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On some results of Moser and of Bangert

Sur quelques résultats de Moser et Bangert

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Abstract

A new proof is given of results in (V. Bangert, AIHP Anal. Nonlin. 6 (1989) 95) on the existence of minimal (in the sense of Giaquinta and Guisti) heteroclinic solutions of a nonlinear elliptic PDE. Bangert's work is based on an earlier paper of Moser (AIHP Anal. Nonlin. 3 (1986) 229). Unlike (V. Bangert, AIHP Anal. Nonlin. 6 (1989) 95), the proof here is variational in nature, and involves the minimization of a 'renormalized' functional. It is meant to be the first step towards finding locally vs. globally minimal solutions of the PDE.

Résumé

Nous donnons une démonstration nouvelle des résultats de Bangert sur l'existence d'une solution minimale (au sens de Giaquinta et Guisti) hétéroclinique d'un EDP elliptique nonlinéaire. Le travail de Bangert est basé sur un article de Moser (AIHP Anal. Nonlin. 3 (1986) 229). Contrairement à (V. Bangert, AIHP Anal. Nonlin. 6 (1989) 95), la démonstration est variationelle en nature, et utilise la minimisation d'une fonctionelle « renormalisée ». C'est une tentative de premier pas pour trouver des solutions minimales localement, plutôt que globalment de l'EDP.

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1. Introduction

Motivated by work of Aubry [3] and of Mather [7] on monotone twist maps, Moser [8] made the first steps towards finding analogues of their theory in the setting of quasilinear elliptic partial differential equations on $\mathbb{R}^n$. He studied the equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \mathcal{F}_{p_i}(x, u, Du) - \mathcal{F}_u(x, u, Du) = 0, \tag{1.1}$$

which arises formally as the Euler equation of the functional

$$\int_{\mathbb{R}^n} \mathcal{F}(x, u, Du)\, \mathrm{d}x, \tag{1.2}$$

where $\mathcal{F}(x, u, p)$ is e.g. C^3 in its arguments and 1-periodic in the components of $x \in \mathbb{R}^n$ and in u. $\mathcal{F}$ further satisfies structural conditions that imply weak solutions of (1.1) are actually classical solutions of the equation [8]. Let $\mathcal{N}_\alpha$ denote the set of solutions of (1.1) that have a prescribed $\alpha \in \mathbb{R}^n$ as rotation vector, that are minimal in the sense of Giaquinta and Guisti [6], and whose graphs viewed on T^{n+1} have no self intersections. The notion of rotation vector here is the extension to $\mathbb{R}^n$ of the usual rotation number used in dynamical systems. Likewise the minimal and non-self intersection properties are the analogues for solutions of (1.1) of properties of monotone twist maps. Among other things, Moser showed $\mathcal{N}_\alpha \neq \emptyset$ for all $\alpha \in \mathbb{R}^n$ and obtained various qualitative and quantitative properties for the members of $\mathcal{N}_\alpha$. E.g. for $\mathcal{N}_0$, he obtained solutions that are 1-periodic in $x_1, \ldots, x_n$. Letting $\mathcal{M}_\alpha$ denote the subset of $\mathcal{N}_\alpha$ whose existence Moser established, he further proved $\mathcal{M}_\alpha$ is an ordered set, i.e. $v, w \in \mathcal{M}_\alpha$ implies $v \equiv w$, $v < w$, or $v > w$.

Bangert carried Moser's analysis further in various ways. Among other things, he showed that whenever $\alpha \in \mathbb{Q}^n$ and $\mathcal{M}_\alpha$ possesses a gap, i.e. there are adjacent $v_0 < w_0$ in $\mathcal{M}_\alpha$, then there is a $U_1 \in \mathcal{N}_\alpha$ which is heteroclinic in an appropriate sense from v_0 to w_0 and likewise another solution of (1.1) heteroclinic from w_0 to v_0. E.g. if $\alpha = 0$ so v_0 and w_0 are adjacent members of $\mathcal{M}_0$, there is a $U_1 \in \mathcal{N}_0$ heteroclinic in x_1 from v_0 to w_0. (Similarly there are members of $\mathcal{N}_0$ heteroclinic in x_i from v_0 to w_0, $2 \leqslant i \leqslant n$.) Moreover if $\mathcal{M}_0^1$ denotes the set of such heteroclinic solutions, $\mathcal{M}_0^1$ is ordered. If there is a gap $v_1 < w_1$ in $\mathcal{M}_0^1$, there is a $U_2 \in \mathcal{M}_0^1$ which is also heteroclinic in x_2 from v_1 to w_1. Further gap conditions lead to yet more complicated heteroclinics.

There are analogues of U_i in the Aubry–Mather Theory: gaps between periodic invariant sets lead to the existence of heteroclinic invariant curves joining the periodic sets. Moreover given any formal chain of such heteroclinic invariant curves, there are actual invariant curves shadowing the chain. Thus it is natural to seek such shadowing solutions of (1.1). In recent years, variational methods have been devised to carry out such constructions in dynamical systems or PDE settings, see e.g. [10,5,1,2,9]. These methods require a variational characterization of the basic solutions such as U_1 above. However Bangert's clever existence argument in [4] is not variational in nature. Therefore as a first step towards constructing more complex solutions of (1.1), in this paper we provide a variational approach to find the type of solutions Bangert discovered. This will only be done for the special but still significant setting of $\alpha = 0$ and $\mathcal{F}(x, u, p) = \frac{1}{2}|p|^2 + F(x, u)$ where

(F_1) $F \in C^2(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$,
(F_2) $F(x, u)$ is 1-periodic in $x_1, \ldots, x_n$ and u, and
(F_3) F is even in $x_1, \ldots, x_n$.

Thus (1.1) becomes

$$-\Delta u + F_u(x, u) = 0. \tag{PDE}$$

A serious difficulty that has to be overcome in the variational approach to (PDE) is that the associated functionals defined on the classes of functions of the heteroclinic types of U_1, U_2, etc. are infinite. Thus to obtain U_1 requires a renormalization, i.e. subtracting an infinite term from the natural functional. Likewise obtaining U_2 requires a second renormalization, etc.

The existence of U_1 will be carried out in Section 2 and its (Giaquinta–Guisti) minimality will be established in Section 3. An inductive argument will then be given in Section 4 to treat the general case.

2. The simplest heteroclinics

In this section, it will be shown how to obtain solutions of (PDE) that are heteroclinic in x_1 from v_0 to w_0 where $v_0 < w_0$ are an adjacent pair of solutions of (PDE) that are 1-periodic in $x_1, \ldots, x_n$.

For ease of notation, set

$$L(u) = \frac{1}{2}|\nabla u|^2 + F(x,u)$$

with F satisfying (F_1)–(F_3). Let

$$\Gamma_0 = \left\{u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n) \mid u \text{ is 1-periodic in } x_1, \ldots, x_n\right\}.$$

For $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{Z}^n$, set $T(\ell) = [\ell_1, \ell_1 + 1] \times \cdots \times [\ell_n, \ell_n + 1]$. Define

$$J_0(u) = \int_{T(0)} L(u)\,\mathrm{d}x.$$

Finally define

$$c_0 = \inf_{u \in \Gamma_0} J_0(u). \tag{2.1}$$

In [3], Moser showed:

Proposition 2.2. *If $\mathcal{M}_0 \equiv \{u \in \Gamma_0 \mid J_0(u) = c_0\}$, then*

1^{o} *$\mathcal{M}_0 \neq \phi$ and if $u \in \mathcal{M}_0$, u is a classical solution of* (*PDE*).
2^{o} *$\mathcal{M}_0$ is an ordered set.*

A further property of $\mathcal{M}_0$ is:

Corollary 2.3. *If $u \in \mathcal{M}_0$, u is even in $x_1, \ldots, x_n$.*

Proof. Set $\zeta_i(x) = (x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n)$, $1 \leqslant i \leqslant n$. If $u \in \mathcal{M}_0$, then $u_i(x) = u(\zeta_i(x)) \in \mathcal{M}_0$ via (F_3). If $u_i \not\equiv u$, by 2^{o} of Proposition 2.2, either (i) $u_i(x) > u(x)$ or (ii) $u_i(x) < u(x)$ for all $x \in \mathbb{R}^n$. If (i) occurs,

$$u_i\big(\zeta_i(x)\big) = u(x) < u_i(x) = u\big(\zeta_i(x)\big), \tag{2.4}$$

a contradiction. Similarly (ii) cannot hold. Thus $u_i \equiv u$, $1 \leqslant i \leqslant n$. □

To continue, assume there is a gap in $\mathcal{M}_0$:

$(*)_0$ There are adjacent $v_0, w_0 \in \mathcal{M}_0$ with $v_0 < w_0$.

Set

$$\widehat{\Gamma}_1 = \{u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n) \mid v_0 \leqslant u \leqslant w_0 \text{ a.e. and } u \text{ is 1-periodic in } x_2, \dots, x_n\}.$$

Now the renormalized functional $J_1(u)$ for this setting can be introduced. For $j \in \mathbb{Z}$ and $k \in \{1, \dots, n\}$, set

$$\tau^k_{-j} u(x) = u(x_1, \dots, x_k + j, \dots, x_n)$$

and for $i \in \mathbb{Z}$, define

$$J_{1,i}(u) = J_0(\tau^1_{-i} u) - c_0. \tag{2.5}$$

Alternatively set $e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$, the usual basis in $\mathbb{R}^n$. Then

$$J_{1,i}(u) = \int\limits_{T(ie_1)} L(u)\,\mathrm{d}x - c_0.$$

Now define

$$J_1(u) = \sum_{i \in \mathbb{Z}} J_{1,i}(u).$$

The next proposition provides the properties of J_1 that will be required here. Hypothesis (F_3) plays its main role in 1°–2°.

Proposition 2.6. *For $u \in \widehat{\Gamma}_1$,*

1° $J_{1,i}(u) \geqslant 0$ *for all $i \in \mathbb{Z}$.*
2° $J_1(u) \geqslant 0$.
3° $\int_{T(ie_1)} L(u)\,\mathrm{d}x \leqslant J_1(u) + c_0$ *for any $i \in \mathbb{Z}$.*
4° J_1 *is weakly lower semicontinuous (lsc) (with respect to $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$) on $\widehat{\Gamma}_1$.*

Proof. For $u \in \widehat{\Gamma}_1$ and $x_1 \in [i + \frac{1}{2}, i + 1]$, set

$$\varphi_i^+(u) = \tau^1_{-i} u \tag{2.7}$$

and for $x_1 \in [i, i + \frac{1}{2}]$, set

$$\varphi_i^-(u) = \tau^1_{-i} u. \tag{2.8}$$

Extend $\varphi_i^\pm(u)$ first as even functions about $x_1 = i + \frac{1}{2}$ and then 1-periodically in x_1. Continuing to denote these extensions by $\varphi_i^\pm(u)$, their definition implies $\varphi_i^\pm(u) \in \Gamma_0$ and

$$J_{1,i}(u) = \frac{1}{2}\big(J_{1,i}(\varphi_i^+(u)) + J_{1,i}(\varphi_i^-(u))\big) \geqslant 0. \tag{2.9}$$

Thus 1° and hence 2° hold. By 1°,

$$J_{1,i}(u) = \int\limits_{T(ie_1)} L(u)\,\mathrm{d}x - c_0 \leqslant J_1(u) \tag{2.10}$$

so 3° is valid. To prove 4°, note first that J_0 is weakly lsc on Γ_0. Let (u_k) be a sequence in $\widehat{\Gamma}_1$ and $u_k \to u$ weakly in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$ as $k \to \infty$. Set

$$J_{1;p,q}(u) \equiv \sum_{p}^{q} J_{1,i}(u).$$

Then by the weak lsc of J_0, (2.5), (2.9) and 1°,

$$
\begin{aligned}
J_{1;p,q}(u) &= \frac{1}{2}\sum_{p}^{q}\bigl(J_{1,i}\bigl(\varphi_i^+(u)\bigr) + J_{1,i}\bigl(\varphi_i^-(u)\bigr)\bigr) \\
&\leqslant \frac{1}{2}\sum_{p}^{q}\Bigl(\varliminf_{k\to\infty} J_{1,i}\bigl(\varphi_i^+(u_k)\bigr) + \varliminf_{k\to\infty} J_{1,i}\bigl(\varphi_i^-(u_k)\bigr)\Bigr) \\
&\leqslant \frac{1}{2}\varliminf_{k\to\infty}\sum_{p}^{q}\bigl(J_{1,i}\bigl(\varphi_i^+(u_k)\bigr) + J_{1,i}\bigl(\varphi_i^-(u_k)\bigr)\bigr) \\
&= \varliminf_{k\to\infty} J_{1;p,q}(u_k) \leqslant \varliminf_{k\to\infty} J_1(u_k).
\end{aligned}
\tag{2.11}
$$

Since (2.11) is valid for all $p \leqslant q \in \mathbb{Z}$,

$$
J_1(u) \leqslant \varliminf_{k\to\infty} J_1(u_k) \tag{2.12}
$$

and 4° holds. □

Remark 2.13. The argument used to prove 1° shows

$$
c_0 = \inf_{u\in W^{1,2}(T(0))} J_0(u)
$$

and if $u \in W^{1,2}(T(0))$ with $J(u) = c_0$, then $u \in \mathcal{M}_0$. See e.g. [9] for a similar argument.

Next the class of functions that will be used to find a solution of (PDE) heteroclinic in x_1 from v_0 to w_0 can be introduced. Set

$$
\Gamma_1 = \bigl\{u \in \widehat{\Gamma}_1 \mid u \leqslant \tau_{-1}^1 u \text{ and } v_0 \not\equiv u \not\equiv w_0\bigr\}.
$$

The members of Γ_1 automatically are heteroclinic in x_1 from v_0 to w_0 (in a weak sense) as the next result shows.

Proposition 2.14. *If $u \in \Gamma_1$ and $J_1(u) < \infty$, then $\tau_j^1 u \to v_0$ and $\tau_{-j}^1 u \to w_0$ weakly in $W^{1,2}(T(0))$ as $j \to \infty$.*

Proof. Since $v_0 \leqslant \tau_j^1(u) \leqslant w_0$ for all $j \in \mathbb{Z}$, 3° of Proposition 2.6 shows $(\tau_j^1 u)$ is bounded in $W^{1,2}(T(0))$. This with the monotonicity property, $u \leqslant \tau_{-1}^1 u$, shows there is a unique $v \in W^{1,2}(T(0))$ such that $\tau_j^1 u \to v$ weakly in $W^{1,2}(T(0))$ and strongly in $L^2(T(0))$ as $j \to \infty$ and

$$
v_0 \leqslant v \leqslant u \leqslant w_0. \tag{2.15}
$$

Since $J_1(u) < \infty$, $J_0(\tau_j^1 u) \to c_0$ as $|j| \to \infty$. But J_0 is weakly lsc so $J_0(v) \leqslant c_0$. Now Remark 2.13 shows $J_0(v) = c_0$ and $v \in \mathcal{M}_0$. Hence (2.15), $(*)_0$, and $u \not\equiv w_0$ imply $v = v_0$. Similarly $\tau_j^1 u \to w_0$ as $j \to -\infty$ weakly in $W^{1,2}(T(0))$. □

Remark 2.16. In fact, by arguments in [9], the convergence of $\tau_j^1 u$ is in $W^{1,2}(T(0))$.

Now to obtain a solution of (PDE) heteroclinic in x_1 from v_0 to w_0, set

$$
c_1 = \inf_{u\in\Gamma_1} J_1(u). \tag{2.17}
$$

Theorem 2.18. *Let F satisfy* (F_1)–(F_3) *and let* $(*)_0$ *hold. Then*

1° *There is a* $U_1 \in \Gamma_1$ *such that* $J_1(U) = c_1$.
2° *Any such* U_1 *is a classical solution of* (*PDE*).
3° U_1 *is heteroclinic* (*in* C^2) *from* v_0 *to* w_0, *i.e.* $\|U_1 - v_0\|_{C^2(T(je_1))} \to 0$ *as* $j \to -\infty$ *and* $\|U_1 - w_0\|_{C^2(T(je_1))} \to 0$ *as* $j \to \infty$.
4° $v_0 < U_1 < \tau_{-1}^1 U_1 < w_0$.
5° $\mathcal{M}_1 \equiv \{u \in \Gamma_1 \mid J_1(u) = c_1\}$ *is an ordered set.*
6° *If* $u \in \mathcal{M}_1$, u *is even in* $x_2, \ldots, x_n$.

Remark 2.19. In Section 3, it will be further shown that U_1 is minimal in the sense of Giaquinta and Guisti.

Proof of Theorem 2.18. Let $(u_k) \subset \Gamma_1$ be a minimizing sequence for (2.17). Normalize (u_k) so that for $i < 0$,

$$\int_{T(ie_1)} u_k \,\mathrm{d}x \leqslant \frac{1}{2} \int_{T(0)} (v_0 + w_0)\,\mathrm{d}x < \int_{T(0)} u_k \,\mathrm{d}x. \tag{2.20}$$

That this can be done follows from Proposition 2.14. By 3° of Proposition 2.6, (u_k) is bounded in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$. Therefore there is a $U_1 \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$, 1-periodic in $x_2, \ldots, x_k$ such that along a subsequence, $u_k \to U_1$ weakly in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$ and strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$. Hence by the properties of u_k, $v_0 \leqslant U_1 \leqslant \tau_{-1} U_1 \leqslant w_0$ and by (2.20), $U_1 \in \Gamma_1$. Thus

$$J_1(U_1) \geqslant c_1. \tag{2.21}$$

On the other hand, 4° of Proposition 2.6 and $U_1 \in \Gamma_1$ imply

$$J_1(U_1) \leqslant c_1. \tag{2.22}$$

Combining (2.21)–(2.22) gives 1° of Theorem 2.18.

To verify 2°–5°, modifications of arguments from [9] will be employed. To prove that U_1 satisfies (PDE), let $z \in \mathbb{R}^n$ and let $B_r(z)$ be a ball of radius r about z with $r < \frac{1}{2}$. For $j \in \mathbb{Z}^n$, set $z_j = z + j$. Let $U^* = U_1$ in $\mathbb{R}^n \setminus \bigcup_{j \in \mathbb{Z}^n} B_r(z_j)$ and $U^* = u_j^*$ in $B_r(z_j)$ where u_j^* minimizes

$$\int_{B_r(z_j)} L(\varphi)\,\mathrm{d}x \tag{2.23}$$

over

$$\mathcal{S}_r(z_j) = \bigl\{\varphi \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n) \mid \varphi = U_1 \text{ in } \mathbb{R}^n \setminus B_r(z_j)\bigr\}. \tag{2.24}$$

As in Lemma 2.4 of [9], there is a $u_j \in \mathcal{S}_r(z_j)$ minimizing (2.23) and any such minimizer of (2.23) is a solution of (PDE) in $B_r(z_j)$. A priori, there may not be a unique minimizer but as in Lemma 2.5 of [9], the set of minimizers is ordered and there is a unique smallest one which is chosen to be u_j^*. Note that

$$\int_{B_r(z_j)} L(U_1)\,\mathrm{d}x \geqslant \int_{B_r(z_j)} L(u_j^*)\,\mathrm{d}x \tag{2.25}$$

for all $j \in \mathbb{Z}^n$. Hence

$$J_1(U_1) \geqslant J_1(U^*). \tag{2.26}$$

We claim $U^* \in \Gamma_1$ and therefore

$$J_1(U^*) \geqslant J_1(U_1). \tag{2.27}$$

But then by (2.25), for all $j \in \mathbb{Z}^n$,

$$\int_{B_r(z_j)} L(U_1)\,\mathrm{d}x = \int_{B_r(z_j)} L(u_j^*)\,\mathrm{d}x.$$

Thus U_1 is a minimizer of (2.23) in $\mathcal{S}_r(z_j)$ and hence a solution of (PDE).

To verify that $U^* \in \Gamma_1$, observe first that the 1-periodicity of U_1 in $x_2, \ldots, x_n$ implies the same for U^*. Thus $U^* \in \Gamma_1$ if (a) $v_0 \leqslant U^* \leqslant w_0$, (b) $U^* \leqslant \tau_{-1}^1 U^*$, and (c) $v_0 \not\equiv U^* \not\equiv w_0$. To prove (a), since $v_0 \leqslant U_1 \leqslant w_0$, it suffices to show $v_0 \leqslant u_j \leqslant w_0$ in $B_r(z_j)$. Suppose e.g. $u_j(y) < v_0(y)$ for some $y \in B_r(z_j)$. Set $\psi_j = \min(v_0, u_j)$. Then $\psi_j = v_0$ in $\mathbb{R}^n \setminus B_r(z_j)$. Let S be a unit n-cube centered at z_j. Then $\psi_j = v_0$ in $S \setminus B_r(z_j)$, $\psi_j = u_j$ near y, and $\psi_j|_S$ extends naturally to $\mathbb{R}^n$ as an element of Γ_0. Continuing to denote this extension by ψ_j,

$$\begin{aligned} c_0 &= J_0(v_0) \leqslant J_0(\psi_j) \\ &= \int_{S\setminus B_r(z_j)} L(v_0)\,\mathrm{d}x + \int_{B_r(z_j)\cap\{v_0\leqslant u_j\}} L(v_0)\,\mathrm{d}x + \int_{B_r(z_j)\cap\{v_0> u_j\}} L(u_j)\,\mathrm{d}x. \end{aligned} \tag{2.28}$$

Hence

$$\int_{B_r(z_j)\cap\{v_0>u_j\}} L(v_0)\,\mathrm{d}x \leqslant \int_{B_r(z_j)\cap\{v_0>u_j\}} L(u_j)\,\mathrm{d}x. \tag{2.29}$$

Set $\chi_j = \max(v_0, u_j)$ so $\chi_j \in \mathcal{S}_r(z_j)$ and

$$\int_{B_r(z_j)} L(u_j)\,\mathrm{d}x \leqslant \int_{B_r(z_j)} L(\chi_j)\,\mathrm{d}x = \int_{B_r(z_j)\cap\{v_0>u_j\}} L(v_0)\,\mathrm{d}x + \int_{B_r(z_j)\cap\{v_0\leqslant u_j\}} L(u_j)\,\mathrm{d}x$$

so

$$\int_{B_r(z_j)\cap\{v_0>u_j\}} L(u_j)\,\mathrm{d}x \leqslant \int_{B_r(z_j)\cap\{v_0>u_j\}} L(v_0)\,\mathrm{d}x. \tag{2.30}$$

Combining (2.29) and (2.30) yields equality in these expressions and returning to (2.28) shows

$$c_0 = J_0(v_0) \leqslant J_0(\psi_j) = J_0(v_0). \tag{2.31}$$

Hence $\psi_j \in \mathcal{M}_0$. But by Proposition 2.2, ψ_j cannot both $= v_0$ in $S \setminus B_r(z_j)$ and $= u_j$ near y. Thus the assumption that $u_j(y) < v_0(y)$ is not tenable and $v_0 \leqslant u_j$. Similarly $u_j \leqslant w_0$ and (a) has been proved.

To obtain (b), suppose not. Then as in the proof of Proposition 2.3 of [9], for some j, there is an $x_0 \in B_r(z_j)$ such that

$$u_{j+e_1}^*(x_0 + e_1) < u_j^*(x_0). \tag{2.32}$$

For $x \in B_{1/2}(z_j)$, set $\psi(x) = u_{j+e_1}^*(x + e_1)$, $\chi = \max(u_j^*, \psi)$, $\zeta = \min(u_j^*, \psi)$. Then $\zeta = u_j^* = U_1 \leqslant \tau_{-1}^1 U_1 = \psi = \chi$ on $B_{1/2}(z_j) \setminus B_r(z_j)$. Thus

$$\int_{B_r(z_j)} \big(L(\zeta) + L(\chi)\big)\,\mathrm{d}x = \int_{B_r(z_j)} \big(L(u_j^*) + L(\psi)\big)\,\mathrm{d}x \tag{2.33}$$

which shows that ζ minimizes (2.23) over $\mathcal{S}_r(z_j)$ and $\tau_1\chi$ minimizes (2.23) over $\mathcal{S}_r(z_j + e_1)$. Therefore by the definitions of U^* and ζ,

$$u_j^*(x_0) \leqslant \zeta(x_0) \leqslant \psi(x_0) = u_{j+e_1}^*(x_0 + e_1) \tag{2.34}$$

contrary to (2.32). Thus (b) is verified. To prove that (c) is valid, suppose not. If $U^* \equiv w_0$, then $U_1 = w_0$ in $\mathbb{R}^n \setminus \bigcup_{j\in\mathbb{Z}^n}(z_j)$ so $\tau_j^1 U_1 \not\to v_0$ in $L^2(T(0))$ as $j \to \infty$, contrary to Proposition 2.14 for U_1. Similarly $U^* \not\equiv v_0$. Thus (a), (b), and (c) hold and U_1 satisfies (PDE). Since any $U \in \mathcal{M}_1$ is trivially the limit of a minimizing sequence for (2.17), the argument just given shows U satisfies (PDE) and 2$^{\text{o}}$ of the theorem is satisfied.

To prove 3$^{\text{o}}$ of Theorem 2.18, observe that $\|U_1 - v_0\|_{L^2(T(ie_1))} \to 0$ as $i \to -\infty$ by Proposition 2.14. Since U_1 and v_0 are bounded in $L^\infty(\mathbb{R}^n)$, local Schauder estimates show $U_1 - v_0$ is bounded in $C^{2,\alpha}_{\text{loc}}(\mathbb{R}^n)$ for any $\alpha \in (0,1)$. Standard interpolation inequalities then yield 3$^{\text{o}}$ for $U_1 - v_0$ and a similar argument applies to $U_1 - w_0$.

To establish 4$^{\text{o}}$–5$^{\text{o}}$, we begin with 5$^{\text{o}}$. Let $V, W \in \mathcal{M}_1$. If 5$^{\text{o}}$ is false, setting $\varphi = \max(V, W)$ and $\psi = \min(V, W)$, then $\varphi(z) = \psi(z)$ for some $z \in \mathbb{R}^n$. Suppose for the moment that $\varphi, \psi \in \Gamma_1$. Arguing as in the proof of Proposition 2.2 [8], for all $i \in \mathbb{Z}$,

$$\int\limits_{T(ie_1)} L(\varphi)\,\mathrm{d}x + \int\limits_{T(ie_1)} L(\psi)\,\mathrm{d}x = \int\limits_{T(ie_1)} L(V)\,\mathrm{d}x + \int\limits_{T(ie_1)} L(W)\,\mathrm{d}x \tag{2.35}$$

and this implies

$$2c_1 \leqslant J_1(\varphi) + J_1(\psi) = J_1(V) + J_1(W) = 2c_1. \tag{2.36}$$

Thus $J_1(\varphi), J_1(\psi) = c_1$ and by 1$^{\text{o}}$–2$^{\text{o}}$, φ and ψ are solutions of (PDE). But $\varphi - \psi \geqslant 0$, $\varphi(z) = \psi(z)$, and $\varphi - \psi$ is a solution of the linear elliptic partial differential equation

$$-\Delta\Phi + A(x)\Phi = 0 \tag{2.37}$$

where

$$\begin{aligned} A(x) &= \frac{F_u(x,\varphi(x)) - F_u(x,\psi(x))}{\varphi(x) - \psi(x)}, \quad \varphi(x) > \psi(x) \\ &= F_{uu}\big(x,\varphi(x)\big), \quad \varphi(x) = \psi(x). \end{aligned}$$

Further writing (2.37) as

$$-\Delta\Phi + \max(A,0)\Phi = -\min(A,0)\Phi \geqslant 0, \tag{2.38}$$

the maximum principle implies $\varphi \equiv \psi$, a contradiction.

To verify that $\varphi, \psi \in \Gamma_1$, it suffices to prove that

$$\tau_{-1}^1 \chi \geqslant \chi \tag{2.39}$$

for $\chi = \varphi, \psi$. First for φ, note that

$$\tau_{-1}^1 \varphi(x) = \varphi(x_1 + 1, x_2, \ldots) = \max\big(V(x_1+1, x_2, \ldots), W(x_1+1, x_2, \ldots)\big). \tag{2.40}$$

If $\tau_{-1}^1\varphi(x) = \tau_{-1}^1 V(x)$, since $\tau_{-1}^1 V(x) \geqslant V(x)$, then by (2.40),

$$\tau_{-1}^1 V(x) \geqslant \tau_{-1}^1 W(x) \geqslant W(x).$$

A similar argument applies if $\tau_{-1}^1\varphi(x) = \tau_{-1}^1 W(x)$. Hence (2.39) holds for φ.

Next to prove (2.39) for ψ, if $\tau_{-1}^1\psi(x) = \tau_{-1}^1 V(x)$ and $\psi(x) = V(x)$, (2.39) is valid while if $\tau_{-1}^1\psi = \tau_{-1}^1 V(x)$ and $\psi(x) = W(x)$,

$$\tau_{-1}^1\psi(x) = \tau_{-1}^1 V(x) \geqslant V(x) \geqslant W(x) = \psi(x).$$

A similar argument obtains if the roles of V and W are reversed. Thus $\varphi, \psi \in \Gamma_1$ and 5$^{\text{o}}$ is proved.

To get 4$^{\text{o}}$, note that

$$v_0 \leqslant U_1 \leqslant \tau_{-1}^1 U_1 \leqslant w_0. \tag{2.41}$$

Now the maximum principle can be used exactly as in (2.37)–(2.38) to get strict inequalities in (2.41). Lastly 6^{o} follows from 5^{o} and the argument of Corollary 2.3.

The proof of Theorem 2.18 is complete. □

3. Minimality of U_1

As was mentioned in the introduction, Moser studied solutions of (1.1) that were minimal in the sense of Giaquinta and Guisti. In this section it will be shown that U_1 is a minimal solution of (PDE) in this sense.

Following [6], U_1 is a minimal solution of (PDE) if for any bounded domain $\Omega \subset \mathbb{R}^n$ with a smooth boundary,

$$\int_{\Omega} L(u)\,\mathrm{d}x \geqslant \int_{\Omega} L(U_1)\,\mathrm{d}x \tag{3.1}$$

for any $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$ with $u = U_1$ in $\mathbb{R}^n \setminus \Omega$. In other words U minimizes $\int_{\Omega} L(\cdot)\,\mathrm{d}x$ over the class of $W^{1,2}(\Omega)$ functions having U_1 as boundary values. The proof of Theorem 2.18 shows that U_1 satisfies (3.1) when Ω is any ball of radius $r < \frac{1}{2}$. To extend this property to the more general class of bounded Ω's with a smooth boundary requires showing that c_0 and c_1 can be characterized as minimizers of functionals in broader classes of functions.

To begin, let $p = (p_1, \ldots, p_n) \in \mathbb{N}^n$ and set

$$\Gamma_0(p) = \left\{u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n) \mid u \text{ is } p_i \text{ periodic in } x_i,\ 1 \leqslant i \leqslant n\right\},$$

$$I_p(u) = \int_0^{p_1} \cdots \int_0^{p_n} L(u)\,\mathrm{d}x,$$

$$c_0(p) = \inf_{u \in \Gamma_0(p)} I_p(u),$$

and

$$\mathcal{M}_0(p) = \left\{u \in \Gamma_0(p) \mid I_p(u) = c_0(p)\right\}.$$

The proof of Proposition 2.2 shows that $\mathcal{M}_0(p) \neq \phi$ and is an ordered set.

Lemma 3.2. $\mathcal{M}_0(p) = \mathcal{M}_0$ (*and therefore* $c_0(p) = (\prod_1^n p_i)c_0$).

Proof. It suffices to show that $\tau^i_{-1}u = u$ for $1 \leqslant i \leqslant n$ and any $u \in \mathcal{M}_0(p)$. If not, since $\mathcal{M}_0(p)$ is ordered, either (i) $\tau^i_{-1}u > u$ or (ii) $\tau^i_{-1}u < u$. If e.g. (i) occurs,

$$u < \tau^i_{-1}u < \cdots < \tau^i_{-p_i}u = u,$$

a contradiction. Similarly (ii) cannot occur and the lemma follows. □

Next it will be shown that there is an analogue of Lemma 3.2 in the setting of Theorem 2.18. Let $\ell = (\ell_2, \ldots, \ell_n) \in \mathbb{N}^{n-1}$ and define

$$\widehat{\Gamma}_1(\ell) = \left\{u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n) \mid v_0 \leqslant u \leqslant w_0 \text{ and } u \text{ is } \ell_i \text{ periodic in } x_i,\ 2 \leqslant i \leqslant n\right\}.$$

Let $p \in \mathbb{N}^n$ with $p = (p_1, \ell)$ and let $i \in \mathbb{Z}$. For $u \in \widehat{\Gamma}_1(\ell)$, define

$$J^p_{1,i}(u) = \sum_{k_2=0}^{\ell_2-1} \cdots \sum_{k_n=0}^{\ell_n-1} \left(\int_{T((ip_1,k))} L(u)\,\mathrm{d}x - c_0 \right)$$

and

$$J_1^p(u) = \sum_{i\in\mathbb{Z}} J_{1,i}^p(u).$$

Observe that with slight modifications in the proof, J_1^p has the same properties on $\widehat{\Gamma}_1(\ell)$ as does J_1 on $\widehat{\Gamma}_1$ given by Proposition 2.6. Set

$$\Gamma_1(p) = \left\{u \in \widehat{\Gamma}_1(\ell) \mid u \leqslant \tau_{-p_1}^1 u \text{ and } v_0 \not\equiv u \not\equiv w_0\right\}.$$

Replacing Γ_1, J_1, and $T(0)$ by $\Gamma_1(p)$, J_1^p, and $\bigcup_{0\leqslant k_r\leqslant p_r} T(k_1e_1 + \cdots + k_ne_n)$, the proof of Proposition 2.14 carries over to the current setting.

Now define

$$c_1(p) = \inf_{u\in\Gamma_1(p)} J_1^p(u).$$

By the above observations, the argument of Theorem 2.18 (with $r < \frac{1}{2}\min_{1\leqslant i\leqslant n} p_i$ now permitted) applies here so

$$\mathcal{M}_1(p) \equiv \left\{u \in \Gamma_1(p) \mid J_1^p(u) = c_1(p)\right\}$$

is a nonempty ordered set of solutions of (PDE). The analogue of Lemma 3.2 in this setting is:

Lemma 3.3. *$\mathcal{M}_1(p) = \mathcal{M}_1$ and $c_1(p) = (\prod_1^n p_i)c_1$.*

Proof. It suffices to show that whenever $u \in \mathcal{M}_1(p)$: (i) $\tau_{-1}^i u = u$, $2 \leqslant i \leqslant n$, and (ii) $\tau_{-1}^1 u \geqslant u$. The proof of (i) is the same as that of Lemma 3.2. For (ii), observe that $\tau_{-1}^1 u \in \mathcal{M}_1(p)$ which is ordered. Hence if (ii) fails, $u > \tau_{-1}^1 u$ so by the definition of $\Gamma_1(p)$,

$$u \leqslant \tau_{-p_1}^1 u < \tau_{-p_1+1}^1 u < \cdots < u,$$

a contradiction. □

Remark 3.4. By (F_2), the replacement of $[ip_1, ip_1 + 1]$ in $T(ip_1, k)$ by $[ip_1 + j, ip_1 + j + 1]$ for any $j \in \mathbb{Z}$ does not effect the above arguments. The same is true if ℓ is replaced by $\ell + q$ for any $q \in \mathbb{R}^{n-1}$.

Theorem 3.5. *Any $U \in \mathcal{M}_1$ is a minimal solution of (PDE) in the sense of Giaquinta and Guisti.*

Proof. To show that (3.1) is satisfied, let $z \in \mathbb{R}^n$ and $r > 0$ such that $\Omega \subset B_r(z)$. Set

$$\mathcal{S}_r(z) = \left\{u \in W_{\mathrm{loc}}^{1,2}(\mathbb{R}^n) \mid u = U \text{ in } \mathbb{R}^n \setminus B_r(z)\right\}.$$

It suffices to prove that

$$\int_{B_r(z)} L(u)\,\mathrm{d}x \geqslant \int_{B_r(z)} L(U)\,\mathrm{d}x \tag{3.6}$$

for any $u \in \mathcal{S}_r(z)$. By Lemma 3.3, $\mathcal{M}_1 = \mathcal{M}_1(p)$ for any $p \in \mathbb{N}^n$. Choose p so that $\min_{1\leqslant i\leqslant n} p_i > 2r$. Further exploiting Remark 3.4, it can be assumed that $\Omega \subset B_r(z) \subset T(p)$. Hence by the proof of Theorem 2.18, U minimizes $\int_{B_r(z)} L(\cdot)\,\mathrm{d}x$ over $\mathcal{S}_r(z)$ and the proof is complete. □

An immediate consequence of Theorem 3.5 is

Corollary 3.7. *U_1 is the unique minimizer of $\int_{B_r(z)} L(\cdot)\,\mathrm{d}x$ in $\mathcal{S}_r(z)$.*

Proof. Suppose $u \in \mathcal{S}_r(z)$ so that

$$\int\limits_{B_r(z)} L(u)\,\mathrm{d}x = \int\limits_{B_r(z)} L(U_1)\,\mathrm{d}x. \tag{3.8}$$

Let $r^* > r$. Then by (3.8),

$$\int\limits_{B_{r^*}(z)} L(u)\,\mathrm{d}x = \int\limits_{B_{r^*}(z)\setminus\overline{B}_r(z)} L(U_1)\,\mathrm{d}x + \int\limits_{B_r(z)} L(u)\,\mathrm{d}x = \int\limits_{B_{r^*}(z)} L(U_1)\,\mathrm{d}x. \tag{3.9}$$

Hence $U_1 \in \mathcal{S}_{r^*}(z)$ and minimizes $\int_{B_{r^*(z)}} L(\cdot)\,\mathrm{d}x$ over $\mathcal{S}_{r^*}(z)$. Again as in Lemma 2.5 of [9], the set of such minimizers is ordered. Since u and U_1 belong to this set and $u = U_1$ in $B_{r^*}(z) \setminus \overline{B}_r(z)$, $u \equiv U_1$. The proof is complete. □

4. The general case

The goal of this section is to show how the results of Sections 1–2 together with induction and minimization arguments can be used to obtain more complex heteroclinic solutions of (PDE) corresponding to those obtained by Bangert [4] via his nonvariational approach.

To give an idea of the inductive procedure at level two, suppose $\mathcal{M}_1$ as obtained in Theorem 1.18 satisfies a gap condition:

$(*)_1$ There are adjacent $v_1 < w_1$ in $\mathcal{M}_1$.

Define the set of functions $\widehat{\Gamma}_2$ via

$$\widehat{\Gamma}_2 = \bigl\{u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n) \mid v_1 \leqslant u \leqslant \tau^1_{-1}u \leqslant w_1 \text{ and } u \text{ is 1-periodic in } x_3, \dots, x_n\bigr\}.$$

The renormalized functional, J_2 on $\widehat{\Gamma}_2$ is defined by

$$J_2(u) = \sum_{i\in\mathbb{Z}} J_{2,i}(u)$$

where

$$J_{2,i}(u) = J_1\bigl(\tau^2_{-i}u\bigr) - c_1.$$

Suppose that J_2 on $\widehat{\Gamma}_2$ has the analogues of the properties of J_1 on $\widehat{\Gamma}_1$ as given by Proposition 2.6 with 3^{o} replaced by

$$\int\limits_{T(\ell_1 e_1+\ell_2 e_2)} L(u)\,\mathrm{d}x \leqslant J_2(u) + c_0 + c_1$$

for any $\ell_1, \ell_2 \in \mathbb{Z}$. Suppose also that Proposition 2.14 is valid with appropriate changes of sub- or superscript 1's to 2's. Setting

$$\Gamma_2 = \bigl\{u \in \widehat{\Gamma}_2 \mid u \leqslant \tau^2_{-1}u \text{ and } v_1 \not\equiv u \not\equiv w_1\bigr\}$$

and

$$c_2 = \inf_{u\in\Gamma_2} J_2(u),$$

the analogue of Theorem 2.18 holds here providing a solution of (PDE) that is heteroclinic in x_1 from v_0 to w_0 and heteroclinic in x_2 from v_1 to w_1. Moreover as in Theorem 3.5, U_2 is a minimal solution of (PDE) in the sense of Giaquinta and Guisti.

Now setting

$$\mathcal{M}_2 = \{u \in \Gamma_2 \mid J_2(u) = c_2\},$$

a gap condition $(*)_2$ can be introduced and the process continues. To carry out the induction argument properly, let $m < n$ and assume the gap condition:

$(*)_i$ There are adjacent $v_i < w_i$ in $\mathcal{M}_i$ holds for $i = 0, \dots, m-1$. Let

$$\widehat{\Gamma}_i = \{u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n) \mid v_{i-1} \leqslant u \leqslant \tau^j_{-1} u \leqslant w_{i-1},\ 1 \leqslant j < i \text{ and } u \text{ is 1-periodic in } x_{i+1}, \dots, x_n\} \quad (4.1)_i$$

for $1 \leqslant i \leqslant m$. The ith renormalized functional, $J_i(u)$, is given by

$$J_i(u) = \sum_{p \in \mathbb{Z}} J_{i,p}(u) \quad (4.2)_i$$

where

$$J_{i,p}(u) = J_{i-1}(\tau^i_{-p} u) - c_{i-1}. \quad (4.3)_i$$

Suppose that J_i on $\widehat{\Gamma}_i$ possesses the following properties for $1 \leqslant i \leqslant m$:

Proposition 4.4$_i$. *For $u \in \widehat{\Gamma}_i$,*

1° $J_{i,p}(u) \geqslant 0$ *for all* $p \in \mathbb{Z}$.
2° $J_i(u) \geqslant 0$.
3° $\int_{T(\sum_1^i \ell_q e_q)} L(u)\,\mathrm{d}x \leqslant J_i(u) + \sum_0^{i-1} c_q$.
4° J_i *is weakly lsc (in* $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$*) on* $\widehat{\Gamma}_i$.

For $1 \leqslant i \leqslant m$, set

$$\Gamma_i = \{u \in \widehat{\Gamma}_i \mid u \leqslant \tau^i_1 u \text{ and } v_{i-1} \not\equiv u \not\equiv w_{i-1}\} \quad (4.5)_i$$

and assume (with the understanding that $\sum_1^0 \ell_q e_q = 0$):

Proposition 4.6$_i$. *If $u \in \Gamma_i$ and $J_i(u) < \infty$, then as $j \to \infty$, $\tau^i_j u \to v_{i-1}$ weakly in $W^{1,2}(T(\sum_1^{i-1} \ell_q e_q))$ for all $\ell_1, \dots, \ell_{i-1} \in \mathbb{Z}$ and as $j \to -\infty$, $\tau^i_j u \to w_{i-1}$ weakly in $W^{1,2}(T(\sum_1^{i-1} \ell_q e_q))$ for all $\ell_1, \dots, \ell_{i-1} \in \mathbb{Z}$.*

Finally define

$$c_i = \inf_{u \in \Gamma_i} J_i(u), \quad 1 \leqslant i \leqslant m, \quad (4.7)_i$$

and assume:

Theorem 4.8$_i$. *Let F satisfy* (F$_1$)–(F$_3$) *and let $(*)_i$ hold. Then*

1° *There is a $U_i \in \Gamma_i$ such that $J_i(U_i) = c_i$.*
2° *Any such U_i is a classical solution of* (*PDE*).

3° *U_i is heteroclinic from v_{i-1} to w_{i-1}:*

$$\|U_i - v_{i-1}\|_{C^2(T(\sum_1^i \ell_k e_k))} \to 0 \quad \text{as } \ell_i \to -\infty \quad \text{and}$$
$$\|U_i - w_{i-1}\|_{C^2(T(\sum_1^i \ell_k e_k))} \to 0 \quad \text{as } \ell_i \to \infty.$$

4° $v_{i-1} < U_i < \tau_{-1}^{j} U_i < w_{i-1}$, $1 \leqslant j \leqslant i$.
5° *$\mathcal{M}_i = \{u \in \Gamma_i \mid J_i(u) = c_i\}$ is an ordered set.*
6° *If $u \in \mathcal{M}_i$, u is even in $x_{i+1}, \dots, x_n$.*

Finally suppose that for $1 \leqslant i \leqslant m$.

Theorem 4.9$_i$. *Any $U \in \mathcal{M}_i$ is a minimal solution of (PDE) in the sense of Giaquinta and Guisti.*

Corollary 4.10$_i$. *U_i is the unique minimizer of $\int_{B_r(z)} L(\cdot)\,\mathrm{d}x$ over $\mathcal{S}_r(z) = \{\varphi \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n) \mid \varphi = U_i$ in $\mathbb{R}^n \setminus B_r(z)\}$.*

With these inductive facts at hand, the results can be extended to level $m+1$. To do so, begin by assuming there is a gap in $\mathcal{M}_m$:

$(*)_m$ There are adjacent $v_m < w_m$ in $\mathcal{M}_m$.

Then with $\widehat{\Gamma}_{m+1}$ defined in $(4.1)_{m+1}$ and J_{m+1} from $(4.2)_{m+1}$, we can give the

Proof of Proposition 4.4$_{m+1}$. To verify 1°_{m+1}–2°_{m+1}, let $u \in \widehat{\Gamma}_{m+1}$ and for $x_{m+1} \in [p + \frac{1}{2}, p+1]$, set

$$\varphi_p^{+}(u) = \tau_{-p}^{m+1} u \tag{4.11}$$

and for $x_{m+1} \in [p, p + \frac{1}{2}]$, set

$$\varphi_p^{-}(u) = \tau_{-p}^{m+1} u. \tag{4.12}$$

Extend these functions to 1-periodic functions in x_{m+1} as in the proof of Proposition 2.6. Then $\varphi_p^{\pm}(u) \in \widehat{\Gamma}_m$ and either (i) $\varphi_p^{\pm}(u) \in \Gamma_m$ or (ii) $\varphi_p^{\pm}(u) \in \{v_m, w_m\}$. If (i) holds,

$$J_{m+1,p}\big(\varphi_p^{\pm}(u)\big) = J_m\big(\varphi_p^{\pm}(u)\big) - c_m \geqslant 0$$

so

$$J_{m+1,p}(u) = \frac{1}{2}\Big(J_{m+1,p}\big(\varphi_p^{+}(u)\big) + J_{m+1,p}\big(\varphi_p^{-}(u)\big)\Big) \geqslant 0$$

and 1°_{m+1} is valid while if (ii) holds, $(4.3)_{m+1}$ and $(4.7)_m$ yield 1°_{m+1} with equality. Now 2°_{m+1} is immediate. Arguing as in the $m=0$ case, using $\varphi_p^{\pm}$,

$$J_m\big(\tau_{-\ell_{m+1}}^{m+1} u\big) \leqslant J_{m+1}(u) + c_m \tag{4.13}$$

follows from 1°_{m+1}–2°_{m+1}. Now applying $(4.3)_m$ and 3° of Proposition 4.4$_m$ gives 3°_{m+1}. Lastly 4° of Proposition 4.4$_m$ and the analogue here of the argument centered around (2.11) yield 4°_{m+1}. □

Next with Γ_{m+1} as provided by $(4.5)_{m+1}$, we have the

Proof of Proposition 4.6$_{m+1}$. Since $J_{m+1}(u) < \infty$, by 3° of Proposition 4.4$_m$, the sequence $(\tau_{\ell}^{m+1} u)_{\ell \in \mathbb{Z}}$ is bounded in $W^{1,2}(T(\sum_1^m \ell_i e_i))$ for each $\ell_1, \dots, \ell_m \in \mathbb{Z}$. Therefore there is a $v \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^m \times [0,1] \times \mathbb{R}^{n-(m+1)})$, 1-periodic in $x_{m+2}, \ldots, x_n$ such that along a subsequence, $\tau_\ell^{m+1} u$ converges weakly to v in $W^{1,2}(T(\sum_1^m \ell_i e_i))$ as $\ell \to \infty$ for each $\ell_1, \ldots, \ell_m \in \mathbb{Z}$. Moreover $u \leqslant \tau_{-1}^{m+1} u$ shows the sequence converges monotonically to v. Hence as $\ell \to \infty$, $\tau_{\ell-1}^{m+1} u \to \tau_{-1}^{m+1} v = v$, i.e. v is 1-periodic in x_{m+1} so $v \in W_{\mathrm{loc}}^{1,2}(\mathbb{R}^n)$. Moreover the corresponding properties for u imply

$$v_m \leqslant v \leqslant \tau_{-1}^i v \leqslant w_m < w_{m-1}, \quad 1 \leqslant j \leqslant m. \tag{4.14}$$

Therefore $v \in \Gamma_m$.

By $(4.2)_{m+1}$–$(4.3)_{m+1}$, 1° of Proposition 4.4_{m+1} and $J_{m+1}(u) < \infty$, as $|p| \to \infty$,

$$J_m\left(\tau_{-p}^{m+1} u\right) \to c_m. \tag{4.15}$$

Observe that if $\varphi_p^\pm(u)$ are as in the proof of Proposition 4.4_{m+1}, $\varphi_p^\pm(u) \to \varphi_0^\pm(v)$ as $p \to \infty$ weakly in $W^{1,2}(T(\sum_1^m \ell_i e_i))$ for each $\ell_1, \ldots, \ell_m \in \mathbb{Z}$. The functions $\varphi_p^\pm(u)$, $\varphi_0^\pm(v)$ belong to $\widehat{\Gamma}_m$. Hence

$$c_m \leqslant J_m\left(\varphi_0^\pm(v)\right) \tag{4.16}$$

and

$$\begin{aligned} c_m \leqslant J_m(v) &= \frac{1}{2} J_m\left(\varphi_0^+(v)\right) + \frac{1}{2} J_m\left(\varphi_0^-(v)\right) \\ &\leqslant \frac{1}{2} \varliminf_{p\to\infty} J_m\left(\varphi_p^+(u)\right) + \frac{1}{2} \varliminf_{p\to\infty} J_m\left(\varphi_p^-(u)\right) \\ &\leqslant \frac{1}{2} \varliminf_{p\to\infty} \left(J_m\left(\varphi_p^+(u)\right) + J_m\left(\varphi_p^-(u)\right)\right) \\ &= \varliminf_{p\to\infty} J_m\left(\tau_{-p}^{m+1} u\right) = c_m \end{aligned} \tag{4.17}$$

via (4.15). Consequently by (4.17), $J_m(v) = c_m$. Therefore (4.14) and $(*)_m$ show $v \in \{v_m, w_m\}$. But $u \in \Gamma_{m+1}$ so v being the monotone limit of $(\tau_{-p}^{m+1} u)$ as $p \to \infty$ implies $v = v_m$. Similarly $\tau_{-p}^{m+1} \to w_m$ as $p \to -\infty$ and the proof of Proposition 4.6_{m+1} is complete. □

Finally defining c_{m+1} via $(4.7)_{m+1}$ brings us the

Proof of Theorem 4.8_{m+1}. Let $(u_m) \subset \Gamma_{m+1}$ be a minimizing sequence for $(4.7)_{m+1}$. Then there is an $M > 0$ such that

$$J_{m+1}(u_k) \leqslant M, \quad k \in \mathbb{N}. \tag{4.18}$$

By Proposition 4.6_{m+1}, u_k can be normalized so that for $\ell < 0$,

$$\int_{T(\ell e_{m+1})} u_k \,\mathrm{d}x \leqslant \frac{1}{2} \int_{T(0)} (v_m + w_m) \,\mathrm{d}x < \int_{T(0)} u_k \,\mathrm{d}x. \tag{4.19}$$

By (4.18) and 3° of Proposition 4.4_{m+1}, (u_k) is bounded in $W_{\mathrm{loc}}^{1,2}(\mathbb{R}^n)$. Therefore there is a $U = U_{m+1} \in W_{\mathrm{loc}}^{1,2}(\mathbb{R}^n)$ such that, along a subsequence, $u_k \to U$ weakly in $W_{\mathrm{loc}}^{1,2}(\mathbb{R}^n)$, strongly in $L_{\mathrm{loc}}^2(\mathbb{R}^n)$, and pointwise a.e. as $k \to \infty$. Hence

$$v_m \leqslant U \leqslant \tau_{-1}^j U \leqslant w_m, \quad 1 \leqslant j \leqslant m+1 \tag{4.20}$$

and U is 1-periodic in $x_{m+2}, \ldots, x_n$. The normalization (4.19) implies

$$\int_{T(\ell e_{m+1})} U \,\mathrm{d}x \leqslant \frac{1}{2} \int_{T(0)} (v_m + w_m) \,\mathrm{d}x \leqslant \int_{T(0)} U \,\mathrm{d}x \tag{4.21}$$

and therefore $v_m \not\equiv U \not\equiv w_m$. Hence $U \in \Gamma_{m+1}$ and

$$J_{m+1}(U) \geqslant c_{m+1}. \tag{4.22}$$

Since (u_k) is a minimizing sequence, by 4° of Proposition 4.4_{m+1},

$$J_{m+1}(U) \leqslant c_{m+1}. \tag{4.23}$$

Thus 1° of Theorem 4.8_{m+1} is valid.

Assuming 2° of Theorem 4.8_{m+1} for the moment, the $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$ bounds for U given by (4.13) together with Proposition 4.6_{m+1} yields 3° of Theorem 4.8_{m+1} with C^2 replaced by L^2. But then the argument of Theorem 2.18 gives convergence in C^2. Likewise, replacing τ^1_{-1} by τ^i_{-1}, $1 \leqslant i \leqslant m+1$, in (2.32)–(2.33) and following sentences shows $\varphi, \psi \in \Gamma_{m+1}$. Then replacing $T(ie_1)$ in (2.27) by $T(\sum_1^{m+1} \ell_i e_i)$ shows $c_{m+1} = J_{m+1}(\varphi) = J_{m+1}(\psi)$ and the reasoning following this implies 4° of Theorem 4.8_{m+1}. Then 5°–6° also follow as earlier.

Lastly to verify 2° of Theorem 4.8_{m+1}, the proof of 2° of Theorem 2.18 can be applied here provided that (a) $v_m \leqslant \varphi_j \leqslant w_m$ for any minimizer φ_j of (2.23) over

$$\mathcal{S}_{r,m+1}(z_j) = \bigl\{\varphi \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n) \mid \varphi = U_{m+1} \text{ in } R^n \setminus B_r(z_j)\bigr\}.$$

(b) $U^* \leqslant \tau^i_{-1} U^*$, $1 \leqslant i \leqslant m+1$, and (c) $v_m \not\equiv U^* \not\equiv w_m$. To prove (a), note that $v_m \leqslant U_{m+1} \leqslant w_m$. Therefore $\psi = \min(\varphi_j, v_m) \in \mathcal{S}_{r,m}(z_j)$ (with $U_m = v_m$) and $\chi = \max(\varphi_j, v_m) \in \mathcal{S}_{r,m+1}(z_j)$. Hence by Theorem 4.9_m,

$$\int_{B_r(z_j)} L(\psi)\,\mathrm{d}x \geqslant \int_{B_r(z_j)} L(v_m)\,\mathrm{d}x \tag{4.24}$$

and by the definition of φ_j,

$$\int_{B_r(z_j)} L(\chi)\,\mathrm{d}x \geqslant \int_{B_r(z_j)} L(\varphi_j)\,\mathrm{d}x. \tag{4.25}$$

Adding (4.24)–(4.25) shows:

$$\begin{aligned}\int_{B_r(z_j)} L(v_m)\,\mathrm{d}x + \int_{B_r(z_j)} L(\varphi_j)\,\mathrm{d}x &= \int_{B_r(z_j)} L(\psi)\,\mathrm{d}x + \int_{B_r(z_j)} L(\chi)\,\mathrm{d}x \\ &\geqslant \int_{B_r(z_j)} L(v_m)\,\mathrm{d}x + \int_{B_r(z_j)} L(\varphi_j)\,\mathrm{d}x.\end{aligned} \tag{4.26}$$

Hence

$$\int_{B_r(z_j)} L(\psi)\,\mathrm{d}x = \int_{B_r(z_j)} L(v_m)\,\mathrm{d}x \tag{4.27}$$

and

$$\int_{B_r(z_j)} L(\chi)\,\mathrm{d}x = \int_{B_r(z_j)} L(\varphi_j)\,\mathrm{d}x.$$

But (4.27) and Corollary 4.10_m imply $\psi \equiv v_m$, i.e. $\varphi_j \geqslant v_m$. Similarly $\varphi_j \leqslant w_m$ and (a) is proved.

To check that (b) holds, we argue exactly as in the proof of the analogous situation in Theorem 2.18 – see (2.32)–(2.34) with e_1 replaced by e_ℓ, $1 \leqslant \ell \leqslant m+1$. Lastly (c) follows the same lines as its analogue in the proof of Theorem 2.18. This completes the proof of Theorem 4.8_{m+1}. □

Next to prove Theorem 4.9_{m+1} requires the extension of Lemmas 3.2 and 3.3 from level m to level $m+1$ and is carried out exactly as earlier. Likewise Corollary 4.10_{m+1} is proved exactly as in Corollary 3.7 and the induction process is complete.

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