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Multiple clustered layer solutions for semilinear Neumann problems on a ball

Solutions multi-singulières pour un problème de Neumann sur la boule unité

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Abstract

We consider the following singularly perturbed Neumann problem

$$\begin{cases} \varepsilon^2 \Delta u - u + f(u) = 0 \quad \text{in } \Omega; \\ u > 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\Omega = B_1(0)$ is the unit ball in $\mathbb{R}^n$, $\varepsilon > 0$ is a small parameter and f is superlinear. It is known that this problem has multiple solutions (spikes) concentrating at some points of $\bar{\Omega}$. In this paper, we prove the existence of radial solutions which concentrate at N spheres $\bigcup_{j=1}^N \{|x| = r_j^\varepsilon\}$, where $1 > r_1^\varepsilon > r_2^\varepsilon > \cdots > r_N^\varepsilon$ are such that $1 - r_1^\varepsilon \sim \varepsilon \log \frac{1}{\varepsilon}$, $r_{j-1}^\varepsilon - r_j^\varepsilon \sim \varepsilon \log \frac{1}{\varepsilon}$, $j = 2, \ldots, N$.

Résumé

On considère le problème de Neumann singulièrement perturbé suivant

$$\begin{cases} \varepsilon^2 \Delta u - u + f(u) = 0 \quad \text{dans } \Omega; \\ u > 0 \quad \text{dans } \Omega \quad \text{et} \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{sur } \partial\Omega, \end{cases}$$

où $\Omega = B_1(0)$ est la boule unité de $\mathbb{R}^n$, $\varepsilon > 0$ est un paramètre petit et f est surlinéaire. Il est bien connu que ce problème possède plusieurs solutions se concentrant en certains points de $\bar{\Omega}$. Dans cet article nous prouvons l'existence de solutions

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radiales qui se concentrent en N sphères $\bigcup_{j=1}^{N}\{|x|=r_j^\varepsilon\}$, où $1>r_1^\varepsilon>r_2^\varepsilon>\cdots>r_N^\varepsilon$ sont tels que $1-r_1^\varepsilon\sim\varepsilon\log\frac{1}{\varepsilon}$, $r_{j-1}^\varepsilon-r_j^\varepsilon\sim\varepsilon\log\frac{1}{\varepsilon}$, $j=2,\ldots,N$.

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1. Introduction

The aim of this paper is to construct a family of multiple layered solutions to the following singularly perturbed elliptic problem

$$\begin{cases}\varepsilon^2\Delta u-u+f(u)=0 & \text{in } \Omega;\\ u>0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial\nu}=0 & \text{on } \partial\Omega,\end{cases} \tag{1.1}$$

where $\Delta=\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, $\Omega=B_1(0)$ is the unit ball in $\mathbb{R}^n$, $\varepsilon>0$ is a small parameter, f is superlinear, and $\nu(x)$ denotes the unit outer normal at $x\in\partial\Omega$. A typical nonlinearity is $f(u)=u^p$, with $p\in(1,+\infty)$.

Problem (1.1) is known as the stationary equation of the Keller–Segel system in chemotaxis [19]. It can also be viewed as a limiting stationary equation of the Gierer–Meinhardt system in biological pattern formation, see [22] for more details.

In the pioneering papers [19,23] and [24], Lin, Ni and Takagi established the existence of a least-energy solution u_ε of (1.1) and showed that, for ε sufficiently small, u_ε has only one local maximum point $P_\varepsilon\in\partial\Omega$. Moreover, $H(P_\varepsilon)\to\max_{P\in\partial\Omega}H(P)$ as $\varepsilon\to0$, where $H(\cdot)$ is the mean curvature of $\partial\Omega$. Such a solution is called boundary spike-layer. Its energy, defined by

$$J_\varepsilon[u]=\frac{\varepsilon^2}{2}\int_\Omega|\nabla u|^2+\frac{1}{2}\int_\Omega u^2-\int_\Omega F(u),\quad u\in H^1(\Omega), \tag{1.2}$$

where $F(u)=\int_0^u f(s)\,ds$, satisfies the following estimate

$$J_\varepsilon[u_\varepsilon]\sim\varepsilon^n. \tag{1.3}$$

(Here and throughout the paper, $A_\varepsilon\sim B_\varepsilon$ means that there exists fixed constants C_1, C_2 such that $C_1\leqslant A_\varepsilon/B_\varepsilon\leqslant C_2$ for ε small.)

Since then, many papers investigated further the solutions of (1.1) concentrating at one or multiple points of $\bar\Omega$ (spike-layers). A general principle is that the location of interior spikes is determined by the distance function from the boundary. We refer the reader to the articles [3,6,9–11,13,14,16,18,26–28,30,31] and references therein. On the other hand, boundary spikes are related to the mean curvature of $\partial\Omega$. This aspect is discussed in the papers [4,8,15,17,25,29,32,33], and references therein. A review of the subject up to 1998 is to be found in [22]. We mention one result which is related to our work here: in [14], it was proved that for any two integers $k\geqslant0$, $l\geqslant0$, $k+l>0$, problem (1.1) possesses a solution u_ε with exactly k interior spikes and l boundary spikes.

However, in all the papers mentioned above, we still have the energy bound (1.3) and the concentration set is zero-dimensional. The question of constructing higher-dimensional concentration sets has been investigated only in recent years. It has been conjectured in [22] that for any $1\leqslant k\leqslant n-1$, problem (1.1) has a solution u_ε which concentrates on a k-dimensional subset of $\bar\Omega$. We mention two recent results that support such a conjecture.

In [20], Malchiodi and Montenegro proved that for $n = 2$ and $f(u) = u^p$, $p \geqslant 2$, there exists a sequence of numbers $\varepsilon_k \to 0$ such that problem (1.1) has a solution u_{ε_k} which concentrates at the boundary of $\partial\Omega$ (or any component of $\partial\Omega$). Such a solution has the following energy bound

$$J_{\varepsilon_k}[u_{\varepsilon_k}] \sim \varepsilon_k^{n-1} = \varepsilon_k. \tag{1.4}$$

In another recent paper [2], Ambrosetti, Malchiodi and Ni proved the existence of a radially symmetric solution u_ε to (1.1) with $f(u) = u^p$, $p > 1$ and $\Omega = B_1(0)$, such that u_ε concentrates at a sphere $\{|x| = r_\varepsilon\}$ with $1 - r_\varepsilon \sim \varepsilon \log \frac{1}{\varepsilon}$. For this solution there holds

$$J_\varepsilon[u_\varepsilon] \sim \varepsilon. \tag{1.5}$$

In this paper we show existence of solutions concentrating at multiple spheres, which cluster near the boundary of $B_1(0)$. Throughout the paper, we assume the following conditions on f

(f1) $f(t) \equiv 0$ for $t > 0$, $f(0) = f'(0) = 0$ and $f \in C^{1+\sigma}[0,\infty) \cap C^2(0,\infty)$,
(f2) the following ODE has a unique solution

$$w'' - w + f(w) = 0 \quad \text{in } \mathbb{R}, \quad w(0) = \max_{y\in\mathbb{R}} w(y), \quad w(y) \to 0 \quad \text{as } |y| \to +\infty. \tag{1.6}$$

Assumption (f2) implies that the following holds: the only solution of the linearization of (1.6) at w

$$v'' - v + f'(w)v = 0 \quad \text{in } \mathbb{R}, \quad v(y) \to 0 \quad \text{for } |y| \to +\infty \tag{1.7}$$

is a multiple of w'. In fact, problem $v'' - v + f'(w)v = 0$ is a second order linear ODE and admits two linearly independent solutions. Since w' is a solution to (1.7), another solution to $v'' - v + f(w)v = 0$ must grow exponentially as $|y| \to +\infty$.

Examples of $f(u)$ include: $f(u) = u^p + \sum_{i=1}^{l} a_i u^{q_i}$, $1 < q_i < p < \infty$. (See [5] and Appendix C of [24].) When $f(u) = u^p$, the function $w(y)$ can be written explicitly and has the form

$$w(y) = \left(\frac{p+1}{2}\right)^{\frac{1}{p-1}} \left(\cosh\left(\frac{p-1}{2}y\right)\right)^{-\frac{2}{p-1}}.$$

Our main result in this paper is the following.

Theorem 1.1. *Let N be a fixed positive integer. Then there exists $\varepsilon_N > 0$ such that for all $\varepsilon < \varepsilon_N$, problem* (1.1) *admits a radially symmetric solution u_ε with the following properties*

(1) *u_ε concentrates at N spheres $\{|x| = r_j^\varepsilon\}$, $j = 1, \dots, N$, with*

$$1 - r_1^\varepsilon \sim \varepsilon \log \frac{1}{\varepsilon}, \qquad r_{j-1}^\varepsilon - r_j^\varepsilon \sim \varepsilon \log \frac{1}{\varepsilon}, \quad j = 2, \dots, N. \tag{1.8}$$

More precisely, we have $u_\varepsilon(r_j^\varepsilon) \to w(0)$, where $w(y)$ is the unique solution of (1.6), *and there exist two constants a and b such that*

$$u_\varepsilon(r) \leqslant a e^{-b \min_{i=1,\dots,N} |r - r_i^\varepsilon|/\varepsilon}. \tag{1.9}$$

(2) *u_ε has the following energy bound*

$$J_\varepsilon[u_\varepsilon] = \omega_{n-1} N \varepsilon I[w] + o(\varepsilon), \tag{1.10}$$

where

$$I[w] = \frac{1}{2}\int_{\mathbb{R}} \left((w')^2 + w^2\right) - \frac{1}{p+1}\int_{\mathbb{R}} F(w),$$

and ω_{n-1} denotes the volume of S^{n-1}.

Theorem 1.1 can also be generalized to singularly perturbed Schroedinger equations of the form

$$\varepsilon^2\Delta u - V(|x|)u + u^p = 0, \quad \text{in } \mathbb{R}^n, \quad u > 0, \quad u \in H^1(\mathbb{R}^n) \tag{1.11}$$

where $p > 1$.

As in [1], the following auxiliary potential

$$M(r) = r^{n-1}V^\theta(r), \quad \theta = \frac{p+1}{p-1} - \frac{1}{2}$$

determines the location of the clustered layers. We state the following result, whose proof is similar to that of Theorem 1.1, see also the computations in [1].

Theorem 1.2. *Assume that $M(r)$ has strict local maximum at $r = \bar{r}$ such that $M''(\bar{r}) < 0$. Then for $\varepsilon > 0$ sufficiently small, problem* (1.11) *admits a radially symmetric solution u_ε which concentrates on N spheres $\{|x| = r_j^\varepsilon\}$, $j = 1, \dots, N$, with*

$$|\bar{r} - r_1^\varepsilon| \sim \varepsilon \log\frac{1}{\varepsilon}, \qquad r_j^\varepsilon - r_{j-1}^\varepsilon \sim \varepsilon \log\frac{1}{\varepsilon}, \quad j = 2, \dots, N. \tag{1.12}$$

More precisely, we have $u_\varepsilon(r_j^\varepsilon) \to w(0)$, where $w(y)$ is the unique solution of (1.6) (*with $f(u) = u^p$*)*, and there exist two constants a and b such that*

$$u_\varepsilon \leqslant a e^{-b \min_{i=1,\dots,N} |r - r_i^\varepsilon|/\varepsilon}. \tag{1.13}$$

Clustered spikes have been shown to exist for (1.1) in the case of general domain Ω. For example, if the boundary mean curvature has a local minimum point P_0, then it is shown [15] that given arbitrary positive integer K, there exists a boundary spike solution with K spikes concentrating near P_0. Interior clustered layers are shown to exist for one-dimensional Allen–Cahn equation with inhomogeneous potential [21] and for bistable nonlinearities with inhomogeneous potential in a unit ball [12]. However, we are not aware of any result on boundary clustered layers in higher dimensions. This paper seems to be the first in dealing with such phenomenon in higher dimensions.

Our approach mainly relies upon a finite dimensional reduction procedure combined with a variational approach. Such a method has been used successfully in many papers, see e.g. [1,2,7,11,13,14]. In particular, we shall follow the one used in [13].

In the rest of section, we introduce some notations for later use.

By the following scaling $x = \varepsilon y$, problem (1.1) is reduced to the ODE

$$\begin{cases} u_{rr} + \frac{n-1}{r}u_r - u + f(u) = 0, & r \in (0, \frac{1}{\varepsilon}), \\ u'(0) = u'(\frac{1}{\varepsilon}) = 0, & u(r) > 0. \end{cases} \tag{1.14}$$

From now on, we shall work with (1.14).

Let $w(y)$ be the unique solution to (1.6). Its energy is given by

$$I[w] := \frac{1}{2}\int_{\mathbb{R}} |w'|^2 + \frac{1}{2}\int_{\mathbb{R}} w^2 - \int_{\mathbb{R}} F(w). \tag{1.15}$$

Set

$$\Omega_\varepsilon = \frac{1}{\varepsilon}B_1(0) = B_{\frac{1}{\varepsilon}}(0), \qquad I_\varepsilon = \left(0, \frac{1}{\varepsilon}\right). \tag{1.16}$$

For $u \in C^2(\Omega_\varepsilon)$ and $u = u(r)$, we have

$$\Delta u = u'' + \frac{n-1}{r}u'. \tag{1.17}$$

For $k \in \mathbb{N}$, we denote by $H_r^k(\Omega_\varepsilon)$ the space of radial functions in $H^k(\Omega_\varepsilon)$. On $H_r^1(\Omega_\varepsilon)$, we define an inner product as follows:

$$(u, v)_\varepsilon = \int_0^{\frac{1}{\varepsilon}} (u'v' + uv) r^{n-1}\, dr. \tag{1.18}$$

Similarly, the inner product on $L_r^2(\Omega_\varepsilon)$ can be defined by

$$\langle u, v\rangle_\varepsilon = \int_0^{\frac{1}{\varepsilon}} (uv) r^{n-1}\, dr. \tag{1.19}$$

We also introduce a new energy functional, which, up to a positive multiplicative constant, is equivalent to J_ε

$$\mathcal{E}_\varepsilon[u] = \frac{1}{2} \int_0^{\frac{1}{\varepsilon}} \big(|u'|^2 + u^2\big) r^{n-1} - \int_0^{\frac{1}{\varepsilon}} F(u) r^{n-1}\, dr, \quad u \in H_r^1(\Omega_\varepsilon). \tag{1.20}$$

Throughout this paper, unless otherwise stated, the letter C will always denote various generic constants which are independent of ε, for ε sufficiently small. The notation $A_\varepsilon \gg B_\varepsilon$ means that $\lim_{\varepsilon\to 0} |B_\varepsilon|/|A_\varepsilon| = 0$, while $A_\varepsilon \ll B_\varepsilon$ means $1/A_\varepsilon \gg 1/B_\varepsilon$.

2. Some preliminary analysis

In this section we introduce a family of approximate solutions to (1.14) and derive some useful estimates.

Let w be the unique solution of (1.6) (see assumption (f2)), and let $t \in (1/2, 1)$. We define

$$\rho_\varepsilon(t) = -w'\left(\frac{1-t}{\varepsilon}\right); \qquad \beta_\varepsilon(r) = e^{r-\frac{1}{\varepsilon}}, \quad r \in \left[0, \frac{1}{\varepsilon}\right], \tag{2.1}$$

and

$$w_{\varepsilon,t}(r) = \left(w\left(r - \frac{t}{\varepsilon}\right) + \rho_\varepsilon(t)\beta_\varepsilon(r)\right)\chi(\varepsilon r), \tag{2.2}$$

where $\chi(s)$ is a smooth cut-off function such that

$$\begin{cases} \chi(s) = 0 & \text{for } s \leqslant 1/8, \\ \chi(s) = 1 & \text{for } s \geqslant 1/4, \\ \chi(s) \in [0,1] & \text{for } s \in [1/8, 1/4]. \end{cases} \tag{2.3}$$

Using ODE analysis, it is standard to see that

$$\begin{cases} w(y) = A_0 e^{-y} + \mathrm{O}(e^{-(1+\sigma)y}); \\ w'(y) = -A_0 e^{-y} + \mathrm{O}(e^{-(1+\sigma)y}); \end{cases} \quad y \geqslant 0, \tag{2.4}$$

where $A_0 > 0$ is a fixed constant and $\sigma > 0$ is given in (f1). This implies that for $\frac{1-t}{\varepsilon} \gg 1$

$$\rho_\varepsilon(t) = A_0 e^{-\frac{1-t}{\varepsilon}} + \mathrm{O}(e^{-(1+\sigma)\frac{1-t}{\varepsilon}}). \tag{2.5}$$

Note that for $r \leqslant \frac{1}{4\varepsilon}$, we have

$$\big|w_{\varepsilon,t}(r)\big| + \big|w'_{\varepsilon,t}(r)\big| + \big|w''_{\varepsilon,t}(r)\big| \leqslant e^{-\frac{1}{4\varepsilon}}. \tag{2.6}$$

Observe also that, by construction, $w_{\varepsilon,t}$ satisfies the Neumann boundary condition, i.e., $w'_{\varepsilon,t}(1/\varepsilon) = 0$. Furthermore, $w_{\varepsilon,t}$ depends smoothly on t as a map with values in $C^2([0, 1/\varepsilon])$.

For simplicity of notation, let

$$w_t(r) := w\left(r - \frac{t}{\varepsilon}\right). \tag{2.7}$$

For $u \in H_r^2(\Omega_\varepsilon)$, we define the operator

$$\mathcal{S}_\varepsilon[u] := u_{rr} + \frac{n-1}{r} u_r - u + f(u). \tag{2.8}$$

We introduce the following set

$$\Lambda = \left\{ \mathbf{t} = (t_1, \ldots, t_N) \;\middle|\; \begin{array}{l} t_N > \frac{1}{2},\, 1 - t_1 \geqslant \eta\varepsilon \log \frac{1}{\varepsilon}, \\ t_{j-1} - t_j > \eta\varepsilon \log \frac{1}{\varepsilon},\ j = 2, \ldots, N \end{array} \right\}, \tag{2.9}$$

where $\eta \in (0, 1/8)$ is a fixed number.

For $\mathbf{t} \in \Lambda$, we define

$$w_{\varepsilon,\mathbf{t}}(r) = \sum_{j=1}^{N} w_{\varepsilon,t_j}(r). \tag{2.10}$$

Before studying the properties of $w_{\varepsilon,\mathbf{t}}$, we need a preliminary lemma.

Lemma 2.1. *For $|t - s| \gg 1$ and $\alpha > \beta > 0$, there holds*

$$w^\alpha(y - t) w^\beta(y - s) = \mathrm{O}\big(w^\beta\big(|t - s|\big)\big), \tag{2.11}$$

$$\int_{\mathbb{R}} w^\alpha(y - t) w^\beta(y - s)\, dy = \big(1 + \mathrm{o}(1)\big) w^\beta\big(|t - s|\big) \int_{\mathbb{R}} w^\alpha(y) e^{-\beta y}\, dy, \tag{2.12}$$

where $\mathrm{o}(1) \to 0$ as $|t - s| \to +\infty$.

Proof. Let $y - t = z$. Formula (2.4) implies

$$w^\beta(y - s) = w^\beta(z + t - s) = A_0 e^{-\beta|z+t-s|} + \mathrm{O}(e^{-\beta(1+\sigma)|z+t-s|}).$$

Multiplying by $w^\alpha(y - t)$ and using the fact that $\beta|z + t - s| + \alpha|z| > \beta|t - s|$, we obtain (2.11).

Regarding the second inequality, assuming $t > s$, we can write

$$\int_{\mathbb{R}} w^\alpha(y - t) w^\beta(y - s)\, dy = A_0 e^{-\beta(t-s)} \int_{\mathbb{R}} w^\alpha(z) e^{-\beta z}\, dz + \int_{\mathbb{R}} w^\alpha(z)\big[w^\beta(z + t - s) - A_0 e^{-\beta(z+t-s)}\big]\, dz.$$

Then the conclusion follows from Lebesgue's Dominated Convergence Theorem. The case $s > t$ can be handled similarly. □

Lemma 2.2. *For ε sufficiently small and $\mathbf{t} \in \Lambda$, we have*

$$\big\|\mathcal{S}_\varepsilon[w_{\varepsilon,\mathbf{t}}]\big\|_{L^\infty} + \varepsilon^{n-1} \int_0^{\frac{1}{\varepsilon}} \big|\mathcal{S}_\varepsilon[w_{\varepsilon,\mathbf{t}}]\big| r^{n-1}\, dr \leqslant C\left[\varepsilon + \sum_{j=1}^{N} \big(\rho_\varepsilon(t_j)\big)^{1+\frac{\sigma}{2}} + \sum_{i \neq j} e^{-\frac{\tau|t_i - t_j|}{\varepsilon}}\right], \tag{2.13}$$

where τ satisfies $\frac{1}{2} < \tau < \frac{1+\sigma}{2}$.

Proof. Using (1.6) it is easy to see that

$$\mathcal{S}_\varepsilon[w_{\varepsilon,\mathbf{t}}] = \frac{n-1}{r} w'_{\varepsilon,\mathbf{t}} + f(w_{\varepsilon,\mathbf{t}}) - \sum_{j=1}^{N} f(w_{t_j}) + \mathrm{O}(e^{-\frac{1}{C\varepsilon}}). \tag{2.14}$$

The first term in right hand side of (2.14) can be estimated as follows

$$\frac{1}{r} w'_{\varepsilon,\mathbf{t}} = \frac{1}{r}\sum_{j=1}^{N}\big(w'_{t_j} + \rho_\varepsilon(t_j)\beta_\varepsilon(r)\big) + \mathrm{O}(e^{-\frac{1}{C\varepsilon}}).$$

From the decay of w and β_ε we deduce that

$$\left\|\frac{1}{r} w'_{\varepsilon,\mathbf{t}}\right\|_\infty + \int_0^{\frac{1}{\varepsilon}} \frac{1}{r}|w'_{\varepsilon,\mathbf{t}}| r^{n-1}\,dr \leqslant C\varepsilon. \tag{2.15}$$

Next, we note that

$$\left| f(w_{\varepsilon,\mathbf{t}}) - \sum_{j=1}^{N} f(w_{t_j})\right| \leqslant S_1 + S_2,$$

where

$$S_1 = \left| f\left(\sum_{j=1}^{N} w_{t_j}\right) - \sum_{j=1}^{N} f(w_{t_j})\right|; \qquad S_2 = \left| f\left(\sum_{j=1}^{N} w_{\varepsilon,t_j}\right) - f\left(\sum_{j=1}^{N} w_{t_j}\right)\right|.$$

To estimate S_1 and S_2, we divide the domain $I_\varepsilon = (0, 1/\varepsilon)$ into the N intervals $I_1, \ldots, I_N$ defined by

$$I_1 = \left[\frac{t_1+t_2}{2\varepsilon}, \frac{1}{\varepsilon}\right), \qquad I_j = \left[\frac{t_j+t_{j+1}}{2\varepsilon}, \frac{t_j+t_{j-1}}{2\varepsilon}\right), \qquad j = 2, \ldots, N-1,$$
$$I_N = \left(0, \frac{t_N + t_{N-1}}{2\varepsilon}\right). \tag{2.16}$$

Then we have

$$|w_{t_i}| \leqslant |w_{t_j}| \quad \text{on } I_j \text{ for } i \neq j. \tag{2.17}$$

We need to apply the following inequality, which can be proved with elementary calculus. Fixed a constant $L > 0$, for each $0 < \tau \leqslant 1$ and for any $\delta \in (0,1)$ there exists a constant $C = C(L, \tau, \delta)$ such that

$$\big|f(x+y) - f(x)\big| \leqslant C|x|^{1+\sigma-\tau}|y|^\tau, \quad \text{provided } \delta|y| < |x| < L. \tag{2.18}$$

Using (f1), (2.17), (2.18) and recalling that $\tau < \frac{1+\sigma}{2}$, we deduce that for $r \in I_1$ there holds

$$S_1 \leqslant C\sum_{j\neq 1}|w_{t_1}|^{1+\sigma-\tau} w_{t_j}^\tau + C\sum_{j\neq 1} w_{t_j}^{1+\sigma} \leqslant C\sum_{j\neq 1}|w_{t_1}|^{1+\sigma-\tau} w_{t_j}^\tau.$$

From (2.11) and (2.12) it follows that

$$\|S_1\|_{L^\infty(I_1)} + \varepsilon^{n-1}\int_{I_1}\big|S_1(r)\big| r^{n-1}\,dr \leqslant C\sum_{j\neq 1} e^{-\tau|t_1-t_j|/\varepsilon}. \tag{2.19}$$

On the other hand, using (2.18) with $\tau = 1$ and the inequality $w_{t_1}^{\sigma/2}(r)\beta_\varepsilon(r) \leqslant C\rho_\varepsilon^{\sigma/2}(t_1)$, we find

$$S_2 \leqslant C w_{t_1}^{\sigma} \left(\sum_{j=1}^{N} \rho_\varepsilon(t_j) \right) \beta_\varepsilon(r) \leqslant C \left(\sum_{j=1}^{N} \rho_\varepsilon(t_j) \right) \bigl(w_{t_1}^{\frac{\sigma}{2}} \beta_\varepsilon(r) \bigr) w_{t_1}^{\frac{\sigma}{2}}$$
$$\leqslant C \left(\sum_{j=1}^{N} \rho_\varepsilon(t_j) \right) \bigl(\rho_\varepsilon(t_1)\bigr)^{\frac{\sigma}{2}} w_{t_1}^{\frac{\sigma}{2}} \leqslant C \bigl(\rho_\varepsilon(t_1)\bigr)^{1+\frac{\sigma}{2}} w_{t_1}^{\frac{\sigma}{2}}.$$

Hence we also get

$$\|S_2\|_{L^\infty(I_1)} + \varepsilon^{n-1} \int_{I_1} \bigl|S_2(r)\bigr| r^{n-1}\, dr \leqslant C \rho_\varepsilon(t_1)^{1+\frac{\sigma}{2}}. \tag{2.20}$$

For $r \in I_j$, $j = 2, \ldots, N-1$, we can estimate similarly

$$S_1 \leqslant C \sum_{i \neq j} |w_{t_j}|^{1+\sigma-\tau} w_{t_i}^{\tau}, \tag{2.21}$$

and

$$S_2 \leqslant C w_{t_j}^{\sigma} \left(\sum_{i=1}^{N} \rho_\varepsilon(t_i) \right) \beta_\varepsilon(r) \leqslant C \bigl(\rho_\varepsilon(t_1)\bigr)^{1+\frac{\sigma}{2}} w_{t_j}^{\frac{\sigma}{2}} \tag{2.22}$$

since $w_{t_j}^{\sigma/2}(r)\beta_\varepsilon(r) \leqslant \rho_\varepsilon^{\sigma/2}(t_j) \leqslant \rho_\varepsilon^{\sigma/2}(t_1)$. On I_N one can use similar estimates and, adding an error term of order $e^{-1/(C\varepsilon)}$. Combining (2.15), (2.19), (2.20) and similar estimates on I_j, $j \geqslant 2$, we obtain (2.13). □

The proof of the next lemma is postponed to Appendix A.

Lemma 2.3. *Let* $\mathbf{t} \in \Lambda$. *Then for* ε *sufficiently small we have*

$$\mathcal{E}_\varepsilon \left[\sum_{j=1}^{N} w_{\varepsilon,t_j} \right] = \sum_{i=1}^{N} \left(\frac{t_i}{\varepsilon} \right)^{n-1} \bigl\{ I[w] - \bigl(A_0^2 + \mathrm{o}(1)\bigr) e^{-2\frac{1-t_i}{\varepsilon}} \bigr\}$$
$$- \sum_{i \neq j} \left(\frac{t_i}{\varepsilon} \right)^{n-1} \bigl(A_0^2 + \mathrm{o}(1)\bigr) e^{-\frac{|t_i - t_j|}{\varepsilon}} + \mathrm{O}(\varepsilon^{2-n}), \tag{2.23}$$

where $A_0 > 0$ *is defined in* (2.4).

3. An auxiliary linear problem

In this section we study a linear theory which allows us to perform the finite-dimensional reduction procedure.

Fix $\mathbf{t} \in \Lambda$. Integrating by parts, one can show that orthogonality to $\frac{\partial w_{\varepsilon,t_j}}{\partial t_j}$ in $H_r^1(\Omega_\varepsilon)$, $j = 1, \ldots, N$, is equivalent to orthogonality in $L^2(\Omega_\varepsilon)$ to the following functions

$$Z_{\varepsilon,t_j} = \Delta \left(\frac{\partial w_{\varepsilon,t_j}}{\partial t_j} \right) - \frac{\partial w_{\varepsilon,t_j}}{\partial t_j}, \quad j = 1, \ldots, N. \tag{3.1}$$

By elementary computations, we obtain

$$\frac{\partial w_{\varepsilon,t_j}}{\partial t_j} = -\frac{1}{\varepsilon} w' \left(r - \frac{t_j}{\varepsilon} \right) + \frac{1}{\varepsilon} w'' \left(\frac{1-t_j}{\varepsilon} \right) \beta_\varepsilon(r) + \mathrm{O}(e^{-\frac{1}{C\varepsilon}}), \tag{3.2}$$

$$Z_{\varepsilon,t_j} = -f'(w_{t_j})\frac{\partial w_{\varepsilon,t_j}}{\partial t_j} - \frac{n-1}{r}\left(\frac{\partial w_{\varepsilon,t_j}}{\partial t_j}\right)' + \mathrm{o}\left(\frac{1}{\varepsilon}\right) = -\frac{1}{\varepsilon} w'_{t_j} f'(w_{t_j}) + \mathrm{o}\left(\frac{1}{\varepsilon}\right), \tag{3.3}$$

where $\mathrm{O}(e^{-1/(C\varepsilon)})$ and $\mathrm{o}(1/\varepsilon)$ are intended both in the C^1 and H_r^1 sense.

In this section, we consider the following linear problem. Given $h \in L^\infty(\Omega_\varepsilon)$, find a function ϕ satisfying

$$\begin{cases} L_\varepsilon[\phi] := \phi'' + \frac{n-1}{r}\phi' - \phi + f'(w_{\varepsilon,\mathbf{t}})\phi = h + \sum_{j=1}^N c_j Z_{\varepsilon,t_j}; \\ \phi'(0) = \phi'(\frac{1}{\varepsilon}) = 0; \quad \langle \phi, Z_{\varepsilon,t_j}\rangle_\varepsilon = 0, \quad j = 1,\dots,N, \end{cases} \tag{3.4}$$

for some constants c_j, $j = 1,\dots,N$. To this purpose, define the norm

$$\|\phi\|_* = \sup_{r\in(0,\frac{1}{\varepsilon})} \left|\phi(r)\right|. \tag{3.5}$$

We have the following result.

Proposition 3.1. *Let ϕ satisfy* (3.4). *Then for ε sufficiently small, we have*

$$\|\phi\|_* \leqslant C\|h\|_* \tag{3.6}$$

where C is a positive constant independent of ε and $\mathbf{t} \in \Lambda$.

Proof. Arguing by contradiction, assume that

$$\|\phi\|_* = 1; \qquad \|h\|_* = \mathrm{o}(1). \tag{3.7}$$

We multiply (3.4) by $\frac{\partial w_{\varepsilon,t_j}}{\partial t_j}$ and integrate over Ω_ε to obtain

$$\sum_{i=1}^N c_i \left\langle Z_{\varepsilon,t_i}, \frac{\partial w_{\varepsilon,t_j}}{\partial t_j}\right\rangle_\varepsilon = -\left\langle h, \frac{\partial w_{\varepsilon,t_j}}{\partial t_j}\right\rangle_\varepsilon + \left\langle \Delta\phi - \phi + f'(w_{\varepsilon,\mathbf{t}})\phi, \frac{\partial w_{\varepsilon,t_j}}{\partial t_j}\right\rangle_\varepsilon. \tag{3.8}$$

From the exponential decay of w one finds

$$\left\langle h, \frac{\partial w_{\varepsilon,t_j}}{\partial t_j}\right\rangle_\varepsilon = \int_0^{\frac{1}{\varepsilon}} h \frac{\partial w_{\varepsilon,t_j}}{\partial t_j} r^{n-1}\,dr = \mathrm{O}\big(\|h\|_*\varepsilon^{-n}\big).$$

Moreover, integrating by parts, using (3.2) and (3.3) we deduce

$$\left\langle \Delta\phi - \phi + f'(w_{\varepsilon,\mathbf{t}})\phi, \frac{\partial w_{\varepsilon,t_j}}{\partial t_j}\right\rangle_\varepsilon = \left\langle Z_{\varepsilon,t_j} + f'(w_{\varepsilon,\mathbf{t}})\frac{\partial w_{\varepsilon,t_j}}{\partial t_j}, \phi\right\rangle_\varepsilon = \mathrm{o}\big(\varepsilon^{-n}\|\phi\|_*\big).$$

From (3.2) and (3.3), we see that

$$\left\langle Z_{\varepsilon,t_i}, \frac{\partial w_{\varepsilon,t_j}}{\partial t_j}\right\rangle_\varepsilon = -\varepsilon^{-n-1}\left(t_i^{n-1}\delta_{ij}\int_{\mathbb{R}} f'(w)(w')^2 + \mathrm{o}(1)\right), \tag{3.9}$$

where δ_{ij} denotes the Kronecker symbol. Note that, using the equation $w''' - w' + f'(w)w' = 0$ we find

$$\int_{\mathbb{R}} f'(w)(w')^2 = \int_{\mathbb{R}} \big((w'')^2 + (w')^2\big) > 0.$$

This shows that the left hand side of Eq. (3.8) is diagonally dominant in the indexes i, j, and hence by (3.7) we have

$$c_i = \mathrm{O}\big(\varepsilon\|h\|_*\big) + \mathrm{o}\big(\varepsilon\|\phi\|_*\big) = \mathrm{o}(\varepsilon), \quad i = 1,\dots,N. \tag{3.10}$$

Also, since we are assuming that $\|h\|_* = \mathrm{o}(1)$ and since $\|Z_{\varepsilon,t_j}\|_* = \mathrm{O}(1/\varepsilon)$, there holds

$$\left\| h + \sum_{j=1}^{N} c_j Z_{\varepsilon,t_j} \right\|_* = \mathrm{o}(1). \tag{3.11}$$

Thus (3.4) yields

$$\begin{cases} \phi'' + \frac{n-1}{r}\phi' - \phi + f'(w_{\varepsilon,\mathbf{t}})\phi = \mathrm{o}(1); \\ \phi'(0) = \phi'(\frac{1}{\varepsilon}) = 0; \quad \langle \phi, Z_{\varepsilon,t_j}\rangle_\varepsilon = 0, \quad j = 1, \dots, N. \end{cases} \tag{3.12}$$

We show that (3.12) is incompatible with our assumption $\|\phi\|_* = 1$. First we claim that, fixed $R > 0$, there holds

$$|\phi| \to 0 \quad \text{on} \quad y \in \bigcup_{j=1}^{N} \left(\frac{t_j}{\varepsilon} - R, \frac{t_j}{\varepsilon} + R \right) \quad \text{as } \varepsilon \to 0, \tag{3.13}$$

where R is any fixed positive constant.

Indeed, assuming the contrary, there exist $\delta_0 > 0$, $j \in \{1, \dots, N\}$ and sequences $\varepsilon_k, \phi_k, y_k \in (t_j/\varepsilon - R, t_j/\varepsilon + R)$ such that ϕ_k satisfies (3.4) and

$$\left|\phi_k(y_k)\right| \geqslant \delta_0. \tag{3.14}$$

Let $\tilde{\phi}_k = \phi_k(y - t_j/\varepsilon_k)$. Then using (3.12) and $\|\phi\|_* = 1$, as $\varepsilon_k \to 0$ $\tilde{\phi}_k$ converges weakly in $H^2_{\mathrm{loc}}(R)$ and strongly in $C^1_{\mathrm{loc}}(R)$ to a bounded function ϕ_0 which satisfies

$$\phi_0'' - \phi_0 + f'(w)\phi_0 = 0 \quad \text{in } \mathbb{R}.$$

Hence ϕ_0 must tend to zero at infinity and so, by (1.7), $\phi_0 = cw'$ for some c. Since $\tilde{\phi}_k \perp Z_{\varepsilon,t_j}$, we conclude that $\int_{\mathbb{R}} \phi_0 f'(w) w'(y) = 0$, which yields $c = 0$. Hence $\phi_0 = 0$ and $\tilde{\phi}_k \to 0$ in $B_{2R}(0)$. This contradicts (3.14), so (3.13) holds true.

Given $\delta > 0$, the decay of w and (3.13) (with R sufficiently large) imply

$$\left\| f'(w_{\varepsilon,\mathbf{t}})\phi \right\|_* \leqslant \delta + \frac{1}{2}\|\phi\|_*. \tag{3.15}$$

Using (3.12) and the Maximum Principle one finds

$$\|\phi\|_* \leqslant \left\| f'(w_{\varepsilon,\mathbf{t}})\phi \right\|_* + \sum_{j=1}^{N} |c_j| \|Z_{\varepsilon,t_j}\|_* + \|h\|_* \leqslant 2\delta + \frac{1}{2}\|\phi\|_*,$$

and hence

$$\|\phi\|_* \leqslant 4\delta < 1$$

if we choose $\delta < 1/4$. This contradicts (3.7). □

Proposition 3.2. *There exists $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ the following property holds true. Given $h \in L^\infty(\Omega_\varepsilon)$, there exists a unique pair $(\phi, c_1, \dots, c_N)$ such that*

$$L_\varepsilon[\phi] = h + \sum_{j=1}^{N} c_j Z_{\varepsilon,t_j}, \tag{3.16}$$

$$\phi'(0) = \phi'\left(\frac{1}{\varepsilon}\right) = 0; \qquad \langle \phi, Z_{\varepsilon,t_j}\rangle_\varepsilon = 0, \quad j = 1, \dots, N. \tag{3.17}$$

Moreover, letting $\mathbf{c} = \{c_j\}_{j=1,\dots,N}$*, we have*

$$\|\phi\|_* + \frac{1}{\varepsilon}|\mathbf{c}| \leqslant C\|h\|_* \tag{3.18}$$

for some positive constant C.

Proof. The bound in (3.18) follows from Proposition 3.1 and (3.10). Let us now prove the existence part. Set

$$\mathcal{H} = \left\{ u \in H^1(\Omega_\varepsilon) \;\middle|\; \left(u, \frac{\partial w_{\varepsilon,\mathbf{t}}}{\partial \mathbf{t}} \right)_\varepsilon = 0 \right\}.$$

Note that, integrating by parts, one has

$$\psi \in \mathcal{H} \quad \text{if and only if} \quad \langle \psi, Z_{\varepsilon,t_j} \rangle_\varepsilon = 0, \quad j = 1, \dots, N.$$

Observe that ϕ solves (3.16) and (3.17) if and only if $\phi \in \mathcal{H}$ satisfies

$$\int_{\Omega_\varepsilon} (\nabla\phi\nabla\psi + \phi\psi) - \left\langle f'(w_{\varepsilon,\mathbf{t}})\phi, \psi \right\rangle_\varepsilon = \langle h, \psi \rangle_\varepsilon, \quad \forall \psi \in \mathcal{H}.$$

This equation can be rewritten as

$$\phi + \mathcal{S}(\phi) = \bar{h} \quad \text{in } \mathcal{H}, \tag{3.19}$$

where $\bar{h}$ is defined by duality and $\mathcal{S} : \mathcal{H} \to \mathcal{H}$ is a linear compact operator.

Using Fredholm's alternative, showing that Eq. (3.19) has a unique solution for each $\bar{h}$, is equivalent to showing that the equation has a unique solution for $\bar{h} = 0$, which in turn follows from Proposition 3.1 and our proof is complete. □

In the following, if ϕ is the unique solution given in Proposition 3.2, we set

$$\phi = \mathcal{A}_\varepsilon(h). \tag{3.20}$$

Note that (3.18) implies

$$\left\|\mathcal{A}_\varepsilon(h)\right\|_* \leqslant C\|h\|_*. \tag{3.21}$$

4. Construction of a natural constraint

In this section we reduce problem (1.14) to a finite-dimensional one. Let M_ε be the N-dimensional manifold defined as

$$M_\varepsilon = \{w_{\varepsilon,\mathbf{t}}\colon \mathbf{t} \in \Lambda\}.$$

For ε small and for $\mathbf{t} \in \Lambda$, we are going to find a function $\phi_{\varepsilon,\mathbf{t}}$ satisfying the two conditions

$$\phi_{\varepsilon,\mathbf{t}} \perp_{H^1_r(\Omega_\varepsilon)} T_{w_{\varepsilon,\mathbf{t}}} M_\varepsilon; \qquad \mathcal{E}'_\varepsilon(w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}) \in T_{w_{\varepsilon,\mathbf{t}}} M_\varepsilon, \tag{4.1}$$

where $T_{w_{\varepsilon,\mathbf{t}}} M_\varepsilon$ denotes the tangential space of M_ε at $w_{\varepsilon,\mathbf{t}}$. This amounts to finding a function ϕ such that for some constants c_j, $j = 1, \dots, N$, the following equation holds true

$$\begin{cases} \Delta(w_{\varepsilon,\mathbf{t}} + \phi) - (w_{\varepsilon,\mathbf{t}} + \phi) + f(w_{\varepsilon,\mathbf{t}} + \phi) = \sum_{j=1}^N c_j Z_{\varepsilon,t_j} \quad \text{in } \Omega_\varepsilon, \\ \phi'(0) = \phi'(\frac{1}{\varepsilon}) = 0, \quad \langle \phi, Z_{\varepsilon,t_j} \rangle_\varepsilon = 0, \quad j = 1, \dots, N. \end{cases} \tag{4.2}$$

Letting

$$\tilde{M}_\varepsilon = \{w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}\colon\ \mathbf{t} \in \Lambda\}, \tag{4.3}$$

we will show that $\tilde{M}_\varepsilon$ is a natural constraint for $\mathcal{E}_\varepsilon$, in the sense that a critical point of $\mathcal{E}_\varepsilon|_{\tilde{M}_\varepsilon}$ is a true critical point of $\mathcal{E}_\varepsilon$.

The first equation in (4.2) can be written as

$$\phi'' + \frac{n-1}{r}\phi' - \phi + f'(w_{\varepsilon,\mathbf{t}})\phi = \big(-\mathcal{S}_\varepsilon[w_{\varepsilon,\mathbf{t}}]\big) + N_\varepsilon[\phi] + \sum_{j=1}^{N} c_j Z_{\varepsilon,t_j},$$

where

$$N_\varepsilon[\phi] = -\big[f(w_{\varepsilon,\mathbf{t}} + \phi) - f(w_{\varepsilon,\mathbf{t}}) - f'(w_{\varepsilon,\mathbf{t}})\phi\big]. \tag{4.4}$$

Lemma 4.1. *For* $\mathbf{t} \in \Lambda$ *and* ε *sufficiently small, we have for* $\|\phi\|_* + \|\phi_1\|_* + \|\phi_2\|_* \leqslant 1$,

$$\big\|N_\varepsilon[\phi]\big\|_* \leqslant C\|\phi\|_*^{1+\sigma}; \tag{4.5}$$

$$\big\|N_\varepsilon[\phi_1] - N_\varepsilon[\phi_2]\big\|_* \leqslant C\big(\|\phi_1\|_*^\sigma + \|\phi_2\|_*^\sigma\big)\|\phi_1 - \phi_2\|_*, \tag{4.6}$$

where σ *is defined in* (f1).

Proof. Inequality (4.5) follows from the mean-value theorem. In fact, for every point in $[0, 1/\varepsilon]$ there holds

$$f(w_{\varepsilon,\mathbf{t}} + \phi) - f(w_{\varepsilon,\mathbf{t}}) = f'(w_{\varepsilon,\mathbf{t}} + \theta\phi)\phi, \quad \theta \in [0,1].$$

Since f' is Holder continuous with exponent σ, we deduce

$$\big|f(w_{\varepsilon,\mathbf{t}} + \phi) - f(w_{\varepsilon,\mathbf{t}}) - f'(w_{\varepsilon,\mathbf{t}})\phi\big| \leqslant C|\phi|^{1+\sigma},$$

which implies (4.5). The proof of (4.6) goes along the same way. □

Proposition 4.2. *For* $\mathbf{t} \in \Lambda$ *and* ε *sufficiently small, there exists a unique* $\phi = \phi_{\varepsilon,\mathbf{t}}$ *such that* (4.2) *holds. Moreover,* $t \mapsto \phi_{\varepsilon,\mathbf{t}}$ *is of class* C^1 *as a map into* $H_r^1(\Omega_\varepsilon)$*, and we have*

$$\|\phi_{\varepsilon,\mathbf{t}}\|_* \leqslant C\left(\varepsilon + \sum_{j=1}^{N} e^{-(1+\frac{\sigma}{2})\frac{1-t_j}{\varepsilon}} + \sum_{i\neq j} e^{-\frac{\tau|t_i-t_j|}{\varepsilon}}\right), \tag{4.7}$$

where $\tau \in (\frac{1}{2}, \frac{1+\sigma}{2})$.

Proof. Let $\mathcal{A}_\varepsilon$ be as defined in (3.20). Then (4.2) can be written as

$$\phi = \mathcal{A}_\varepsilon\big[\big(-\mathcal{S}_\varepsilon[w_{\varepsilon,\mathbf{t}}]\big) + N_\varepsilon[\phi]\big]. \tag{4.8}$$

Let r be a positive (large) number, and set

$$\mathcal{F}_r = \left\{\phi \in H_r^1(\Omega_\varepsilon)\colon\ \|\phi\|_* \leqslant r\left(\varepsilon + \sum_{j=1}^{N} e^{-(1+\frac{\sigma}{2})\frac{1-t_j}{\varepsilon}} + \sum_{i\neq j} e^{-\frac{\tau|t_i-t_j|}{\varepsilon}}\right)\right\}.$$

Define now the map $\mathcal{B}_\varepsilon : \mathcal{F}_r \to H_r^1(\Omega_\varepsilon)$ as

$$\mathcal{B}_\varepsilon(\phi) = \mathcal{A}_\varepsilon\big[\big(-\mathcal{S}_\varepsilon[w_{\varepsilon,\mathbf{t}}]\big) + N_\varepsilon[\phi]\big].$$

Solving (4.2) is equivalent to finding a fixed point for $\mathcal{B}_\varepsilon$. By Lemma 4.1, for ε sufficiently small and r large we have

$$\|\mathcal{B}_\varepsilon[\phi]\|_* \leqslant C\|\mathcal{S}_\varepsilon[w_{\varepsilon,\mathbf{t}}]\|_* + C\|N_\varepsilon[\phi]\|_* < r\left(\varepsilon + \sum_{j=1}^{n} e^{-(1+\frac{\sigma}{2})\frac{1-t_j}{\varepsilon}} + \sum_{i\neq j} e^{-\frac{\tau|t_i-t_j|}{\varepsilon}}\right);$$

$$\|\mathcal{B}_\varepsilon[\phi_1] - \mathcal{B}_\varepsilon[\phi_2]\|_* \leqslant C\|N_\varepsilon[\phi_1] - N_\varepsilon[\phi_2]\|_* < \frac{1}{2}\|\phi_1 - \phi_2\|_*,$$

which shows that $\mathcal{B}_\varepsilon$ is a contraction mapping on $\mathcal{F}_r$. Hence there exists a unique $\phi \in \mathcal{F}_r$ such that (4.2) holds.

Now we come to the differentiability of ϕ. Consider the following map $H_\varepsilon : \Lambda \times H_r^1(\Omega_\varepsilon) \times \mathbb{R}^N \to H_r^1(\Omega_\varepsilon) \times \mathbb{R}^N$ of class C^1

$$H_\varepsilon(\mathbf{t}, \phi, \mathbf{c}) = \begin{pmatrix} (\Delta - 1)^{-1}(S_\varepsilon[w_{\varepsilon,\mathbf{t}} + \phi]) - \sum_{j=1}^N c_j \frac{\partial w_{\varepsilon,\mathbf{t}}}{\partial t_j} \\ (\phi, \frac{\partial w_{\varepsilon,\mathbf{t}}}{\partial t_1})_\varepsilon \\ \vdots \\ (\phi, \frac{\partial w_{\varepsilon,\mathbf{t}}}{\partial t_N})_\varepsilon \end{pmatrix}, \tag{4.9}$$

where $v = (\Delta - 1)^{-1}(u)$ is defined as the unique solution of

$$v'' + \frac{n-1}{r}v' - v = u, \quad v'(0) = v'\left(\frac{1}{\varepsilon}\right) = 0.$$

Eq. (4.2) is equivalent to $H_\varepsilon(\mathbf{t}, \phi, \mathbf{c}) = 0$. We know that, given $\mathbf{t} \in \Lambda$, there is a unique local solution $\phi_{\varepsilon,\mathbf{t}}, c_{\varepsilon,\mathbf{t}}$ obtained with the above procedure. We prove that the linear operator

$$\left.\frac{\partial H_\varepsilon(\mathbf{t}, \phi, \mathbf{c})}{\partial(\phi, \mathbf{c})}\right|_{(\mathbf{t}, \phi_{\varepsilon,\mathbf{t}}, \mathbf{c}_{\varepsilon,\mathbf{t}})} : H_r^1(\Omega_\varepsilon) \times \mathbb{R}^N \to H_r^1(\Omega_\varepsilon) \times \mathbb{R}^N$$

is invertible for ε small. Then the C^1-regularity of $t \mapsto (\phi_{\varepsilon,\mathbf{t}}, c_{\varepsilon,\mathbf{t}})$ follows from the Implicit Function Theorem. Indeed we have

$$\left.\frac{\partial H_\varepsilon(\mathbf{t}, \phi, \mathbf{c})}{\partial(\phi, \mathbf{c})}\right|_{(\mathbf{t}, \phi_{\varepsilon,\mathbf{t}}, \mathbf{c}_{\varepsilon,\mathbf{t}})}[\psi, \mathbf{d}] = \begin{pmatrix} (\Delta - 1)^{-1}(S'[w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}](\psi)) - \sum_{j=1}^N d_j \frac{\partial w_{\varepsilon,\mathbf{t}}}{\partial t_j} \\ (\psi, \frac{\partial w_{\varepsilon,\mathbf{t}}}{\partial t_1})_\varepsilon \\ \vdots \\ (\psi, \frac{\partial w_{\varepsilon,\mathbf{t}}}{\partial t_N})_\varepsilon \end{pmatrix}.$$

Since $\|\phi_{\varepsilon,\mathbf{t}}\|_*$ is small, the same proof as in that of Proposition 3.1 shows that $\frac{\partial H_\varepsilon(\mathbf{t},\phi,\mathbf{c})}{\partial(\phi,\mathbf{c})}|_{(\mathbf{t},\phi_{\varepsilon,\mathbf{t}},\mathbf{c}_{\varepsilon,\mathbf{t}})}$ is invertible for ε small.

This concludes the proof of Proposition 4.2. □

5. Energy computation for reduced energy functional

In this section we expand the quantity

$$\mathcal{K}_\varepsilon(\mathbf{t}) := \varepsilon^{n-1}\mathcal{E}_\varepsilon[w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}] : \Lambda \to R \tag{5.1}$$

in ε and $\mathbf{t}$, where $\phi_{\varepsilon,\mathbf{t}}$ is given by Proposition 4.2.

Lemma 5.1. *For $\mathbf{t} \in \Lambda$ and ε sufficiently small, we have*

$$\begin{aligned}\mathcal{K}_\varepsilon(\mathbf{t}) &= \varepsilon^{n-1}\mathcal{E}_\varepsilon[w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}] \\ &= I[w]\sum_{j=1}^N t_j^{n-1} - \left(A_0^2 + o(1)\right)\sum_{j=1}^N t_j^{n-1} e^{-2\frac{1-t_j}{\varepsilon}} - \left(A_0^2 + o(1)\right)\sum_{i\neq j} t_i^{n-1} e^{-\frac{|t_i-t_j|}{\varepsilon}} + O(\varepsilon).\end{aligned} \tag{5.2}$$

Proof. It is sufficient to show that

$$\mathcal{K}_\varepsilon(\mathbf{t}) = \varepsilon^{n-1}\mathcal{E}_\varepsilon[w_{\varepsilon,\mathbf{t}}] + \mathrm{o}\left(\sum_{j=1}^{N} e^{-2\frac{1-t_j}{\varepsilon}} + \sum_{i\neq j} e^{-\frac{|t_i-t_j|}{\varepsilon}}\right) + \mathrm{O}(\varepsilon),$$

and to apply Lemma 2.3. To this end, we write

$$\varepsilon^{1-n}\mathcal{K}_\varepsilon(\mathbf{t}) = \mathcal{E}_\varepsilon[w_{\varepsilon,\mathbf{t}}] + K_1 + K_2 - K_3,$$

where

$$K_1 = \int_0^{\frac{1}{\varepsilon}} \left[w'_{\varepsilon,\mathbf{t}}\phi'_{\varepsilon,\mathbf{t}} + w_{\varepsilon,\mathbf{t}}\phi_{\varepsilon,\mathbf{t}} - f(w_{\varepsilon,\mathbf{t}})\phi_{\varepsilon,\mathbf{t}}\right] r^{n-1}\,dr;$$

$$K_2 = \frac{1}{2}\int_0^{\frac{1}{\varepsilon}} \left[|\phi'_{\varepsilon,\mathbf{t}}|^2 + |\phi_{\varepsilon,\mathbf{t}}|^2 - f'(w_{\varepsilon,\mathbf{t}})\phi^2_{\varepsilon,\mathbf{t}}\right] r^{n-1}\,dr;$$

$$K_3 = \int_0^{\frac{1}{\varepsilon}} \left[F(w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}) - F(w_{\varepsilon,\mathbf{t}}) - f(w_{\varepsilon,\mathbf{t}})\phi_{\varepsilon,\mathbf{t}} - \frac{1}{2} f'(w_{\varepsilon,\mathbf{t}})\phi^2_{\varepsilon,\mathbf{t}}\right] r^{n-1}\,dr.$$

Integrating by parts and using Lemmas 2.2 and 4.2, we have

$$\begin{aligned} |K_1| &= \left|\int_0^{\frac{1}{\varepsilon}} \mathcal{S}_\varepsilon[w_{\varepsilon,\mathbf{t}}]\phi_{\varepsilon,\mathbf{t}} r^{n-1}\,dr\right| \leqslant C\|\phi_{\varepsilon,\mathbf{t}}\|_* \int_0^{\frac{1}{\varepsilon}} \left|\mathcal{S}_\varepsilon[w_{\varepsilon,\mathbf{t}}]\right| r^{n-1}\,dr \\ &\leqslant C\varepsilon^{1-n}\left(\varepsilon^2 + \sum_{j=1}^{N}\big(\rho_\varepsilon(t_j)\big)^{2+\sigma} + \sum_{i\neq j} e^{-2\tau|t_i-t_j|/\varepsilon}\right). \end{aligned} \tag{5.3}$$

To estimate K_2, we note that $\phi_{\varepsilon,\mathbf{t}}$ satisfies

$$\Delta\phi_{\varepsilon,\mathbf{t}} - \phi_{\varepsilon,\mathbf{t}} + f(w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}) - f(w_{\varepsilon,\mathbf{t}}) + \mathcal{S}_\varepsilon[w_{\varepsilon,\mathbf{t}}] = \sum_{j=1}^{N} c_j Z_{\varepsilon,t_j}. \tag{5.4}$$

Multiplying (5.4) by $\phi_{\varepsilon,\mathbf{t}} r^{n-1}$ and integrating over I_ε, we obtain

$$\begin{aligned} \int_{I_\varepsilon} \mathcal{S}_\varepsilon[w_{\varepsilon,\mathbf{t}}]\phi_{\varepsilon,\mathbf{t}} r^{n-1}\,dr &= \int_{I_\varepsilon} \big(|\phi'_{\varepsilon,\mathbf{t}}|^2 + |\phi_{\varepsilon,\mathbf{t}}|^2 - f'(w_{\varepsilon,\mathbf{t}})\phi^2_{\varepsilon,\mathbf{t}}\big) r^{n-1}\,dr \\ &\quad + \int_{I_\varepsilon} \left[f(w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}) - f(w_{\varepsilon,\mathbf{t}}) - f'(w_{\varepsilon,\mathbf{t}})\phi_{\varepsilon,\mathbf{t}}\right]\phi_{\varepsilon,\mathbf{t}} r^{n-1}\,dr. \end{aligned} \tag{5.5}$$

Hence we find

$$2K_2 = -\int_{I_\varepsilon} \left[f(w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}) - f(w_{\varepsilon,\mathbf{t}}) - f'(w_{\varepsilon,\mathbf{t}})\phi_{\varepsilon,\mathbf{t}}\right]\phi_{\varepsilon,\mathbf{t}} r^{n-1}\,dr + \int_{I_\varepsilon} \mathcal{S}_\varepsilon[w_{\varepsilon,\mathbf{t}}]\phi_{\varepsilon,\mathbf{t}} r^{n-1}\,dr.$$

From the Taylor's formula, we get

$$\left|f(w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}) - f(w_{\varepsilon,\mathbf{t}}) - f'(w_{\varepsilon,\mathbf{t}})\phi_{\varepsilon,\mathbf{t}}\right| \leqslant C|\phi_{\varepsilon,\mathbf{t}}|^{1+\sigma},$$

so we deduce

$$|K_2| \leqslant C \int_{I_\varepsilon} |\phi_{\varepsilon,\mathbf{t}}|^{2+\sigma} r^{n-1}\, dr + C\|\phi_{\varepsilon,\mathbf{t}}\|_* \int_{I_\varepsilon} \left|\mathcal{S}_\varepsilon[w_{\varepsilon,\mathbf{t}}]\right| r^{n-1}\, dr.$$

From the exponential decay of w and (3.10), one finds that $\phi_{\varepsilon,\mathbf{t}}(r)$ satisfies

$$\phi''_{\varepsilon,\mathbf{t}} + \frac{n-1}{r}\phi'_{\varepsilon,\mathbf{t}} - \phi_{\varepsilon,\mathbf{t}} + f(w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}) - f(w_{\varepsilon,\mathbf{t}}) = \mathrm{O}\left(\sum_{j=1}^N e^{-\frac{|r-t_j|}{\varepsilon}}\right), \quad \phi'_{\varepsilon,\mathbf{t}}(0) = \phi'_{\varepsilon,\mathbf{t}}\left(\frac{1}{\varepsilon}\right) = 0.$$

From (5.4) and a comparison principle, we obtain

$$\left|\phi_{\varepsilon,\mathbf{t}}(r)\right| \leqslant C \sum_{j=1}^N e^{-\frac{1}{C}\frac{|r-t_j|}{\varepsilon}} \tag{5.6}$$

for some $C < 1$.

Using (5.6), we get

$$|K_2| \leqslant C\varepsilon^{1-n}\left(\varepsilon^2 + \sum_{j=1}^N \left(\rho_\varepsilon(t_j)\right)^{2+\sigma} + \sum_{i\neq j} e^{-2\tau|t_i-t_j|/\varepsilon}\right). \tag{5.7}$$

From the Holder continuity of f' we deduce

$$\left|F(w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}) - F(w_{\varepsilon,\mathbf{t}}) - f(w_{\varepsilon,\mathbf{t}})\phi_{\varepsilon,\mathbf{t}} - \frac{1}{2} f'(w_{\varepsilon,\mathbf{t}})\phi^2_{\varepsilon,\mathbf{t}}\right| \leqslant C|\phi_{\varepsilon,\mathbf{t}}|^{2+\sigma},$$

so, again, it follows that

$$|K_3| \leqslant C\varepsilon^{1-n}\left(\varepsilon^2 + \sum_{j=1}^N \left(\rho_\varepsilon(t_j)\right)^{2+\sigma} + \sum_{i\neq j} e^{-2\tau|t_i-t_j|/\varepsilon}\right). \tag{5.8}$$

Combining with (2.23) of Lemma 2.2, we obtain the conclusion. □

6. Proof of Theorem 1.1

In this section we prove Theorem 1.1. Fix $\mathbf{t} \in \bar{\Lambda}$ and let $\phi_{\varepsilon,\mathbf{t}}$ be the solution given by Lemma 3.4. Let also $\mathcal{K}_\varepsilon(\mathbf{t})$ denote the reduced energy functional defined by (5.1).

Proposition 6.1. *For ε small, the following maximization problem*

$$\sup\left\{\mathcal{K}_\varepsilon(\mathbf{t}):\ \mathbf{t} \in \Lambda\right\} \tag{6.1}$$

has a solution $\mathbf{t}^\varepsilon$ in the interior of Λ.

Proof. Since $\mathcal{K}_\varepsilon(\mathbf{t})$ is continuous in $\mathbf{t}$, it achieves a maximum in $\bar{\Lambda}$. Let $\mathbf{t}^\varepsilon$ be a maximum point. We claim that $\mathbf{t}^\varepsilon \in \Lambda$.

Let us argue by contradiction and assume that $\mathbf{t}^\varepsilon \in \partial\Lambda$. Then from the definition of Λ, there are three possibilities: either $1 - t_1 = \eta\varepsilon \log\frac{1}{\varepsilon}$, or there exists $j \geqslant 2$ such that $|t_j - t_{j-1}| = \min_{i\neq j}|t_i - t_j| = \eta\varepsilon\log\frac{1}{\varepsilon}$, or $t_N = \frac{1}{2}$.

In the first case, we have

$$I[w]t_1^{n-1} - \left(A_0^2 + \mathrm{o}(1)\right)e^{-2\frac{1-t_1}{\varepsilon}} = I[w]\left(1 - \eta\varepsilon\log\frac{1}{\varepsilon}\right)^{n-1} - A_0^2 e^{-2\eta\log\frac{1}{\varepsilon}} + \mathrm{o}(\varepsilon^{2\eta}) \leqslant I[w] - \frac{A_0^2}{2}\varepsilon^{2\eta}.$$

Hence we obtain

$$\mathcal{K}_\varepsilon(\mathbf{t}^\varepsilon) \leqslant NI[w] - \frac{A_0^2}{2}\varepsilon^{2\eta} \tag{6.2}$$

since $\eta < \frac{1}{8}$.

In the second case, there holds

$$\mathcal{K}_\varepsilon(\mathbf{t}^\varepsilon) \leqslant I[w]\sum_{j=1}^N t_j^{n-1} - \big(A_0^2 + \mathrm{o}(1)\big)\sum_{j=1}^N e^{-2\frac{1-t_j}{\varepsilon}} t_j^{n-1} - A_0^2\varepsilon^\eta \leqslant NI[w] - \frac{A_0^2}{2}\varepsilon^\eta. \tag{6.3}$$

In the last case, we have

$$\mathcal{K}_\varepsilon(\mathbf{t}^\varepsilon) \leqslant I[w](N - 1 + t_N^{n-1}) + \mathrm{o}(1) \leqslant I[w]\left(N - 1 + \left(\frac{1}{2}\right)^{n-1}\right) + \mathrm{o}(1). \tag{6.4}$$

On the other hand, by choosing $t_j = 1 - j\varepsilon\log\frac{1}{\varepsilon}$, $j = 1, \dots, N$, we obtain

$$\sum_{j=1}^N t_j^{n-1} = 1 - \frac{N(N+1)(n-1)}{2}\varepsilon\log\frac{1}{\varepsilon} + \mathrm{O}\left(\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^2\right);$$

$$\rho_\varepsilon(t_1)\beta_\varepsilon(t_1) \sim e^{-2\log\frac{1}{\varepsilon}} \sim \varepsilon^2; \qquad e^{-\frac{|t_{j-1}-t_j|}{\varepsilon}} = \varepsilon, \tag{6.5}$$

and hence

$$\mathcal{K}_\varepsilon(\mathbf{t}^\varepsilon) \geqslant NI[w] - \frac{N(N+1)(n-1)^2}{2}\varepsilon\log\frac{1}{\varepsilon} + \mathrm{O}(\varepsilon)$$

which contradicts (6.2), or (6.3), or (6.4). This completes the proof of Proposition 6.1. □

Proof of Theorem 1.1. By Proposition 4.2, there exists ε_N such that for $\varepsilon < \varepsilon_N$ we have a C^1 map $\mathbf{t} \mapsto \phi_{\varepsilon,\mathbf{t}}$ from $\bar{\Lambda}$ into $C^2(I_\varepsilon)$ such that

$$\mathcal{S}_\varepsilon[w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}] = \sum_{j=1}^N c_j Z_{\varepsilon,t_j} \tag{6.6}$$

for some constants $c_j \in \mathbb{R}$, which also are of class C^1 in $\mathbf{t}$.

By Proposition 6.1, there exists $\mathbf{t}^\varepsilon \in \Lambda$ achieving the maximum of $\mathcal{K}_\varepsilon : t \to \mathcal{E}_\varepsilon[w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}]$. Let $u_\varepsilon = \sum_{i=1}^N w_{\varepsilon,t_i} + \phi_{\varepsilon,\mathbf{t}} = w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}$. Then we have

$$\partial_{t_i}|_{\mathbf{t}=\mathbf{t}^\varepsilon}\mathcal{K}_\varepsilon(\mathbf{t}^\varepsilon) = 0, \quad i = 1, \dots, N,$$

and hence

$$\int_{I_\varepsilon} \big[\nabla u_\varepsilon \nabla\partial_{t_i}(w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}) + u_\varepsilon\partial_{t_i}(w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}) - f(u_\varepsilon)\partial_{t_i}(w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}})\big]\big|_{\mathbf{t}=\mathbf{t}^\varepsilon} r^{n-1}\,dr = 0.$$

Therefore, by (6.6) we find

$$\sum_{j=1}^N c_j \int_{I_\varepsilon} \big(Z_{\varepsilon,t_j}\partial_{t_i}(w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}})\big) r^{n-1}\,dr = 0. \tag{6.7}$$

Differentiating the equation $\langle\phi, Z_{\varepsilon,t_j}\rangle_\varepsilon = 0$ with respect to t_j, we get

$$\langle\partial_{t_i}\phi, Z_{\varepsilon,t_j}\rangle_\varepsilon = -\langle\phi, \partial_{t_i}Z_{\varepsilon,t_j}\rangle_\varepsilon = \mathrm{O}\big(\|\phi\|_*\big)\varepsilon^{-n-1}.$$

Using (3.9), we see that (6.7) is diagonally dominant in the coefficients $\{c_i\}$, which implies $c_j = 0$ for $j = 1, \dots, N$. Hence $u_\varepsilon = w_{\varepsilon,\mathbf{t}^\varepsilon} + \phi_{\varepsilon,\mathbf{t}^\varepsilon}$ is a solution of (1.1).

By our construction and the Maximum Principle, one can easily check that $u_\varepsilon > 0$ in Ω. Moreover $\varepsilon^{n-1}\mathcal{E}_\varepsilon(u_\varepsilon) \to NI(w)$ as $\varepsilon \to 0$, and u_ε has only N local maximum points $s_1^\varepsilon, \dots, s_N^\varepsilon$. By the structure of u_ε we see that (up to a permutation) $s_i^\varepsilon - t_i^\varepsilon = \mathrm{o}(1)$. This proves Theorem 1.1. □

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Appendix A

In this appendix we expand the quantity $\mathcal{E}_\varepsilon[\sum_{j=1}^N w_{\varepsilon,t_j}]$ as a function of ε and $\mathbf{t}$. Integrating by parts we get

$$\begin{aligned}
\mathcal{E}_\varepsilon\left[\sum_{j=1}^N w_{\varepsilon,t_j}\right] &= \frac{1}{2}\int_{I_\varepsilon}\left(|w'_{\varepsilon,\mathbf{t}}|^2 + |w_{\varepsilon,\mathbf{t}}|^2\right)r^{n-1}\,dr - \int_{I_\varepsilon} F(w_{\varepsilon,\mathbf{t}})r^{n-1}\,dr \\
&= \frac{1}{2}\int_{I_\varepsilon}\left[-\frac{n-1}{r}w'_{\varepsilon,\mathbf{t}} + \sum_{i=1}^N f(w_{t_i})\right]w_{\varepsilon,\mathbf{t}}r^{n-1}\,dr - \int_{I_\varepsilon} F(w_{\varepsilon,\mathbf{t}})r^{n-1}\,dr + \mathrm{O}(e^{-\frac{1}{C\varepsilon}}) \\
&= E_1 + E_2 + E_3 + E_4 + \mathrm{O}(e^{-\frac{1}{C\varepsilon}}),
\end{aligned}$$

where

$$E_1 = -\frac{1}{2}\int_{I_\varepsilon}\frac{n-1}{r}w'_{\varepsilon,\mathbf{t}}w_{\varepsilon,\mathbf{t}}r^{n-1}\,dr;$$

$$E_2 = \sum_{j=1}^N\left[\frac{1}{2}\int_{I_\varepsilon} f(w_{t_j})w_{\varepsilon,t_j}r^{n-1}\,dr - \int_{I_\varepsilon} F(w_{\varepsilon,t_j})r^{n-1}\,dr\right];$$

$$E_3 = \frac{1}{2}\sum_{i\neq j}\int_{I_\varepsilon} f(w_{t_i})w_{\varepsilon,t_j}r^{n-1}\,dr; \qquad E_4 = \int_{I_\varepsilon}\left[F\left(\sum_{j=1}^N w_{\varepsilon,t_j}\right) - \sum_{j=1}^N F(w_{\varepsilon,t_j})\right]r^{n-1}\,dr.$$

Integrating by parts and using the exponential decay of w we obtain

$$E_1 = -\frac{(n-1)}{2}\int_{I_\varepsilon} r^{n-2}w'_{\varepsilon,\mathbf{t}}w_{\varepsilon,\mathbf{t}}\,dr = \frac{(n-1)(n-2)}{2}\int_0^{\frac{1}{\varepsilon}} r^{n-3}w^2_{\varepsilon,\mathbf{t}}\,dr = \mathrm{O}(\varepsilon^{3-n}).$$

The term E_2 can be decomposed as

$$E_2 = \sum_{j=1}^N\left[\int_{I_\varepsilon}\left(\frac{1}{2}f(w_{t_j})w_{t_j} - F(w_{t_j})\right)r^{n-1}\,dr + \frac{1}{2}\int_{I_\varepsilon} f(w_{t_j})(w_{\varepsilon,t_j} - w_{t_j})r^{n-1}\,dr\right]$$

$$+\sum_{j=1}^{N}\int_{I_\varepsilon}\big(F(w_{t_j})-F(w_{\varepsilon,t_j})\big)r^{n-1}\,dr.$$

We have the following estimates, where (f1) and the decay of w are used

$$\int_{I_\varepsilon}\left(\frac{1}{2}f(w_{t_j})w_{t_j}-F(w_{t_j})\right)r^{n-1}\,dr=\left(\frac{t_j}{\varepsilon}\right)^{n-1}\left(I[w]-\int_{\mathbb{R}\setminus I_\varepsilon}\left(\frac{1}{2}f(w_{t_j})w_{t_j}-F(w_{t_j})\right)dr\right)$$
$$+\int_{I_\varepsilon}\left(\frac{1}{2}f(w_{t_j})w_{t_j}-F(w_{t_j})\right)\left[r^{n-1}-\left(\frac{t_j}{\varepsilon}\right)^{n-1}\right]dr$$
$$=\left(\frac{t_j}{\varepsilon}\right)^{n-1}I[w]+\mathrm{O}(\varepsilon^{1-n}e^{-(2+\sigma)\frac{1-t_j}{\varepsilon}})+\mathrm{O}(\varepsilon^{2-n});$$

$$\int_{I_\varepsilon}f(w_{t_j})(w_{\varepsilon,t_j}-w_{t_j})r^{n-1}\,dr=\left(\frac{t_j}{\varepsilon}\right)^{n-1}\rho_\varepsilon(t_j)\left[\int_{\mathbb{R}}\beta_\varepsilon(r)f(w_{t_j})\,dr-\int_{\mathbb{R}\setminus I_\varepsilon}\beta_\varepsilon(r)f(w_{t_j})\,dr\right]+\mathrm{O}(\varepsilon^{2-n}).$$

The first term on the right-hand side can be written as

$$\int_{\mathbb{R}}\beta_\varepsilon(r)f(w_{t_j})\,dr=e^{-\frac{1-t_j}{\varepsilon}}\big(\gamma_0+\mathrm{o}(1)\big),$$

where

$$\gamma_0=\int_{\mathbb{R}}f(w)e^y\,dy=\int_{\mathbb{R}}(w-w'')e^y\,dy. \tag{6.8}$$

The second term on the right-hand side can be estimated in the following way

$$\left|\int_{\mathbb{R}\setminus I_\varepsilon}\beta_\varepsilon(r)f(w_{t_j})\,dr\right|\leqslant C\int_{\frac{1}{\varepsilon}}^{\infty}e^{r-\frac{1}{\varepsilon}}e^{-(1+\sigma)(r-\frac{t_j}{\varepsilon})}+\mathrm{O}(e^{-\frac{1}{C\varepsilon}})$$
$$\leqslant Ce^{-(1+\sigma)\frac{1-t_j}{\varepsilon}}+\mathrm{O}(e^{-\frac{1}{C\varepsilon}}).$$

The last term of E_2 can be estimated as follows. Using the Taylor's formula and (2.18), we get

$$F(w_{t_j})-F(w_{\varepsilon,t_j})=-f(w_{t_j})\beta_\varepsilon\rho(t_j)+\mathrm{O}\big(\beta_\varepsilon^2\rho(t_j)^2w_{t_j}^\sigma\big).$$

Hence, reasoning as above we find

$$\int_{I_\varepsilon}\big(F(w_{t_j})-F(w_{\varepsilon,t_j})\big)r^{n-1}\,dr=-\left(\frac{t_j}{\varepsilon}\right)^{n-1}\int_{I_\varepsilon}f(w_{t_j})\beta_\varepsilon(r)\rho(t_j)\,dr+\mathrm{O}(e^{-(2+\sigma)\frac{1-t_j}{\varepsilon}})+\mathrm{O}(\varepsilon^{2-n})$$
$$=-\left(\frac{t_j}{\varepsilon}\right)^{n-1}e^{-2\frac{1-t_j}{\varepsilon}}\gamma_0+\mathrm{O}(e^{-(2+\sigma)\frac{1-t_j}{\varepsilon}})+\mathrm{O}(\varepsilon^{2-n}).$$

In conclusion, from (2.4) we obtain

$$E_2=\sum_{j=1}^{N}\left(\frac{t_j}{\varepsilon}\right)^{n-1}\left(I[w]-\frac{1}{2}A_0\big(\gamma_0+\mathrm{o}(1)\big)e^{-2\frac{1-t_j}{\varepsilon}}\right)+\mathrm{O}(\varepsilon^{2-n}). \tag{6.9}$$

Regarding the term E_3, we can write

$$\int_{I_\varepsilon} f(w_{t_i}) w_{\varepsilon,t_j} r^{n-1}\,dr = \left(\frac{t_i}{\varepsilon}\right)^{n-1} \int_{I_\varepsilon} f(w_{t_i}) w_{\varepsilon,t_j}\,dr + \int_{I_\varepsilon} f(w_{t_i}) w_{\varepsilon,t_j} \left[r^{n-1} - \left(\frac{t_i}{\varepsilon}\right)^{n-1}\right] dr.$$

There holds

$$\begin{aligned}\int_{I_\varepsilon} f(w_{t_i}) w_{\varepsilon,t_j}\,dr &= \int_{I_\varepsilon} f(w_{t_i}) w_{t_j}\,dr + \rho_\varepsilon(t_j) \int_{I_\varepsilon} f(w_{t_i}) \beta_\varepsilon(r) + \mathrm{O}(e^{-\frac{1}{C\varepsilon}}) \\ &= A_0\big(\gamma_0 + \mathrm{o}(1)\big) e^{-\frac{|t_i - t_j|}{\varepsilon}} + A_0\big(\gamma_0 + \mathrm{o}(1)\big) e^{-\frac{|t_i + t_j - 1|}{\varepsilon}} + \mathrm{O}(e^{-\frac{1}{C\varepsilon}}) \\ &= A_0\big(\gamma_0 + \mathrm{o}(1)\big) e^{-\frac{|t_i - t_j|}{\varepsilon}} + \mathrm{O}(e^{-\frac{1}{C\varepsilon}}).\end{aligned}$$

Reasoning as above one also finds

$$\left|\int_{I_\varepsilon} f(w_{t_i}) w_{\varepsilon,t_j} \left[r^{n-1} - \left(\frac{t_i}{\varepsilon}\right)^{n-1}\right] dr\right| \leqslant C\varepsilon^{2-n}.$$

In conclusion we deduce

$$E_3 = \frac{1}{2} \sum_{i \neq j} A_0\big(\gamma_0 + \mathrm{o}(1)\big) e^{-\frac{|t_i - t_j|}{\varepsilon}} + \mathrm{O}(e^{2-n}).$$

About E_4, recall that

$$E_4 = \int_{I_\varepsilon} \left[F\left(\sum_{j=1}^N w_{\varepsilon,t_j}\right) - \sum_{j=1}^N F(w_{\varepsilon,t_j})\right] r^{n-1}\,dr.$$

Similarly to the proof of Lemma 2.2, we divide $I_\varepsilon = (0, 1/\varepsilon)$ into N parts as in (2.16).

On I_i, $i = 1, \dots, N-1$, we have $w_{\varepsilon,t_j} \leqslant w_{t_i}$, so we can use Taylor's formula and (2.18) to get

$$F\left(\sum_{j=1}^N w_{\varepsilon,t_j}\right) - \sum_{j=1}^N F(w_{\varepsilon,t_j}) = f(w_{t_i})\left[\sum_{j \neq i} w_{\varepsilon,t_j} + \beta_\varepsilon \rho_\varepsilon(t_i)\right] + \mathrm{O}(w_{t_i}^\sigma)\left[\sum_{j \neq i} w_{\varepsilon,t_j}^2 + \beta_\varepsilon^2 \rho_\varepsilon(t_i)^2\right].$$

On I_N, we again add some error terms of order $e^{-1/(C\varepsilon)}$.

Hence, integrating on I_i and using the exponential decay of w, we get

$$\begin{aligned}&\int_{I_i} \left[F\left(\sum_{j=1}^N w_{\varepsilon,t_j}\right) - \sum_{j=1}^N F(w_{\varepsilon,t_j})\right] r^{n-1}\,dr \\ &\quad = \left(\frac{t_i}{\varepsilon}\right)^{n-1} A_0\big(\gamma_0 + \mathrm{o}(1)\big) e^{-2\frac{1-t_i}{\varepsilon}} + \left(\frac{t_i}{\varepsilon}\right)^{n-1} \sum_{j \neq i} A_0\big(\gamma_0 + \mathrm{o}(1)\big) e^{-\frac{|t_i - t_j|}{\varepsilon}} + \mathrm{O}(\varepsilon^{2-n}).\end{aligned}$$

Combining the estimates for E_1, E_2, E_3 and E_4, we obtain

$$\begin{aligned}\mathcal{E}_\varepsilon\left[\sum_{j=1}^N w_{\varepsilon,t_j}\right] &= \sum_{i=1}^N \left(\frac{t_i}{\varepsilon}\right)^{n-1} \left\{I[w] - \frac{1}{2} A_0\big(\gamma_0 + \mathrm{o}(1)\big) e^{-2\frac{1-t_i}{\varepsilon}}\right\} \\ &\quad - \frac{1}{2} \sum_{i \neq j} \left(\frac{t_i}{\varepsilon}\right)^{n-1} A_0\big(\gamma_0 + \mathrm{o}(1)\big) e^{-\frac{|t_i - t_j|}{\varepsilon}} + \mathrm{O}(\varepsilon^{2-n}).\end{aligned} \tag{6.10}$$

Finally we note that

$$\gamma_0 = \int_{\mathbb{R}} (w - w'')e^y\,dy = \lim_{R\to+\infty} \int_{-R}^{R} (w - w'')e^y$$
$$= \lim_{R\to+\infty} \big(e^R w(R) + e^{-R} w(-R) - e^R w'(R) - e^{-R} w'(-R)\big) = 2A_0 > 0 \tag{6.11}$$

where A_0 is defined at (2.4). □

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