\Sigma, f, \mathcal{E}, A)$ is *periodic* if any $p \in \Sigma$ is a periodic point of f.

Let (V, ω) be a symplectic vector space and $W \subseteq V$ a vector subspace of V. Then the *symplectic complement* of W is given by

$$W^\omega = \big\{x \in V : \omega(x,w) = 0, \text{ for all } w \in W\big\}.$$

It is easy to see that W^ω is also a vector space and, by definition,

(1) If $W \subset W^\omega$, then W is an *isotropic subspace*.
(2) If $W \cap W^\omega = 0$, then W is a *symplectic subspace*.
(3) If $W = W^\omega$, then W is *Lagrangian subspace*.

Let $(\Sigma, f, \mathcal{E}, A)$ be a diagonalizable periodic linear symplectic system such that $\mathcal{E}(x) = \mathbb{R}^{2N}$. In what follows, we indicate by $\lambda_1(x) < \lambda_2(x) < \cdots < \lambda_{2N}(x)$ their eigenvalues and we denote by $E_1(x) \prec E_2(x) \prec \cdots \prec E_{2N}(x)$ their respective eigenspaces.

We know that if λ is an eigenvalue of a symplectic transformation then λ^{-1} is also an eigenvalue. So, if $\lambda_1 < \lambda_2 < \cdots < \lambda_{2N}$ are $2N$ distinct eigenvalues of a symplectic transformation then $\lambda_{i^*} := \lambda_{2N-i+1} = \lambda_i^{-1}$.

We say that $\mathcal{B} = \{e_1, e_2, \ldots, e_{2N}\}$ is a *symplectic basis* for a symplectic vector space (V, ω) if $\mathcal{B}$ is a basis of V and

$$\omega(e_i, e_j) = \begin{cases} 0 & \text{for } j \neq i^*, \\ 1 & \text{for } j = i^* > i. \end{cases}$$

The fact that ω is an anti-symmetric 2-form implies that $\omega(e_{i^*}, e_i) = -1$ for $i^* > i$.

Let $(\Sigma, f, \mathcal{E}, A)$ be a diagonalizable symplectic linear system. Suppose that $\{e_1(p), e_2(p), \ldots, e_{2N}(p)\}$ is a symplectic basis of $\mathcal{E}(p)$ then $\{A(e_1(p)), A(e_2(p)), \ldots, A(e_{2N}(p))\}$ is a symplectic basis of $\mathcal{E}(f(p))$. For simplicity of notations, we omit the dependence on the point of vectors e_i.

Lemma 4.1. *Let $(\Sigma, f, \mathcal{E}, A)$ be diagonalizable periodic linear symplectic system with distinct eigenvalues $\lambda_1 < \cdots < \lambda_{2N}$ and $E_1 \prec \cdots \prec E_{2N}$ be the corresponding eigenspaces. There exists a symplectic basis $\{e_1, \ldots, e_{2N}\}$ constituted by eigenvectors. Moreover, for $j \neq i^*$, $E_i \oplus E_j$ is an isotropic subspace and $E_i \oplus E_{i^*}$ is a symplectic subspace.*

Proof. Let $n = n(p)$ be the period of $p \in \Sigma$. Let $e_i \in E_i$ and $e_j \in E_j$ be eigenvectors of A^n with respective eigenvalues λ_i and λ_j. Then,

$$\omega\big(A^n(e_i), A^n(e_j)\big) = \lambda_i \lambda_j \omega(e_i, e_j).$$

On the other hand, since A is a linear symplectic system, we have

$$\omega\big(A^n(e_i), A^n(e_j)\big) = \omega(e_i, e_j).$$

Thus, if $j \neq i^*$ then $\lambda_i \lambda_j \neq 1$ and consequently $\omega(e_i, e_j) = 0$. Moreover, as ω is non-degenerate we have $\omega(e_i, e_{i^*}) \neq 0$. So, normalizing the vectors e_i, $1 \leqslant i \leqslant N$, we can choose a new basis constituted by eigenvectors of A, which we still denote by e_i, such that $\omega(e_i, e_{i^*}) = 1$ for all $1 \leqslant i \leqslant N$.

Other claims in the statement are direct consequence of definitions of symplectic and isotropic subspaces. □

Let $E_j \oplus E_k$ be a vector subspace of a diagonalizable symplectic vector space. Any small symplectic perturbation of $A|_{E_j \oplus E_k}$ is called a *symplectic perturbation along* $E_j \oplus E_k$. The next lemma asserts that any symplectic perturbation along $E_j \oplus E_k$ can be realized as the restriction of a symplectic perturbation of A. More precisely,

Lemma 4.2 ((Symplectic realization)). *Let $(\Sigma, f, \mathcal{E}, A)$ be a diagonalizable periodic linear system as above. Given any $\varepsilon > 0$ and $1 \leqslant j < k \leqslant 2N$, every ε-symplectic perturbation B of*

$$A|_{E_j \oplus E_k} : E_j \oplus E_k \to E_j \oplus E_k$$

along the orbit of $x \in \Sigma$ is the restriction of a symplectic ε-perturbation $\tilde{A}$ of A such that $\tilde{A}|_{E_i} = A|_{E_i}$ for $i \neq j, k, j^*, k^*$.

Proof. If $E_j \oplus E_k$ is a symplectic subspace, that is $k = j^*$, then we define $\tilde{A}|_{E_j \oplus E_{j^*}} = B$, $\tilde{A}|_{E_i} = A$, for $i \neq j, j^*$ and we extend it linearly. In that way, we have

$$\omega\big(\tilde{A}(e_j), \tilde{A}(e_{j^*})\big) = \omega\big(B(e_j), B(e_{j^*})\big) = \omega(e_j, e_{j^*}) = 1,$$

and for $i \neq j, j^*$, we get

$$\omega\big(\tilde{A}(e_i), \tilde{A}(e_j)\big) = \omega\big(A(e_i), \alpha e_j + \beta e_{j^*}\big) = 0.$$

Moreover, for $r, s \neq j, j^*$, we have

$$\omega\big(\tilde{A}(e_s), \tilde{A}(e_r)\big) = \omega\big(A(e_s), A(e_r)\big) = \omega(e_s, e_r).$$

Therefore, in this case, $\tilde{A}$ is symplectic.

Now, we suppose $k \neq j^*$. An important feature we have to take account is the way we extend B to $E_{j^*} \oplus E_{k^*}$. Once we have made it, we define $\tilde{A}$ equal to A when restricted to the others subspace E_i, $i \neq j, k, j^*, k^*$. Finally, we extend linearly this operator to other vectors.

There are constants $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$B(e_j) = \alpha e_j + \beta e_k,$$
$$B(e_k) = \gamma e_j + \delta e_k.$$

Let us suppose $1 \leqslant j < k \leqslant N$. We construct a symplectic linear system $\tilde{A}$ in this case. In the other cases are completely analogous, by changing conveniently the sign of the constant in $\widetilde{B}$ bellow. We denote $\Delta = \alpha\delta - \beta\gamma$ and we define

$$\widetilde{B}(e_{j^*}) = \frac{\delta}{\Delta} e_{j^*} - \frac{\gamma}{\Delta} e_{k^*},$$
$$\widetilde{B}(e_{k^*}) = -\frac{\beta}{\Delta} e_{j^*} + \frac{\alpha}{\Delta} e_{k^*},$$

and extend $\widetilde{B}$ linearly to $E_{j^*} \oplus E_{k^*}$.

Then, we define $\tilde{A}$ as follows:

- $\tilde{A}|_{E_i} = A|_{E_i}$, $i \notin \{j, k, j^*, k^*\}$,
- $\tilde{A}|_{E_j \oplus E_k} = B$,
- $\tilde{A}|_{E_{j^*} \oplus E_{k^*}} = \widetilde{B}$,

and extend $\tilde{A}$ linearly.

Note that, if B in $E_j \oplus E_k$ is a rotation then the perturbation $\widetilde{B}$ is the same rotation in $E_{j*} \oplus E_{k*}$.

To verify that $\tilde{A}$ is a symplectic linear system, it is enough to show that

$$\omega\big(\tilde{A}(e_r), \tilde{A}(e_s)\big) = \omega(e_r, e_s) \quad \text{for any } 1 \leqslant r, s \leqslant 2N.$$

Let us begin by the case when $r = j, s = j^*$:

$$\begin{aligned}
\omega\big(\tilde{A}(e_j), \tilde{A}(e_{j^*})\big) &= \omega\left(\alpha e_j + \beta e_k, \frac{\delta}{\Delta} e_{j^*} - \frac{\gamma}{\Delta} e_{k^*}\right) \\
&= \frac{\alpha\delta}{\Delta}\omega(e_j, e_{j^*}) - \frac{\beta\gamma}{\Delta}\omega(e_k, e_{k^*}) - \frac{\alpha\gamma}{\Delta}\omega(e_j, e_{k^*}) + \frac{\beta\delta}{\Delta}\omega(e_k, e_{j^*}) \\
&= \frac{\alpha\delta}{\Delta} - \frac{\beta\gamma}{\Delta} = 1 = \omega(e_j, e_{j^*}).
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\omega\big(\tilde{A}(e_{j^*}), \tilde{A}(e_j)\big) = \omega(e_{j^*}, e_j), \\
&\omega\big(\tilde{A}(e_k), \tilde{A}(e_{k^*})\big) = \omega(e_k, e_{k^*}), \\
&\omega\big(\tilde{A}(e_{k^*}), \tilde{A}(e_k)\big) = \omega(e_{k^*}, e_k).
\end{aligned}$$

If $r = j, s = k^*$, then

$$\begin{aligned}
\omega\big(\tilde{A}(e_j), \tilde{A}(e_{k^*})\big) &= \omega\left(\alpha e_j + \beta e_k, -\frac{\beta}{\Delta} e_{j^*} + \frac{\alpha}{\Delta} e_{k^*}\right) \\
&= \frac{-\alpha\beta}{\Delta}\omega(e_j, e_{j^*}) + \frac{\alpha\beta}{\Delta}\omega(e_k, e_{k^*}) + \frac{\alpha^2}{\Delta}\omega(e_j, e_{k^*}) - \frac{\beta^2}{\Delta}\omega(e_k, e_{j^*}) \\
&= 0 = \omega(e_j, e_{k^*}).
\end{aligned}$$

Analogously, we have

$$\omega\big(\tilde{A}(e_k), \tilde{A}(e_{j^*})\big) = 0 = \omega(e_k, e_{j^*}).$$

Hence,

$$\begin{aligned}
&\omega\big(\tilde{A}(e_{j^*}), \tilde{A}(e_k)\big) = 0 = \omega(e_{j^*}, e_k), \\
&\omega\big(\tilde{A}(e_{k^*}), \tilde{A}(e_j)\big) = 0 = \omega(e_{k^*}, e_j).
\end{aligned}$$

The remaining cases are direct consequence of the fact that $\tilde{A}(e_i) = A(e_i)$ belongs to E_i, if $i \neq j, j^*, k, k^*$. This completes the proof. □

The following lemma is used in the next sections.

Lemma 4.3. *Let $(\Sigma, f, \mathcal{E}, A)$ be a periodic diagonalizable linear system. If $\widetilde{E}_j(p) \in E_i(p) \oplus E_j(p)$ is close to $E_j(p)$ then there exists a symplectic perturbation p of the identity map such that $\mathsf{p}(E_i(p)) = E_i(p)$ and $\mathsf{p}(\widetilde{E}_j(p)) = E_j(p)$.*

Proof. By Lemma 4.2 it is enough to define p symplectic on $E_i(p) \oplus E_j(p)$ close to the identity map.

Let us suppose $E_i(p) \oplus E_j(p)$ is an isotropic subspace ($j \neq i^*$). Given $e_i \in E_i$ and $\tilde{e}_j \in \widetilde{E}_j$, we define $\mathsf{p}(e_i) = e_i$ and $\mathsf{p}(\tilde{e}_j)$ the projection of $\tilde{e}_j$ over E_j and extend p to $E_i(p) \oplus E_j(p)$ linearly. Since $\widetilde{E}_j$ is close to E_j, p is close to the identity map. Moreover, there exists $e_j \in E_j$ such that $\tilde{e}_j = \alpha e_i + \beta e_j$, for some constants α, β. Then

$$\omega(\mathsf{p}(e_i), \mathsf{p}(\tilde{e}_j)) = 0 = \alpha\omega(e_i, e_i) + \beta\omega(e_i, e_j) = \omega(e_i, \tilde{e}_j).$$

On the other hand, let $E_i(p) \oplus E_j(p)$ be a symplectic subspace. We take $e_i \in E_i$, $e_j \in E_j$ such that $\omega(e_i, e_j) = 1$. Let $\tilde{e}_j \in \widetilde{E}_j$ be close to e_j Then, there exists constants r close to zero and s close to 1 such that $\tilde{e}_j = re_i + se_j$. Hence, $\{e_i, \tilde{e}_j\}$ is a basis of $E_i(p) \oplus E_j(p)$ and $s = \omega(e_i, \tilde{e}_j)$. We define p on this base as follows:

$$\mathsf{p}(\tilde{e}_j) = e_j \quad \text{and} \quad \mathsf{p}(e_i) = se_i.$$

Therefore,

$$\omega(\mathsf{p}(e_i), \mathsf{p}(\tilde{e}_j)) = \omega(\omega(e_i, \tilde{e}_j)e_i, e_j) = \omega(e_i, \tilde{e}_j).$$

So, in both cases p is symplectic perturbation of the identity map. □

4.1. Symplectic transitions

Here we recall an important notion introduced in [5]: the concept of transitions. In this work we are dealing with the systems which admit transitions. In order to introduce this important notion let us begin with an example, see [5, Section 1.4]. Suppose P and Q are saddles of the same index linked by transverse intersection of their invariant manifolds. The existence of a Markov partition shows that for any fixed finite sequence of times there is a periodic point expending alternately the times of the sequence close to P and Q, respectively. Moreover, the transition time between a neighborhood of P and a neighborhood of Q can be chosen bounded. This property allow us to scatter in the whole homoclinic class of P some properties of the periodic points of this class.

Now, we introduce the concept of *linear systems with transitions* as in [5]. Given a set $\mathcal{A}$, a *word* with letters in $\mathcal{A}$ is a finite sequence of elements of $\mathcal{A}$. The product of the word $[a] = [a_1, \ldots, a_n]$ by $[b] = [b_1, \ldots, b_m]$ is the word $[a_1, \ldots, a_n, b_1, \ldots, b_m]$. We say a word is *not a power* if $[a] \neq [b]^k$ for every word $[b]$ and $k > 1$.

Let $(\Sigma, f, \mathcal{E}, A)$ be a periodic linear system of dimension $2N$, that is, all $x \in \Sigma$ is a periodic point for f with period $n = n(x)$. We denote M_A the product $A^n(x)$ of A along the orbit of x.

If (Σ, f, A) is a periodic symplectic linear system of matrices in $SP(2N, \mathbb{R})$, then for any $x \in \Sigma$ we write,

$$[M]_A(x) = (A(f^{n-1}(x)), \ldots, A(x)),$$

where n is period of x. The matrix $M_A(x)$ is the product of the words $[M]_A(x)$.

Definition 4.4 *((Definition* 1.6 *of* [5])*).* Given $\varepsilon > 0$, a periodic linear system $(\Sigma, f, \mathcal{E}, A)$ *admits ε-transitions* if for every finite family of points $x_1, \ldots, x_n = x_1 \in \Sigma$ there is an orthonormal system of coordinates of the linear bundle $\mathcal{E}$ so that $(\Sigma, f, \mathcal{E}, A)$ can now be considered as a system of matrices (Σ, f, A), and for any $(i, j) \in \{1, \ldots, n\}^2$ there exist $k(i, j) \in \mathbb{N}$ and a finite word $[t^{i,j}] = (t_1^{i,j}, \ldots, t_{k(i,j)}^{i,j})$ of matrices in $SP(2N, \mathbb{R})$, satisfying the following properties:

(1) For every $m \in \mathbb{N}$, $\iota = (i_1, \ldots, i_m) \in \{1, \ldots, n\}^m$, and $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$ consider the word

$$[W(\iota, \alpha)] = [t^{i_1,i_m}][M_A(x_{i_m})]^{\alpha_m}[t^{i_m,i_{m-1}}][M_A(x_{i_{m-1}})]^{\alpha_{m-1}} \cdots [t^{i_2,i_1}][M_A(x_{i_1})]^{\alpha_1},$$

where the word $w(\iota,\alpha) = ((x_{i_1},\alpha_1),\dots,(x_{i_m},\alpha_m))$ with letters in $M \times \mathbb{N}$ is not a power. Then there is $x(\iota,\alpha) \in \Sigma$ such that

- The length of $[W(\iota,\alpha)]$ is the period of $x(\iota,\alpha)$.
- The word $[M]_A(x(\iota,\alpha))$ is ε-close to $[W(\iota,\alpha)]$ and there is an ε-symplectic perturbation $\tilde{A}$ of A such that the word $[M]_{\tilde{A}}(x(\iota,\alpha))$ is $[W(\iota,\alpha)]$.

(2) One can choose $x(\iota,\alpha)$ such that the distance between the orbit of $x(\iota,\alpha)$ and any point x_{i_k} is bounded by some function of α_k which tends to zero as α_k goes to infinity.

Given ι, α as above, the word $[t^{i,j}]$ is an ε-transition from x_j to x_i. We call *ε-transition matrices* the matrices $T_{i,j}$ which are product of the letters composing $[t^{i,j}]$. We say a periodic linear system *admits transitions* if for any $\varepsilon > 0$ it admits ε-transitions.

The following lemma gives an example of linear systems with symplectic transitions. It is a symplectic version of [5, Lemma 1.9] and its proof, based on the existence of Markov Partitions, is analogous the proof of [5].

Lemma 4.5. *Let f be a symplectic diffeomorphism and let P be a hyperbolic saddle of index k (dimension of its stable manifold). The derivative Df induces a continuous periodic symplectic linear system on the set Σ of hyperbolic saddles in the homoclinic class $H(P, f)$ of index k and homoclinically related to P.*

Remark 4.6. We have some good properties that we use during the next sections. Consider points $x_1,\dots,x_n = x_1 \in \Sigma$ and ε-transitions $[t^{i,j}]$ from x_j to x_i. Then

(1) for every positive $\alpha \geqslant 0$ and $\beta \geqslant 0$ the word

$$\big([M]_A(x_i)\big)^{\alpha}\big[t^{i,j}\big]\big([M]_A(x_j)\big)^{\beta}$$

is also an ε-transition from $x_j$to x_i.

(2) for any i, j and k the word $[t^{i,j}][t^{j,k}]$ is an ε-transition from x_k to x_i.

The following lemma whose proof is analogous of [5, Lemma 1.10] states that every periodic symplectic system with transitions can be approximated by a diagonalizable systems defined on a dense subset of Σ. We emphasize that this lemma is also true for symplectic case.

Lemma 4.7. *Let $(\Sigma, f, \mathcal{E}, A)$ be a periodic linear symplectic system with transition. Then for any $\varepsilon > 0$ there is a diagonalizable symplectic ε-perturbation $\tilde{A}$ of A defined on a dense invariant subset $\widetilde{\Sigma}$ of Σ.*

Remark 4.8. We remark that the diagonalizable system near to A as required in the above lemma is not necessarily continuous, but it does not matter in the way we apply this lemma.

5. A dichotomy for symplectic linear systems

In this section, we reduce the study of the dynamics of symplectic diffeomorphisms in Theorem 2.1 to a problem on symplectic linear systems in Proposition 5.3. We split the proof of this proposition in two propositions (Propositions 5.5 and 5.6) whose proofs is given in Sections 6 and 7, see [5, Section 2.1] for more details.

An important tool to make the interplay between a dichotomy for diffeomorphisms and for linear symplectic systems is a symplectic version of Franks' lemma.

Lemma 5.1 ((Symplectic Franks' lemma)). *Let $f \in \mathrm{Diff}^1_\omega(M)$ and E a finite f-invariant set. Assume that B is a small symplectic perturbation of Df along E. Then for every neighborhood V of E there is a symplectic* *diffeomorphism g arbitrarily C^1-close to f coinciding with f on E and out of V, and such that Dg is equal to B on E.*

Remark 5.2. If the symplectic diffeomorphism in Lemma 5.1 belongs to $\mathrm{Diff}^r_\omega(M)$, $1 \leqslant r \leqslant \infty$, then the corresponding perturbed diffeomorphism can be taken in $\mathrm{Diff}^r_\omega(M)$. That is because the perturbation envolves exponential function and some bump functions which are C^∞. However, it is important to note that the perturbed diffeomorphism is just C^1-close to the initial one.

The following proposition is a result that provides a dichotomy for symplectic linear systems.

Proposition 5.3 *((Main proposition)). For any $\varepsilon > 0$, $N \in \mathbb{N}$ and $K > 0$ there is $\ell > 0$ such that any continuous periodic $2N$-dimensional linear system $(\Sigma, f, \mathcal{E}, A)$ bounded by K (i.e. $\|A\| < K$) and having symplectic transitions satisfies the following,*

- *either A admits an ℓ-dominated splitting,*
- *or there are a symplectic ε-perturbation $\tilde{A}$ of A and a point $x \in \Sigma$ such that $M_{\tilde{A}}(X)$ is the identity matrix.*

In the proof of this theorem we use the following notions.

Definition 5.4 *((Definition 2.2 of [5])).* Let $M \in GL(N, \mathbb{R})$ be a linear isomorphism of $\mathbb{R}^N$ such that M has some complex eigenvalue λ, i.e., $\lambda \in \mathbb{C} \setminus \mathbb{R}$. We say λ has *rank* $(i, i+1)$ if there is a M-invariant splitting of $\mathbb{R}^N$, $F \oplus G \oplus H$, such that:

- every eigenvalue σ of $M|_F$ (resp. $M|_H$) has modulus $|\sigma| < |\lambda|$ (resp. $|\sigma| > |\lambda|$),
- $\dim(F) = i - 1$ and $\dim(H) = N - i - 1$,
- the plane G is the eigenspace of λ.

We say a periodic linear system $(\Sigma, f, \mathcal{E}, A)$ *has a complex eingenvalue of rank* $(i, i+1)$ if there is $x \in \Sigma$ such that the matrix $M_A(x)$ has a complex eigenvalue of rank $(i, i+1)$.

We split the proof of Proposition 5.3 into the following two results, which are symplectic version of [5, Propositions 2.4 and 2.5]:

Proposition 5.5. *For every $\varepsilon > 0$, $N \in \mathbb{N}$ and $K > 0$ there is $\ell \in \mathbb{N}$ satisfying the following property: Let $(\Sigma, f, \mathcal{E}, A)$ be a continuous periodic $2N$-dimensional linear system with symplectic transitions such that its norm $\|A\|$ is bounded by K. Assume that there exists $i \in \{1, \ldots, 2N-1\}$ such that every symplectic ε-perturbation $\tilde{A}$ of A has no complex eigenvalues of rank $(i, i+1)$. Then $(\Sigma, f, \mathcal{E}, A)$ admits an ℓ-dominated splitting $F \oplus G$, $F \prec_\ell G$, with* $\dim(F) = i$.

Proposition 5.6. *Let $(\Sigma, f, \mathcal{E}, A)$ be a periodic linear system with symplectic transitions. Given $\varepsilon > \varepsilon_0 > 0$ assume that, for any $i \in \{1, \ldots, 2N-1\}$, there is a symplectic ε_0-perturbation of A having a complex eigenvalue of rank $(i, i+1)$. Then there are a symplectic ε-perturbation $\tilde{A}$ of A and $x \in \Sigma$ such that $M_{\tilde{A}}$ is the identity matrix.*

The proof of Proposition 5.5 follows from 2-dimensional arguments of Mañé [11] and higher dimensional arguments of Bonatti, Díaz and Pujals in [5]. However, when we change a symplectic linear system along a subspace by a symplectic perturbation, it produces an effect along it conjugated symplectic subspace. So, in many time, we have to deal with 4-dimensional arguments instead of Mañé's 2-dimensional arguments.

Although our arguments are based on [5], all perturbations we have to perform are symplectic. For this reason, we have to be more careful and introduce new techniques.

6. Proof of Proposition 5.5

The main ideas of the proof of Proposition 5.5 is to use the argument of Mañé in 2-dimensional case and reduction techniques in [5]. In fact, by symplectic nature of our systems, the problem will not be reduced to a 2-dimensional problem. Roughly speaking, the problem will be reduced to a problem in a 4-dimensional subspace.

So, first of all, let us recall the 2-dimensional version of Proposition 5.5.

Proposition 6.1 *((*Mañé*)). Given any K and $\varepsilon > 0$ there is $\ell \in \mathbb{N}$ such that for every 2-dimensional linear system $(\Sigma, f, \mathcal{E}, A)$, with norm $\|A\|$ bounded by K and such that the matrices $M_A(x)$ preserve the orientation,*

(1) *either A admits an ℓ-dominated splitting,*
(2) *or there are an ε-perturbation B of A and $x \in \Sigma$ such that $M_B(x)$ has a complex (non-real) eigenvalue.*

Remark 6.2. If A in the above proposition preserves a non-degenerate form ω defined on $\mathcal{E}$ then it is possible to choose B ε-close to A also preserving ω. Recall that, in 2-dimensional setting, if $\det(B) = 1$ then B preserves ω. In order to construct such B suppose that $\det(\widetilde{B}) \neq 1$ where $\widetilde{B}$ comes from the above proposition, we just substitute $\widetilde{B}$ by $B = \widetilde{B}/\det(\widetilde{B})$. The fact that the determinant map is continuous and $\det(A) = 1$ implies that B is close to A.

6.1. Dimension reduction

To generalize Proposition 6.1 to higher dimensions, Bonatti, Diaz and Pujals in [5] introduced dominated splittings for quotient space. Here we just recall the definitions, for more details we recommend the reader to see [5, Section 4].

Let $(\Sigma, f, \mathcal{E}, A)$ be a linear system and F an invariant subbundle of $\mathcal{E}$ with constant dimension. We denote by A_F the restriction of A to F and by A/F the quotient of A along F endowed with the metric of orthogonal complement $F^{\perp}$ of F; i.e., given a class $[v]$ we let $\|[v]\| = |v_F^{\perp}|$.

An important statement proved in [5] is the following.

Lemma 6.3 *((*Lemma 4.4 of [5]*)). For any $K > 0$ and $\ell \in \mathbb{N}$, there exists L with the following property: Given any linear system $(\Sigma, f, \mathcal{E}, A)$ such that $\|A\|$ is bounded by K with an invariant splitting $E \oplus F \oplus G$, one has*

(1) *$E \prec_\ell F$ and $E/F \prec_\ell G/F \Rightarrow E \prec_L F \oplus G$.*
(2) *$F \prec_\ell G$ and $E/F \prec_\ell G/F \Rightarrow E \oplus F \prec_L G$.*

Using the previous lemma, the proof of Proposition 5.5 follows from Lemma 6.4 besides a inductive process completely analogous to that in [5, Lemmas 5.2 and 5.3]. In fact the next lemma is a symplectic version of [5, Lemma 5.1] and it can be understood as a 2-dimensional version of Proposition 5.5 and in it proof we have to take account that we are deal with symplectic systems. Therefore, we give a complete proof of this statement. We omit the inductive argument necessary to the proof of Proposition 5.5, since it is similar to the mentioned work.

Lemma 6.4. *Given $K > 0$ and $\varepsilon > 0$ there is $\ell \in \mathbb{N}$ such that for any diagonalizable linear periodic system $(\Sigma, f, \mathcal{E}, A)$ of dimension $2N$ and bounded by K, and any $1 \leqslant i \leqslant 2N - 1$ one has*

- *Either there is an ε-perturbation of A having a complex eigenvalue of rank $(i, i+1)$,*
- *or for every $j \leqslant i \leqslant k$*

$$E_j/(E_{j+1} \oplus \cdots \oplus E_k) \prec_\ell E_{k+1}/(E_{j+1} \oplus \cdots \oplus E_k).$$

Proof. Suppose that $\lambda_1 < \cdots < \lambda_N < \lambda_{N+1} < \cdots < \lambda_{2N}$ are the eigenvalues of A. As A is symplectic we have $\lambda_i = (\lambda_{2N-i+1})^{-1} = \lambda_{i^*}^{-1}$.

Fix $\varepsilon > 0$ and let ℓ be the dominance constant in Proposition 6.1. If $E_j/(E_{j+1} \oplus \cdots \oplus E_k) \prec_\ell E_{k+1}/(E_{j+1} \oplus \cdots \oplus E_k)$ we are done. Otherwise, by Proposition 6.1, we perturb the quotient to collapse the eigenvalues λ_j and λ_{k+1}. Moreover, by Lemma 4.2, this perturbation of the quotient gives a perturbation $\tilde{A}$ of A having a pair of eigenvalues $\tilde{\lambda}_j = \tilde{\lambda}_{k+1}$, which are respectively continuation of λ_j and λ_{k+1} and preserving the eigenvalues of the restriction of A to $E_{j+1} \oplus \cdots \oplus E_k$.

Consider a symplectic isotopy A_t, $0 \leqslant t \leqslant 1$, such that $A_0 = A$, $A_1 = \tilde{A}$ and denote $\lambda_{j,t}$ and $\lambda_{k+1,t}$ the continuations of λ_j and λ_{k+1} at time t. Let us assume that $\lambda_{j,t} \leqslant \lambda_{k+1,t}$ for every $0 \leqslant t < 1$. We analyze the following cases:

(1) $\lambda_i \leqslant 1 \leqslant \lambda_{i+1}$,
(2) $\lambda_i < \lambda_{i+1} < 1$,
(3) $1 < \lambda_i < \lambda_{i+1}$.

Observe that, as this perturbation is symplectic, the eigenvalues $\tilde{\lambda}_{j^*}$ and $\tilde{\lambda}_{(k+1)^*}$ also will collapse. So, we get $\tilde{A}$ having the same eigenvalues λ_s, for $s \notin \{j, j^*, k+1, (k+1)^*\}$ of A and $\tilde{\lambda}_j = \tilde{\lambda}_{k+1}$, $\tilde{\lambda}_{(k+1)^*} = \tilde{\lambda}_{j^*}$. Furthermore, when we get a complex eigenvalue of rank $(i, i+1)$, we also get a complex eigenvalue of rank $((i+1)^*, i^*)$. Hence, the proof of item (2) yields to the proof of item (3).

In order to proof item (1), note that we must have $i = N$. It is because $\lambda_i \leqslant 1$ and $\lambda_{i+1} = \lambda_i^* \geqslant 1$ are consecutive eigenvalues of a symplectic system. Moreover, since A is diagonalizable then λ_N and $\lambda_{N+1} = \lambda_N^*$ cannot assume the value 1. For the proof of this item, we have the following alternatives:

(1.a) $\tilde{\lambda}_j = \tilde{\lambda}_{k+1} < \lambda_N$. So, there exists $0 \leqslant t \leqslant 1$ such that $\lambda_{k+1,t} = \lambda_N$ and $\lambda_{j,t} < \lambda_N$. Hence, $\lambda_{(k+1)^*,t} = \lambda_{N+1} > 1$. Therefore, there exists $t' < t$ such that $\lambda_{k+1,t'} = \lambda_{(k+1)^*,t'} = 1$. Then, we perturb $A_{t'}$ to get a complex eigenvalue of rank $(i, i+1)$.
(1.b) $\tilde{\lambda}_j = \tilde{\lambda}_{k+1} > \lambda_{N+1}$. This case is similar to the previous one.
(1.c) $\lambda_N < \tilde{\lambda}_j = \tilde{\lambda}_{k+1} < \lambda_{N+1}$. Recall that $\lambda_N < 1 < \lambda_{N+1}$. So, there exists t such that either $\lambda_{j,t} = \lambda_{j^*,t} = 1$ or $\lambda_{k+1,t} = \lambda_{(k+1)^*,t} = 1$. In both cases we obtain a complex eigenvalue of rank $(i, i+1)$ after a small perturbation of A_t.

Now, we consider the second case where $\lambda_i < \lambda_{i+1} < 1$. Again we take account the following alternatives:

(2.a) $\tilde{\lambda}_j = \tilde{\lambda}_{k+1} < \lambda_i$. If $\lambda_{k+1} < \lambda_{(i+1)^*}$ then there exists $0 \leqslant t \leqslant 1$ such that $\lambda_{k+1,t} = \lambda_i$. After a small perturbation of A_t, we get a complex eigenvalue of rank $(i, i+1)$. Otherwise, $\lambda_{k+1} > \lambda_{(i+1)^*}$ implies $\lambda_{(k+1)^*} < \lambda_i$. Then, there exists $0 \leqslant t' < 1$ such that either $\lambda_{j,t'} < \lambda_{(k+1)^*,t'} = \lambda_{i+1}$ (recall that $\tilde{\lambda}_{(k+1)^*} > 1$) or $\lambda_{(k+1)^*,t'} < \lambda_{j,t'} = \lambda_{i+1}$ (there are no reason to $\lambda_{j,t}$ remains less than λ_{i+1} during all the isotopy). In both cases we perturb slightly $A_{t'}$ to produce a complex eigenvalue of rank $(i, i+1)$.
(2.b) $\tilde{\lambda}_j = \tilde{\lambda}_{k+1} > \lambda_{i+1}$. If $\lambda_{k+1} < \lambda_{(i+1)^*}$, let $0 \leqslant t' \leqslant 1$ be the smallest t such that or $\lambda_i < \lambda_{j,t} = \lambda_{(k+1)*,t} < \lambda_{i+1}$ or $\lambda_{j,t} = \lambda_{i+1} < \lambda_{(k+1)*,t}$ or $\lambda_{j,t} < \lambda_i = \lambda_{(k+1)*,t}$. After a small perturbation of $A_{t'}$, we get a complex eigenvalue of rank $(i, i+1)$. Otherwise, $\lambda_{k+1} > \lambda_{(i+1)^*}$ implies $\lambda_{(k+1)^*} < \lambda_i$. Then, there exists $0 \leqslant t' \leqslant 1$ such that either $\lambda_{(k+1)^*,t'} < \lambda_{j,t'} = \lambda_{i+1}$ or $\lambda_{j,t'} < \lambda_{(k+1)^*,t'} = \lambda_{i+1}$. In both cases we perturb slightly $A_{t'}$ to produce a complex eigenvalue of rank $(i, i+1)$.

(2.c) $\lambda_i < \tilde{\lambda}_j = \tilde{\lambda}_{k+1} < \lambda_{i+1}$. If $\lambda_{k+1} > \lambda_{(i+1)^*}$ then $\lambda^*_{k+1} < \lambda_i$. Then, there exists $0 \leqslant t' \leqslant 1$ such that either $\lambda_{j,t'} < \lambda_{(k+1)^*,t'} = \lambda_i$ or $\lambda_{(k+1)^*,t'} < \lambda_{j,t'} = \lambda_i$. In both cases we perturb slightly $A_{t'}$ to produce a complex eigenvalue of rank $(i, i+1)$. Otherwise, $\lambda_{k+1} < \lambda_{(i+1)^*}$ implies that a small perturbation of $\tilde{A}$ gives us a complex eigenvalue of rank $(i, i+1)$.

This completes the proof. □

End of the proof of Proposition 5.5. Observe that Lemma 6.4 is for diagonalizable systems. But, now we use Lemma 4.7 to end the proof of Proposition 5.5.

Assume that there are $\varepsilon > 0$ and $i \in \{1, \ldots, 2N-1\}$ such that every ε-perturbation of A has no complex eigenvalue of rank $(i, i+1)$. Choose a sequence $\varepsilon_n < \varepsilon/2$ converging to zero. As the system (Σ, f, A) has transition, using Lemma 4.7, we get a dense subset Σ_n and diagonalizable ε_n-perturbation B_n of A defined on Σ_n. Then, we apply Lemma 6.4 for B_n: there is an integer $L > 0$ such that every B_n admits an L-dominated splitting $E_n \oplus F_n$ with $\dim(E_n) = i$. Finally, as Σ_n are dense and $\|B_n - A\| \to 0$, we conclude that A admits an L-dominated splitting $E \oplus F$ with $\dim(E) = i$. □

7. Proof of Proposition 5.6

After getting periodic points with complex eigenvalues with rank $(i, i+1)$ the idea in the proof of Proposition 5.6 is to use the symplectic transitions to multiply matrices corresponding to different points of Σ having complex eigenvalues of different ranks. Observe that as f is symplectic a periodic point with complex eigenvalue of rank $(i, i+1)$ is also of rank $((i+1)^*, i^*)$.

The next proposition is a symplectic version of Lemma 5.4 of [5].

Proposition 7.1. *Let $(\Sigma, f, \mathcal{E}, A)$ be a continuous periodic symplectic linear system with symplectic transitions. Fix $\varepsilon_0 > 0$ and assume that a symplectic ε_0-perturbation of A has a complex eigenvalue of rank $(i, i+1)$ for some $i \in \{1, \ldots, 2N-1\}$. Then for every $0 < \varepsilon_1 < \varepsilon_0$ there is a point $p \in \Sigma$ such that for every $1 \leqslant i < 2N$ there is a symplectic ε_1-transition $[t^i]$ from p to itself with the following properties*:

- *There exists a symplectic ε_1-perturbation $[M]_{\tilde{A}}(p)$ of the word $[M]_A(p)$ such that the corresponding matrix $[M]_{\tilde{A}}(p)$ has only real positive eigenvalues with multiplicity 1. Denote by $\tilde{\lambda}_1 < \cdots < \tilde{\lambda}_{2N}$ such eigenvalues and by $E_i(p)$ their respective (1-dimensional) eigenspaces.*
- *There is a symplectic $(\varepsilon_0 + \varepsilon_1)$-perturbation $[\tilde{t}^i]$ of the transition $[t^i]$ such that the corresponding matrix $\widetilde{T}^i$ satisfies*
  - *$\widetilde{T}^i(E_j(p)) = E_j(p)$ if $j \notin \{i, i+1, i^*, (i+1)^*\}$,*
  - *$\widetilde{T}^i(E_i(p)) = E_{i+1}(p)$ and $\widetilde{T}^i(E_{i+1}(p)) = E_i(p)$.*

A key tool in the proof of Proposition 7.1 is symplectic transitions constructed in Proposition 7.2. These transitions preserve the dominated splitting corresponding to two different periodic points.

Let $p, p_i \in \Sigma$ such that

$$\mathcal{E}(p) = F_1 \prec \cdots \prec F_{2N},$$

and

$$\mathcal{E}(p_i) = E_1 \prec \cdots \prec E_{(i+1)^*,i^*} \prec \cdots \prec E_{i,i+1} \prec \cdots \prec E_{2N},$$

where F_i and E_i are 1-dimensional eigenspaces and $E_{i,i+1}$ and $E_{(i+1)^*,i^*}$ are 2-dimensional eigenspaces corresponding to the complex eigenvalues. We fix a symplectic basis $\{f_1, \ldots, f_{2N}\}$ for $\mathcal{E}(p)$. The eigenvalues corre-

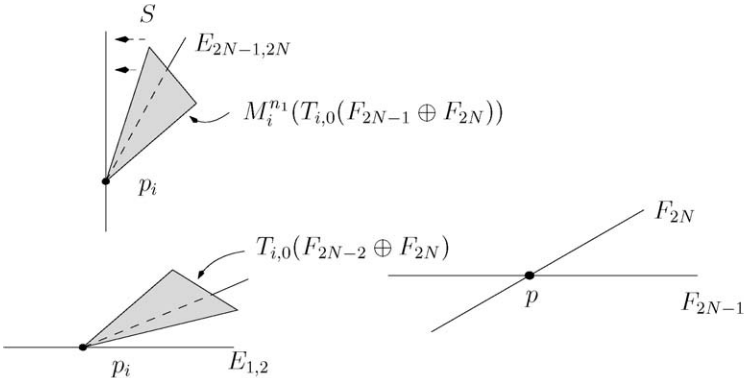

Fig. 1. Constructing a transition whose matrix sends $F_{2N-1} \oplus F_{2N}$ to $E_{2N-1,2N}$.

sponding to $E_{(i+1)^*,i^*}$ are inverse of the eigenvalues corresponding to $E_{i,i+1}$. We only deal with the case $i > N$, that is when $E_{i,i+1}$ appears after $E_{(i+1)^*,i^*}$ in the above dominated splitting. The another case is completely similar to this case.

Proposition 7.2. *There exists a symplectic transition* $[t_{i,0}]$ *from* p *to* p_i *such that its matrix* $T_{i,0}$ *satisfies the following*:

- $T_{i,0}(F_j) = E_j$, *if* $j \neq i, i+1, i^*, (i+1)^*$,
- $T_{i,0}(F_i \oplus F_{i+1}) = E_{i,i+1}$, *and*
- $T_{i,0}(F_{(i+1)^*} \oplus F_{i^*}) = E_{(i+1)^*,i^*}$.

There is also a symplectic transition $[t_{0,i}]$ *from* p_i *to* p *with a similar properties.*

To prove the proposition we state two auxiliary lemmas.

Lemma 7.3. *Under the hypotheses of Proposition* 7.2, *if* $i = 2N-1$ *there exists a symplectic transition from* p *to* p_i *such that its matrix* T *satisfies the following properties*:

- $T(F_{2N-1} \oplus F_{2N}) = E_{2N-1,2N}$ *and* $T(F_1 \oplus F_2) = E_{1,2}$,
- $T(F_3 \oplus \cdots \oplus F_{2N-2}) = E_3 \oplus \cdots \oplus E_{2N-2}$.

Proof. Suppose that $t_{i,0}$ is an arbitrary transition from p to p_i. Let M_i and M denote respectively $M_A(p_i)$ and $M_A(p)$. After a small symplectic perturbation we may suppose

$$T_{i,0}(F_{2N-1} \oplus F_{2N}) \nsubseteq E_{1,2} \oplus E_3 \oplus \cdots \oplus E_{2N-2}.$$

By domination, for n_1 sufficiently large $M_i^{n_1} \circ T_{i,0}(F_{2N-1} \oplus F_{2N})$ is close enough to $E_{2N-1,2N}$. Now by another small symplectic perturbation S we may send $M_i^{n_1} \circ T_{i,0}(F_{2N-1} \oplus F_{2N})$ inside $E_{2N-1,2N}$. Let $T_1 = S \circ M_i^{n_1} \circ T_{i,0}$. Now, using again the dominated splitting, there is n_2 sufficiently large such that $T_2^{-1}(E_{1,2}) := M^{-n_2} \circ T_1^{-1}(E_{1,2})$ is close enough to $F_1 \oplus F_2$. It is possible to choose $v_1, v_2 \in E_{1,2}$ such that

$$T_2^{-1}(v_1) = f_1 + \sum_{j=3}^{2N} \varepsilon_j f_j \quad \text{and} \quad T_2^{-1}(v_2) = f_2 + \sum_{j=3}^{2N} \tilde{\varepsilon}_j f_j.$$

Indeed, let $V_{f_2} := F_1 \oplus F_3 \oplus \cdots \oplus F_{2N}$. As $\dim(V_{f_2}) = 2N-1$ and $\dim(T_1^{-1}(E_{1,2})) = 2$ there exists $w := f_1 + \varepsilon_3 f_3 + \cdots + \varepsilon_{2N} f_{2N} \in V_{f_2} \cap T_1^{-1}(E_{1,2})$. Since w is close enough to $F_1 \oplus F_2$ it comes out that ε_i, $i = 3, \ldots, 2N$, are arbitrarily small. So, it is enough to take $v_1 = T_2(w)$. Similarly we get v_2 satisfying the assertion.

Define a symplectic linear map $\tilde{I}$ as follows.

$$\tilde{I}(f_i) = \begin{cases} T_2^{-1}(v_i) & \text{if } i = 1, 2, \\ f_i & \text{if } i = 2N, 2N-1, \\ f_i + \varepsilon_{i^*} f_{2N} + \tilde{\varepsilon}_{i^*} f_{2N-1} & \text{if } 3 \leqslant i \leqslant N, \\ f_i - \varepsilon_{i^*} f_{2N} - \tilde{\varepsilon}_{i^*} f_{2N-1} & \text{if } N < i \leqslant 2N-2. \end{cases}$$

Let us check that $\tilde{I}$ is a symplectic transformation. It is enough to verify that $\omega(\tilde{I}(f_j), \tilde{I}(f_k)) = \omega(f_j, f_k)$ for all $1 \leqslant j, k \leqslant 2N$, $j \neq k$. We have the following cases:

(i) Let $j = 1, 2$ and $3 \leqslant k \leqslant N$. One just prove for $j = 1$ (the case $j = 2$ is analogous). We have

$$\begin{aligned} \omega\big(\tilde{I}(f_1), \tilde{I}(f_k)\big) &= \omega\left(f_1 + \sum_{j=3}^{2N} \varepsilon_j f_j,\, f_k + \varepsilon_{k^*} f_{2N} + \tilde{\varepsilon}_{k^*} f_{2N-1}\right) \\ &= \omega(f_1, \varepsilon_{k^*} f_{2N}) + \omega(\varepsilon_{k^*} f_{k^*}, f_k) = \varepsilon_{k^*} - \varepsilon_{k^*} = 0 = \omega(f_1, f_k). \end{aligned}$$

(ii) Let $j = 1, 2$ and $N < k \leqslant 2N-2$. Again we just give the proof for $j = 1$. We have

$$\begin{aligned} \omega\big(\tilde{I}(f_1), \tilde{I}(f_k)\big) &= \omega\left(f_1 + \sum_{i=3}^{2N} \varepsilon_j f_j,\, f_k - \varepsilon_{k^*} f_{2N} - \tilde{\varepsilon}_{k^*} f_{2N-1}\right) \\ &= \omega(f_1, -\varepsilon_{k^*} f_{2N}) + \omega(\varepsilon_{k^*} f_{k^*}, f_k) = -\varepsilon_{k^*} + \varepsilon_{k^*} = 0 = \omega(f_1, f_k). \end{aligned}$$

(iii) Let $3 \leqslant j, k \leqslant 2N-2$. In this case we have

$$\omega\big(\tilde{I}(f_j), \tilde{I}(f_k)\big) = \omega(f_j \pm \varepsilon_{j^*} f_{2N} \pm \tilde{\varepsilon}_{j^*} f_{2N-1}, f_k \pm \varepsilon_{k^*} f_{2N} \pm \tilde{\varepsilon}_{k^*} f_{2N-1}) = \omega(f_j, f_k).$$

(iv) Let $j = 1, 2$ and $k = 2N-1, 2N$. Let us just show this case for the when $j = 1$ and $k = 2N$:

$$\omega\big(\tilde{I}(f_1), \tilde{I}(f_{2N})\big) = \omega\left(f_1 + \sum_{i=3}^{2N} \varepsilon_i f_i,\, f_{2N}\right) = \omega(f_1, f_{2N}).$$

This complete the proof that $\tilde{I}$ is symplectic.

Hence, $T := T_2 \circ M_p^{n_2} \circ \tilde{I}$ maps $F_1 \oplus F_2$ and $F_{2N-1} \oplus F_{2N}$ respectively to $E_{1,2}$ and $E_{2N-1,2N}$.

It remains to prove that

$$T(F_3 \oplus \cdots \oplus F_{2N-2}) = E_3 \oplus \cdots \oplus E_{2N-2}.$$

Take $2 < j < 2N-1$ and assume that $T(f_j) \notin E_3 \oplus \cdots \oplus E_{2N-2}$. So, there is $v \in E_{1,2} \cup E_{2N-1,2N}$ such that $\omega(T(f_j), v) \neq 0$. Let $w := T^{-1}(v) \in (F_1 \oplus F_2) \cup (F_{2N-1} \oplus F_{2N})$. Then,

$$0 = \omega(f_j, w) = \omega\big(T(f_j), T(w)\big) = \omega\big(T(f_j), v\big) \neq 0,$$

which is a contradiction. This completes the proof of the lemma. □

Lemma 7.4. *Under the hypotheses of Proposition* 7.2, *if* $i < 2N-1$ *there exists a symplectic transition* T *from* p *to* p_i *with the following properties*:

- $T(F_{2N}) = E_{2N}$ *and* $T(F_1) = E_1$,
- $T(F_2 \oplus \cdots \oplus F_{2N-1}) = E$, *where* $E = E_2 \oplus \cdots \oplus E_{2N-1}$ *or* $E_{2,3} \oplus E_4 \oplus \cdots \oplus E_{2N-3} \oplus E_{2N-2,2N-1}$.

Proof. Let $t_{i,0}$ be an arbitrary symplectic ε-transition from p to p_i. After a small symplectic perturbation we may suppose $T_{i,0}(F_{2N}) \notin E_1 \oplus E_2 \oplus \cdots \oplus E_{2N-1}$. So, by the dominated splitting, for n_1 sufficiently large $M_i^{n_1} \circ T_{i,0}(F_{2N})$ is close enough to E_{2N}. Now, by another small symplectic perturbation S we may send $M_i^{n_i} \circ T_{i,0}(F_{2N})$ inside E_{2N}. Let $T_1 = S \circ M_i^{n_1} \circ T_{i,0}$. Then $T_1(F_{2N}) = E_{2N}$. All perturbations are symplectic, but we do not know whether $T_1(F_1) = E_1$ or not.

Let $\{f_1, \ldots, f_{2N}\}$ be a symplectic basis for $\mathcal{E}(P)$ and $e_{2N} = T_1(f_{2N})$. For any $e_1 \in E_1$ we have

$$\omega\big(T_1^{-1}(e_{2N}), T_1^{-1}(e_1)\big) = \omega(e_{2N}, e_1) \neq 0.$$

So, we conclude that $T_1^{-1}(E_1)$ has a non-null component in the F_1 direction. Now, using the dominated splitting, for n_2 sufficiently large we have $T_2^{-1}(e_1) := M^{-n_2} \circ T_1^{-1}(e_1)$ is close enough to f_1 for some $e_1 \in E_1$. More precisely for some $e_1 \in E_1$ we can write:

$$T_2^{-1}(e_1) = f_1 + \varepsilon_2 f_2 + \varepsilon_3 f_3 + \cdots + \varepsilon_{2N} f_{2N}$$

where $f_j \in F_j$ and ε_i are small enough whenever n_2 is sufficiently large.

Now we define a symplectic perturbation of the identity map defined on the basis $\{f_1, f_2, \ldots, f_{2N}\}$ as follows:

$$\tilde{I}(f_i) = \begin{cases} f_1 + \sum_{i=2}^{2N} \varepsilon_i f_i & \text{if } i = 1, \\ f_{2N} & \text{if } i = 2N, \\ f_i + \varepsilon_{i^*} f_{2N} & \text{if } 1 < i \leqslant N, \\ f_i - \varepsilon_{i^*} f_{2N} & \text{if } N < i < 2N. \end{cases}$$

Let us verify that $\tilde{I}$ is a symplectic transformation. It is enough to verify that $\omega(\tilde{I}(f_j), \tilde{I}(f_k)) = \omega(f_j, f_k)$ for all $1 \leqslant j, k \leqslant 2N$, $j \neq k$. We have the following cases:

(i) Let $j = 1$ and $1 < k < N$. Observe that for $k \leqslant N$ we have $\omega(f_{k^*}, f_k) = -1$. So, we have

$$\begin{aligned} \omega\big(\tilde{I}(f_1), \tilde{I}(f_k)\big) &= \omega\left(f_1 + \sum_{i=2}^{2N} \varepsilon_i f_i,\ f_k + \varepsilon_{k^*} f_{2N}\right) = \omega(f_1, \varepsilon_{k^*} f_{2N}) + \omega(\varepsilon_{k^*} f_{k^*}, f_k) \\ &= -\varepsilon_{k^*} + \varepsilon_{k^*} = 0 = \omega(f_1, f_k). \end{aligned}$$

(ii) Let $j = 1$, $N < k < 2N$. Note that for $k > N$, $\omega(f_{k^*}, f_k) = 1$. Then,

$$\begin{aligned} \omega\big(\tilde{I}(f_1), \tilde{I}(f_k)\big) &= \omega\left(f_1 + \sum_{i=2}^{2N} \varepsilon_i f_i,\ f_k - \varepsilon_{k^*} f_{2N}\right) = \omega(f_1, -\varepsilon_{k^*} f_{2N}) + \omega(\varepsilon_{k^*} f_{k^*}, f_k) \\ &= \varepsilon_{k^*} - \varepsilon_{k^*} = 0 = \omega(f_1, f_k). \end{aligned}$$

(iii) Let $1 < j, k < 2N$. In this case we have

$$\omega\big(\tilde{I}(f_j), \tilde{I}(f_k)\big) = \omega(f_j \pm \varepsilon_{j^*} f_{2N}, f_k \pm \varepsilon_{k^*} f_{2N}) = \omega(f_j, f_k).$$

(iv) Let $j = 1, k = 2N$. Then

$$\omega\big(\tilde{I}(f_1), \tilde{I}(f_{2N})\big) = \omega\left(f_1 + \sum_{i=2}^{2N} \varepsilon_i f_i,\ f_{2N}\right) = \omega(f_1, f_{2N}).$$

Hence, $T := T_2 \circ \tilde{I}$ is a symplectic transition from p to p_i with $T(F_{2N}) = E_{2N}$ and $T(F_1) = E_1$. Moreover, the fact that T is symplectic implies that for all $1 < j < 2N$

$$\omega\big(T(f_j), T(f_{2N})\big) = \omega\big(T(f_j), T(f_1)\big) = 0.$$

This implies that $T(f_j)$ cannot have coordinates in the e_1 and e_{2N} directions. Therefore, T sends the eigenspace $F_2 \oplus \cdots \oplus F_{2N-1}$ into E. □

Proof of Proposition 7.2. If $i = 2N - 1$ we apply Lemma 7.3 and then use the method in the proof of Lemma 7.4 inductively to get the transition $T_{i,0}$.

Otherwise, if $i < 2N - 1$ we apply Lemma 7.4 successively $2N - i - 1$ times to reduce the problem to the above case. □

Proof of Proposition 7.1. The proof of this proposition follows from the same arguments of [5, Lemma 5.4]. Let us outline the main steps of the proof and point out how to complete the proof in the symplectic case.

After a small symplectic perturbation we may suppose that M_p is diagonalizable for some $p \in \Sigma$ and there is $p_i \in \Sigma$ such that p_i has complex eigenvalue of rank $(i, i+1)$. Let

$$(1) \quad \mathcal{E}(p) = F_1 \prec \cdots \prec F_{2N}$$

and

$$(2) \quad \mathcal{E}(p_i) = E_1 \prec \cdots \prec E_{(i+1)^*,i^*} \prec \cdots \prec E_{i,i+1} \prec \cdots \prec E_{2N}.$$

By Proposition 7.2 we can construct a symplectic transition $[t^i] := [t_{0,i}][t_{i,0}]$ from p to itself whose matrix T^i preserves all subspaces F_j, $j \neq i, i+1, i^*, (i+1)^*$, $F_i \oplus F_{i+1}$, and $F_{i^*} \oplus F_{(i+1)^*}$.

Using again Proposition 7.2 and the complex eigenvalues inside $E_{i,i+1}$ we can obtain a new transition $[\tilde{t}^i]$ such that $\widetilde{T}^i$ interchanges F_i and F_{i+1}. This transition can be constructed exactly as in [5, Lemma 5.7] using Lemmas 4.3 and 4.2. The unique difference is that to complete the proof in the symplectic case we should prove that $\widetilde{T}^i$ interchanges F_{i^*} and $F_{(i+1)^*}$ too.

Given a non-zero vector $f_{(i+1)} \in F_{(i+1)}$, let $f_i = \widetilde{T}^i(f_{(i+1)})$. We show that $\widetilde{T}^i(f_{i^*}) \in F_{(i+1)^*}$ for any $f_{i^*} \in F_{i^*}$.

Since $F_{i^*} \oplus F_{(i+1)^*}$ is preserved by $\widetilde{T}^i$, there exists constants $r, s \in \mathbb{R}$ such that $\widetilde{T}^i(f_{i^*}) = r f_{i^*} + s f_{(i+1)^*}$. Then,

$$0 = \omega(f_{i^*}, f_{(i+1)}) = \omega\big(\widetilde{T}^i(f_{i^*}), \widetilde{T}^i(f_{i+1})\big) = \omega(r f_{i^*} + s f_{(i+1)^*}, f_i) = r\omega(f_{i^*}, f_i).$$

The fact that $\omega(f_{i^*}, f_i) \neq 0$ implies $r = 0$. Therefore $\widetilde{T}^i(f_{i^*})$ belongs to $F_{(i+1)^*}$. Similarly we can prove that $\widetilde{T}^i(F_{(i+1)^*}) = F_{i^*}$. □

End of proof of Proposition 5.6. In Proposition 7.1 we construct transitions $[\tilde{t}^i]$ whose action on the finite set $\{F_i(p)\}_{1 \leqslant j \leqslant 2N}$ of eigenspaces of $M_{\tilde{A}}(p)$ is the transposition $(i, i+1)$ which interchanges $E_i(p)$ and $E_{i+1}(p)$. In what follows a combinatorial argument exactly as done in [5] shows that after a small perturbation we obtain a totally elliptic periodic point. We verify that the arguments in their paper works also in the symplectic case.

Given $0 \leqslant k < 2N$ denote by σ_k the cyclic permutation defined by $\sigma_k(E_j(p)) = E_{j+k}(p)$, where the sum $i + j$ is considered in the cyclic group $\mathbb{Z}/(2N)\mathbb{Z}$.

As any permutation is a composition of transpositions, for every $0 \leqslant k < 2N$ there exists an element $[\widetilde{S}_k]$ in the semi-group generated by transitions $[\tilde{t}^i]$ such that if $\widetilde{S}_k$ is the matrix corresponding to the word $[\widetilde{S}_k]$ then one has $\widetilde{S}_k(E_j(p)) = E_{j+k}(p)$.

Let $[S_k]$ be the word of matrices corresponding to the perturbation $[\widetilde{S}_k]$ in the semi-group generated by the initials $[t^j]$. As the $[t^j]$ are ε_1-transitions from p to itself, any word in the semi-group generated by the $[t^j]$, in particular the $[S_k]$ is also an ε_1-transition from p to itself. Let us write $[S_0] = [S_{2N}]$ the empty word whose corresponding matrix is the identity.

By definition of transitions, for any $n \in \mathbb{N}$ there is a point $x_n \in \Sigma$ such that the word $[M]_A(x_n)$ is ε_1-close to the word $[W_n]$ corresponding to the matrix W_n defined by

$$W_n := W_{2N-1,n} \circ \cdots \circ W_{1,n} \circ W_{0,n},$$

where $W_{i,n} := S_{2N-i} \circ (M_A(p))^n \circ S_i \circ S_{2N-i} S_i$.

We know that for any i, the matrix $\widetilde{S}_{2N-i} \circ \widetilde{S}_i$ acts trivially on the set of spaces $\{E_j(p)\}$. Let us denote by

- $\tilde{\lambda}_k$ the eigenvalues of $M_{\tilde{A}}(p)$ corresponding to E_k,
- $\mu_{i,j}$ the eigenvalue of $\widetilde{S}_{2N-i} \circ \widetilde{S}_i$ corresponding to the $E_j(p)$.

Consequently, for every j and any $n \in \mathbb{N}$ the space $E_j(p)$ is an eigenspace of the matrix

$$\widetilde{W}_{i,n} := \widetilde{S}_{2N-i} \circ \left(M_{\tilde{A}}(p)\right)^n \circ \widetilde{S}_i \circ \widetilde{S}_{2N-i}\widetilde{S}_i,$$

whose corresponding eigenvalue is $\mu_{i,j}^2 \tilde{\lambda}_{i+j}^n$.

As the transitions are symplectic we have $\tilde{\lambda}_i \tilde{\lambda}_{i^*} = 1$ and $\mu_{i,j}\mu_{i,j^*} = 1$. It comes out that

$$\prod_{i=1}^{2N} \tilde{\lambda}_i = 1$$

and

$$C_j := \prod_{i=0}^{2N-1} \mu_{i,j}^2 = \left(\prod_{i=0}^{2N-1} \mu_{i,j^*}^2\right)^{-1} = C_{j^*}^{-1}.$$

The word $[\widetilde{W}_n]$ corresponding to the matrix defined as

$$\widetilde{W}_n := \widetilde{W}_{2N-1,n} \circ \cdots \circ \widetilde{W}_{1,n} \circ \widetilde{W}_{0,n}$$

is $(\varepsilon_0 + \varepsilon_1)$-close to $[W_n]$ and so it is an $\varepsilon_0 + 2\varepsilon_1$-perturbation of the word $[M]_A(x_n)$. So we conclude that $E_j(p)$ is an eigenspace of $\widetilde{W}_n$ with eigenvalue C_j.

Observe that C_j are not necessarily close to 1. Consider B_n matrices having $E_j(p)$ as eigenspaces and $(C_j)^{-1/n}$ as their eigenvalues. Observe that B_n is symplectic too and $(B_n)^n = \widetilde{W}_n^{-1}$. Denote by $[M]_{\hat{A}}(p)$ the word obtained from $[M]_{\tilde{A}}(p)$ by replacing its first letter $\tilde{A}(p)$ by $\tilde{A}(p) \circ B_n$. For n large enough this new word is an ϵ_1-perturbation of $[M]_{\tilde{A}}(p)$, so by item (i) of Proposition 7.1 it is also $2\epsilon_1$-perturbation of $[M]_A(p)$. As B_n commutes with $M_{\tilde{A}}(p)$ we get

$$\left(M_{\tilde{A}}(p) \circ B_n\right)^n = M_{\tilde{A}}^n(p) \circ \widetilde{W}_n^{-1}.$$

So, the word $[\widehat{W}_n]$ obtained by changing the initial subword $[M]_{\tilde{A}}^n(p)$ of $[\widetilde{W}_n]$ by $[M]_{\hat{A}}(p)$ is $(\epsilon_0 + 2\epsilon_1) < \epsilon$ close to the word $[M]_A(x_n)$ and its corresponding matrix $\widehat{W}_n = \widetilde{W}_n \circ \widetilde{W}_n^{-1} = \mathrm{Id}$. This completes the proof.

References

[1] A. Arbieto, C. Matheus, A pasting lemma I: the case of vector fields, Preprint, IMPA, 2003.

[2] M.-C. Arnaud, The generic symplectic C^1-diffeomorphisms of four-dimensional symplectic manifolds are hyperbolic, partially hyperbolic or have a completely elliptic periodic point, Ergodic Theory Dynam. Systems 22 (6) (2002) 1621–1639.

[3] J. Bochi, M. Viana, Lyapunov exponents: How frequently are dynamical systems hyperbolic?, Preprint IMPA, 2003.

[4] C. Bonatti, L.J. Díaz, Nonhyperbolic transitive diffeomorphisms, Ann. of Math. 143 (1996) 357–396.

[5] C. Bonatti, L.J. Díaz, E. Pujals, A C^1-generic dichotomy for diffeomorphisms: weak forms of hyperbolicity or infinitely many sinks or sources, Ann. of Math. 157 (2) (2003) 355–418.

[6] C. Bonatti, M. Viana, SRB measures for partially hyperbolic systems whose central direction is mostly contracting, Israel J. Math. 115 (2000) 157–193.

[7] K. Burns, C. Pugh, M. Shub, A. Wilkinson, Recent results about stable ergodicity, Proc. Sympos. Amer. Math. Soc. 69 (2001) 327–366.

[8] B. Dacorogna, J. Moser, On a partial differential equation involving the Jacobian determinant, Ann. Inst. H. Poincaré Anal. Non Linéaire 7 (1) (1990) 1–26.

[9] L.J. Díaz, E. Pujals, R. Ures, Partial hyperbolicity and robust transitivity, Acta Math. 183 (1999) 1–43.
[10] R. Mañé, Contributions to the stability conjecture, Topology 17 (1978) 383–396.
[11] R. Mañé, An ergodic closing lemma, Ann. of Math. 116 (1982) 503–540.
[12] R. Mañé, Oseledec's theorem from the generic viewpoint, in: Proceedings of the International Congress of Mathematicians, vols. 1, 2, Warsaw, 1983, PWN, Warsaw, 1984, pp. 1269–1276.
[13] S. Newhouse, Quasi-elliptic periodic points in conservative dynamical systems, Amer. J. Math. 99 (5) (1976) 1061–1087.
[14] M. Shub, Topologically transitive diffeomorphisms on T^4, in: Lecture Notes in Math., vol. 206, Springer-Verlag, 1971, p. 39.
[15] A. Tahzibi, Stably ergodic systems which are not partially hyperbolic, Israel J. Math. 24 (204) (2004) 315–342.
[16] T. Vivier, Flots robustament transitif sur des variété compactes, C. R. Math. Acad. Sci. Paris 337 (12) (2003) 791–796.
[17] Z. Xia, Homoclinic points in symplectic and volume-preserving diffeomorphisms, Comm. Math. Phys. 177 (2) (1996) 435–449.
[18] E. Zehnder, A note on smoothing symplectic and volume preserving diffeomorphisms, in: Lecture Notes in Math., vol. 597, Springer-Verlag, 1977, pp. 828–854.