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# Regularity for degenerate elliptic problems via $p$ -harmonic approximation

## Régularité pour problèmes elliptiques dégénérés via approximation $p$ -harmonique

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### Abstract

We introduce a new method to prove regularity of solutions to certain degenerate elliptic problems. The method is based on the  $p$ -harmonic approximation lemma, recently proved by the authors in [F. Duzaar, G. Mingione, The  $p$ -harmonic approximation and the regularity of  $p$ -harmonic maps, *Calc. Var.*, 2004, in press], that allows to approximate functions with  $p$ -harmonic functions in the same way as the classical harmonic approximation lemma (going back to De Giorgi) does via harmonic functions. The method presented here also bypasses certain difficulties arising when treating some degenerate and singular problems with a weak structure, such as degenerate and singular quasiconvex integrals, and provides transparent and elementary proofs. © 2004 Elsevier SAS. All rights reserved.

### Résumé

Nous introduisons une nouvelle méthode pour prouver la régularité des solutions de certains problèmes elliptiques dégénérés. La méthode se base sur le lemme d'approximation  $p$ -harmonique (prouvé récemment par les auteurs dans [F. Duzaar, G. Mingione, The  $p$ -harmonic approximation and the regularity of  $p$ -harmonic maps, Preprint Dip. Mat. Univ. Parma, 2002] qui permet d'approximer des fonctions par des fonctions  $p$ -harmoniques de la même façon où le lemme d'approximation harmonique classique (qui remonte à De Giorgi) le permettait avec des fonctions harmoniques. Le méthode présentée ici évite aussi des difficultés qui apparaissent dans certains problèmes dégénérés et singuliers avec structure faible, comme des intégrales quasiconvexes dégénérés et singuliers, et fournit des preuves élémentaires et transparentes. © 2004 Elsevier SAS. All rights reserved.

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## 1. Introduction

It is a historically well established fact that regularity methods from Geometric Measure Theory inspired the implementation of powerful techniques for regularization of solutions to nonlinear elliptic systems of partial differential equations. This started with the papers by Morrey, Giusti and Miranda [17,18,13] after the pioneering work of De Giorgi and Almgren for the regularity of minimal surfaces and minimizing varifolds, respectively. Recently, a more elementary proof of regularity of minimizers of elliptic integrals in Geometric Measure Theory has been proposed by Duzaar and Steffen [10] on the basis of the  $\mathcal{A}$ -harmonic approximation method, which is inspired again by the original methods of De Giorgi and later used by Simon [19,20]. The advantages of such a method, apart from the considerable technical simplifications, consist of the possibility to get optimal regularity results for solutions; moreover the optimal regularity is achieved for boundary value problems too. Following the tradition outlined at the beginning, the method was successfully transferred to the parametric case: in [7] and [6] it allowed to get optimal regularity results for the solutions to elliptic systems and almost minimizers of solutions to quasiconvex integrals thus giving a new elegant treatment of the regularity, yielding optimal regularity results, also for boundary value problems (see [14]). In this setting the main technical tool is the  $\mathcal{A}$ -harmonic approximation lemma (see Lemma 3 below). This lemma states, roughly speaking, that if a map  $f$  is approximately a solution to a linear elliptic system with constant coefficients in the sense of (3.5), then it is possible to find a true solution of such a system, say  $h$ , which is  $L^2$  close to  $f$  in the sense of (3.6).

The search for the degenerate analog of De Giorgi's harmonic approximation lemma (see for instance the version by Simon in [19,20]) ended with the paper [9], where the authors were able to show that a similar approximation lemma can be proven when replacing the Laplacian operator with the  $p$ -Laplacian operator: therefore replacing, in the approximation, harmonic functions with  $p$ -harmonic functions; the lemma, in a suitable scaled version, is also presented below (see Lemma 5). This, in a first stage, allowed to extend Simon's treatment of regularity of harmonic maps to  $p$ -harmonic maps (see again [9]). It is worth remarking that, although the proof of the classical harmonic approximation lemma (and therefore of the  $\mathcal{A}$ -harmonic approximation lemma) rests on simple weak compactness arguments, the proof of the  $p$ -harmonic approximation lemma involves the use of some approximations results via the Hardy–Littlewood Maximal Function plus subtle truncations and selection arguments. The difficulties are essentially due to the nonlinearity of the  $p$ -Laplacian operator (see [9] for the proof).

The aim of this paper is now twofold. First, we want to show how the two mentioned lemmata really link and form a unitary tool that allows to treat general, non-degenerate and degenerate problems in an elementary and transparent way. In doing so we shall achieve our second goal, that is the treatment of a family of quasiconvex functionals exhibiting a certain degenerate structure; such type of functionals, as far as we know, have not been treated up to now, from the point of view of the regularity. Moreover, we shall do that avoiding the use of tools like Reverse Hölder inequalities and Gehring's lemma. A typical model of such functionals is the following ( $U$  being a domain in  $\mathbb{R}^n$ ):

$$\int_U f(Du) dx = \int_U |Du|^p + g(Du) dx, \quad p > 1 \quad (1.1)$$

where  $g: \mathbb{R}^{nN} \rightarrow \mathbb{R}$  is a  $C^2$  quasiconvex function with  $p$ -growth

$$0 \leq g(A) \leq L(1 + |A|^p)$$

satisfying suitable smoothness assumptions. For instance,  $g$  can be a function vanishing on a ball centered at the origin. Moreover, it may also happen that  $g$  and/or its second derivatives, vanish on other large portions of  $\mathbb{R}^{nN}$ . In this way, the function  $f$  only satisfies the degenerate form of strict quasiconvexity:

$$\lambda \int_{(0,1)^n} (|A|^2 + |D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2 dx \leq \int_{(0,1)^n} f(A + D\varphi) - f(A) dx, \quad \lambda > 0, \quad (1.2)$$

for any  $A \in \mathbb{R}^{nN}$  and any smooth function  $\varphi$  with compact support in  $(0, 1)^n$ . More general functionals of the type

$$\int_U f(Du) dx$$

are then allowed here, prescribing, for the function  $f$  a degenerate behavior of  $p$ -Laplacian type at the origin (see assumption (H4) below). As a consequence, models of the type in (1.1) are covered.

For minimizers of such degenerate functionals we prove partial  $C^{1,\alpha}$  regularity, that is the Hölder continuity of the gradient  $Du$  outside a negligible closed set, for some exponent  $\alpha \in (0, 1)$ . We remind the reader that the importance of quasiconvexity in the Calculus of Variations stems from the fact that it is a necessary and sufficient condition for lower semicontinuity (see [17,1]). Our result extends results originally developed by Evans [11] and then extended up to optimal assumptions in [2,5], to the case of degenerate quasiconvex functionals. In the before mentioned papers condition (1.2) is replaced by its non-degenerate analog

$$\lambda \int_{(0,1)^n} (1 + |A|^2 + |D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2 dx \leq \int_{(0,1)^n} f(A + D\varphi) - f(A) dx, \quad \lambda > 0, \tag{1.3}$$

and therefore functionals of the type in (1.1) are ruled out. We would like to point out that, also thanks to our methods, delicate cases can be treated here. For instance, we cover the case of functionals with subquadratic growth:  $1 < p < 2$ ; that is, when the functional is *singular*. The problem of regularity in this case, raised after the examples of quasiconvex functions with subquadratic growth given by Šverák (see [21]), presented technical difficulties, and its complete treatment in the non-degenerate case was achieved by Carozza, Fusco and Mingione in [5] (see also [8] for the case of almost minimizers). As far as we know, the only papers dealing with singular functionals with subquadratic growth are [3] and [15]; these papers are devoted to convex functionals with special (diagonal) structure and the techniques used there, which are different from the ones usually employed in the non-degenerate case, strongly rely on certain tools such as weak Harnack inequalities, higher differentiability and approximation procedures; all these things are not available here, since we deal only with quasiconvex functionals. Nevertheless, by some careful estimations via the  $p$ -harmonic approximation lemma, we are able to find quite an elementary way to overcome these difficulties that also allows to avoid the use of Gehring’s lemma.

Finally, we discuss some of the technical aspects of our proofs. The analysis of the regularity of minimizers proceeds along a very natural path. Indeed, in order to achieve the partial Hölder continuity of the gradient, a standard method is to obtain an estimate for the growth of a certain quantity, traditionally called “excess”, see (2.4). In order to get such an estimate, which is already valid both for harmonic and  $p$ -harmonic functions, a local comparison argument will be used. More precisely, if the average of the gradient of the minimizer is not very small compared to the excess, then the problem behaves like a non-degenerate one, the minimizer turns out to be locally “approximately harmonic” and it can be compared to a suitable harmonic function via the  $\mathcal{A}$ -approximation lemma (see Lemma 3); therefore the desired growth estimate follows. If not, that is, if the average of the gradient is suitably small, then the problem really behaves like a degenerate one and the solution will be “approximately  $p$ -harmonic”: it will be compared to a suitable  $p$ -harmonic function (see Lemma 5); the estimate will follow again. Finally, the two cases will match via a delicate iteration procedure implemented in Section 5 (Lemma 13). This iteration procedure shows how the method of  $\mathcal{A}$ -harmonic approximation and the one of  $p$ -harmonic approximation perfectly combine in order to build a unified, powerful tool.

## 2. Notation and statement of the result

In the following,  $c$  will denote a positive constant, possibly varying from expression to expression; most peculiar occurrences will be emphasized properly; we shall denote  $B_\varrho(x_0) := \{x \in \mathbb{R}^n : |y - x_0| < \varrho\}$ ; when no ambiguity

will arise, or when the center will not be important in the context, we shall also denote  $B_\varrho(x_0) \equiv B_\varrho$ . Adopting a similar convention about the centers, if  $g \in L^1(B_\varrho(x_0))$  we shall put:

$$(g)_\varrho \equiv (g)_{x_0, \varrho} := \int_{B_\varrho(x_0)} g(x) dx.$$

Throughout the paper we consider functionals of the form

$$\mathcal{F}(u) = \int_U f(Du) dx, \quad u \in W^{1,p}(U, \mathbb{R}^N), \quad p > 1, \quad p \neq 2, \quad (2.1)$$

where  $U$  is an open domain in  $\mathbb{R}^n$ ,  $n, N$  being integers such that  $n \geq 2$ ,  $N > 1$  and  $f: \mathbb{R}^{nN} \rightarrow \mathbb{R}$  satisfies the following structure conditions:

(H1)  $f \in C^2(\mathbb{R}^{nN})$  if  $p > 2$  and  $f \in C^2(\mathbb{R}^{nN} \setminus \{0\})$  if  $1 < p < 2$ .

(H2) (growth condition) there exists  $\Lambda \in (1, +\infty)$  such that for all  $A \in \mathbb{R}^{nN}$  we have

$$|D^2 f(A)| \leq \Lambda |A|^{p-2} \quad (|A| \neq 0 \text{ if } 1 < p < 2).$$

(H3) (Hölder continuity of second derivatives) there exist a constant  $0 < L < \infty$  and some Hölder exponent

$$\alpha \in \begin{cases} (0, \min(1, p-2)) & \text{if } p > 2, \\ (0, 2-p) & \text{if } 1 < p < 2, \end{cases}$$

such that for all  $A, B \in \mathbb{R}^{nN}$  we have in the case  $p > 2$

$$|D^2 f(A) - D^2 f(B)| \leq L(|A|^2 + |B|^2)^{\frac{p-2-\alpha}{2}} |A - B|^\alpha,$$

whereas in the subquadratic case  $1 < p < 2$  we have

$$|D^2 f(A) - D^2 f(B)| \leq L|A|^{p-2}|B|^{p-2}(|A|^2 + |B|^2)^{\frac{2-p-\alpha}{2}} |A - B|^\alpha,$$

provided  $|A| \neq 0 \neq |B|$ .

(H4) ( $p$ -Laplacian type behaviour at 0) we have

$$\lim_{t \downarrow 0} \frac{Df(tA)}{t^{p-1}} = |A|^{p-2} A$$

uniformly in  $\{A \in \mathbb{R}^{nN}: |A| = 1\}$ .

(H5) (degenerate quasiconvexity) the function  $f$  is (strictly) degenerate quasiconvex, i.e. there exists a constant  $\lambda > 0$  such that

$$\int_{B_\varrho(x_0)} (f(A + D\varphi) - f(A)) dx \geq \lambda \int_{B_\varrho(x_0)} (|A|^2 + |D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2 dx$$

for  $B_\varrho(x_0) \Subset U$ ,  $A \in \mathbb{R}^{nN}$  and  $\varphi \in C_0^1(B_\varrho(x_0), \mathbb{R}^N)$ .

Let us briefly comment on the assumptions. We first note that (H2) implies the growth conditions

$$|Df(A)| \leq \Lambda |A|^{p-1}, \quad |f(A)| \leq |f(0)| + \Lambda |A|^p \quad (2.2)$$

for  $A \in \mathbb{R}^{nN}$ . Assumption (H3) is quite common for degenerate integrals (see [3,15]) while, of course, taking  $\alpha$  small enough to satisfy the condition imposed in (H3) entails no loss of generality. Finally, we have to distinguish the formulation of the Hölder continuity between the cases  $p > 2$  and  $1 < p < 2$ , referring to the degenerate model

case  $f(A) := |A|^p$  (which does not satisfy the first condition in (H3) when  $p < 2$ ). Of course we restrict to the case  $p \neq 2$ ; this case is non-degenerate and it has already been treated [11,2]. Assumption (H4) serves to prescribe the type of degeneration of the functional: the ellipticity of  $f$  degenerates at the origin as  $|A|^p$ . Finally, hypothesis (H5) implies that for all  $\xi \in \mathbb{R}^N, \eta \in \mathbb{R}^n$  we have

$$D^2 f(A)(\eta \otimes \xi, \eta \otimes \xi) \geq 2\lambda |A|^{p-2} |\xi|^2 |\eta|^2, \quad |A| \neq 0. \tag{2.3}$$

We can state our regularity result.

**Theorem 1.** *Let  $u \in W^{1,p}(U, \mathbb{R}^N)$  be local minimizer of the functional  $\mathcal{F}$ , under the assumptions (H1)–(H5). Then there exists  $\alpha = \alpha(n, N, p) \in (0, 1)$  and an open subset  $U_0 \subset U$  such that:*

$$Du \in C^{0,\alpha}(U_0, \mathbb{R}^{nN}), \quad |U \setminus U_0| = 0.$$

Now, it is well known that for non-degenerate quasiconvex integrals (that is those ones satisfying (1.3) rather than (H5)), the Hölder continuity exponent of the gradient can be picked arbitrarily close to 1:  $Du \in C^{0,\beta}(U_0, \mathbb{R}^{nN})$  for any  $\beta \in (0, 1)$  (see for instance [3,5]), while here we can reach only a certain exponent  $\alpha$ . This is unavoidable, since the regularity of  $p$ -harmonic functions themselves does not go beyond this degree. Anyway, our proof allows a finer analysis on the degree of regularity of the gradient in that if the gradient “stays” far from the origin (the zone where the problem becomes degenerate) in a suitable asymptotic sense, then the regularity exhibited by the minimizer is a bit higher; in particular it does not depend on the one found via the estimates for the solutions to the  $p$ -Laplacian system (see Lemma 1). To be precise, we recall that a regular point  $x_0 \in \mathbb{R}^n$  is a point such that  $Du$  is Hölder continuous in a neighborhood of  $x_0$ . Let us introduce the following notation, with  $V(B) = |B|^{(p-2)/2} B$ , for a function  $v \in W^{1,p}(U, \mathbb{R}^N)$  and  $B_r(x_0) \Subset U$ :

$$\begin{aligned} \Phi(v; r) &\equiv \Phi(v; x_0, r) = \int_{B_r} |(Dv)_r|^{p-2} |Dv - (Dv)_r|^2 + |Dv - (Dv)_r|^p dx \quad \text{if } p > 2, \\ \Phi(v; r) &\equiv \Phi(v; x_0, r) = \int_{B_r} |V(Dv) - (V(Dv))_r|^2 dx \quad \text{if } 1 < p < 2. \end{aligned} \tag{2.4}$$

**Theorem 2.** *Let  $u \in W^{1,p}(U, \mathbb{R}^N)$  be local minimizer of the functional  $\mathcal{F}$ , under the assumptions (H1)–(H5) and  $p > 2$ . Let  $\mathcal{R}(u)$  denote the set of regular points of  $u$ . Then*

$$\mathcal{R}(u) = \{x_0 \in U: \liminf_{r \rightarrow 0} \Phi(u; x_0, r) = 0\}. \tag{2.5}$$

Moreover, if  $x_0 \in \mathcal{R}(u)$  and

$$\limsup_{r \rightarrow 0} \frac{|(Du)_{x_0,r}|^p}{\Phi(u; x_0, r)} = +\infty, \tag{2.6}$$

then there exists a  $\sigma > 0$  such that  $Du \in C^{0,\beta}(B_\sigma(x_0), \mathbb{R}^{nN})$  for any  $\beta \in (0, 2/p)$ . Furthermore, if  $Du(x_0) \neq 0$ ,  $Du \in C^{0,\beta}(B_\sigma(x_0), \mathbb{R}^{nN})$  for any  $\beta \in (0, 1)$ .

**Theorem 3.** *Let  $u \in W^{1,p}(U, \mathbb{R}^N)$  be local minimizer of the functional  $\mathcal{F}$ , under the assumptions (H1)–(H5) and  $1 < p < 2$ . Let  $\mathcal{R}(u)$  denote the set of regular points of  $u$ . Then*

$$\mathcal{R}(u) = \{x_0 \in U: \liminf_{r \rightarrow 0} \Phi(u; x_0, r) = 0\}.$$

Moreover, if  $x_0 \in \mathcal{R}(u)$  and

$$\limsup_{r \rightarrow 0} \frac{|(V(Du))_{x_0,r}|^2}{\Phi(u; x_0, r)} = +\infty,$$

then there exists a  $\sigma > 0$  such that  $Du \in C^{0,\beta}(B_\sigma(x_0), \mathbb{R}^{nN})$  for any  $\beta \in (0, 1)$ . In particular, this holds for the regular points  $x_0$  such that  $Du(x_0) \neq 0$ .

For a more precise statement see also Remark 1 at the very end of the paper.

### 3. Preliminary results

We shall widely use the functions  $V, V_\mu : \mathbb{R}^k \rightarrow \mathbb{R}^k$

$$V(B) = |B|^{\frac{p-2}{2}} B, \quad V_\mu(B) = (\mu^2 + |B|^2)^{\frac{p-2}{4}} B \quad \text{for } B \in \mathbb{R}^k, k \in \mathbb{N}, \mu \geq 0.$$

The following lemma collects some algebraic properties of the functions  $V_\mu$  and  $V$ .

**Lemma 1.** *There exists  $c = c(k, p) > 1$  such that, for any  $B, C \in \mathbb{R}^k$ :*

$$c^{-1}(|B|^2 + |C|^2)^{\frac{p-2}{4}} |B - C| \leq |V(B) - V(C)| \leq c(|B|^2 + |C|^2)^{\frac{p-2}{4}} |B - C|; \quad (3.1)$$

moreover, the following Young type inequality is satisfied, for any  $\mu \geq 0$ :

$$(\mu + |B|^2)^{\frac{p-2}{2}} |B||C| \leq c(|V_\mu(B)|^2 + |V_\mu(C)|^2). \quad (3.2)$$

Furthermore,

$$|V_\mu(B + C)| \leq c(p)(|V_\mu(B)| + |V_\mu(C)|); \quad |V_\mu(tB)| \leq \max\{t, t^{p/2}\} |V_\mu(B)| \quad \forall t > 0. \quad (3.3)$$

Finally, in the case  $1 < p < 2$  there exists  $c = c(p) > 1$ , independent of  $\mu \geq 0$ , such that:

$$c^{-1} \min\{|B|, |B|^{p/2}\} \leq |V_1(B)| \leq c \min\{|B|, |B|^{p/2}\}; \quad |V_\mu(B)| \leq |B|^{p/2}. \quad (3.4)$$

The inequality in (3.1) can be retrieved from [3], Lemma 2.2, while the one in (3.2) can be easily adapted from [4], Lemma 2.3 (in this paper the proof is given for  $\mu = 1$ ; the general case  $\mu > 0$  can be obtained via a simple scaling argument while the case  $\mu = 0$  reduces to the standard Young's inequality). The last facts are from [5], Lemma 2.1; the proofs are presented there in the case  $\mu = 1$ ; the general case follows in a similar way. We want to emphasize that in the following we shall use repeatedly the function  $V_\mu$  with various domains, i.e. for various values of  $k \in \mathbb{N}$  (usually  $k = N$  and  $k = nN$ ), also in the same formula.

The next algebraic fact can be retrieved again from [3], Lemma 2.1.

**Lemma 2.** *For every  $t \in (-1/2, 0)$  and  $\mu \geq 0$  we have*

$$1 \leq \frac{\int_0^1 (\mu^2 + |A + s(\tilde{A} - A)|^2)^t}{(\mu^2 + |A|^2 + |\tilde{A}|^2)^t} \leq \frac{8}{2t + 1}$$

for any  $A, \tilde{A} \in \mathbb{R}^{nN}$ , not both zero if  $\mu = 0$ .

In the following we shall collect a few preliminary lemmata we shall use in our proofs. The first one is the  $\mathcal{A}$ -approximation lemma, whose proof can be found in [10]:

**Lemma 3** ( $\mathcal{A}$ -harmonic approximation). *There exists a positive function  $\delta(n, N, \lambda, \Lambda, \varepsilon) \leq 1$  with the following property: Whenever  $\mathcal{A}$  is a bilinear form on  $\mathbb{R}^{nN}$  which is elliptic in the sense of Legendre–Hadamard with*

ellipticity constant  $\lambda > 0$  and upper bound  $\Lambda, \varepsilon$  and  $\varrho$  are given positive numbers, and  $v \in W^{1,2}(B_\varrho, \mathbb{R}^N)$  with  $\int_{B_\varrho} |Dv|^2 dx \leq 1$  is approximatively  $\mathcal{A}$ -harmonic in the sense that

$$\left| \int_{B_\varrho} \mathcal{A}(Dv, D\varphi) dx \right| \leq \delta(n, N, \lambda, \Lambda, \varepsilon) \sup_{B_\varrho} |D\varphi| \tag{3.5}$$

holds for all  $\varphi \in C_0^1(B_\varrho, \mathbb{R}^N)$ , then there exists an  $\mathcal{A}$ -harmonic function  $h \in W^{1,2}(B_\varrho, \mathbb{R}^N)$  such that

$$\int_{B_\varrho} |Dh|^2 dx \leq 1 \quad \text{and} \quad \varrho^{-2} \int_{B_\varrho} |v - h|^2 dx \leq \varepsilon. \tag{3.6}$$

Of course, a function  $h$  on  $B_\varrho$  is termed an  $\mathcal{A}$ -harmonic function iff:

$$\int_{B_\varrho} \mathcal{A}(Dh, D\varphi) dx = 0 \quad \forall \varphi \in C_0^1(B_\varrho, \mathbb{R}^N).$$

The following variant of the  $\mathcal{A}$ -approximation lemma can be retrieved from [8]:

**Lemma 4** ( *$\mathcal{A}$ -harmonic approximation,  $V_1$ -version*). *There exists a positive function  $\delta(n, N, p, \lambda, \Lambda, \varepsilon) \leq 1$  with the following property: Whenever  $\mathcal{A}$  is a bilinear form on  $\mathbb{R}^{nN}$  which is elliptic in the sense of Legendre–Hadamard with ellipticity constant  $\lambda > 0$  and upper bound  $\Lambda, \varepsilon$  and  $\varrho$  are given positive numbers, and  $v \in W^{1,p}(B_\varrho, \mathbb{R}^N)$  with  $\int_{B_\varrho} |V_1(Dv)|^2 dx \leq s^2 \leq 1$  is approximatively  $\mathcal{A}$ -harmonic in the sense that*

$$\left| \int_{B_\varrho} \mathcal{A}(Dv, D\varphi) dx \right| \leq s\delta \sup_{B_\varrho} |D\varphi|$$

holds for all  $\varphi \in C_0^1(B_\varrho, \mathbb{R}^N)$ , then there exists an  $\mathcal{A}$ -harmonic function  $h \in W^{1,p}(B_\varrho, \mathbb{R}^N)$  such that, for an absolute constant  $c_0 = c_0(p) \geq 1$

$$\int_{B_\varrho} |V_1(Dh)|^2 dx \leq c_0 \quad \text{and} \quad \int_{B_\varrho} \left| V_1\left(\frac{v - sh}{\varrho}\right) \right|^2 dx \leq c_0 s^2 \varepsilon.$$

The next lemma is the degenerate variant of the  $\mathcal{A}$ -approximation lemma, where a linear operator with constant coefficients is replaced by the degenerate  $p$ -Laplacian operator; the proof can be found in [9] (again a  $p$ -harmonic function  $h \in W^{1,p}(B_\varrho, \mathbb{R}^N)$  will be a solution of the  $p$ -Laplacian system in  $B_\varrho$ ).

**Lemma 5** ( *$p$ -harmonic approximation*). *For any  $\varepsilon > 0$  there exists a positive constant  $\delta \in (0, 1]$ , depending only on  $n, N, p$  and  $\varepsilon$ , such that the following is true: whenever  $w \in W^{1,p}(B_\varrho, \mathbb{R}^N)$  with  $\int_{B_\varrho} |Dw|^p dx \leq 1$  is approximatively  $p$ -harmonic in the sense that*

$$\left| \int_{B_\varrho} |Dw|^{p-2} Dw \cdot D\varphi dx \right| \leq \delta \sup_{B_\varrho} |D\varphi|$$

holds for all  $\varphi \in C_0^1(B_\varrho, \mathbb{R}^N)$ , then there exists a  $p$ -harmonic function  $h \in W^{1,p}(B_\varrho, \mathbb{R}^N)$  such that

$$\int_{B_\varrho} |Dh|^p dx \leq 1 \quad \text{and} \quad \varrho^{-p} \int_{B_\varrho} |w - h|^p dx \leq \varepsilon.$$

From [16] (Lemma 2 worked out for  $p \neq 2$ ) we recall the following facts; suppose  $u \in L^2(B_\varrho(x_0), \mathbb{R}^N)$ , then we denote by  $P_{x_0, \varrho}$  the unique affine function minimizing  $P \mapsto \int_{B_\varrho(x_0)} |u - P|^2$  amongst all affine functions  $P : \mathbb{R}^n \rightarrow \mathbb{R}^N$ . Note that  $P_{x_0, \varrho}(x) = u_{x_0, \varrho} + Q_{x_0, \varrho}(x - x_0)$  where  $Q_{x_0, \varrho} = \frac{n+2}{\varrho^2} \int_{B_\varrho(x_0)} u(x) \otimes (x - x_0) dx$  is the momentum of  $u$ . Then, the following properties hold:

**Lemma 6.** *Let  $p \geq 2$ . There exists a constant  $c = c(n, p)$  such that the following assertions hold: for every  $u \in L^p(B_\varrho(x_0), \mathbb{R}^N)$  we have*

$$|Q_{x_0, \varrho} - Q_{x_0, \theta\varrho}|^p \leq \frac{c}{(\theta\varrho)^p} \int_{B_{\theta\varrho}(x_0)} |u - P_{x_0, \varrho}|^p dx. \tag{3.7}$$

For every  $u \in W^{1,p}(B_\varrho(x_0), \mathbb{R}^N)$  we have

$$|Q_{x_0, \varrho} - (Du)_{x_0, \varrho}|^p \leq c \int_{B_\varrho(x_0)} |Du - (Du)_{x_0, \varrho}|^p dx. \tag{3.8}$$

The next lemma is an iteration result; the case  $1 < p < 2$  can be inferred directly from [5]; the argument is based on (3.3) and works for any  $p > 1$ .

**Lemma 7.** *Let  $0 < \vartheta < 1$ ,  $a, b \geq 0$ ,  $A \in \mathbb{R}^{nN}$ ,  $v \in L^p(B_\varrho(x_0), \mathbb{R}^N)$  and  $g : [\varrho/2, \varrho] \rightarrow [0, \infty)$  be a bounded function satisfying*

$$g(t) \leq \vartheta g(s) + a \int_{B_\varrho(x_0)} \left| V_{|A|} \left( \frac{v}{s-t} \right) \right|^2 dx + b,$$

for all  $\varrho/2 \leq t < s \leq \varrho$ . Then there exists a constant  $c$  depending only on  $\vartheta$  and  $p$  such that

$$g\left(\frac{\varrho}{2}\right) \leq c \left( a \int_{B_\varrho(x_0)} \left| V_{|A|} \left( \frac{v}{\varrho} \right) \right|^2 dx + b \right).$$

The following version of Uhlenbeck’s result can be found in [12] and [3], according to the cases  $p > 2$  and  $1 < p < 2$ .

**Proposition 1.** *There exist constants  $c \geq 1$  and  $\gamma \in (0, 1)$ , depending only on  $n, N$  and  $p > 1$  with the following property: Whenever  $h \in W^{1,p}(U, \mathbb{R}^N)$  is a solution of*

$$\int_U |Dh|^{p-2} Dh \cdot D\varphi dx = 0 \quad \text{for all } \varphi \in C_c^1(U, \mathbb{R}^N),$$

and  $B_R(x_0) \Subset U$  then, for any  $0 < r \leq R$

$$\sup_{B_{R/2}(x_0)} |Dh|^p \leq c \int_{B_R(x_0)} |Dh|^p dx, \quad \Phi(h; x_0, r) \leq c \left(\frac{r}{R}\right)^{2\gamma} \Phi(h; x_0, R), \tag{3.9}$$

where  $\Phi(h; x_0, r)$  has been defined in (2.4).

The first step in proving a partial regularity theorem for  $\mathcal{F}$ -minimizing functions is to establish a suitable Caccioppoli-type-inequality. The following version is tailored to our needs and differs from the ones in [11,2,5]



in that it is stated in terms of  $V_{|A|}$ , that is, taking into account the possible degeneracy of the strict quasiconvexity (H5) when  $|A|$  approaches 0.

**Proposition 2.** *Let  $u \in W^{1,p}(U, \mathbb{R}^N)$  be  $\mathcal{F}$ -minimizing in  $U$ . There exists a constant  $c = c(p, \lambda, \Lambda)$  such that for every ball  $B_\rho(x_0) \Subset U$ ,  $\xi \in \mathbb{R}^N$ ,  $A \in \mathbb{R}^{nN}$*

$$\int_{B_{\rho/2}} |V_{|A|}(Du - A)|^2 dx \leq c \int_{B_\rho} \left| V_{|A|} \left( \frac{u - \xi - A(x-x_0)}{\rho} \right) \right|^2 dx. \tag{3.10}$$

**Proof.** Let  $B_\rho(x_0) \Subset U$ ,  $\xi \in \mathbb{R}^N$  and  $A \in \mathbb{R}^{nN}$  be fixed. W.l.o.g. we may assume that  $x_0 = 0$ . We choose  $\rho/2 \leq t < s \leq \rho$  and a standard cut-off function and  $\eta \in C_0^\infty(B_\rho, [0, 1])$  such that  $\eta \equiv 1$  on  $B_t$ ,  $\eta \equiv 0$  outside  $B_s$ , and  $|\nabla \eta| \leq 2(s-t)^{-1}$ . We set  $v = u - \xi - Ax$ ,  $\varphi = \eta v$ ,  $\psi = (1 - \eta)v$ . Then it turns out that  $D\varphi + D\psi = Dv = Du - A$  and

$$|D\varphi|^p \leq 2^{2p-1} \left( |Dv|^p + \left| \frac{v}{s-t} \right|^p \right) \quad \text{and} \quad |D\psi|^p \leq 2^{2p-1} \left( |Dv|^p + \left| \frac{v}{s-t} \right|^p \right). \tag{3.11}$$

Using the hypothesis (H5) we find

$$\begin{aligned} & \lambda \int_{B_s} (|A|^2 + |D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2 dx \\ & \leq \int_{B_s} (f(Du - D\psi) - f(Du)) dx \\ & \quad + \int_{B_s} (f(Du) - f(Du - D\varphi)) dx + \int_{B_s} (f(A + D\psi) - f(A)) dx \\ & =: \text{I} + \text{II} + \text{III}. \end{aligned} \tag{3.12}$$

The  $\mathcal{F}$ -minimality of  $u$  implies  $\text{II} \leq 0$ . To estimate  $\text{I} + \text{III}$  we note that

$$\begin{aligned} \text{I} + \text{III} &= \int_{B_s} \int_0^1 [Df(A + \tau D\psi) - Df(A)] d\tau D\psi dx \\ & \quad + \int_{B_s} \int_0^1 [Df(A) - Df(A + Dv - \tau D\psi)] d\tau D\psi dx = \text{I}' + \text{III}' \end{aligned}$$

with the obvious labelling. To estimate  $\text{I}'$  we use the bound (H2) and Lemma 2 twice

$$\begin{aligned} |\text{I}'| &= \left| \int_{B_s} \int_0^1 \int_0^1 D^2 f(A + s\tau D\psi)(\tau D\psi, D\psi) ds d\tau dx \right| \\ & \leq c\Lambda \int_{B_s} \int_0^1 (|A| + |\tau D\psi|)^{p-2} \tau |D\psi|^2 d\tau dx \leq c\Lambda \int_{B_s \setminus B_t} |V_{|A|}(D\psi)|^2 dx, \end{aligned} \tag{3.13}$$

where the constant  $c$  depends only on  $p$ . To estimate  $\text{III}'$  we use the assumption (H2), again Lemma 2 and Young's inequality for  $V_{|A|}$ , i.e. (3.2), and finally (3.3), in order to obtain

$$\begin{aligned}
|\text{III}'| &= \left| \int_{B_s} \int_0^1 \int_0^1 D^2 f(A + s(Dv - \tau D\psi))(Dv - \tau D\psi, D\psi) ds d\tau dx \right| \\
&\leq \Lambda \int_{B_s} \int_0^1 \int_0^1 |A + s(Dv - \tau D\psi)|^{p-2} |Dv - \tau D\psi| |D\psi| ds d\tau dx \\
&\leq c\Lambda \int_{B_s} \int_0^1 (|A| + |Dv - \tau D\psi|)^{p-2} |Dv - \tau D\psi| |D\psi| d\tau dx \\
&\leq c\Lambda \int_{B_s \setminus B_t} |V_{|A|}(Dv - \tau D\psi)|^2 + |V_{|A|}(D\psi)|^2 d\tau dx \\
&\leq c(p)\Lambda \int_{B_s \setminus B_t} |V_{|A|}(Dv)|^2 + |V_{|A|}(D\psi)|^2 d\tau dx. \tag{3.14}
\end{aligned}$$

*Warning!* The first identities in (3.13) and (3.14) need to be justified in the singular case  $1 < p < 2$ , since the argument of the second derivatives  $D^2 f$  could be 0; see the *justification* at the end of Section 4.

Combining the last and the second last estimate with (3.12) we arrive at

$$\lambda \int_{B_s} (|A|^2 + |D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2 dx \leq c \int_{B_s \setminus B_t} |V_{|A|}(Dv)|^2 + |V_{|A|}(D\psi)|^2 dx, \tag{3.15}$$

where  $c = c(p, \Lambda)$ . We estimate  $V_{|A|}(D\psi)$  distinguishing two cases using the fact that the function  $t \rightarrow (\mu + t^2)^{(p-2)/4} t$  is increasing; therefore  $|V_{|A|}(B)| \leq |V_{|A|}(C)|$  provided  $|B| \leq |C|$ . Using this simple fact together with (3.3) it follows

$$|V_{|A|}(D\psi)| \leq c(p) |V_{|A|}((1 - \eta)Dv)| + |V_{|A|}(\nabla\eta \otimes v)| \leq c(p) \left[ |V_{|A|}(Dv)| + \left| V_{|A|}\left(\frac{v}{s-t}\right) \right| \right].$$

Using the last inequality in (3.15), and recalling that  $D\varphi = Dv$  on  $B_t$  we find

$$\int_{B_t} |V_{|A|}(Dv)|^2 dx \leq c \int_{B_s \setminus B_t} |V_{|A|}(Dv)|^2 + \left| V_{|A|}\left(\frac{v}{s-t}\right) \right|^2 dx,$$

where  $c = c(p, \lambda, \Lambda)$ . “Filling the hole” on the right-hand side, i.e. adding  $c \int_{B_t} |V_{|A|}(Dv)|^2 dx$  on both sides of the previous inequality we finally deduce

$$\int_{B_t} |V_{|A|}(Du - A)|^2 dx \leq \vartheta \int_{B_s} |V_{|A|}(Du - A)|^2 dx + \int_{B_s} \left| V_{|A|}\left(\frac{u - \xi - Ax}{s-t}\right) \right|^2 dx,$$

for all  $\varrho/2 \leq t < s \leq \varrho$ ; here we have set  $\vartheta := c(1 + c)^{-1} < 1$ . Note that  $\vartheta$  depends only on  $p, \lambda$  and  $\Lambda$ . The result then follows from Lemma 7.  $\square$

We conclude the section with a Poincaré-type inequality involving the function  $V_\mu$ . In the case  $\mu = 1$  similar inequalities have been found in [5] and, in a sharp way, in [8].

**Lemma 8.** *Let  $p \in (1, 2)$  and  $u \in W^{1,p}(B_\varrho, \mathbb{R}^N)$ ,  $B_\varrho \subset U$  then:*

$$\left( \int_{B_\varrho} \left| V_\mu \left( \frac{u - (u)_\varrho}{\varrho} \right) \right|^{p^\#} dx \right)^{1/p^\#} \leq c(n, N, p) \left( \int_{B_\varrho} |V_\mu(Du)|^2 dx \right)^{1/2},$$

where  $p^\# := \frac{2n}{n-p}$ . In particular, the previous inequality is valid with  $p^\#$  replaced by 2. The constant  $c(n, N, p)$  is independent of  $\mu \geq 0$ .

**Proof.** The proof can be achieved following the arguments in [8], Theorem 2; therefore, we shall only sketch it. Since there exists a constant  $c = c(p) \geq 1$  such that  $c^{-1}(W_\mu(t))^2 \leq (V_\mu(t))^2 \leq c(W_\mu(t))^2$  whenever  $t \geq 0$  and  $W_\mu(t) := (\mu + t)^{(p-2)/2}t$ , we can reduce ourselves to prove the lemma with  $V_\mu$  replaced by  $W_\mu$ . At this point we are exactly in the setting of [8], Theorem 2, since the function  $t \rightarrow (W_\mu(t))^{2/p}$  is convex as soon as  $p \geq 1$  and  $\mu \geq 0$ ; we just have to replace the function  $W^{2/p}$  considered in [8] by our  $W_\mu^{2/p}$ , and the statement follows.  $\square$

#### 4. Approximate $\mathcal{A}$ -harmonicity and $p$ -harmonicity

We fix some notation we shall use in this section; for a ball  $B_\varrho(x_0) \Subset U$ , a function  $u \in W^{1,p}(B_\varrho(x_0), \mathbb{R}^N)$  and a linear function  $A \in \mathbb{R}^{nN}$  we define (compare (2.4))

$$\begin{aligned} \Phi(x_0, \varrho, A) &= \int_{B_r} |A|^{p-2} |Du - A|^2 + |Du - A|^p dx \quad \text{if } p > 2, \\ \Phi(x_0, \varrho, A) &= \int_{B_r} |V(Du) - V(A)|^2 dx \quad \text{if } 1 < p < 2. \end{aligned} \tag{4.1}$$

**Lemma 9** (Approximate  $\mathcal{A}$ -harmonicity). *There exists a constant  $c_1$  depending on  $p$  and  $L$  in the case  $p > 2$  and on  $n, N, p, L$  and  $\Lambda$  in the case  $1 < p < 2$  such that for every  $u \in W^{1,p}(U, \mathbb{R}^N)$  that is  $\mathcal{F}$ -minimizing in  $U$ , every ball  $B_\varrho(x_0) \Subset U$  and every  $A \in \mathbb{R}^{nN}$  such that  $|A| \neq 0 \neq \Phi(x_0, \varrho, A)$ , we have:*

$$\left| \int_{B_\varrho(x_0)} \frac{D^2 f(A)}{|A|^{p-2}} \left( |A|^{\frac{p-2}{2}} \frac{Du - A}{\sqrt{\Phi}}, D\varphi \right) dx \right| \leq c_1 \left[ \left( \frac{\Phi}{|A|^p} \right)^{\frac{p-2}{2p}} + \left( \frac{\Phi}{|A|^p} \right)^{\frac{\alpha}{2}} \right] \sup_{B_\varrho(x_0)} |D\varphi|$$

for all  $\varphi \in C_0^1(B_\varrho(x_0), \mathbb{R}^N)$ . Here we have abbreviated  $\Phi(x_0, \varrho, A)$  by  $\Phi$ .

**Proof.** *Case  $p > 2$ .* We write  $B_\varrho$  instead of  $B_\varrho(x_0)$ . Moreover we shall often abbreviate  $\Phi(x_0, \varrho, A) \equiv \Phi(\varrho)$ . For  $\varphi \in C_0^1(B_\varrho, \mathbb{R}^N)$  we assume w.l.o.g. that  $|D\varphi| \leq 1$ . Using the Euler–Lagrange equation  $\int_{B_\varrho} Df(Du)D\varphi dx = 0$  and the obvious identity  $\int_{B_\varrho} Df(A)D\varphi dx = 0$  (note that  $Df(A)$  is constant) and finally, using the Hölder continuity assumption in (H3), we deduce:

$$\left| \int_{B_\varrho} D^2 f(A)(Du - A, D\varphi) dx \right| = \left| \int_{B_\varrho} \int_0^1 (D^2 f(A) - D^2 f(A + t(Du - A)))(Du - A, D\varphi) dt dx \right| =: S$$

$$\leq L \int_{B_\rho} |Du - A|^{1+\alpha} (|A|^2 + |Du - A|^2)^{\frac{p-2-\alpha}{2}} dx. \tag{4.2}$$

Using the elementary estimate  $(a + b)^\gamma \leq 2^\gamma (a^\gamma + b^\gamma)$  for  $a, b \geq 0$  and  $\gamma > 0$  with  $\gamma = \frac{p-2-\alpha}{2}$ ,  $a = |A|$ ,  $b = |Du - A|$ , and Hölder’s inequality the right-hand side of (4.2) can be estimated as follows:

$$\begin{aligned} & L \int_{B_\rho} |Du - A|^{1+\alpha} (|A|^2 + |Du - A|^2)^{\frac{p-2-\alpha}{2}} dx \\ & \leq 2^{\frac{p-2-\alpha}{2}} L \int_{B_\rho} (|A|^{p-2-\alpha} |Du - A|^{1+\alpha} + |Du - A|^{p-1}) dx \\ & \leq 2^{\frac{p-2}{2}} L \left[ \left( \int_{B_\rho} |Du - A|^p dx \right)^{1-\frac{1}{p}} + |A|^{\frac{p-2}{2}-\frac{p\alpha}{2}} \left( \int_{B_\rho} |A|^{p-2} |Du - A|^2 dx \right)^{\frac{1+\alpha}{2}} \right]. \end{aligned}$$

Dividing by  $|A|^{(p-2)/2} \sqrt{\Phi}$  we arrive at

$$\left| \int_{B_\rho} \frac{D^2 f(A)}{|A|^{p-2}} \left( |A|^{\frac{p-2}{2}} \frac{Du - A}{\sqrt{\Phi}}, D\varphi \right) dx \right| \leq c_1 \left[ \left( \frac{\Phi}{|A|^p} \right)^{\frac{p-2}{2p}} + \left( \frac{\Phi}{|A|^p} \right)^{\frac{\alpha}{2}} \right],$$

where we have set  $c_1 = 2^{(p-2)/2} L$ . This proves the lemma in the case  $p > 2$ .

Case  $1 < p < 2$ . We proceed as in the previous case. Our main effort is in estimating  $S$ , appearing in (4.2); therefore, we distinguish the two cases in which  $|Du(x) - A| < |A|$  or in which the opposite inequality  $|Du(x) - A| \geq |A|$  holds. We abbreviate  $B_\rho^+ = \{x \in B_\rho : |Du(x) - A| \geq |A|\}$  and  $B_\rho^- = \{x \in B_\rho : |Du(x) - A| < |A|\}$ . Moreover we denote by  $I(x, t)$  the integrand appearing in the second line of (4.2). We first consider the case in which  $x \in B_\rho^+$ . Using the bound for the second derivative in (H2) and Lemma 2 we see that for  $x \in B_\rho^+$  we have

$$\int_0^1 |I(x, t)| dt \leq c(p, \Lambda) (|A|^{p-2} + (|A|^2 + |Du(x)|^2)^{\frac{p-2}{2}}) |Du(x) - A|.$$

Warning! As for Proposition 2, also the previous estimate, and (4.2), must be justified in the singular case  $1 < p < 2$ . See the *justification* at the end of this section.

Integrating with respect to  $x$  over  $B_\rho^+$  yields:

$$|B_\rho|^{-1} \int_{B_\rho^+} \int_0^1 |I(x, t)| dt dx \leq c(p, \Lambda) (I_1 + I_2), \tag{4.3}$$

with the obvious labelling for  $I_1$  and  $I_2$ . To estimate  $I_2$  we note that  $|Du(x) - A| \leq |Du(x) - A|^{1+\alpha} |A|^{-\alpha}$  for  $x \in B_\rho^+$ . Then using Hölder’s inequality we obtain

$$\begin{aligned} |B_\rho|^{-1} I_2 & \leq |A|^{-\alpha} \int_{B_\rho} (|A|^2 + |Du|^2)^{\frac{p-2}{2}} |Du - A|^{1+\alpha} dx \\ & \leq |A|^{-\alpha} \left( \int_{B_\rho} (|A|^2 + |Du|^2)^{\frac{p-2}{2}} |Du - A|^2 dx \right)^{\frac{1+\alpha}{2}} \left( \int_{B_\rho} (|A|^2 + |Du|^2)^{\frac{p-2}{2}} dx \right)^{\frac{1-\alpha}{2}} \end{aligned}$$

$$\begin{aligned}
 &\leq |A|^{\frac{p-2}{2}-\frac{p\alpha}{2}} \left( \int_{B_\varrho} (|A|^2 + |Du|^2)^{\frac{p-2}{2}} |Du - A|^2 dx \right)^{\frac{1+\alpha}{2}} \\
 &\leq c(n, N, p) |A|^{\frac{p-2}{2}-\frac{p\alpha}{2}} \left( \int_{B_\varrho} |V(Du) - V(A)|^2 dx \right)^{\frac{1+\alpha}{2}} \\
 &\leq c |A|^{\frac{p-2}{2}-\frac{p\alpha}{2}} \Phi^{\frac{1+\alpha}{2}}.
 \end{aligned} \tag{4.4}$$

We note that we used (3.1) in the second last line. To estimate  $I_1$  we proceed as follows: We first use Hölder’s inequality, and for  $x \in B_\varrho^+$  then the elementary estimate

$$|A|^2 + |Du(x)|^2 \leq 2|Du(x) - A|^2 + 3|A|^2 \leq 5|Du(x) - A|^2 \tag{4.5}$$

to deduce

$$\begin{aligned}
 |B_\varrho|^{-1} I_1 &\leq \frac{|A|^{p-2}}{|B_\varrho|} \left( \int_{B_\varrho^+} (|A|^2 + |Du|^2)^{\frac{p-2}{2}} |Du - A|^2 dx \right)^{\frac{1}{2}} \left( \int_{B_\varrho^+} (|A|^2 + |Du|^2)^{\frac{2-p}{2}} dx \right)^{\frac{1}{2}} \\
 &\leq c(p) \frac{|A|^{p-2}}{|B_\varrho|} \left( \int_{B_\varrho^+} (|A|^2 + |Du|^2)^{\frac{p-2}{2}} |Du - A|^2 dx \right)^{\frac{1}{2}} \left( \int_{B_\varrho^+} |Du - A|^{2-p} dx \right)^{\frac{1}{2}}.
 \end{aligned} \tag{4.6}$$

To estimate the second integral of the right-hand side of the previous inequality we use (3.1) and (4.5) on the set  $B_\varrho^+$  to obtain

$$|Du - A| \leq c(p) (|A|^2 + |Du|^2)^{\frac{2-p}{4}} |V(Du) - V(A)| \leq c(p) |Du - A|^{\frac{2-p}{2}} |V(Du) - V(A)|$$

so that it follows

$$|Du - A| \leq c(p) |V(Du) - V(A)|^{\frac{2}{p}} \text{ on } B_\varrho^+.$$

Hence, using this last estimate and Hölder’s inequality in (4.6), we have

$$\begin{aligned}
 |B_\varrho|^{-1} I_1 &\leq c(p) |A|^{p-2} |B_\varrho|^{-1} \left( \int_{B_\varrho^+} (|A|^2 + |Du|^2)^{\frac{p-2}{2}} |Du - A|^2 dx \right)^{\frac{1}{2}} \left( \int_{B_\varrho^+} |V(Du) - V(A)|^{\frac{2(2-p)}{p}} dx \right)^{\frac{1}{2}} \\
 &\leq c(p) |A|^{p-2} \left( \int_{B_\varrho} (|A|^2 + |Du|^2)^{\frac{p-2}{2}} |Du - A|^2 dx \right)^{\frac{1}{2}} \left( \int_{B_\varrho} |V(Du) - V(A)|^2 dx \right)^{\frac{2-p}{2p}}.
 \end{aligned}$$

Recalling once again (3.1), we deduce

$$|B_\varrho|^{-1} I_1 \leq c(p) |A|^{p-2} \left( \int_{B_\varrho} |V(Du) - V(A)|^2 dx \right)^{\frac{1}{p}}.$$

Combining the estimates for  $I_1$  and  $I_2$  we finally obtain

$$|B_\varrho|^{-1} \int_{B_\varrho^+} \int_0^1 |I(x, t)| dt dx \leq c(n, N, p, \Lambda) [ |A|^{\frac{p-2}{2}-\frac{p\alpha}{2}} \Phi(\varrho)^{\frac{1+\alpha}{2}} + |A|^{p-2} \Phi(\varrho)^{\frac{1}{p}} ]. \tag{4.7}$$

Next we treat the case in which we have to integrate over  $B_\varrho^-$ . Here we use the hypothesis (H3) to estimate the integrand  $I(x, t)$  as for  $x \in B_\varrho^-, 0 \leq t \leq 1$ , as follows:

$$\begin{aligned} |I(x, t)| &\leq L|A|^{p-2} |A + t(Du(x) - A)|^{p-2} (|A|^2 + |A + t(Du(x) - A)|^2)^{\frac{2-p-\alpha}{2}} |Du(x) - A|^{1+\alpha} \\ &\leq 5L|A|^{-\alpha} |A + t(Du(x) - A)|^{p-2} |Du(x) - A|^{1+\alpha}. \end{aligned} \quad (4.8)$$

Using Lemma 2, we see that

$$\int_0^1 |I(x, t)| dt \leq c(p, L) |A|^{-\alpha} (|A|^2 + |Du(x)|^2)^{\frac{p-2}{2}} |Du(x) - A|^{1+\alpha} \quad (4.9)$$

for  $x \in B_\varrho^-$ .

*Warning!* Also in this case the inequalities in (4.8) and (4.9) must be justified. Look at the end of Section 4, once again.

Integrating with respect to  $x$  over  $B_\varrho^-$  we obtain, proceeding exactly as for (4.4) and using again (3.1):

$$|B_\varrho|^{-1} \int_{B_\varrho^-} \int_0^1 |I(x, t)| dt dx \leq c(n, N, p, L) |A|^{\frac{p-2}{2} - \frac{p\alpha}{2}} \Phi^{\frac{1+\alpha}{2}}.$$

Combining this with (4.7) we finally arrive at

$$S \leq c(n, N, p, L, \Lambda) [ |A|^{\frac{p-2}{2} - \frac{p\alpha}{2}} \Phi(\varrho)^{\frac{1+\alpha}{2}} + |A|^{p-2} \Phi(\varrho)^{\frac{1}{p}} ].$$

Merging this last inequality with (4.2) and dividing by  $|A|^{(p-2)/2} \sqrt{\Phi}$  completes the proof of the lemma also in the subquadratic case.  $\square$

In order to treat the *degenerate case*, i.e. the case in which we expect the minimizer to behave in a neighborhood of a certain point approximately like a solution of the  $p$ -Laplacian system, we define

$$\Psi(x_0, \varrho) = \int_{B_\varrho(x_0)} |Du|^p dx.$$

From Section 2 we recall that hypothesis (H4), i.e. the assumption that the integrand  $f$  behaves like the  $p$ -Laplacian at the origin, implies that there exists a function  $\eta: (0, \infty) \rightarrow (0, \infty)$  such that for  $\mu > 0$  we have

$$|Df(A) - |A|^{p-2} A| \leq \mu |A|^{p-1} \quad \text{for any } A \in \mathbb{R}^{nN} \text{ with } |A| \leq \eta(\mu).$$

**Lemma 10** (Approximate  $p$ -harmonicity). *There exists a constant  $c_2 = c_2(\Lambda)$  such that for every  $u \in W^{1,p}(U, \mathbb{R}^N)$  that is  $\mathcal{F}$ -minimizing in  $U$ , every ball  $B_\varrho(x_0) \Subset U$ , every  $A \in \mathbb{R}^{nN}$  and every  $\mu > 0$  we have:*

$$\left| \int_{B_\varrho(x_0)} |Du|^{p-2} Du \cdot D\varphi dx \right| \leq c_2 \left[ \mu \cdot \Psi^{1-\frac{1}{p}} + \frac{\Psi}{\eta(\mu)} \right] \sup_{B_\varrho(x_0)} |D\varphi|,$$

for all  $\varphi \in C_0^1(B_\varrho(x_0), \mathbb{R}^N)$ . Here we have abbreviated  $\Psi(x_0, \varrho)$  by  $\Psi$ .

**Proof.** Again we write  $B_\varrho$  instead of  $B_\varrho(x_0)$  and assume w.l.o.g. that  $\varphi \in C_0^1(B_\varrho, \mathbb{R}^N)$  satisfies  $|D\varphi| \leq 1$  in  $B_\varrho$ . Using the Euler–Lagrange equation for  $u$  on  $B_\varrho$ , i.e. the fact that  $\int_{B_\varrho} Df(Du) D\varphi dx = 0$ , we obtain

$$\left| \int_{B_\varrho} |Du|^{p-2} \cdot Du D\varphi dx \right| = \left| \int_{B_\varrho} (Df(Du) - |Du|^{p-2} Du) \cdot D\varphi dx \right|.$$

To estimate the right-hand side of the previous identity we distinguish between the cases where  $|Du| \leq \eta(\mu)$  and  $|Du| > \eta(\mu)$ . On  $B_\rho \cap \{|Du| \leq \eta(\mu)\}$  we have

$$|B_\rho|^{-1} \left| \int_{B_\rho \cap \{|Du| \leq \eta(\mu)\}} (Df(Du) - |Du|^{p-2} Du) \cdot D\varphi \, dx \right| \leq \mu \int_{B_\rho} |Du|^{p-1} \, dx \leq \mu \left( \int_{B_\rho} |Du|^p \, dx \right)^{1-\frac{1}{p}}.$$

Next, we first note that

$$|B_\rho \cap \{|Du| > \eta(\mu)\}| \eta(\mu)^p \leq \int_{B_\rho} |Du|^p \, dx. \tag{4.10}$$

On  $B_\rho \cap \{|Du| > \eta(\mu)\}$  we use the bound  $|Df(A)| \leq \Lambda|A|^{p-1}$ , Hölder’s inequality and (4.10) to infer

$$\begin{aligned} & |B_\rho|^{-1} \left| \int_{B_\rho \cap \{|Du| > \eta(\mu)\}} (Df(Du) - |Du|^{p-2} Du) \cdot D\varphi \, dx \right| \\ & \leq (\Lambda + 1) |B_\rho|^{-1} \int_{B_\rho \cap \{|Du| > \eta(\mu)\}} |Du|^{p-1} \, dx \\ & \leq (\Lambda + 1) |B_\rho|^{-1} |B_\rho \cap \{|Du| > \eta(\mu)\}|^{\frac{1}{p}} \left( \int_{B_\rho} |Du|^p \, dx \right)^{1-\frac{1}{p}} \\ & \leq \frac{\Lambda + 1}{\eta(\mu)} \int_{B_\rho} |Du|^p \, dx. \end{aligned}$$

Collecting terms yields

$$\left| \int_{B_\rho} |Du|^{p-2} Du \cdot D\varphi \, dx \right| \leq \mu \left( \int_{B_\rho} |Du|^p \, dx \right)^{1-\frac{1}{p}} + \frac{\Lambda + 1}{\eta(\mu)} \cdot \int_{B_\rho} |Du|^p \, dx,$$

which proves the assertion of the lemma with  $c_2 = \Lambda + 1$ .  $\square$

*Justification* of the linearization procedures in the case  $1 < p < 2$ . Let us start from the estimate of the terms I’ and III’, immediately before (3.13). We give the description for I’. First, we can confine ourselves to those  $x \in B_s$  such that  $|A|$  and  $|D\psi(x)|$  are not simultaneously 0, otherwise the integrand is 0, since, by (H4),  $Df(0) = 0$ . Therefore for such  $x$  we start justifying the identity:

$$Df(A + \tau D\psi(x)) - Df(A) = \int_0^1 D^2 f(A + s\tau D\psi(x)) \, ds \tau D\psi(x), \tag{4.11}$$

that we used in (3.13). We consider the function  $[0, 1] \ni s \mapsto g(s) = Df(A + s\tau D\psi(x)) \in \mathbb{R}^{nN}$ . We first note that (4.11) trivially holds, if the segment  $[A, \tau D\psi(x)]$  does not contain the origin of  $\mathbb{R}^{nN}$  because then  $g(s)$  is differentiable with respect to  $s$  on  $[0, 1]$ . Therefore we can assume that there is one parameter value  $\tilde{s} \in [0, 1]$  such that  $A + \tilde{s}\tau D\psi(x) = 0$  (note that both  $x$  and  $\tau$  are fixed). We first assume that  $\tilde{s} \in (0, 1)$ . Then,  $g(s)$  is differentiable on  $[0, \tilde{s})$  and  $(\tilde{s}, 1]$  and for any  $0 < \varepsilon < \min\{\tilde{s}, 1 - \tilde{s}\}$ , the following formulae are valid:

$$g(1) - g(\tilde{s} + \varepsilon) = \int_{\tilde{s} + \varepsilon}^1 D^2 f(A + s\tau D\psi(x)) ds \tau D\psi(x),$$

$$g(\tilde{s} - \varepsilon) - g(0) = \int_0^{\tilde{s} - \varepsilon} D^2 f(A + s\tau D\psi(x)) ds \tau D\psi(x).$$

At this point, we note that the function  $g$  is continuous and so we can recover (4.11) from the previous identities after letting  $\varepsilon \rightarrow 0$ , since the integrals do converge due to the growth condition (H2), i.e.  $|D^2 f(A + s\tau D\psi(x))| \leq \Lambda |A + s\tau D\psi(x)|^{p-2}$ , and  $p - 2 > -1$  (note that the integral considered in (4.11) is actually singular). The subsequent estimate in (3.13) is then justified in a more straightforward way, via Lemma 2; this gives a pointwise inequality for the integrand, therefore ensuring the finiteness of the integral. The cases  $\tilde{s} = 0$  resp.  $\tilde{s} = 1$  are similar. The procedure for III' is similar at this point. The same arguments apply to the justification of the first identity in (4.2) in the case  $1 < p < 2$ . We finally justify (4.8) and (4.9). Observe that in this case  $|A| \neq 0$ , therefore for any  $x \in B_\varrho^-$  there exists at most one  $\tilde{s} \in (0, 1]$  such that  $|A + \tilde{s}(Du(x) - A)| = 0$ . It follows (recall  $x$  is fixed) that the inequality stated in (4.8) holds a.e. with respect to  $t \in [0, 1]$ . Then we can integrate first with respect to  $t$  and then with respect to  $x$ ; the convergence of the resulting integral in (4.9) then follows via Lemma 2.

## 5. Proof of the theorems

For the sake of clearness and in order to show a larger spreading of the  $\mathcal{A}$ ,  $p$ -harmonic approximation techniques, we shall separate the cases  $p > 2$  and  $1 < p < 2$ . We warn the reader on the following convention. We have defined both in (2.4) and (4.1) two similar quantities, with a similar notation. We shall use them without ambiguity, since in the lemmata below the choice of  $A$  (see (4.1)) will be always such that the two quantities will coincide.

### 5.1. The super-quadratic case $p > 2$

**Proposition 3.** For  $0 < \beta < 1$  there exist constants  $\theta = \theta(n, N, p, \lambda, \Lambda, \beta) \in (0, 1/4]$  and  $\varepsilon_0 = \varepsilon_0(n, N, p, \lambda, \Lambda, L, \alpha, \beta) > 0$  such that the following is true: Whenever  $u \in W^{1,p}(U, \mathbb{R}^N)$  is  $\mathcal{F}$ -minimizing in  $U$  such that for some ball  $B_\varrho(x_0) \Subset U$  the smallness condition

$$\Phi(x_0, \varrho, (Du)_\varrho) < \varepsilon_0 |(Du)_{x_0, \varrho}|^p \tag{5.1}$$

is satisfied, then the following growth condition holds:

$$\Phi(x_0, \theta\varrho, (Du)_{\theta\varrho}) \leq \theta^{2\beta} \Phi(x_0, \varrho, (Du)_\varrho). \tag{5.2}$$

**Proof.** Without loss of generality we take  $x_0 = 0$ . We write  $\Phi(\varrho)$  instead of  $\Phi(x_0, \varrho, (Du)_\varrho)$ . Moreover, we assume  $\Phi(\varrho) > 0$ , otherwise the conclusion of the lemma holds trivially; it follows from (5.1), that  $|(Du)_\varrho| > 0$ . We define

$$w(x) = |(Du)_\varrho|^{\frac{p-2}{2}} \frac{u(x) - (Du)_\varrho x}{\sqrt{\Phi(\varrho)}} \quad \text{for } x \in B_\varrho. \tag{5.3}$$

From the very definition of both  $\Phi$  and  $w$  we find

$$\int_{B_\varrho} |Dw|^2 dx = |(Du)_\varrho|^{p-2} \int_{B_\varrho} |Du - (Du)_\varrho|^2 dx \Phi(\varrho)^{-1} \leq 1. \tag{5.4}$$



Moreover from Lemma 9 we infer that (note that  $|(Du)_\varrho| > 0$ )

$$\left| \int_{B_\varrho} \frac{D^2 f((Du)_\varrho)}{|(Du)_\varrho|^{p-2}} (Dw, D\varphi) dx \right| \leq c_1 \left[ \left( \frac{\Phi(\varrho)}{|(Du)_\varrho|^p} \right)^{\frac{p-2}{2p}} + \left( \frac{\Phi(\varrho)}{|(Du)_\varrho|^p} \right)^{\frac{\alpha}{2}} \right] \sup_{B_\varrho} |D\varphi| \tag{5.5}$$

for any  $\varphi \in C_1(B_\varrho, \mathbb{R}^N)$ . Let  $\varepsilon > 0$  (to be chosen later) and  $\delta = \delta(n, N, \lambda, \Lambda, \varepsilon) \in (0, 1]$  from the  $\mathcal{A}$ -harmonic approximation Lemma 3. Now we assume that

$$c_1 \left[ \left( \frac{\Phi(\varrho)}{|(Du)_\varrho|^p} \right)^{\frac{p-2}{2p}} + \left( \frac{\Phi(\varrho)}{|(Du)_\varrho|^p} \right)^{\frac{\alpha}{2}} \right] < \delta, \tag{5.6}$$

where  $c_1 = c_1(p, L)$  is the constant from Lemma 9. Then  $w$  is approximatively harmonic with respect to the bilinear form  $\mathcal{A} := |(Du)_\varrho|^{2-p} D^2 f((Du)_\varrho)$  which is elliptic in the sense of Legendre–Hadamard with ellipticity constant  $2\lambda$  (see (2.3)) and upper bound  $\Lambda$  (see (H2)). Therefore, by Lemma 3, we find  $h \in W^{1,2}(B_\varrho, \mathbb{R}^N)$ ,  $\mathcal{A}$ -harmonic, such that

$$\int_{B_\varrho} |Dh|^2 dx \leq 1 \quad \text{and} \quad \varrho^{-2} \int_{B_\varrho} |w - h|^2 dx \leq \varepsilon. \tag{5.7}$$

Being an  $\mathcal{A}$ -harmonic functions,  $h$  also satisfies the estimate

$$\varrho^{-2} \sup_{B_{\varrho/2}} |Dh|^2 + \sup_{B_{\varrho/2}} |D^2 h|^2 \leq \frac{c_3}{\varrho^2} \int_{B_\varrho} |Dh|^2 dx \leq \frac{c_3}{\varrho^2}, \tag{5.8}$$

with  $c_3 = c_3(n, N, \lambda, \Lambda)$  (without loss of generality we take  $c_3 \geq 1$ ). For  $\theta \in (0, 1/4]$  to be specified later we can therefore apply Taylor’s theorem to  $h$  at 0 to deduce

$$\sup_{x \in B_{2\theta\varrho}} |h(x) - h(0) - Dh(0)x|^2 \leq c_3 \varrho^{-2} (2\theta\varrho)^4 = 16c_3\theta^4\varrho^2. \tag{5.9}$$

Thus we have, using the triangle inequality together with (5.7) and (5.9),

$$(2\theta\varrho)^{-2} \int_{B_{2\theta\varrho}} |w(x) - h(0) - Dh(0)x|^2 dx \leq 2(2\theta\varrho)^{-2} [(2\theta)^{-n}\varrho^2\varepsilon + 16c_3\theta^4\varrho^2] = 2^{-n-1}\theta^{-n-2}\varepsilon + 8c_3\theta^2.$$

We now set  $\varepsilon = \theta^{n+4}$ . Then, recalling the definition of  $w$  we obtain

$$\begin{aligned} & (2\theta\varrho)^{-2} \int_{B_{2\theta\varrho}} |u(x) - (Du)_\varrho x - \sqrt{|(Du)_\varrho|^{2-p} \Phi(\varrho)} (h(0) + Dh(0)x)|^2 dx \\ & \leq \frac{1}{2} c (\theta^{-n-2}\varepsilon + \theta^2) |(Du)_\varrho|^{2-p} \Phi(\varrho) = c\theta^2 |(Du)_\varrho|^{2-p} \Phi(\varrho) \end{aligned} \tag{5.10}$$

where the constant  $c$  depends only on  $n, N, \lambda$  and  $\Lambda$ . Denoting by  $P$  the affine function minimizing  $\int_{B_{2\theta\varrho}} |u - Q|^2 dx$  amongst all the affine functions  $Q$  (see Section 3), we easily deduce from (5.10)

$$(2\theta\varrho)^{-2} \int_{B_{2\theta\varrho}} |u - P_{2\theta\varrho}|^2 dx \leq c\theta^2 |(Du)_\varrho|^{2-p} \Phi(\varrho) \tag{5.11}$$

with  $c = c(n, N, \lambda, \Lambda)$ . We next derive an estimate for the term  $(2\theta\varrho)^{-p} \int_{B_{2\theta\varrho}} |u - P_{2\theta\varrho}|^p dx$  which is needed for the application of Caccioppoli’s inequality. For this we let  $p^*$  be the usual Sobolev conjugate (that is  $p^* := \frac{np}{n-p}$  if

$p < n$  and  $p^* :=$  “any exponent  $> p$ ” if  $p \geq n$ ). We then find  $t \in (0, 1)$  such that  $\frac{1}{p} = (1 - t)\frac{1}{2} + t\frac{1}{p^*}$ . With this choice of  $t$  we use in turn the  $L^p$ -interpolation inequality, the definition of  $P_{2\theta\varrho}$ , the estimate found in (5.11) and Sobolev’s–Poincaré inequality, obtaining

$$\begin{aligned} \int_{B_{2\theta\varrho}} |u - P_{2\theta\varrho}|^p dx &\leq \left( \int_{B_{2\theta\varrho}} |u - P_{2\theta\varrho}|^2 dx \right)^{(1-t)\frac{p}{2}} \left( \int_{B_{2\theta\varrho}} |u - P_{2\theta\varrho}|^{p^*} dx \right)^{t\frac{p}{p^*}} \\ &\leq c\varrho^p \theta^{(2-t)p} |(Du)_\varrho|^{(2-p)\frac{(1-t)p}{2}} \Phi(\varrho)^{\frac{(1-t)p}{2}} \left( \int_{B_{2\theta\varrho}} |D(u - P_{2\theta\varrho})|^p dx \right)^t, \end{aligned} \tag{5.12}$$

where the constant  $c$  depends on  $n, N, p, \lambda$  and  $\Lambda$ . The last factor in the right-hand side of (5.12) can be estimated as follows: We denote by  $P_\varrho$  the unique affine function which minimizes  $P \mapsto \int_{B_\varrho} |u - P|^2 dx$ . Using in turn Minkowski’s inequality, (3.7), Poincaré’s inequality and (3.8) we obtain

$$\begin{aligned} \left( \int_{B_{2\theta\varrho}} |D(u - P_{2\theta\varrho})|^p dx \right)^{\frac{1}{p}} &\leq \left( \int_{B_{2\theta\varrho}} |D(u - P_\varrho)|^p dx \right)^{\frac{1}{p}} + |B_{2\theta\varrho}|^{\frac{1}{p}} |DP_{2\theta\varrho} - DP_\varrho| \\ &\leq |B_\varrho|^{\frac{1}{p}} \left( \int_{B_\varrho} |D(u - P_\varrho)|^p dx \right)^{\frac{1}{p}} + \theta^{-1} |B_\varrho|^{\frac{1}{p}} \left( c\varrho^{-p} \int_{B_\varrho} |u - P_\varrho|^p dx \right)^{\frac{1}{p}} \\ &\leq c\theta^{-1} |B_\varrho|^{\frac{1}{p}} \left( \int_{B_\varrho} |D(u - P_\varrho)|^p dx \right)^{\frac{1}{p}} \\ &\leq c\theta^{-1} |B_\varrho|^{\frac{1}{p}} \left( \int_{B_\varrho} |Du - (Du)_\varrho|^p dx \right)^{\frac{1}{p}} \leq c\theta^{-1} |B_\varrho|^{\frac{1}{p}} \Phi(\varrho)^{\frac{1}{p}}, \end{aligned}$$

with  $c = c(n, p)$ . Inserting this in (5.12) we find

$$(2\theta\varrho)^{-p} \int_{B_{2\theta\varrho}} |u - P_{2\theta\varrho}|^p dx \leq c\theta^{p-(n+2p)t} \left( \frac{\Phi(\varrho)}{|(Du)_\varrho|^p} \right)^{\frac{p-2}{2}(1-t)} \Phi(\varrho),$$

where the constant  $c$  depends only on  $n, N, p, \lambda$  and  $\Lambda$ . We now assume that the following additional smallness condition is satisfied:

$$\theta^{p-(n+2p)t} \left( \frac{\Phi(\varrho)}{|(Du)_\varrho|^p} \right)^{\frac{p-2}{2}(1-t)} \leq \theta^2. \tag{5.13}$$

Then, we arrive at

$$(2\theta\varrho)^{-p} \int_{B_{2\theta\varrho}} |u - P_{2\theta\varrho}|^p dx \leq c\theta^2 \Phi(\varrho), \tag{5.14}$$

where  $c = c(n, N, p, \lambda, \Lambda)$ . Combining (5.11) and (5.14) we obtain

$$(2\theta\varrho)^{-2} \int_{B_{2\theta\varrho}} |(Du)_\varrho|^{p-2} |u - P_{2\theta\varrho}|^2 dx + (2\theta\varrho)^{-p} \int_{B_{2\theta\varrho}} |u - P_{2\theta\varrho}|^p dx \leq c\theta^2 \Phi(\varrho), \tag{5.15}$$

with a constant  $c$  having the same dependencies as the constant in (5.14). Next, we want to replace in (5.15) the term  $|(Du)_\varrho|^{p-2}$  by  $|DP_\varrho|^{p-2}$ . Using (3.8) and the definition of  $\Phi(\varrho)$

$$\begin{aligned} |DP_{2\theta\varrho} - (Du)_\varrho| &\leq |DP_{2\theta\varrho} - (Du)_{2\theta\varrho}| + |(Du)_{2\theta\varrho} - (Du)_\varrho| \\ &\leq c \left( \int_{B_{2\theta\varrho}} |Du - (Du)_{2\theta\varrho}|^2 dx \right)^{\frac{1}{2}} + \left( \int_{B_{2\theta\varrho}} |Du - (Du)_\varrho|^2 dx \right)^{\frac{1}{2}} \\ &\leq c\theta^{-\frac{n}{2}} \left( \int_{B_\varrho} |Du - (Du)_\varrho|^2 dx \right)^{\frac{1}{2}} \leq c\theta^{-\frac{n}{2}} \left( \frac{\Phi(\varrho)}{|(Du)_\varrho|^p} \right)^{\frac{1}{2}} |(Du)_\varrho|. \end{aligned}$$

Note that the constant  $c$  depends only on  $n$ ; we assume without loss of generality that  $c \geq 2$  (this will be very convenient later). Imposing the smallness condition

$$c\theta^{-\frac{n}{2}} \left( \frac{\Phi(\varrho)}{|(Du)_\varrho|^p} \right)^{\frac{1}{2}} \leq \frac{1}{2}, \tag{5.16}$$

we see that  $2|DP_{2\theta\varrho} - (Du)_\varrho| \leq |(Du)_\varrho|$ . Hence, we can replace  $|(Du)_\varrho|^{p-2}$  in (5.15) by  $|DP_{2\theta\varrho}|^{p-2}$  enlarging the constant  $c$  on the right-hand side by a factor  $2^{p-2}$ :

$$(2\theta\varrho)^{-2} \int_{B_{2\theta\varrho}} |DP_{2\theta\varrho}|^{p-2} |u - P_{2\theta\varrho}|^2 dx + (2\theta\varrho)^{-p} \int_{B_{2\theta\varrho}} |u - P_{2\theta\varrho}|^p dx \leq c\theta^2 \Phi(\varrho), \tag{5.17}$$

where the constant  $c$  has the same dependencies as the constant in (5.15). Now, we apply Caccioppoli’s inequality (3.10) with  $\xi = P_{2\theta\varrho}(0)$  and  $A = DP_{2\theta\varrho}$  on the ball  $B_{2\theta\varrho}$  to the left-hand side of (5.17); it follows

$$\int_{B_{\theta\varrho}} |DP_{2\theta\varrho}|^{p-2} |Du - DP_{2\theta\varrho}|^2 + |Du - DP_{2\theta\varrho}|^p dx \leq c\theta^2 \Phi(\varrho),$$

where  $c = c(n, N, p, \lambda, \Lambda)$ . Obviously this implies

$$\int_{B_{\theta\varrho}} |DP_{2\theta\varrho}|^{p-2} |Du - Du_{\theta\varrho}|^2 dx + \inf_{A \in \mathbb{R}^{nN}} \int_{B_{\theta\varrho}} |Du - A|^p dx \leq c\theta^2 \Phi(\varrho). \tag{5.18}$$

To obtain the desired excess-decay estimate from (5.18) we have to replace the term  $|DP_{2\theta\varrho}|^{p-2}$  by  $|(Du)_{\theta\varrho}|^{p-2}$ . This can be achieved by first replacing  $DP_{2\theta\varrho}$  by  $DP_{\theta\varrho}$  and then  $DP_{\theta\varrho}$  by  $(Du)_{\theta\varrho}$ . The occurring error-term in the first replacement can be estimated using (3.7) and (5.14)

$$\begin{aligned} &\int_{B_{\theta\varrho}} |DP_{2\theta\varrho} - DP_{\theta\varrho}|^{p-2} |Du - (Du)_{\theta\varrho}|^2 dx \\ &\leq c \left( (\theta\varrho)^{-p} \int_{B_{\theta\varrho}} |u - P_{2\theta\varrho}|^p dx \right)^{\frac{p-2}{p}} \int_{B_{\theta\varrho}} |Du - (Du)_{\theta\varrho}|^2 dx \\ &\leq c\theta^{-n} \left( (2\theta\varrho)^{-p} \int_{B_{2\theta\varrho}} |u - P_{2\theta\varrho}|^p dx \right)^{\frac{p-2}{p}} \int_{B_\varrho} |Du - (Du)_\varrho|^2 dx \\ &\leq c\theta^2 \theta^{-n-\frac{4}{p}} \left( \frac{\Phi(\varrho)}{|(Du)_\varrho|^p} \right)^{\frac{p-2}{p}} \Phi(\varrho) \leq c\theta^2 \Phi(\varrho), \end{aligned} \tag{5.19}$$

where  $c = c(n, N, p, \lambda, \Lambda)$ . The last estimate has been performed assuming that

$$\theta^{-n-\frac{4}{p}} \left( \frac{\Phi(\varrho)}{|(Du)_{\varrho}|^p} \right)^{\frac{p-2}{2}} \leq 1. \quad (5.20)$$

The second replacement, i.e. the one of  $DP_{\theta\varrho}$  by  $(Du)_{\theta\varrho}$ , is possible via (3.8) (with  $p = 2$  there)

$$\begin{aligned} & \int_{B_{\theta\varrho}} |DP_{\theta\varrho} - (Du)_{\theta\varrho}|^{p-2} |Du - (Du)_{\theta\varrho}|^2 dx \\ & \leq c \left( \int_{B_{\theta\varrho}} |Du - (Du)_{\theta\varrho}|^2 dx \right)^{\frac{p-2}{2}} \int_{B_{\theta\varrho}} |Du - (Du)_{\theta\varrho}|^2 dx \\ & \leq c\theta^{-\frac{np}{2}} \left( \int_{B_{\varrho}} |Du - (Du)_{\varrho}|^2 dx \right)^{\frac{p}{2}} = c\theta^{-\frac{np}{2}} \left( \frac{\Phi(\varrho)}{|(Du)_{\varrho}|^p} \right)^{\frac{p-2}{2}} \Phi(\varrho) \leq c\theta^2 \Phi(\varrho), \end{aligned} \quad (5.21)$$

where  $c = c(n, p)$ . Again, the last estimate is true, provided we assume

$$\theta^{-\frac{np}{2}} \left( \frac{\Phi(\varrho)}{|(Du)_{\varrho}|^p} \right)^{\frac{p-2}{2}} \leq \theta^2. \quad (5.22)$$

Combining (5.18) with (5.19) and (5.21) we obtain

$$\int_{B_{\theta\varrho}} |(Du)_{\theta\varrho}|^{p-2} |Du - (Du)_{\theta\varrho}|^2 dx + \inf_{A \in \mathbb{R}^{nN}} \int_{B_{\theta\varrho}} |Du - A|^p dx \leq c\theta^2 \Phi(\varrho), \quad (5.23)$$

where  $c = c(n, N, p, \lambda, \Lambda)$ , as for (5.18). Next, we note that the inf in the second term of the left-hand side of the previous inequality is achieved by a unique  $A_0 \in \mathbb{R}^{nN}$ , and in a standard way, we can replace  $A_0$  by  $(Du)_{\theta\varrho}$  in (5.23) enlarging the constant on the right-hand side by a factor  $2^p$ . Recalling the definition of  $\Phi(\theta\varrho)$ , we finally arrive at  $\Phi(\theta\varrho) \leq c_e \theta^2 \Phi(\varrho)$ , where  $c_e = c_e(n, N, p, \lambda, \Lambda)$ . Now, given  $\beta \in (0, 1)$  we choose  $\theta \in (0, 1/4]$  such that  $c_e \theta^2 = \theta^{2\beta}$ ; note that  $\theta$  depends only on  $n, N, p, \lambda, \Lambda$  and  $\beta$ . For later purposes we also assume that  $\theta^{2\beta} 2^p < 1$ . This fixes  $\varepsilon = \theta^{n+4}$ , i.e.  $\varepsilon = \varepsilon(n, N, p, \lambda, \Lambda, \beta)$ , and  $\delta = \delta(n, N, \lambda, \Lambda, \varepsilon)$ . With these specifications all the smallness assumptions (5.6), (5.13), (5.16), (5.20) and (5.22) are satisfied, if we require  $\Phi(\varrho) \leq \varepsilon_0 |(Du)_{\varrho}|^p$ , with a sufficiently small constant  $\varepsilon_0$ ; note that  $\varepsilon_0$  depends only on  $n, N, p, \lambda, \Lambda, L, \alpha$  and  $\beta$ . This proves the assertion of the lemma.  $\square$

We are now going to iterate Proposition 3. Starting with  $u$  and  $\mathcal{F}$  satisfying the hypothesis of Proposition 3 and taking  $x_0 = 0$  without loss of generality, we see that

$$\Phi(\theta\varrho) \leq \theta^{2\beta} \Phi(\varrho) \leq \theta^{2\beta} \varepsilon_0 |(Du)_{\varrho}|^p.$$

From the elementary estimate

$$|(Du)_{\varrho}| \leq \theta^{-\frac{n}{2}} \left( \int_{B_{\varrho}} |Du - (Du)_{\varrho}|^2 dx \right)^{\frac{1}{2}} + |(Du)_{\theta\varrho}| \leq \theta^{-\frac{n}{2}} \left( \frac{\Phi(\varrho)}{|(Du)_{\varrho}|^p} \right)^{\frac{1}{2}} |(Du)_{\varrho}| + |(Du)_{\theta\varrho}| \quad (5.24)$$

we conclude

$$\left( 1 - \theta^{-\frac{n}{2}} \left( \frac{\Phi(\varrho)}{|(Du)_{\varrho}|^p} \right)^{\frac{1}{2}} \right) |(Du)_{\varrho}| \leq |(Du)_{\theta\varrho}|.$$

Recalling the smallness condition (5.16) we see that  $(\dots) \geq 1/2$ ; hence  $|(Du)_\varrho| \leq 2|(Du)_{\theta\varrho}|$ , which implies (by the choice of  $\theta$  at the end of the proof of Proposition 3)  $\Phi(\theta\varrho) \leq \theta^{2\beta}\varepsilon_0 2^p |(Du)_{\theta\varrho}|^p < \varepsilon_0 |(Du)_{\theta\varrho}|^p$ . Hence, the starting hypotheses of Proposition 3 are also satisfied on the ball  $B_{\theta\varrho}$  (i.e.  $u, \mathcal{F}$  satisfy on  $B_{\theta\varrho}$  the same set of conditions as on  $B_\varrho$ ). Therefore we can proceed by induction and easily deduce that  $\Phi(\theta^k\varrho) \leq \theta^{2k\beta}\Phi(\varrho)$  for any  $k \in \mathbb{N}$ . By means of a standard iteration procedure (see for instance [12]) this leads us to the following *excess-decay lemma*:

**Lemma 11.** *Assume that the hypotheses of Proposition 3 are satisfied. Then for any  $0 < r \leq \varrho$  we have, with  $c = c(n, N, p, \lambda, \Lambda, \beta)$ ,*

$$\Phi(x_0, r, (Du)_r) \leq c \left(\frac{r}{\varrho}\right)^{2\beta} \Phi(x_0, \varrho, (Du)_\varrho).$$

We now turn our attention to the *degenerate case*. By  $\gamma \in (0, 1)$  we go on denoting the Hölder exponent from the excess decay estimate (3.9) from Proposition 1. Then we have:

**Lemma 12.** *For  $0 < \tilde{\gamma} < 2\gamma/p$  and  $\chi > 0$  there exist constants  $\varepsilon_1 = \varepsilon_1(n, N, p, \lambda, \Lambda, \gamma, \tilde{\gamma}, \eta(\cdot), \chi) > 0$  and  $\tau = \tau(n, N, p, \lambda, \Lambda, \gamma, \tilde{\gamma}, \chi) \in (0, 1/4]$  such that the following is true: Whenever  $u \in W^{1,p}(U, \mathbb{R}^n)$  is  $\mathcal{F}$ -minimizing in  $U$  such that for some ball  $B_\varrho(x_0) \Subset U$  we have*

$$\chi |(Du)_{x_0, \varrho}|^p \leq \Phi(x_0, \varrho, (Du)_\varrho) \quad \text{and} \quad \chi |(Du)_{x_0, \tau\varrho}|^p \leq \Phi(x_0, \tau\varrho, (Du)_{\tau\varrho}) \tag{5.25}$$

and the smallness condition

$$\Phi(x_0, \varrho, (Du)_\varrho) < \varepsilon_1 \tag{5.26}$$

is satisfied, then  $\Phi(x_0, \tau\varrho, (Du)_{\tau\varrho}) \leq \tau^{\tilde{\gamma}}\Phi(x_0, \varrho, (Du)_\varrho)$ .

**Proof.** Once again we take  $x_0 = 0$  without loss of generality and adopt again the usual abbreviation  $\Phi(\varrho)$ . Since  $|(Du)_\varrho|^p \leq \chi^{-1}\Phi(\varrho)$ , we have

$$\Psi(\varrho) \leq 2^{p-1} \int_{B_\varrho} |Du - (Du)_\varrho|^p dx + 2^{p-1} |(Du)_\varrho|^p \leq c_4\Phi(\varrho), \tag{5.27}$$

where we have abbreviated  $c_4 = 2^{p-1}(1 + \chi^{-1})$ . From Lemma 10 (i.e. the approximative  $p$ -harmonicity) and (5.27), we have for any  $\mu > 0$  and for any  $\varphi \in C_0^1(B_\varrho, \mathbb{R}^N)$ :

$$\left| \int_{B_\varrho} |Du|^{p-2} Du \cdot D\varphi dx \right| \leq c_2 (c_4\Phi(\varrho))^{\frac{p-1}{p}} \left[ \mu + \frac{(c_4\Phi(\varrho))^{\frac{1}{p}}}{\eta(\mu)} \right] \sup_{B_\varrho} |D\varphi|.$$

Introducing, on  $B_\varrho$ , the scaled function  $w := (c_4\Phi(\varrho))^{-1/p}u$ , we deduce

$$\left| \int_{B_\varrho} |Dw|^{p-2} Dw \cdot D\varphi dx \right| \leq c_2 \left[ \mu + \frac{(c_4\Phi(\varrho))^{\frac{1}{p}}}{\eta(\mu)} \right] \sup_{B_\varrho} |D\varphi|,$$

and

$$\int_{B_\varrho} |Dw|^p dx = (c_4\Phi(\varrho))^{-1} \int_{B_\varrho} |Du|^p dx = (c_4\Phi(\varrho))^{-1} \Psi(\varrho) \leq 1;$$

here we have used (5.27) in the last inequality. Now let  $0 < \tau \leq 1/4$  (to be specified later) be given and define  $\varepsilon = \tau^{n+p+\gamma}$ . By  $\delta = \delta(n, N, p, \varepsilon) \in (0, 1]$  we denote the associated constant from Lemma 5. We then fix  $\mu > 0$  such that

$$c_2 \mu \leq \frac{\delta}{2}. \quad (5.28)$$

Note that  $\mu = \mu(\Lambda, \delta)$ . This fixes  $\eta(\mu)$ . Assuming that

$$c_2 \frac{(c_4 \Phi(\varrho))^{1/p}}{\eta(\mu)} \leq \frac{\delta}{2}, \quad (5.29)$$

we see that  $w$  satisfies the hypothesis of Lemma 5, i.e. we have  $\int_{B_\varrho} |Dw|^p dx \leq 1$  and

$$\left| \int_{B_\varrho} |Dw|^{p-2} Dw \cdot D\varphi dx \right| \leq \delta \sup_{B_\varrho} |D\varphi|$$

for any  $\varphi \in C_0^1(B_\varrho, \mathbb{R}^N)$ . We now apply the  $p$ -harmonic approximation lemma, i.e. Lemma 5, to obtain for a given  $\varepsilon = \tau^{n+p+\gamma} > 0$  a  $p$ -harmonic function  $h \in W^{1,p}(B_\varrho, \mathbb{R}^N)$  such that

$$\int_{B_\varrho} |Dh|^p dx \leq 1 \quad \text{and} \quad \varrho^{-p} \int_{B_\varrho} |w - h|^p dx \leq \varepsilon = \tau^{n+p+\gamma}. \quad (5.30)$$

Then, using Poincaré's inequality, (5.30) and (3.9), we obtain

$$\begin{aligned} & (2\tau\varrho)^{-p} \int_{B_{2\tau\varrho}} |w - (h)_{2\tau\varrho} - (Dh)_{2\tau\varrho} x|^p dx \\ & \leq c \left[ (2\tau\varrho)^{-p} \int_{B_{2\tau\varrho}} |w - h|^p dx + (2\tau\varrho)^{-p} \int_{B_{2\tau\varrho}} |h - (h)_{2\tau\varrho} - (Dh)_{2\tau\varrho} x|^p dx \right] \\ & \leq c \left[ \tau^{-n-p} \varrho^{-p} \int_{B_\varrho} |w - h|^p dx + \int_{B_{2\tau\varrho}} |Dh - (Dh)_{2\tau\varrho}|^p dx \right] \\ & \leq c [\tau^{-n-p} \varepsilon + \tau^\gamma \Phi(h; \varrho)] = c \tau^\gamma [1 + \Phi(h; \varrho)], \end{aligned}$$

with  $c = c(n, N, p)$ ; here  $\Phi(h; \varrho)$  is the excess on  $B_\varrho$  of the  $p$ -harmonic approximation  $h$  associated to  $w$  via (5.30). In view of  $|(Dh)_\varrho| \leq (\int_{B_\varrho} |Dh|^p dx)^{1/p} \leq 1$  we infer that  $\Phi(h; \varrho) \leq c(p)$ . Inserting this into the previous estimate and recalling also the definition of  $w$  we deduce

$$(2\tau\varrho)^{-p} \int_{B_{2\tau\varrho}} |u - P|^p dx \leq c c_4 \tau^\gamma \Phi(\varrho) \quad (5.31)$$

where  $c = c(n, N, p)$  and we abbreviated

$$\mathbb{R}^n \ni x \mapsto P(x) := (c_4 \Phi(\varrho))^{1/p} ((h)_{2\tau\varrho} + (Dh)_{2\tau\varrho} x) \in \mathbb{R}^N.$$

From Uhlenbeck's theorem (see Proposition 1) we infer

$$|DP| = (c_4 \Phi(\varrho))^{1/p} |(Dh)_{2\tau\varrho}| \leq (c_4 \Phi(\varrho))^{1/p} \sup_{2\tau\varrho} |Dh| \leq c (c_4 \Phi(\varrho))^{1/p},$$

where the constant  $c$  depends once again only on  $n, N$  and  $p$ . Using this and (5.31) we find

$$\begin{aligned}
 & (2\tau\varrho)^{-2} \int_{B_{2\tau\varrho}} |DP|^{p-2} |u - P|^2 dx \\
 & \leq c(c_4\Phi(\varrho))^{\frac{p-2}{p}} \left( (2\tau\varrho)^{-p} \int_{B_{2\tau\varrho}} |u - P|^p dx \right)^{\frac{2}{p}} \leq cc_4\tau^{\frac{2\gamma}{p}} \Phi(\varrho) = cc_4\tau^{\frac{2\gamma}{p}} \Phi(\varrho),
 \end{aligned} \tag{5.32}$$

where  $c = c(n, N, p)$ . Combining (5.31) and (5.32) with Caccioppoli’s inequality (see Lemma 2) and the elementary inequality  $\tau^\gamma \leq \tau^{2\gamma/p}$  we arrive at

$$\int_{B_{\tau\varrho}} |DP|^{p-2} |Du - DP|^2 + |Du - DP|^p dx \leq cc_4\tau^{\frac{2\gamma}{p}} \Phi(\varrho), \tag{5.33}$$

where  $c = c(n, N, p, \lambda, \Lambda)$ . Finally, by means of a standard argument we replace  $DP$  in (5.33) by  $(Du)_{\tau\varrho}$  in the above integrals; we obtain

$$\int_{B_{\tau\varrho}} |DP|^{p-2} |Du - (Du)_{\tau\varrho}|^2 + |Du - (Du)_{\tau\varrho}|^p dx \leq cc_4\tau^{\frac{2\gamma}{p}} \Phi(\varrho). \tag{5.34}$$

Now we use the second condition in (5.25); for  $\sigma > 0$  we have, via Young’s and Hölder’s inequality

$$\begin{aligned}
 \Phi(\tau\varrho) &= \int_{B_{\tau\varrho}} |(Du)_{\tau\varrho}|^{p-2} |Du - (Du)_{\tau\varrho}|^2 dx + \int_{B_{\tau\varrho}} |Du - (Du)_{\tau\varrho}|^p dx \\
 &\leq \left( \frac{\Phi(\tau\varrho)}{\chi} \right)^{\frac{p-2}{p}} \int_{B_{\tau\varrho}} |Du - (Du)_{\tau\varrho}|^2 dx + \int_{B_{\tau\varrho}} |Du - (Du)_{\tau\varrho}|^p dx \\
 &\leq \sigma\Phi(\tau\varrho) + (\sigma^{-\frac{p-2}{2}} \chi^{-\frac{p-2}{2}} + 1) \int_{B_{\tau\varrho}} |Du - (Du)_{\tau\varrho}|^p dx.
 \end{aligned}$$

Choosing  $\sigma = \frac{1}{2}$  we obtain

$$\Phi(\tau\varrho) \leq 2^{\frac{p}{2}} (\chi^{-\frac{p-2}{2}} + 1) \int_{B_{\tau\varrho}} |Du - (Du)_{\tau\varrho}|^p dx \leq cc_4 (\chi^{-\frac{p-2}{2}} + 1) \tau^{\frac{2\gamma}{p}} \Phi(\varrho);$$

here we have used (5.34) in the last inequality. We note that  $c = c(n, N, p, \lambda, \Lambda)$ . Now, given  $\tilde{\gamma} \in (0, 2\gamma/p)$  we fix  $\tau \in (0, 1/4]$  such that (recall the definition of  $c_4 = 2^{p-1}(1 + \chi^{-1})$ )

$$c2^{p-1}(1 + \chi^{-1})(1 + \chi^{-\frac{p-2}{2}}) \tau^{\frac{2\gamma}{p}} \leq \tau^{\tilde{\gamma}}.$$

Note that  $\tau = \tau(n, N, p, \lambda, \Lambda, \gamma, \tilde{\gamma}, \chi)$ . This fixes  $\delta = \delta(n, N, p, \tau^{n+p+\gamma})$ . Furthermore,  $\mu$  and hence also  $\eta(\mu)$  are determined by (5.28). With these specifications all constants (i.e.  $c_2, c_4 = 2^{p-1}(1 + \chi^{-1}), \eta(\mu) > 0$  and  $\delta$ ) in (5.29) and (5.28) are fixed; in summary, denoting by  $\Rightarrow$  each determination of the constants, we have that  $(\chi, \tilde{\gamma}) \Rightarrow \tau \Rightarrow \varepsilon \Rightarrow \delta \Rightarrow \mu \Rightarrow \eta(\mu)$ . We note that (5.29) is equivalent to the smallness assumption

$$\Phi(\varrho) \leq \frac{\chi}{2^{p-1}(1 + \chi)} \left( \frac{\delta\eta(\mu)}{2c_2} \right)^p. \tag{5.35}$$

Hence, the smallness assumption is satisfied, if we require (5.26) with a sufficiently small constant  $\varepsilon_1 > 0$  possessing the indicated dependencies stated in the formulation of the lemma. The proof is now complete.  $\square$

**Lemma 13** (excess-decay). *For any  $\beta \in (0, \gamma/p)$  there exists a positive constant  $\varepsilon_1 = \varepsilon_1(n, N, p, \lambda, \Lambda, \gamma, \beta, \alpha, \eta(\cdot)) > 0$  such that the following is true: Whenever  $u \in W^{1,p}(U, \mathbb{R}^N)$  is  $\mathcal{F}$ -minimizing in  $U$  such that*

$$\Phi(x_0, \varrho, (Du)_\varrho) < \varepsilon_1 \quad (5.36)$$

for some ball  $B_\varrho(x_0) \Subset U$ , then we have, for a constant  $c = c(n, N, p, \lambda, \Lambda, L, \alpha, \beta, \gamma) < +\infty$

$$\Phi(x_0, r, (Du)_r) \leq c \left(\frac{r}{\varrho}\right)^{2\beta} \Phi(x_0, \varrho, (Du)_\varrho) \quad \text{whenever } 0 < r \leq \varrho. \quad (5.37)$$

**Proof.** As usual, we shall abbreviate  $\Phi(x_0, \varrho, (Du)_\varrho)$  by  $\Phi(\varrho)$ . We first take  $\beta$  in Proposition 3 such that  $0 < \beta < \gamma/p$  where  $\gamma$  is from Proposition 1, (3.9). This fixes the constant  $\varepsilon_0 = \varepsilon_0(n, N, p, \lambda, \Lambda, L, \alpha, \beta) > 0$  from Proposition 3. Then, we choose  $\tilde{\gamma} \in (0, 2\gamma/p)$  in Lemma 12 such that  $\tilde{\gamma} = 2\beta$  (which is possible by our choice of  $\beta$ ). Furthermore, we let  $\chi = \varepsilon_0$ . This fixes the constants  $\varepsilon_1 = \varepsilon_1(n, N, p, \lambda, \Lambda, \gamma, \tilde{\gamma}, \eta(\cdot), \varepsilon_0) > 0$  and  $\tau = \tau(n, N, p, \lambda, \Lambda, \gamma, \tilde{\gamma}, \varepsilon_0) \in (0, 1/4]$ . Therefore, by (5.36), we are assuming that the smallness condition (5.26) from Lemma 12 is fulfilled, i.e. we have

$$\Phi(\varrho) < \varepsilon_1. \quad (5.38)$$

Now we introduce the following set of natural numbers:

$$\mathbb{S} := \{n \in \mathbb{N} : \Phi(\tau^n \varrho) \geq \varepsilon_0 |(Du)_{\tau^n \varrho}|^p \text{ and } \Phi(\tau^{n+1} \varrho) \geq \varepsilon_0 |(Du)_{\tau^{n+1} \varrho}|^p\}$$

and we distinguish two cases

*Case  $\mathbb{S} = \mathbb{N}$ .* In this case we prove, by induction, that, for any  $n \in \mathbb{N}$ :

$$\Phi(\tau^n \varrho) < \varepsilon_1, \quad \Phi(\tau^n \varrho) \leq \tau^{n\tilde{\gamma}} \Phi(\varrho). \quad (5.39)$$

For  $n = 0$  (5.39) trivially follows by (5.38). Suppose now that (5.39) holds for  $n \in \mathbb{N}$ ; since  $n \in \mathbb{S}$  (recall that in this case  $\mathbb{S} = \mathbb{N}$ ) we can apply Lemma 12 and deduce that:  $\Phi(\tau^{n+1} \varrho) \leq \tau^{\tilde{\gamma}} \Phi(\tau^n \varrho) \leq \tau^{(n+1)\tilde{\gamma}} \Phi(\varrho)$ , and trivially  $\Phi(\tau^{n+1} \varrho) \leq \tau^{\tilde{\gamma}} \Phi(\tau^n \varrho) < \tau^{\tilde{\gamma}} \varepsilon_1 < \varepsilon_1$ . Therefore (5.39) holds for  $n + 1$ . By induction, (5.39) is valid now for any  $n \in \mathbb{N}$ . We are now ready to prove (5.37) in this case. First, let us recall the following elementary fact (see for instance [12]):

$$\Phi(\vartheta \varrho) \leq c(p) \vartheta^{-n} \Phi(\varrho), \quad \forall 0 < \vartheta < 1. \quad (5.40)$$

Now let  $s \in \mathbb{N}$  be such that  $r \in (\tau^{s+1} \varrho, \tau^s \varrho]$ ; it follows, via (5.40) and (5.39)

$$\Phi(r) \leq c(p) \left(\frac{r}{\tau^s \varrho}\right)^{-n} \Phi(\tau^s \varrho) \leq \frac{c(p)}{\tau^n} \tau^{s\tilde{\gamma}} \Phi(\varrho) \leq \frac{c(p)}{\tau^{n+\tilde{\gamma}}} \left(\frac{r}{\varrho}\right)^{\tilde{\gamma}} \Phi(\varrho) \quad (5.41)$$

and (5.37) follows in this case taking into account the dependence upon the various constants exhibited by  $\tau$  and  $\tilde{\gamma}$ .

*Case  $\mathbb{S} \neq \mathbb{N}$ .* Therefore there exists  $m := \min \mathbb{N} \setminus \mathbb{S}$ ; by the definition of  $m$  we have

$$\Phi(\tau^{m+1} \varrho) < \varepsilon_0 |(Du)_{\tau^{m+1} \varrho}|^p. \quad (5.42)$$

Actually the last information is an immediate consequence of the definition of  $m$  when  $m \geq 1$ . In the case  $m = 0$  it is possible that (5.42) is not immediately available. In that case it must be that  $\Phi(\varrho) < \varepsilon_0 |(Du)_\varrho|^p$ ; therefore we can apply Proposition 3 (see the discussion immediately after its proof) to get that  $\Phi(\tau \varrho) < \varepsilon_0 |(Du)_{\tau \varrho}|^p$ , that is, (5.42) holds also in the case  $m = 0$ . Iterating as in the case  $\mathbb{S} = \mathbb{N}$  to get (5.39), but now up the integer  $m \in \mathbb{N}$  (again: no iteration when  $m = 0$ ), we get

$$\Phi(\tau^s \varrho) \leq \tau^{s\tilde{\gamma}} \Phi(\varrho), \quad \forall s \leq m. \quad (5.43)$$



Now, by (5.42) we can apply Lemma 11, therefore when  $r \in (0, \tau^{m+1}\varrho]$ , we obtain, thanks to (5.40) and (5.43) with  $s = m$

$$\Phi(r) \leq c \left(\frac{r}{\tau^{m+1}\varrho}\right)^{\tilde{\gamma}} \Phi(\tau^{m+1}\varrho) \leq \frac{c}{\tau^{(m+1)\tilde{\gamma}+n}} \left(\frac{r}{\varrho}\right)^{\tilde{\gamma}} \Phi(\tau^m\varrho) \leq \frac{c}{\tau^{\tilde{\gamma}+n}} \left(\frac{r}{\varrho}\right)^{\tilde{\gamma}} \Phi(\varrho), \tag{5.44}$$

the constant  $c$  appearing in the previous inequality comes from Lemma 11, therefore  $c = c(n, N, p, \lambda, \Lambda, \beta)$ . Hence the dependencies prescribed for the constant in (5.37) follows taking into account again the dependencies exhibited by  $\tau$  and  $\tilde{\gamma}$ . So, in order to prove (5.37) in general it remains to be proven in the case  $r \in (\tau^{m+1}\varrho, \varrho]$ ; therefore again let  $s \leq m$  be such that  $r \in (\tau^{s+1}\varrho, \tau^s\varrho]$ ; using (5.43) we can argue as in the proof of (5.41) to finish the proof of the assertion. Observe that a crucial point in the preceding argument is that the integer  $m$  (which depends on the point  $x_0$  and cannot be controlled) does not reflect in the constant  $c$  appearing in (5.44). This is avoided by the use of (5.43); it is exactly this point where the  $p$ -harmonic approximation scheme and the  $A$ -harmonic approximation match.  $\square$

**Proof of Theorem 1.** *Case  $p > 2$ .* The proof follows in a standard way from the decay estimate of Lemma 13 and Campanato’s integral characterization of Hölder continuity (see for instance [11] and [2]). In particular, the usual key observation in partial regularity is that (5.36) is a so called *open condition*, that is, if (5.36) is satisfied (for the fixed radius  $\varrho$ ) at the point  $x_0$ , it is automatically satisfied in a small neighborhood of  $x_0$ , say the ball  $B_\sigma(x_0)$ . Therefore the local Hölder continuity of the gradient on an open subset with full measure follows as well as the inclusion stated in (2.5).  $\square$

**Proof of Theorem 2.** Let  $x_0$  be a regular point such that (2.6) is satisfied and select  $\beta \in (0, 1)$ ; this determines, according to Proposition 3, the choice of  $\varepsilon_0$  in (5.1), which essentially depends on  $\beta$ . Therefore, by (2.6), there exists a radius  $\varrho > 0$  such that (5.1) is satisfied; moreover, we note that also (5.1) is an *open condition*; so, we can find  $\sigma > 0$  such that (5.1) is satisfied with  $x_0$  replaced by  $y$ , for any  $y \in B_\sigma(x_0)$ . At this point we can apply Lemma 11, which is valid at any  $y \in B_\sigma(x_0)$ . This implies, via Campanato’s characterization of Hölder continuity, the fact that  $Du$  is Hölder continuous (in  $B_\sigma(x_0)$ ) with exponent  $2\beta/p$ . Moreover, suppose that  $|Du(x_0)| \neq 0$ , then (possibly decreasing  $\sigma$ ) since  $Du$  is already continuous, we can assume that  $|(Du)_{y,\varrho}| \neq 0$  for any  $B_\varrho(y) \subset B_\sigma(x_0)$ , therefore (look at the structure of the excess function  $\Phi$ ) this implies, again by Campanato’s characterization of Hölder continuity, that  $Du$  is Hölder continuous (in  $B_\sigma(x_0)$ ) with exponent  $\beta$ .  $\square$

5.2. *The sub-quadratic case  $1 < p < 2$*

**Proposition 4.** *For  $0 < \beta < 1$  there exist constants  $\theta = \theta(n, N, p, \lambda, \Lambda, \beta) \in (0, 1/4]$  and  $\varepsilon_0 = \varepsilon_0(n, N, p, \lambda, \Lambda, L, \alpha, \beta) > 0$  such that the following is true: Whenever  $u \in W^{1,p}(U, \mathbb{R}^N)$  is  $\mathcal{F}$ -minimizing in  $U$  such that for some ball  $B_\varrho(x_0) \Subset U$  the smallness condition*

$$\Phi(x_0, \varrho, (Du)_\varrho) < \varepsilon_0 |(V(Du))_{x_0,\varrho}|^2 \tag{5.45}$$

*is satisfied, then the following growth condition holds:*

$$\Phi(x_0, \theta\varrho, (Du)_{\theta\varrho}) \leq \theta^{2\beta} \Phi(x_0, \varrho, (Du)_\varrho). \tag{5.46}$$

**Proof.** We recall that here  $\Phi(x_0, \varrho, (Du)_\varrho) := \int_{B_\varrho(x_0)} |V(Du) - (V(Du))_{x_0,\varrho}|^2 dx$ . As in the case  $p > 2$ , we assume  $x_0 = 0$ ; we write  $\Phi(\varrho)$  instead of  $\Phi(x_0, \varrho, (Du)_\varrho)$ . Finally, we assume  $|(V(Du))_\varrho| \neq 0 \neq \Phi(\varrho)$ ; otherwise the conclusion of the lemma trivially holds. We then choose  $0 \neq A \in \mathbb{R}^{nN}$  according to

$$V(A) = |A|^{\frac{p-2}{2}} A = \int_{B_\varrho} |Du|^{\frac{p-2}{2}} Du dx = (V(Du))_\varrho,$$

and note that  $|A|^p = |(V(Du))_\varrho|^2$ . With this specific choice of  $A$  we now define

$$v(x) = \frac{u(x) - Ax}{|A|}.$$

Then from Lemma 9 we infer that for any  $\varphi \in C_0^1(B_\varrho, \mathbb{R}^N)$ :

$$\begin{aligned} & \left| \int_{B_\varrho} \frac{D^2 f(A)}{|A|^{p-2}} \left( \frac{Du - A}{|A|}, D\varphi \right) dx \right| \\ & \leq c_2 \left( \frac{\Phi(\varrho)}{|(V(Du))_\varrho|^2} \right)^{\frac{1}{2}} \left[ \left( \frac{\Phi(\varrho)}{|(V(Du))_\varrho|^2} \right)^{\frac{2-p}{2p}} + \left( \frac{\Phi(\varrho)}{|(V(Du))_\varrho|^2} \right)^{\frac{\alpha}{2}} \right] \sup_{B_\varrho(x_0)} |D\varphi|. \end{aligned}$$

Moreover we have, by (3.1)

$$\begin{aligned} \int_{B_\varrho} |V_1(Dv)|^2 dx &= \int_{B_\varrho} \left( 1 + \left| \frac{Du - A}{|A|} \right|^2 \right)^{\frac{p-2}{2}} \left| \frac{Du - A}{|A|} \right|^2 dx \\ &= |A|^{-p} \int_{B_\varrho} (|A|^2 + |Du - A|^2)^{\frac{p-2}{2}} |Du - A|^2 dx \\ &\leq c(p) |A|^{-p} \int_{B_\varrho} (|A|^2 + |Du|^2)^{\frac{p-2}{2}} |Du - A|^2 dx \\ &\leq c(p) |(V(Du))_\varrho|^{-2} \int_{B_\varrho} |V(Du) - V(A)|^2 dx \\ &= c(p) |(V(Du))_\varrho|^{-2} \int_{B_\varrho} |V(Du) - (V(Du))_\varrho|^2 dx \\ &= c(p) \frac{\Phi(\varrho)}{|(V(Du))_\varrho|^2}. \end{aligned}$$

Here we have used the elementary estimate  $|A|^2 + |A - B|^2 \geq \frac{1}{3}(|A|^2 + |B|^2)$ . Now, for  $\theta \in (0, 1/4]$  such that  $\theta^\beta \leq 1/2$  to be specified later, we set  $\varepsilon = \theta^{n+4}$ . With  $\delta = \delta(n, N, p, \lambda, \Lambda, \theta) \in (0, 1]$  we denote the constant from Lemma 4, corresponding to the quantities  $n, N, p, \lambda, \Lambda$  and the particular choice of  $\varepsilon$ . Therefore, if we let

$$s^2 = c(p) \frac{\Phi(\varrho)}{|(V(Du))_\varrho|^2}$$

and if we assume that the following smallness assumptions are satisfied

$$\frac{c_1}{c(p)} \left[ \left( \frac{\Phi(\varrho)}{|(V(Du))_\varrho|^2} \right)^{\frac{2-p}{2p}} + \left( \frac{\Phi(\varrho)}{|(V(Du))_\varrho|^2} \right)^{\frac{\alpha}{2}} \right] \leq \delta; \quad c(p) \frac{\Phi(\varrho)}{|(V(Du))_\varrho|^2} \leq 1 \quad (5.47)$$

we see that the function  $v$  and the bilinear form  $\mathcal{A} := |A|^{2-p} D^2 f(A)$  fulfill the hypothesis of Lemma 4. Therefore we can find a function  $h \in W^{1,p}(B_\varrho, \mathbb{R}^N)$  which is  $\mathcal{A}$ -harmonic on  $B_\varrho$  satisfying

$$\int_{B_\varrho} |V_1(Dh)|^2 dx \leq c_0 \quad \text{and} \quad \int_{B_\varrho} \left| V_1 \left( \frac{v-h}{\varrho} \right) \right|^2 dx \leq s^2 \varepsilon, \quad (5.48)$$

where  $c_0$  is the constant appearing in Lemma 4. Using (3.3) we deduce that

$$\begin{aligned} & \int_{B_{2\theta Q}} \left| V_1 \left( \frac{v - s(h(0) + Dh(0)x)}{2\theta Q} \right) \right|^2 dx \\ & \leq c(p) \left[ \int_{B_{2\theta Q}} \left| V_1 \left( \frac{v - sh}{2\theta Q} \right) \right|^2 dx + \int_{B_{2\theta Q}} \left| V_1 \left( s \frac{h - h(0) - Dh(0)x}{2\theta Q} \right) \right|^2 dx \right]. \end{aligned}$$

To estimate the right-hand side we proceed as follows: The first term is estimated by using the second inequality in (3.3) (with  $t = (2\theta)^{-1}$ ), (5.48) and the particular choice of  $\varepsilon$ :

$$\int_{B_{2\theta Q}} \left| V_1 \left( \frac{v - sh}{2\theta Q} \right) \right|^2 dx \leq (2\theta)^{-n-2} \int_{B_Q} \left| V_1 \left( \frac{v - sh}{Q} \right) \right|^2 dx \leq 2^{-n-2} \theta^{-n-2} s^2 \varepsilon = c(n) \theta^2 s^2.$$

Using (3.4), Taylor’s theorem applied to  $h$  on  $B_{2\theta Q}$ , the a-priori estimate (see [5,8])

$$\sup_{B_{Q/2}} |Dh| + Q \sup_{B_{Q/2}} |D^2h| \leq c(n, N, \lambda, \Lambda) \int_{B_Q} |Dh| dx \tag{5.49}$$

and the elementary estimate

$$\int_{B_Q} |Dh| dx \leq c(p) \int_{B_Q} |V_1(Dh)|^2 dx + 1 \leq c(p)c_0 + 1, \tag{5.50}$$

which is a consequence of (3.4) and (5.48), we obtain, again using (3.4) and Taylor’s formula,

$$\begin{aligned} \int_{B_{2\theta Q}} \left| V_1 \left( s \frac{h - h(0) - Dh(0)x}{2\theta Q} \right) \right|^2 dx & \leq s^2 \int_{B_{2\theta Q}} \left| \frac{h - h(0) - Dh(0)x}{2\theta Q} \right|^2 dx \\ & \leq \frac{s^2}{4\theta^2 Q^2} \sup_{B_{2\theta Q}} |h(x) - h(0) - Dh(0)x|^2 \leq c\theta^2 s^2, \end{aligned}$$

where  $c = c(n, N, p, \lambda, \Lambda)$ . Combining the last estimates and recalling the definition of  $s$  we arrive at

$$\int_{B_{2\theta Q}} \left| V_1 \left( \frac{v - s(h(0) + Dh(0)x)}{2\theta Q} \right) \right|^2 \leq c\theta^2 \frac{\Phi(Q)}{|(V(Du))_Q|^2},$$

where  $c$  has the same dependencies as before. Recalling the definitions of  $V_1$  and  $v$ , as well as the choice of  $A$ , i.e.  $V(A) = (V(Du))_Q$ , we obtain

$$\int_{B_{2\theta Q}} |V_{|A|}(B(x))|^2 := \int_{B_{2\theta Q}} \left| V_{|A|} \left( \frac{u - Ax - s|A|(h(0) + Dh(0)x)}{2\theta Q} \right) \right|^2 \leq c\theta^2 \Phi(Q), \tag{5.51}$$

with the obvious abbreviation  $B(x)$ . Next we are going to use Caccioppoli’s inequality to estimate the left-hand side of the previous inequality from below. However, the direct application of Lemma 2 is not possible. We first have to replace  $V_{|A|}(B(x))$  by  $V_{|A+\gamma|A|Dh(0)|}(B(x))$  in the integral of the left-hand side. In order to do so, we observe that

$$|A|(1 - \tilde{c}s) \leq |A + s|A|Dh(0)| \leq |A|(1 + \tilde{c}s)$$

since, by (5.49) and (5.50)

$$|Dh(0)| \leq c(n, N, \lambda, \Lambda) \int_{B_\varrho} |Dh| dx \leq \tilde{c}$$

where  $\tilde{c} = \tilde{c}(n, N, p, \lambda, \Lambda)$ ; with no loss of generality we assume that  $\tilde{c} \geq 4$ . Therefore, if we impose  $\tilde{c}s \leq 1/2$ , that is, by the definition of  $s$ ,

$$\tilde{c}^2 \frac{\Phi(\varrho)}{|(V(Du))_\varrho|^2} < \min \left\{ \frac{1}{4}, \theta^n \right\} \tag{5.52}$$

(the reason for the presence of  $\theta^n$  will become clear later) we obtain

$$\begin{aligned} & \left| |V_{|A|}(B)|^2 - |V_{|A+s|A|Dh(0)}(B)|^2 \right| \\ & \leq c(p)|B|^2 \sup_{t \in [(1-\tilde{c}s)|A|, (1+\tilde{c}s)|A|]} (t^2 + |B|^2)^{\frac{p-3}{2}} \left| |A| - |A+s|A|Dh(0)| \right| \\ & \leq c(p)\tilde{c}s|B|^2 \left( (1-\tilde{c}s)^2|A|^2 + |B|^2 \right)^{\frac{p-3}{2}} |A| \\ & \leq c(p)|V_{|A|}(B)|^2. \end{aligned}$$

This and (5.51) imply in particular that, for  $c = c(n, N, p, \lambda, \Lambda)$ ,

$$\int_{B_{2\theta\varrho}} |V_{|A+s|A|Dh(0)}(B(x))|^2 dx \leq c \int_{B_{2\theta\varrho}} |V_{|A|}(B(x))|^2 dx \leq c\theta^2\Phi(\varrho).$$

We are now able to apply Caccioppoli’s inequality, i.e. Lemma 2, to obtain

$$\int_{B_{\theta\varrho}} |V_{|A+s|A|Dh(0)}(Du - A - s|A|Dh(0))|^2 dx \leq c\theta^2\Phi(\varrho), \tag{5.53}$$

where the constant  $c$  depends only on  $n, N, p, \lambda$  and  $\Lambda$ . Now observe that (3.1) implies:

$$\begin{aligned} & \left| |Du|^{\frac{p-2}{2}} Du - |A+s|A|Dh(0)|^{\frac{p-2}{2}} (A+s|A|Dh(0)) \right|^2 \\ & \leq c(p) \left( |A+s|A|Dh(0)|^2 + |Du|^2 \right)^{\frac{p-2}{2}} |Du - (A+s|A|Dh(0))|^2 \\ & \leq c(p) \left( |A+s|A|Dh(0)|^2 + |Du - (A+s|A|Dh(0))|^2 \right)^{\frac{p-2}{2}} |Du - (A+s|A|Dh(0))|^2. \end{aligned} \tag{5.54}$$

Therefore, using (5.53) and the previous estimate we see that

$$\Phi(\theta\varrho) = \int_{B_{\theta\varrho}} |V(Du) - (V(Du))_{\theta\varrho}|^2 dx \leq \int_{B_{\theta\varrho}} |V(Du) - V(A + \gamma|A|Dh(0))|^2 dx \leq c\theta^2\Phi(\varrho),$$

where  $c = c(n, N, p, \lambda, \Lambda)$ . Now we can argue as in the case  $p > 2$ : given  $\beta \in (0, 1)$  we choose  $\theta \in (0, 1/4]$  such that  $c\theta^2 = \theta^{2\beta}$ ; note that  $\theta$  then depends on  $n, N, p, \lambda, \Lambda$  and  $\beta$ . This fixes  $\varepsilon = \theta^{n+4}$ , i.e.  $\varepsilon = \varepsilon(n, N, p, \lambda, \Lambda, \beta)$ , and of course also  $\delta = \delta(n, N, p, \lambda, \Lambda, \varepsilon)$ . With these specifications the smallness assumptions we have imposed during the proof are satisfied, if we require  $\Phi(\varrho) \leq \varepsilon_0|(V(Du))_\varrho|^2$  with a sufficiently small constant  $\varepsilon_0 = \varepsilon_0(n, N, p, \lambda, \Lambda, L, \alpha, \beta) > 0$  in order to meet both (5.47) and (5.52). This proves the claim of the lemma.  $\square$

We now iterate Proposition 4; we shall sketch the arguments here, since they are very similar to the ones employed for the case  $p > 2$ . We start with  $u, \mathcal{F}$  satisfying the hypothesis of Proposition 4 for some ball

$B_\varrho(x_0) \subseteq U$ . We take once again  $x_0$  to be the origin of  $\mathbb{R}^n$  without loss of generality. Then the conclusion of the lemma states that  $\Phi(\theta\varrho) \leq \theta^{2\beta}\Phi(\varrho) < \theta^{2\beta}\varepsilon_0|(V(Du))_\varrho|^2$ . As for (5.24) we get

$$\left(1 - \theta^{-\frac{n}{2}} \left(\frac{\Phi(\varrho)}{|(V(Du))_\varrho|^2}\right)^{\frac{1}{2}}\right) |(V(Du))_\varrho| \leq |(V(Du))_{\theta\varrho}|.$$

Since we can choose  $\theta$  such that  $\theta^\beta \leq 1/2$  we see, using also (5.52) (recall that  $\tilde{c} \geq 4$ ; this is the reason for the presence of  $\theta^n$  in (5.52)), that  $\Phi(\theta\varrho) < \varepsilon_0|(V(Du))_{\theta\varrho}|^2$ , i.e. the initial smallness condition which is needed to apply Proposition 4 is also satisfied on the ball  $B_{\theta\varrho}$ . Therefore we can proceed by induction to deduce easily:  $\Phi(\theta^k\varrho) \leq \theta^{2k\beta}\Phi(\varrho)$  for any  $k \in \mathbb{N}$ . With a standard argument this leads us to the following *excess-decay-lemma*:

**Lemma 14.** *Assume that the hypotheses of Proposition 4 are satisfied. Then for any  $0 < r \leq \varrho$  we have, for  $c = c(n, N, p, \lambda, \Lambda, \beta)$ ,*

$$\Phi(x_0, r, (Du)_r) \leq c \left(\frac{r}{\varrho}\right)^{2\beta} \Phi(x_0, \varrho, (Du)_\varrho).$$

We now treat the *degenerate case* in the subquadratic case  $1 < p < 2$ .

**Lemma 15.** *For  $0 < \tilde{\gamma} < \gamma$  and  $\chi > 0$  there exist constants  $\varepsilon_1 = \varepsilon_1(n, N, p, \lambda, \Lambda, \gamma, \tilde{\gamma}, \eta(\cdot), \chi) > 0$  and  $\tau = \tau(n, N, p, \lambda, \Lambda, \gamma, \tilde{\gamma}, \chi) \in (0, 1/4]$  such that the following is true: Whenever  $u \in W^{1,p}(U, \mathbb{R}^N)$  is  $\mathcal{F}$ -minimizing in  $U$  and  $B_\varrho(x_0) \subseteq U$  is a ball such that*

$$\chi |(V(Du))_{x_0, \varrho}|^2 \leq \Phi(x_0, \varrho, (Du)_\varrho) \tag{5.55}$$

and such that furthermore

$$\Phi(x_0, \varrho, (Du)_\varrho) \leq \varepsilon_1, \tag{5.56}$$

is satisfied, then

$$\Phi(x_0, \tau\varrho, (Du)_{\tau\varrho}) \leq \tau^{2\tilde{\gamma}} \Phi(x_0, \varrho, (Du)_\varrho).$$

**Proof.** We will sketch most of the arguments here, since they are similar to the ones used in the proof of Lemma 12. We take once again  $x_0 = 0$ . Since  $|(V(Du))_\varrho|^2 \leq \chi^{-1}\Phi(\varrho)$ , we have

$$\begin{aligned} \Psi(\varrho) &= \int_{B_\varrho} |Du|^p dx \leq 2 \int_{B_\varrho} |V(Du) - (V(Du))_\varrho|^2 dx + 2|(V(Du))_\varrho|^2 \\ &\leq 2(1 + \chi^{-1})\Phi(\varrho) =: c_4\Phi(\varrho), \end{aligned}$$

where this time we have abbreviated  $c_4 = 2(1 + \chi^{-1})$ . Now we proceed exactly as in the proof of Lemma 12 up to (5.30), taking this time  $\varepsilon := \tau^{n+p+2\gamma}$  (recall that  $\gamma \in (0, 1)$  is the Hölder exponent from the excess decay estimate (3.9) from Proposition 1). Therefore we determine a  $p$ -harmonic function  $h \in W^{1,p}(B_\varrho, \mathbb{R}^N)$  such that

$$\int_{B_\varrho} |Dh|^p dx \leq 1 \quad \text{and} \quad \varrho^{-p} \int_{B_\varrho} |w - h|^p dx \leq \varepsilon = \tau^{n+p+2\gamma}. \tag{5.57}$$

We now choose  $A_{2\tau\varrho} \in \mathbb{R}^{nN}$  according to

$$|A_{2\tau\varrho}|^{\frac{p-2}{2}} A_{2\tau\varrho} = \int_{B_{2\tau\varrho}} |Dh|^{p-2} Dh dx = (V(Dh))_{2\tau\varrho}.$$

Observe that if the mean value is zero, there is nothing to choose. Then using in turn (3.3), (3.4), Poincaré's inequality from Lemma 8, (3.1), the estimate for the  $L^p$ -distance between  $w$  and  $h$  on  $B_\varrho$ , the excess-decay estimate from Proposition 1 in the sub-quadratic case, and the bound  $\int_{B_\varrho} |Dh|^p dx \leq 1$  we deduce that

$$\begin{aligned} & \int_{B_{2\tau\varrho}} \left| V_{|A_{2\tau\varrho}|} \left( \frac{w - (h)_{2\tau\varrho} - A_{2\tau\varrho}x}{2\tau\varrho} \right) \right|^2 dx \\ & \leq c(p) \left[ \int_{B_{2\tau\varrho}} \left| V_{|A_{2\tau\varrho}|} \left( \frac{w - h}{2\tau\varrho} \right) \right|^2 dx + \int_{B_{2\tau\varrho}} \left| V_{|A_{2\tau\varrho}|} \left( \frac{h - (h)_{2\tau\varrho} - A_{2\tau\varrho}x}{2\tau\varrho} \right) \right|^2 dx \right] \\ & \leq c(p) \left[ \int_{B_{2\tau\varrho}} \left| \frac{w - h}{2\tau\varrho} \right|^p dx + \int_{B_{2\tau\varrho}} |V_{|A_{2\tau\varrho}|}(Dh - A_{2\tau\varrho})|^2 dx \right] \\ & \leq c(n, N, p) \left[ (2\tau)^{-n-p\varepsilon} + \int_{B_{2\tau\varrho}} |V(Dh) - V(A_{2\tau\varrho})|^2 dx \right] \\ & \leq c(n, N, p) \left[ \tau^{2\gamma} + \int_{B_{2\tau\varrho}} |V(Dh) - (V(Dh))_{2\tau\varrho}|^2 dx \right] \\ & \leq c(n, N, p) \left[ \tau^{2\gamma} + \tau^{2\gamma} \int_{B_\varrho} |V(Dh) - (V(Dh))_\varrho|^2 dx \right] \\ & \leq c(n, N, p) \tau^{2\gamma}. \end{aligned}$$

Recalling the definition of  $w$ , the previous estimate yields

$$\int_{B_{2\tau\varrho}} \left| V_{|(c_4\Phi(\varrho))^{1/p}A_{2\tau\varrho}|} \left( \frac{u - (c_4\Phi(\varrho))^{1/p}((h)_{2\tau\varrho} + A_{2\tau\varrho}x)}{2\tau\varrho} \right) \right|^2 dx \leq c(n, N, p)c_4\tau^{2\gamma}\Phi(\varrho).$$

An application of Lemma 2, i.e. Caccioppoli's inequality and the use of (3.1) in a way similar to (5.54), yields

$$\begin{aligned} \Phi(\tau\varrho) & \leq \int_{B_{\tau\varrho}} |V(Du) - V((c_4\Phi(\varrho))^{1/p}A_{2\tau\varrho})|^2 dx \\ & \leq c(n, N, p) \int_{B_{\tau\varrho}} |V_{|(c_4\Phi(\varrho))^{1/p}A_{2\tau\varrho}|}(Du - (c_4\Phi(\varrho))^{1/p}A_{2\tau\varrho})|^2 dx \\ & \leq c(n, N, p, \lambda, \Lambda) \int_{B_{2\tau\varrho}} \left| V_{|(c_4\Phi(\varrho))^{1/p}A_{2\tau\varrho}|} \left( \frac{u - (c_4\Phi(\varrho))^{1/p}((h)_{2\tau\varrho} + A_{2\tau\varrho}x)}{2\tau\varrho} \right) \right|^2 dx \\ & \leq c(n, N, p, \lambda, \Lambda)c_4\tau^{2\gamma}\Phi(\varrho). \end{aligned}$$

Now, given  $\tilde{\gamma} \in (0, \gamma)$  we fix  $\tau \in (0, 1/4]$  such that  $2c(1 + \chi^{-1})\tau^{2\tilde{\gamma}} \leq \tau^{2\gamma}$ , where  $c$  is the constant from the previous estimate. Then,  $\tau = \tau(n, N, p, \lambda, \Lambda, \gamma, \tilde{\gamma}, \chi)$ . This fixes the constant  $\delta = \delta(n, N, p, \tau^{n+p+2\gamma})$ . Moreover,  $\mu$  is determined by  $c_2\mu \leq \delta/2$  (to fulfill the analog of (5.28) in our case); note that in this way also

$\mu = \mu(n, N, p, \lambda, \Lambda, \gamma, \tilde{\gamma}, \chi)$  via the corresponding choice for the analog of (5.29). Hence, also  $\eta(\mu)$  is now fixed. The smallness condition imposed on  $\Phi(\varrho)$  is then equivalent to

$$\Phi(\varrho) \leq \frac{\chi}{2(1+\chi)} \left( \frac{\delta\eta(\mu)}{2c_2} \right)^p =: \varepsilon_1.$$

Note that  $\varepsilon_1$  admits the indicated dependencies stated in the formulation of the lemma. This finally proves the assertion.  $\square$

**Proof of Theorem 1.** *Case*  $1 < p < 2$ . The proof follows as for the case  $p > 2$ . Indeed a lemma similar to Lemma 13 can be derived, combining Lemmata 14 and 12 in the same way as for the case  $p > 2$ . A consideration of the structure of the excess functional  $\Phi(v; x_0, \varrho)$  in this case yields the partial regularity of the function  $x \rightarrow V(Du(x))$ , i.e.  $V(Du)$  is Hölder continuous in an open subset  $U_0 \subset U$  of full measure. In turn, this implies the Hölder continuity of  $Du$  (with the same exponent); for this last implication see [9], Lemma 2.4.  $\square$

**Proof of Theorem 3.** The proof follows as for the case  $p > 2$  noting, as mentioned before, that if  $V(Du)$  is Hölder continuous with exponent  $\beta$  then so is  $Du$ .  $\square$

**Remark 1.** From the proofs of the Theorems 2 and 3 follows the more precise statement (which we give for instance in the case  $1 < p < 2$ ): “for any  $\beta \in (0, 1)$  there exists  $M = M(n, N, p, \lambda, \Lambda, \beta) = M(\beta)$  such that if:

$$\limsup_{r \rightarrow 0} \frac{|(Du)_{x_0, r}|^p}{\Phi(u; x_0, r)} \geq M,$$

then  $Du$  is Hölder continuous in a neighborhood of  $x_0$ , with exponent  $\beta$ ”. That is, the local degree of regularity of solutions depends in a quantitative way on the speed of degeneration.

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