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Heat flow, Brownian motion and Newtonian capacity

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Abstract

Let K be a compact, non-polar set in $\mathbb{R}^m\,(m \geqslant 3)$ and let u be the unique weak solution of $\Delta u = \frac{\partial u}{\partial t}$ on $\mathbb{R}^m \backslash K \times (0,\infty)$, $u(x;0)=0$ on $\mathbb{R}^m \backslash K$ and $u(x;t)=1$ for all x on the boundary of K and for all $t>0$. The asymptotic behaviour of $u(x;t)$ as t tends to infinity is obtained up to order $\mathrm{O}(t^{-m/2})$.

Résumé

Soit K un ensemble compact, non-polaire dans $\mathbb{R}^m\,(m \geqslant 3)$ et soit u l'unique solution faible de $\Delta u = \frac{\partial u}{\partial t}$ sur $\mathbb{R}^m \backslash K \times (0,\infty)$, $u(x;0)=0$ sur $\mathbb{R}^m \backslash K$ et $u(x;t)=1$ pour tout x sur la frontière de K et tout $t>0$. On obtient le comportement asymptotique de $u(x,t)$ quand t tend vers l'infini avec un reste $\mathrm{O}(t^{-m/2})$.

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Keywords: Heat flow; Brownian motion; Newtonian capacity

1. Introduction

Let K be a compact, non-polar set in Euclidean space $\mathbb{R}^m\,(m \geqslant 3)$ with boundary ∂K and let $u : \mathbb{R}^m \backslash K \times [0,\infty) \to \mathbb{R}$ be the unique weak solution of

$$\Delta u = \frac{\partial u}{\partial t}, \quad x \in \mathbb{R}^m \backslash K,\ t > 0, \tag{1}$$

with boundary condition

$$u(x;t) = 1, \quad x \in \partial K,\ t > 0, \tag{2}$$

and initial condition

$$u(x;0) = 0, \quad x \in \mathbb{R}^m \backslash K. \tag{3}$$

It is well known that

$$\lim_{t\to\infty} u(x;t) = h_K(x), \quad x \in \mathbb{R}^m \backslash K, \tag{4}$$

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where h_K is the unique function which is harmonic on $\mathbb{R}^m\backslash K$, which equals 1 on the regular points of K, and which vanishes at infinity.

S.C. Port [8], [10, pp. 64, 65] proved that if K is a compact and non-polar set in $\mathbb{R}^m\,(m \geqslant 3)$ then for $t \to \infty$

$$u(x;t) = h_K(x) - \left(\frac{m}{2}-1\right)^{-1}(4\pi)^{-m/2}C(K)\big(1-h_K(x)\big)t^{(2-m)/2} + \mathrm{o}\big(t^{(2-m)/2}\big), \tag{5}$$

where $C(K)$ is the Newtonian capacity of K.

Formula (5) was first proved by A. Joffe [7] in the special case where $m = 3$ and where K has positive Lebesgue measure $|K|$. Subsequently F. Spitzer [12, p. 114] proved formula (5) for arbitrary compact, non-polar sets in $\mathbb{R}^3$ and obtained the asymptotic behaviour of the total amount of heat $E_K(t)$ in $\mathbb{R}^m\backslash K$ at time t defined by

$$E_K(t) = \int\limits_{\mathbb{R}^m\backslash K} u(x;t)\,\mathrm{d}x. \tag{6}$$

He showed that for $m = 3$ and $t \to \infty$

$$E_K(t) = C(K)t + \frac{1}{2\pi^{3/2}}C(K)^2t^{1/2} + \mathrm{o}\big(t^{1/2}\big). \tag{7}$$

J.-F. Le Gall [4–6] and Port [11] obtained refinements of (7) and extensions to $m \geqslant 4$ and $m = 2$ without the use of (5). Port also obtained the large t behaviour of u in the case where K is a non-polar compact set in $\mathbb{R}^2$ [9].

The main result of this paper concerns the analysis of the remainder estimate $\mathrm{o}(t^{(2-m)/2})$ in (5). For $m \geqslant 5$ we show that this remainder can be improved to $\mathrm{O}(t^{-m/2})$. A new term of order $(\log t)/t^2$ shows up for $m = 4$ before we recover the remainder $\mathrm{O}(t^{-2})$. A remarkable cancellation of two terms of order t^{-1} and four terms of order $(\log t)/t^{3/2}$ takes place for $m = 3$, resulting in the sharp remainder $\mathrm{O}(t^{-3/2})$.

Theorem 1. *Let K be a compact and non-polar set in $\mathbb{R}^m$.*

(i) *If $m = 3, 5, 6, \ldots$ then for $x \in \mathbb{R}^m\backslash K$ and $t \to \infty$*

$$u(x;t) = h_K(x) - \left(\frac{m}{2}-1\right)^{-1}(4\pi)^{-m/2}C(K)\big(1-h_K(x)\big)t^{(2-m)/2} + \mathrm{O}\big(t^{-m/2}\big). \tag{8}$$

(ii) *If $m = 4$ then for $x \in \mathbb{R}^4\backslash K$ and $t \to \infty$*

$$u(x;t) = h_K(x) - (4\pi)^{-2}C(K)\big(1-h_K(x)\big)t^{-1} + 2(4\pi)^{-4}C(K)^2\big(1-h_K(x)\big)\frac{\log t}{t^2} + \mathrm{O}\big(t^{-2}\big). \tag{9}$$

(iii) *The remainder in* (8) *is sharp for a ball in $\mathbb{R}^3$.*

(iv) *The remainder $\mathrm{O}(t^{-m/2})$ in* (8) *and* (9) *is uniform in x on compact subsets of $\mathbb{R}^m\backslash K$.*

The results described in Theorem 1 have an equivalent probabilistic formulation. Let $(B(s), s \geqslant 0; \mathbb{P}_x, x \in \mathbb{R}^m)$ be a Brownian motion with generator Δ. For $x \in \mathbb{R}^m$ we define the first hitting time of K by

$$T_K = \inf\big\{s \geqslant 0\colon\ B(s) \in K\big\}, \tag{10}$$

and $T_K = +\infty$ if the infimum is taken over the empty set. It is a classical result that

$$u(x;t) = \mathbb{P}_x[T_K < t], \quad x \in \mathbb{R}^m,\ t > 0, \tag{11}$$

where we have extended both u and h_K to all of $\mathbb{R}^m$ by putting $u \equiv h_K \equiv 1$ on K. For $x \in \mathbb{R}^m\,(m \geqslant 3)$ we define the last exit time of K by

$$L_K = \sup\big\{s \geqslant 0\colon\ B(s) \in K\big\}, \tag{12}$$

and $L_K = +\infty$ if the supremum is taken over the empty set. The law of L_K is given by [10, p. 61]

$$\mathbb{P}_x[L_K < t] = \int\limits_0^t \mathrm{d}s \int \mu_K(\mathrm{d}y)p(x,y;s), \tag{13}$$

where

$$p(x,y;s)=(4\pi s)^{-m/2}\mathrm{e}^{-|x-y|^2/(4s)}, \tag{14}$$

and where μ_K is the equilibrium measure supported on K with

$$\int \mu_K(\mathrm{d}y)=C(K). \tag{15}$$

It follows that

$$h_K(x)=\mathbb{P}_x[T_K<\infty]=\mathbb{P}_x[L_K<\infty]=c_m\int \mu_K(\mathrm{d}y)|x-y|^{2-m}, \tag{16}$$

where

$$c_m=4^{-1}\pi^{-m/2}\Gamma\big((m-2)/2\big). \tag{17}$$

Since

$$\mathbb{P}_x[t<L_K<\infty]=\int_t^\infty \mathrm{d}s\int \mu_K(\mathrm{d}y)p(x,y;s), \tag{18}$$

and

$$(4\pi s)^{-m/2}\big(1-|x-y|^2/(4s)\big)\leqslant p(x,y;s)\leqslant (4\pi s)^{-m/2}, \tag{19}$$

we have that

$$\mathbb{P}_x[t<L_K<\infty]=\left(\frac{m}{2}-1\right)^{-1}(4\pi)^{-m/2}C(K)t^{(2-m)/2}+\mathrm{O}\big(t^{-m/2}\big). \tag{20}$$

Using (11), (16) and (20) we can rewrite (8), (9) as follows.

Proposition 2. *Let K be a compact and non-polar set in $\mathbb{R}^m$.*

(i) *If $m=3,5,6,\ldots$ then for $x\in\mathbb{R}^m\backslash K$ and $t\to\infty$*

$$\mathbb{P}_x[t<T_K<\infty]=\mathbb{P}_x[T_K=\infty]\mathbb{P}_x[t<L_K<\infty]+\mathrm{O}\big(t^{-m/2}\big). \tag{21}$$

(ii) *If $m=4$ then for $x\in\mathbb{R}^4\backslash K$ and $t\to\infty$*

$$\mathbb{P}_x[t<T_K<\infty]=\mathbb{P}_x[T_K=\infty]\mathbb{P}_x[t<L_K<\infty]-2(4\pi)^{-4}C(K)^2\mathbb{P}_x[T_K=\infty]\frac{\log t}{t^2}+\mathrm{O}\big(t^{-2}\big). \tag{22}$$

It is well known [4, p. 392] that if $m=3$ and $K=B(0;R)$ (the closed ball with center 0 and radius R) then for $|x|\geqslant R$

$$\mathbb{P}_x[t<T_{B(0;R)}<\infty]=\int_t^\infty \mathrm{d}s\,\big(4\pi s^3\big)^{-1/2}\frac{R(|x|-R)}{|x|}\,\mathrm{e}^{-(|x|-R)^2/(4s)}. \tag{23}$$

Moreover for a ball $B(0;R)$ in $\mathbb{R}^3$ the corresponding equilibrium measure is concentrated on $\partial B(0;R)$ and proportional to the surface measure, with constant of proportionality equal to R^{-1}. This gives by (18)

$$\mathbb{P}_x[T_{B(0;R)}=\infty]=\frac{|x|-R}{|x|}, \tag{24}$$

and

$$\mathbb{P}_x[t<L_{B(0;R)}<\infty]=\int_t^\infty \mathrm{d}s\,(4\pi s)^{-1/2}|x|^{-1}\big(1-\mathrm{e}^{-|x|R/s}\big)\mathrm{e}^{-(|x|-R)^2/(4s)}. \tag{25}$$

It is a straightforward computation to show that, by (23)–(25), for $m = 3$

$$\mathbb{P}_x[t < T_{B(0;R)} < \infty] = \mathbb{P}_x[T_{B(0;R)} = \infty]\mathbb{P}_x[t < L_{B(0;R)} < \infty] + \frac{1}{6\pi^{1/2}}\mathbb{P}_x[T_{B(0;R)} = \infty]|x|R^2 t^{-3/2} + \mathrm{O}(t^{-5/2}). \tag{26}$$

This proves the assertion in Theorem 1(iii).

The main stratagem which permeates the proof of Proposition 2 is to replace T_K by L_K at "every possible opportunity" and to use the strong Markov property to control terms like $\mathbb{P}_x[T_K < t < L_K]$. For a different application of these techniques we refer to the study of the expected volume of a Wiener sausage in $\mathbb{R}^3$ associated to the compact set K [4]. There Spitzer's formula (7) was improved up to order $\mathrm{O}(t^{-1/2})$ proving a conjecture by M. Kac. See [1–3,13] for more recent applications.

It turns out that a single application of the strong Markov property (Proposition 4) supplemented by additional estimates (Lemma 3) is sufficient to prove Proposition 2 for $m \geqslant 5$. However, for $m = 4$ or $m = 3$ the strong Markov property has to be applied twice respectively six times (Propositions 5 and 8). The reason is that for $m = 3$ two non-trivial terms of order t^{-1} and four non-trivial terms of order $(\log t)/t^{3/2}$ contribute to $\mathbb{P}_x[t < T_K < \infty]$. Lengthy calculations using the above techniques finally result in the cancellation of these non-trivial terms. Such a cancellation does not take place for $m = 4$, and this results in the $(\log t)/t^2$ contribution in (9).

The analysis of the $\mathrm{O}(t^{-m/2})$ remainder in Proposition 2 is complicated since the distribution of the random variable $B(T_K)$ on the regular part of ∂K enters at each application of the strong Markov property. Unlike the special case of a ball in $\mathbb{R}^3$ we do not expect a simple improvement of the remainder.

This paper is organized as follows. In Section 2 we prove some basic estimates (Lemma 3) which will be used throughout the paper. Proposition 4 is the key estimate from which Proposition 2 follows for $m \geqslant 5$. In Section 3 we use Proposition 4 to obtain a further refinement (Proposition 5) from which Proposition 2 follows for $m = 4$. Finally in Section 4 we complete the proof of Proposition 2 for $m = 3$ by refining Proposition 5 (Proposition 8). The proof of Proposition 8 follows the same strategy as the proof of Proposition 5, and has been omitted.

2. Proof of Proposition 2 for $m \geqslant 5$

It is convenient to introduce some further notation. For $c \in \mathbb{R}^m$ and K compact in $\mathbb{R}^m$ we define

$$R(c) = \inf\{\rho \geqslant 0\colon\ K \subset B(c;\rho)\}, \tag{27}$$

where $B(c;\rho)$ is the closed ball with center c and radius ρ. Let

$$R = \inf\{R(c)\colon\ c \in \mathbb{R}^m\}. \tag{28}$$

The infima in (27) and (28) are attained and we may assume without loss of generality that the latter is attained at $c = 0$.

Lemma 3. *Let K be a compact and non-polar set in $\mathbb{R}^m$ $(m \geqslant 3)$. Then for $0 < s < t < \infty$*

$$\mathbb{P}_x[t < T_K < \infty] \leqslant \mathbb{P}_x[t < L_K < \infty] \leqslant 1 \wedge \left(\frac{m}{2} - 1\right)^{-1} (4\pi)^{-m/2} C(K) t^{(2-m)/2}, \tag{29}$$

$$\mathbb{P}_x[s < L_K < t] \leqslant 1 \wedge \left(\frac{m}{2} - 1\right)^{-1} (4\pi)^{-m/2} C(K)\left(s^{(2-m)/2} - t^{(2-m)/2}\right), \tag{30}$$

and for $z \in K$

$$\left|\mathbb{P}_x[t < L_K < \infty] - \mathbb{P}_z[t < L_K < \infty]\right| \leqslant 1 \wedge C_{x,K} t^{-m/2}, \tag{31}$$

where

$$C_{x,K} = (|x| + R)(|x| + 3R)C(K). \tag{32}$$

For any Borel set E of $[0,t]$

$$\int_E \mathrm{d}s \int \mu_K(\mathrm{d}y)p(x,y;t-s) \leqslant 1. \tag{33}$$

Let $T>0$ be arbitrary. There exists a constant C depending on T and on K such that for all $t>T$, $0<s<t$ and $x \in \mathbb{R}^m$

$$\mathbb{P}_x[s<T_K<t] \leqslant C\big(T(t-T)^{-m/2} \vee (t-s)s^{-m/2}\big). \tag{34}$$

Proof. Estimate (29) follows immediately from the fact that $L_K \geqslant T_K$ and (18), (19).

Estimate (30) follows from

$$\mathbb{P}_x[s<L_K<t] = \int_s^t \mathrm{d}\tau \int \mu_K(\mathrm{d}y)p(x,y;\tau), \tag{35}$$

and the bound in the right-hand side of (19).

To prove (31) we note that by (18)

$$\begin{aligned}
\big|\mathbb{P}_x[t<L_K<\infty]-\mathbb{P}_z[t<L_K<\infty]\big| &\leqslant \int_t^\infty \mathrm{d}s\,(4\pi s)^{-m/2}\int \mu_K(\mathrm{d}y)\big|\mathrm{e}^{-|x-y|^2/(4s)}-\mathrm{e}^{-|z-y|^2/(4s)}\big| \\
&\leqslant \int_t^\infty \mathrm{d}s\,(4\pi s)^{-m/2}(4s)^{-1}\int \mu_K(\mathrm{d}y)\big||x-y|^2-|z-y|^2\big| \\
&\leqslant t^{-m/2}\int \mu_K(\mathrm{d}y)\big(|x|+|z|\big)\big(|x|+|z|+2|y|\big) \\
&\leqslant C_{x,K}t^{-m/2}
\end{aligned} \tag{36}$$

since both y and $z \in K \subset B(0;R)$.

Since p is non-negative

$$\begin{aligned}
\int_E \mathrm{d}s \int \mu_K(\mathrm{d}y)p(x,y;t-s) &\leqslant \int_{[0,t]} \mathrm{d}s \int \mu_K(\mathrm{d}y)p(x,y;t-s) \\
&= \mathbb{P}_x[L_K<t] \leqslant 1.
\end{aligned} \tag{37}$$

This proves (33).

The proof of (34) relies on the following [4,11,12]. For $m \geqslant 3$

$$\int_{\mathbb{R}^m} \mathrm{d}y\,\mathbb{P}_y[T_K<t] = C(K)t + \mathrm{o}(t), \quad t \to \infty. \tag{38}$$

Hence there exists T_1 such that for all $t \geqslant T_1$

$$\int_{\mathbb{R}^m} \mathrm{d}y\,\mathbb{P}_y[T_K<t] \leqslant 2C(K)t. \tag{39}$$

By the Markov property at time s we have that

$$\mathbb{P}_x[s<T_K<t] = \int_{\mathbb{R}^m} \mathrm{d}y\, p_{\mathbb{R}^m\setminus K}(x,y;s)\mathbb{P}_y[T_K<t-s], \tag{40}$$

where $p_{\mathbb{R}^m\setminus K}(\cdot,\cdot;\cdot)$ is the Dirichlet heat kernel for the open set $\mathbb{R}^m \setminus K$ (i.e. the transition density of Brownian motion with killing on K). By domain monotonicity of the Dirichlet heat kernel

$$p_{\mathbb{R}^m\setminus K}(x,y;s) \leqslant p(x,y;s) \leqslant (4\pi s)^{-m/2}. \tag{41}$$

We first consider the case $t-s>T_1$. Then by (39)–(41)

$$\mathbb{P}_x[s<T_K<t]\leqslant(4\pi s)^{-m/2}\int\limits_{\mathbb{R}^m}\mathrm{d}y\,\mathbb{P}_y[T_K<t-s]$$
$$\leqslant 2(4\pi s)^{-m/2}C(K)(t-s). \tag{42}$$

Next suppose that $T<T_1$ and $t-s\in[T,T_1]$. Then by monotonicity

$$\int\limits_{\mathbb{R}^m}\mathrm{d}y\,\mathbb{P}_y[T_K<t-s]\leqslant\int\limits_{\mathbb{R}^m}\mathrm{d}y\,\mathbb{P}_y[T_K<T_1]$$
$$\leqslant 2C(K)T_1\leqslant 2C(K)\frac{T_1}{T}(t-s), \tag{43}$$

and

$$\mathbb{P}_x[s<T_K<t]\leqslant 2(4\pi s)^{-m/2}C(K)\frac{T_1}{T}(t-s). \tag{44}$$

Combining (42) and (44) we obtain that

$$\mathbb{P}_x[s<T_K<t]\leqslant Cs^{-m/2}(t-s),\quad t-s\geqslant T, \tag{45}$$

with C given by

$$C=2(4\pi)^{-m/2}C(K)\left(1\vee\frac{T_1}{T}\right). \tag{46}$$

By (45)

$$\mathbb{P}_x[s<T_K<t]\leqslant\mathbb{P}_x[t-T<T_K<t]\leqslant CT(t-T)^{-m/2},\quad t-s\leqslant T, \tag{47}$$

and (34) follows from (45)–(47). □

Proposition 4. *Let K be a compact and non-polar set in $\mathbb{R}^m\,(m\geqslant 3)$. Then for $t\to\infty$*

$$\mathbb{P}_x[t<T_K<\infty]=\mathbb{P}_x[T_K=\infty]\,\mathbb{P}_x[t<L_K<\infty]+\mathbb{P}_x[t<T_K<\infty]\mathbb{P}_x[t<L_K<\infty]$$
$$-\int_0^t\mathrm{d}s\,\mathbb{P}_x[s<T_K<t]\int\mu_K(\mathrm{d}y)p(x,y;t-s)+\mathrm{O}\big(t^{-m/2}\big). \tag{48}$$

Proof. Note that

$$\mathbb{P}_x[t<T_K<\infty]=\mathbb{P}_x[t<L_K<\infty]-\mathbb{P}_x[T_K<t<L_K]. \tag{49}$$

By the strong Markov property

$$\mathbb{P}_x[T_K<t<L_K]=E_x\left\{\int_0^t 1_{T_K\in\mathrm{d}s}\mathbb{P}_{B(T_K)}[t-s<L_K<\infty]\right\}. \tag{50}$$

Using Lemma 3, (31) with $z=B(T_K)$

$$\big|\mathbb{P}_{B(T_K)}[t-s<L_K<\infty]-\mathbb{P}_x[t-s<L_K<\infty]\big|\leqslant 1\wedge C_{x,K}(t-s)^{-m/2}. \tag{51}$$

If we can show that

$$E_x\left\{\int_0^t 1_{T_K\in\mathrm{d}s}\big(1\wedge C_{x,K}(t-s)^{-m/2}\big)\right\}=\mathrm{O}\big(t^{-m/2}\big), \tag{52}$$

then, by (50)–(52),

$$\begin{aligned}\mathbb{P}_x[T_K < t < L_K] &= E_x\left\{\int_0^t 1_{T_K\in \mathrm{d}s}\mathbb{P}_x[t-s < L_K < \infty]\right\} + \mathrm{O}\big(t^{-m/2}\big)\\ &= \int_0^t \mathrm{d}s\,\frac{\mathrm{d}}{\mathrm{d}s}\big(\mathbb{P}_x[T_K < s] - \mathbb{P}_x[T_K < t]\big)\mathbb{P}_x[t-s < L_K < \infty] + \mathrm{O}\big(t^{-m/2}\big)\\ &= \mathbb{P}_x[T_K < t]\mathbb{P}_x[t < L_K < \infty]\\ &\quad + \int_0^t \mathrm{d}s\,\mathbb{P}_x[s < T_K < t]\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{P}_x[t-s < L_K < \infty] + \mathrm{O}\big(t^{-m/2}\big).\end{aligned} \tag{53}$$

This implies Proposition 4 since, by (18),

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{P}_x[t-s < L_K < \infty] = \int \mu_K(\mathrm{d}y)\,p(x,y;t-s). \tag{54}$$

To prove (52) we note that

$$\begin{aligned}&E_x\left\{\int_0^t 1_{T_K\in \mathrm{d}s}\big(1\wedge C_{x,K}(t-s)^{-m/2}\big)\right\}\\ &\quad= \int_0^t \mathrm{d}s\,\frac{\mathrm{d}}{\mathrm{d}s}\big(\mathbb{P}_x[T_K < s] - \mathbb{P}_x[T_K < t]\big)\big(1\wedge C_{x,K}(t-s)^{-m/2}\big)\\ &\quad= \mathbb{P}_x[T_K < t]\big(1\wedge C_{x,K}t^{-m/2}\big) + \frac{m}{2}C_{x,K}\int_0^{t^*} \mathrm{d}s\,\mathbb{P}_x[s < T_K < t](t-s)^{-(m+2)/2},\end{aligned} \tag{55}$$

where

$$t^* = (t-T)\vee 0, \tag{56}$$

and

$$T = C_{x,K}^{2/m}. \tag{57}$$

The first term in the right-hand side of (55) is $\mathrm{O}(t^{-m/2})$. To estimate the second term in the right-hand side of (55) we suppose that $t > T$ and use Lemma 3 with $T = C_{x,K}^{2/m}$ to obtain that

$$\begin{aligned}&\int_0^{t^*} \mathrm{d}s\,\mathbb{P}_x[s < T_K < t](t-s)^{-(m+2)/2}\\ &\quad\leqslant \int_0^{(t-T)/2} \mathrm{d}s\,(t-s)^{-(m+2)/2} + \int_{(t-T)/2}^{t-T} \mathrm{d}s\,Cs^{-m/2}(t-s)^{-m/2}\\ &\quad\leqslant \big((t+T)/2\big)^{-m/2} + C\big((t-T)/2\big)^{-m/2}\int_0^{t-T} \mathrm{d}s\,(t-s)^{-m/2} = \mathrm{O}\big(t^{-m/2}\big). \qquad \square\end{aligned} \tag{58}$$

We conclude this section with the proof of Proposition 2 for $m \geqslant 5$. By Lemma 3, (29), the second term in the right-hand side of (48) is $\mathrm{O}(t^{2-m})$ and hence is $\mathrm{O}(t^{-m/2})$ for $m \geqslant 4$. By (19) and (15) we have for any $z \in \mathbb{R}^m$

$$\int \mu_K(\mathrm{d}y)\,p(z,y;t-s) \leqslant C(K)t^{-m/2}, \quad s\in[0,t/2]. \tag{59}$$

Hence, by (29) and (59), we have for $m \geqslant 5$

$$\int_0^{t/2} \mathrm{d}s\, \mathbb{P}_x[s < T_K < t] \int \mu_K(\mathrm{d}y) p(x, y; t-s) \leqslant C(K) t^{-m/2} \int_0^{t/2} \mathrm{d}s \left(1 \wedge C(K) s^{1-m/2}\right) = \mathrm{O}\left(t^{-m/2}\right). \tag{60}$$

By (34) we have for $m \geqslant 5$

$$\int_{t/2}^{t-T} \mathrm{d}s\, \mathbb{P}_x[s < T_K < t] \int \mu_K(\mathrm{d}y) p(x, y; t-s) \leqslant C \int_{t/2}^{t-T} \mathrm{d}s\, s^{-m/2} C(K)(t-s)^{1-m/2} = \mathrm{O}\left(t^{-m/2}\right). \tag{61}$$

By (34) and (33) for $E = [t-T, t]$ we have

$$\int_{t-T}^{t} \mathrm{d}s\, \mathbb{P}_x[s < T_K < t] \int \mu_K(\mathrm{d}y) p(x, y; t-s) \leqslant CT(t-T)^{-m/2} = \mathrm{O}\left(t^{-m/2}\right). \tag{62}$$

By (60)–(62) and Proposition 4 we conclude that (21) holds for $m \geqslant 5$. □

3. Proof of Proposition 2 for $m = 4$

The proof of Proposition 2 for $m = 4$ and $m = 3$ relies on the asymptotic analysis of the third term in the right-hand side of (48).

Proposition 5. *Let K be a compact and non-polar set in $\mathbb{R}^m$ ($m \geqslant 3$). Then for $t \to \infty$*

$$\int_0^t \mathrm{d}s\, \mathbb{P}_x[s < T_K < t] \int \mu_K(\mathrm{d}y) p(x, y; t-s)$$
$$= \mathbb{P}_x[T_K = \infty] \int_0^t \mathrm{d}s\, \mathbb{P}_x[s < L_K < t] \int \mu_K(\mathrm{d}y) p(x, y; t-s) + \sum_{i=1}^{4} A_i + \mathrm{O}\left(t^{-m/2}\right), \tag{63}$$

where

$$A_1 = \int_0^t \mathrm{d}s\, \mathbb{P}_x[s < T_K < t] \mathbb{P}_x[t-s < L_K < \infty] \int \mu_K(\mathrm{d}y) p(x, y; t-s), \tag{64}$$

$$A_2 = \int_0^t \mathrm{d}s\, \mathbb{P}_x[s < T_K < \infty] \mathbb{P}_x[s < L_K < t] \int \mu_K(\mathrm{d}y) p(x, y; t-s), \tag{65}$$

$$A_3 = \int_0^t \mathrm{d}s \int_s^t \mathrm{d}\tau\, \mathbb{P}_x[\tau < T_K < t] \int \mu_K(\mathrm{d}z) p(x, z; t-\tau) \int \mu_K(\mathrm{d}y) p(x, y; t-s), \tag{66}$$

$$A_4 = \int_0^t \mathrm{d}s \int_0^s \mathrm{d}\tau\, \mathbb{P}_x[\tau < T_K < s] \int \mu_K(\mathrm{d}z)\left(p(x, z; t-\tau) - p(x, z; s-\tau)\right) \int \mu_K(\mathrm{d}y) p(x, y; t-s). \tag{67}$$

Proof. Since

$$\mathbb{P}_x[s < T_K < t] = \mathbb{P}_x[s < L_K < t] + \mathbb{P}_x[T_K < t < L_K] - \mathbb{P}_x[T_K < s < L_K], \tag{68}$$

we have that the left-hand side of (63) equals

$$\int_0^t \mathrm{d}s\, \mathbb{P}_x[s < L_K < t] \int \mu_K(\mathrm{d}y) p(x, y; t-s)$$
$$+ \int_0^t \mathrm{d}s\, \big(\mathbb{P}_x[T_K < t < L_K] - \mathbb{P}_x[T_K < s < L_K]\big) \int \mu_K(\mathrm{d}y) p(x, y; t-s). \tag{69}$$

By the strong Markov property we can write the second term in (69) as

$$\int_0^t \mathrm{d}s\, E_x\Bigg\{ \int_0^t 1_{T_K \in \mathrm{d}\tau} \mathbb{P}_{B(T_K)}[t-\tau < L_K < \infty]$$
$$- \int_0^s 1_{T_K \in \mathrm{d}\tau} \mathbb{P}_{B(T_K)}[s-\tau < L_K < \infty]\Bigg\} \int \mu_K(\mathrm{d}y) p(x, y; t-s). \tag{70}$$

First we show that we can replace $B(T_K)$ in (70) by x at a cost $\mathrm{O}(t^{-m/2})$. By (52)

$$\int_0^t \mathrm{d}s\, E_x\Bigg\{ \int_0^t 1_{T_K \in \mathrm{d}\tau} \big(1 \wedge C_{x,K}(t-\tau)^{-m/2}\big)\Bigg\} \int \mu_K(\mathrm{d}y) p(x, y; t-s)$$
$$\leqslant E_x\Bigg\{ \int_0^t 1_{T_K \in \mathrm{d}\tau} \big(1 \wedge C_{x,K}(t-\tau)^{-m/2}\big)\Bigg\} = \mathrm{O}\big(t^{-m/2}\big). \tag{71}$$

Moreover by (55)

$$\int_0^t \mathrm{d}s\, E_x\Bigg\{ \int_0^s 1_{T_K \in \mathrm{d}\tau} \big(1 \wedge C_{x,K}(s-\tau)^{-m/2}\big)\Bigg\} \int \mu_K(\mathrm{d}y) p(x, y; t-s)$$
$$= \int_0^t \mathrm{d}s\, \mathbb{P}_x[T_K < s]\big(1 \wedge C_{x,K} s^{-m/2}\big) \int \mu_K(\mathrm{d}y) p(x, y; t-s)$$
$$+ \int_0^t \mathrm{d}s \int_0^{s^*} \mathrm{d}\tau\, \mathbb{P}_x[\tau < T_K < s] C_{x,K} \frac{m}{2}(s-\tau)^{-(m+2)/2} \int \mu_K(\mathrm{d}y) p(x, y; t-s), \tag{72}$$

where

$$s^* = (s-T) \vee 0. \tag{73}$$

By (59)

$$\int_0^{t/2} \mathrm{d}s\, \mathbb{P}_x[T_K < s]\big(1 \wedge C_{x,K} s^{-m/2}\big) \int \mu_K(\mathrm{d}y) p(x, y; t-s) \leqslant C(K) t^{-m/2} \Bigg(\int_0^\infty \mathrm{d}s\, \big(1 \wedge C_{x,K} s^{-m/2}\big)\Bigg)$$
$$= \mathrm{O}\big(t^{-m/2}\big). \tag{74}$$

By (33) with $E = [t/2, t]$

$$\int_{t/2}^{t} \mathrm{d}s\, \mathbb{P}_x[T_K < s]\big(1 \wedge C_{x,K} s^{-m/2}\big) \int \mu_K(\mathrm{d}y) p(x,y;t-s) \leqslant C_{x,K}(t/2)^{-m/2} \int_E \mathrm{d}s \int \mu_K(\mathrm{d}y) p(x,y;t-s)$$
$$= \mathrm{O}\big(t^{-m/2}\big). \tag{75}$$

To estimate the second term in the right-hand side of (72) we have that the contribution from $s \in [T, 2T]$ is bounded by

$$\int_T^{2T} \mathrm{d}s \int_0^{s-T} \mathrm{d}\tau\, C_{x,K} \frac{m}{2}(s-\tau)^{-(m+2)/2} \int \mu_K(\mathrm{d}y) p(x,y;t-s) \leqslant \int_T^{2T} \mathrm{d}s \int \mu_K(\mathrm{d}y) p(x,y;t-s)$$
$$\leqslant C(K) T (t-2T)^{-m/2}. \tag{76}$$

The interval $[2T, t/2]$ contributes at most, by (34) and (59),

$$\begin{aligned}
&\int_{2T}^{t/2} \mathrm{d}s \int_0^{(s-T)/2} \mathrm{d}\tau\, C_{x,K} \frac{m}{2}(s-\tau)^{-(m+2)/2} \int \mu_K(\mathrm{d}y) p(x,y;t-s) \\
&\quad + \int_{2T}^{t/2} \mathrm{d}s \int_{(s-T)/2}^{s-T} \mathrm{d}\tau\, C C_{x,K} \frac{m}{2} \tau^{-m/2}(s-\tau)^{-m/2} \int \mu_K(\mathrm{d}y) p(x,y;t-s) \\
&\leqslant C(K) C_{x,K} t^{-m/2} \int_{2T}^{t/2} \mathrm{d}s \int_{-\infty}^{(s-T)/2} \mathrm{d}\tau\, \frac{m}{2}(s-\tau)^{-(m+2)/2} \\
&\quad + C C(K) C_{x,K} t^{-m/2} \int_{2T}^{t/2} \mathrm{d}s \big((s-T)/2\big)^{-m/2} \int_{-\infty}^{s-T} \mathrm{d}\tau\, \frac{m}{2}(s-\tau)^{-m/2} \\
&= \mathrm{O}\big(t^{-m/2}\big).
\end{aligned} \tag{77}$$

The interval $[t/2, t]$ contributes at most, by (33) and (34),

$$\sup_{t/2<s<t} \left\{ \int_0^{(s-T)/2} \mathrm{d}\tau\, C_{x,K} \frac{m}{2}(s-\tau)^{-(m+2)/2} + \int_{(s-T)/2}^{s-T} \mathrm{d}\tau\, C C_{x,K} \frac{m}{2} \tau^{-m/2}(s-\tau)^{-m/2} \right\}$$
$$\leqslant \sup_{t/2<s<t} \big\{ C_{x,K}\big((s+T)/2\big)^{-m/2} + 3C C_{x,K}\big((s-T)/2\big)^{-m/2} T^{(2-m)/2} \big\} = \mathrm{O}\big(t^{-m/2}\big). \tag{78}$$

By (74)–(78) we conclude that the right-hand side of (72) is $\mathrm{O}(t^{-m/2})$. Then, by Lemma 3, (31), (71) we have that the expression in (70) equals

$$\begin{aligned}
&\int_0^t \mathrm{d}s \left\{ \int_0^t \mathrm{d}\tau \frac{\mathrm{d}}{\mathrm{d}\tau}\big(\mathbb{P}_x[T_K < \tau] - \mathbb{P}_x[T_K < t]\big) \mathbb{P}_x[t-\tau < L_K < \infty] \right. \\
&\quad \left. - \int_0^s \mathrm{d}\tau \frac{\mathrm{d}}{\mathrm{d}\tau}\big(\mathbb{P}_x[T_K < \tau] - \mathbb{P}_x[T_K < s]\big) \mathbb{P}_x[s-\tau < L_K < \infty] \right\} \int \mu_K(\mathrm{d}y) p(x,y;t-s) + \mathrm{O}\big(t^{-m/2}\big) \\
&= \int_0^t \mathrm{d}s \left\{ \int_s^t \mathrm{d}\tau \frac{\mathrm{d}}{\mathrm{d}\tau}\big(\mathbb{P}_x[T_K < \tau] - \mathbb{P}_x[T_K < t]\big) \mathbb{P}_x[t-\tau < L_K < \infty] \right.
\end{aligned}$$

$$- \int_0^s d\tau \frac{d}{d\tau}\big(\mathbb{P}_x[T_K < \tau] - \mathbb{P}_x[T_K < s]\big)\mathbb{P}_x[s-\tau < L_K < t-\tau]\Big\} \int \mu_K(dy)p(x,y;t-s) + O\big(t^{-m/2}\big)$$

$$= -\mathbb{P}_x[T_K < \infty] \int_0^t ds\, \mathbb{P}_x[s < L_K < t] \int \mu_K(dy)p(x,y;t-s) + \sum_{i=1}^4 A_i + O\big(t^{-m/2}\big), \tag{79}$$

after two integrations by parts. Proposition 5 follows by (69) and (79). □

Below we obtain the asymptotic behaviour of the first term in the right-hand side of (63).

Lemma 6. *Let K be a compact and non-polar set in $\mathbb{R}^4$. Then for $t \to \infty$*

$$\int_0^t ds\, \mathbb{P}_x[s < L_K < t] \int \mu_K(dy)p(x,y;t-s) = 2(4\pi)^{-4}C(K)^2 \frac{\log t}{t^2} + O\big(t^{-2}\big). \tag{80}$$

Proof. By (35)

$$\int_0^T ds\, \mathbb{P}_x[s < L_K < t] \int \mu_K(dy)p(x,y;t-s) \leqslant \int_0^T ds \int \mu_K(dy)p(x,y;t-s)$$
$$\leqslant C(K)T(t-T)^{-2}. \tag{81}$$

By (33)

$$\int_{t-T}^t ds\, \mathbb{P}_x[s < L_K < t] \int \mu_K(dy)p(x,y;t-s) \leqslant P_x[t-T < L_K < t] \int_{t-T}^t ds \int \mu_K(dy)p(x,y;t-s)$$
$$\leqslant \mathbb{P}_x[t-T < L_K < t] \leqslant C(K)T/\big(t(t-T)\big). \tag{82}$$

Furthermore by (35) and (19)

$$\int_T^{t-T} ds\, \mathbb{P}_x[s < L_K < t] \int \mu_K(dy)p(x,y;t-s) \leqslant (4\pi)^{-4}C(K)^2 \int_T^{t-T} ds\, \big(s^{-1} - t^{-1}\big)(t-s)^{-2}$$
$$= 2(4\pi)^{-4}C(K)^2 \frac{\log t}{t^2} + O\big(t^{-2}\big), \tag{83}$$

which proves the upper bound in (80). To prove the lower bound in (80) we have by (35) and (19)

$$\int_0^t ds\, \mathbb{P}_x[s < L_K < t] \int \mu_K(dy)p(x,y;t-s)$$
$$\geqslant \int_T^{t-T} ds \int_s^t d\tau\, (4\pi\tau)^{-2} \int \mu_K(dz)\left(1 - \frac{|x-z|^2}{4\tau}\right) \int \mu_K(dy)p(x,y;t-s). \tag{84}$$

Since

$$\int_T^{t-T} ds \int_s^t d\tau\, (4\pi\tau)^{-2} \int \mu_K(dz)\frac{|x-z|^2}{4\tau} \int \mu_K(dy)p(x,y;t-s)$$
$$\leqslant C(K)^2\big(|x|+R\big)^2 \int_T^{t-T} ds \int_s^\infty d\tau\, \tau^{-3}(t-s)^{-2} = O\big(t^{-2}\big), \tag{85}$$

we have that the left-hand side of (84) is bounded below by

$$\int_T^{t-T} \mathrm{d}s \int_s^t \mathrm{d}\tau\, (4\pi\tau)^{-2} C(K) \int \mu_K(\mathrm{d}y) p(x,y;t-s) + \mathrm{O}\big(t^{-2}\big)$$

$$\geqslant (4\pi)^{-4} C(K)^2 \int_T^{t-T} \mathrm{d}s\, (t-s)^{-2}\big(s^{-1}-t^{-1}\big) - C(K)^2\big(|x|+R\big)^2 \int_T^{t-T} \mathrm{d}s\, (t-s)^{-3}\big(s^{-1}-t^{-1}\big) + \mathrm{O}\big(t^{-2}\big)$$

$$= 2(4\pi)^{-4} C(K)^2 \frac{\log t}{t^2} + \mathrm{O}\big(t^{-2}\big). \tag{86}$$

The lower bound in (80) follows from the estimates in (84)–(86). □

We conclude this section with the proof of Proposition 2 for $m=4$. By (29) we have that the second term in the right-hand side of (48) is $\mathrm{O}(t^{-2})$. Below we will show that $A_i = \mathrm{O}(t^{-2})$ for $i = 1,\dots,4$ and $t\to\infty$. This implies Theorem 1 for $m=4$ by Propositions 4, 5 and Lemma 6.

The contribution from $s\in[0,T]$ to A_1 in (64) is bounded by $C(K)T(t-T)^{-2}$. Similarly by (33) with $E=[t-T,t]$ and (34) the contribution from $s\in[t-T,t]$ is bounded by

$$\mathbb{P}_x[t-T<T_K<t] \int_{t-T}^{t} \mathrm{d}s \int \mu_K(\mathrm{d}y) p(x,y;t-s) \leqslant CC(K)T/\big(t(t-T)\big). \tag{87}$$

The contribution from $s\in[T,t/2]$ is bounded, using (29), by

$$\int_T^{t/2} \mathrm{d}s\, C(K)^3 s^{-1}(t-s)^{-3} = \mathrm{O}\bigg(\frac{\log t}{t^3}\bigg), \tag{88}$$

and the contribution from $s\in[t/2,t-T]$ is bounded, using (34) and (29), by

$$\int_{t/2}^{t-T} \mathrm{d}s\, CC(K)^2 s^{-2}(t-s)^{-2} = \mathrm{O}\big(t^{-2}\big). \tag{89}$$

This proves that $A_1 = \mathrm{O}(t^{-2})$.

The contribution from $s\in[0,T]$ to A_2 is bounded by $C(K)T(t-T)^{-2}$ and the contribution from $s\in[t-T,t]$ to A_2 is bounded, using (29), by $C(K)^2(t-T)^{-2}$.

Finally, the contribution from $s\in[T,t-T]$ is bounded, using (29), (30), by

$$\int_T^{t-T} \mathrm{d}s\, C(K)^3 s^{-1}\big(s^{-1}-t^{-1}\big)(t-s)^{-2} = \mathrm{O}\big(t^{-2}\big). \tag{90}$$

This proves that $A_2 = \mathrm{O}(t^{-2})$.

The contribution from $s\in[0,t/2]$ to A_3 is bounded, using Lemma 3 and (59), by

$$t^{-2}\int_0^{t/2} \mathrm{d}s \left\{ \int_s^{t/2} \mathrm{d}\tau\, \frac{C(K)^3}{\tau(t-\tau)^2} + \int_{t/2}^{t-T} \mathrm{d}\tau\, \frac{CC(K)^2}{\tau^2(t-\tau)} + \int_{t-T}^{t} \mathrm{d}\tau\, \frac{CC(K)T}{(t-T)^2} \int \mu_K(\mathrm{d}z) p(x,z;t-\tau) \right\} = \mathrm{O}\bigg(\frac{\log t}{t^3}\bigg). \tag{91}$$

The contribution from $s\in[t/2,t]$ to A_3 is bounded, using (34), by

$$\int_{t/2}^{t-T} \mathrm{d}s \int_s^t \mathrm{d}\tau\, \frac{C(t-\tau)}{\tau^2} \int \mu_K(\mathrm{d}z) p(x,z;t-\tau) \int \mu_K(\mathrm{d}y) p(x,y;t-s)$$

$$
\begin{aligned}
&+\frac{CT}{(t-T)^2}\int_{t-T}^{t}\mathrm{d}s\int_{t-T}^{t}\mathrm{d}\tau\int\mu_K(\mathrm{d}z)p(x,z;t-\tau)\int\mu_K(\mathrm{d}y)p(x,y;t-s)\\
&\leqslant\frac{4C}{t^2}\biggl(\int_{-\infty}^{t}\mathrm{d}\tau\,(t-\tau)^{1/2}\int\mu_K(\mathrm{d}z)p(x,z;t-\tau)\biggr)^2+\frac{CT}{(t-T)^2}\\
&=\mathrm{O}\bigl(t^{-2}\bigr),
\end{aligned}
\tag{92}
$$

where we have used that for $m=4$

$$
\begin{aligned}
&\int_0^\infty\mathrm{d}\tau\,\tau^{1/2}\int\mu_K(\mathrm{d}y)p(x,y;\tau)=\frac{1}{8\pi^{3/2}}\int\mu_K(\mathrm{d}y)|x-y|^{-1}\\
&\quad\leqslant\frac{1}{8\pi^{3/2}}\biggl(\int\mu_K(\mathrm{d}y)|x-y|^{-2}\biggr)^{1/2}\biggl(\int\mu_K(\mathrm{d}y)\biggr)^{1/2}\leqslant C(K)^{1/2}.
\end{aligned}
\tag{93}
$$

This proves that $A_3=\mathrm{O}(t^{-2})$.

The contribution from $s\in[0,2T]$ to A_4 is bounded by

$$
\frac{2C(K)T}{(t-2T)^2}\biggl(\int_{-\infty}^{s}\mathrm{d}\tau\int\mu_K(\mathrm{d}z)p(x,z;s-\tau)+\int_{-\infty}^{t}\mathrm{d}\tau\int\mu_K(\mathrm{d}y)p(x,y;t-\tau)\biggr)=\mathrm{O}\bigl(t^{-2}\bigr).
\tag{94}
$$

The contribution from $s\in[2T,t/2]$ to A_4 is bounded by

$$
\begin{aligned}
&C(K)t^{-2}\int_{2T}^{t/2}\mathrm{d}s\biggl\{\int_0^T\mathrm{d}\tau\,\frac{2C(K)}{(s-\tau)^2}+\int_T^{s/2}\mathrm{d}\tau\,\frac{2C(K)^2}{\tau(s-\tau)^2}+\int_{s/2}^{s-T}\mathrm{d}\tau\,\frac{2CC(K)}{\tau^2(s-\tau)}+\int_{s-T}^{s}\mathrm{d}\tau\,\frac{CT}{(s-T)^2}\\
&\quad\times\int\mu_K(\mathrm{d}z)\bigl(p(x,z;s-\tau)+p(x,z;t-\tau)\bigr)\biggr\}=\mathrm{O}\bigl(t^{-2}\bigr),
\end{aligned}
\tag{95}
$$

where we have used that $P_x[\tau<T_K<s]$ is bounded on the intervals $[0,T]$, $[T,s/2]$, $[s/2,s-T]$ and $[s-T,s]$ by 1, $C(K)/\tau$, $C(s-\tau)/\tau^2$ and $CT/(s-T)^2$ respectively.

To bound the contribution from $s\in[t/2,t]$ to A_4 we use that uniformly in x,z,s,τ and t

$$
\bigl|p(x,z;s-\tau)-p(x,z;t-\tau)\bigr|\leqslant(s-\tau)^{-2}\wedge(t-s)(s-\tau)^{-3}\wedge(t-s)^{1/2}(s-\tau)^{-5/2}.
\tag{96}
$$

First of all the contribution from the rectangle $\{(s,\tau)\colon\ t/2<s<t,\ 0<\tau<T\}$ to A_4 is bounded by

$$
\int_{t/2}^{t}\mathrm{d}s\int_0^T\mathrm{d}\tau\,\frac{2C(K)}{(s-\tau)^2}\int\mu_K(\mathrm{d}y)p(x,y;t-s)\leqslant\frac{2C(K)T}{(t/2-T)^2}\int_{t/2}^{t}\mathrm{d}s\int\mu_K(\mathrm{d}y)p(x,y;t-s)=\mathrm{O}\bigl(t^{-2}\bigr).
\tag{97}
$$

Secondly, by Lemma 3 and (96), (93)

$$
\begin{aligned}
&\int_{t/2}^{t}\mathrm{d}s\int_T^{s/2}\mathrm{d}\tau\,\mathbb{P}_x[\tau<T_K<s]\int\mu_K(\mathrm{d}z)\bigl|p(x,z;t-\tau)-p(x,z;s-\tau)\bigr|\int\mu_K(\mathrm{d}y)p(x,y;t-s)\\
&\quad\leqslant\int_{t/2}^{t}\mathrm{d}s\int_T^{s/2}\mathrm{d}\tau\,\frac{C(K)^2(t-s)^{1/2}}{\tau(s-\tau)^{5/2}}\int\mu_K(\mathrm{d}y)p(x,y;t-s)\\
&\quad\leqslant C(K)^2\biggl(\frac{t}{4}\biggr)^{-5/2}\int_{t/2}^{t}\mathrm{d}s\,(t-s)^{1/2}\log\biggl(\frac{s}{2T}\biggr)\int\mu_K(\mathrm{d}y)p(x,y;t-s)
\end{aligned}
$$

$$\leqslant C(K)^{5/2}\left(\frac{t}{4}\right)^{-5/2}\log\left(\frac{t}{2T}\right). \tag{98}$$

Thirdly, by Lemma 3 and (96), (93)

$$\int_{t/2}^{t} \mathrm{d}s \int_{s/2}^{s-T} \mathrm{d}\tau\, \mathbb{P}_x[\tau < T_K < s] \int \mu_K(\mathrm{d}z)\big|p(x,z;t-\tau) - p(x,z;s-\tau)\big| \int \mu_K(\mathrm{d}y) p(x,y;t-s)$$
$$\leqslant \int_{t/2}^{t} \mathrm{d}s \int_{s/2}^{s-T} \mathrm{d}\tau\, \frac{CC(K)(t-s)^{1/2}}{\tau^2(s-\tau)^{3/2}} \int \mu_K(\mathrm{d}y) p(x,y;t-s)$$
$$\leqslant 16CC(K)t^{-2} \int_0^t \mathrm{d}s\,(t-s)^{1/2} \int \mu_K(\mathrm{d}y) p(x,y;t-s)$$
$$\times \int_{-\infty}^{s-T} \mathrm{d}\tau\,(s-\tau)^{-3/2} \leqslant 32CC(K)^{3/2}T^{-1/2}t^{-2}. \tag{99}$$

Finally, by Lemma 3,

$$\int_{t/2}^{t} \mathrm{d}s \int_{s-T}^{s} \mathrm{d}\tau\, \mathbb{P}_x[\tau < T_K < s] \int \mu_K(\mathrm{d}z)\big|p(x,z;t-\tau) - p(x,z;s-\tau)\big| \int \mu_K(\mathrm{d}y) p(x,y;t-s)$$
$$\leqslant \int_{t/2}^{t} \mathrm{d}s\, \frac{CT}{(s-T)^2} \int_{s-T}^{s} \mathrm{d}\tau \int \mu_K(\mathrm{d}z)\big(p(x,z;t-\tau) + p(x,z;s-\tau)\big) \int \mu_K(\mathrm{d}y) p(x,y;t-s)$$
$$\leqslant \frac{8CT}{(t-2T)^2}. \tag{100}$$

This completes the proof of $A_4 = \mathrm{O}(t^{-2})$ and hence of Proposition 2 for $m = 4$. □

4. Proof of Proposition 2 for $m = 3$

Throughout this section we assume that $m = 3$. By Propositions 4 and 5

$$\mathbb{P}_x[t < T_K < \infty] = \big(1 - \mathbb{P}_x[t < L_K < \infty]\big)^{-1} \mathbb{P}_x[T_K = \infty]\mathbb{P}_x[t < L_K < \infty]$$
$$- \big(1 - \mathbb{P}_x[t < L_K < \infty]\big)^{-1} \mathbb{P}_x[T_K = \infty] \int_0^t \mathrm{d}s\, \mathbb{P}_x[s < L_K < t]$$
$$\times \int \mu_K(\mathrm{d}y) p(x,y;t-s) - \big(1 - \mathbb{P}_x[t < L_K < \infty]\big)^{-1} \sum_{i=1}^{4} A_i + \mathrm{O}\big(t^{-3/2}\big). \tag{101}$$

By (20)

$$\big(1 - \mathbb{P}_x[t < L_K < \infty]\big)^{-1} \mathbb{P}_x[T_K = \infty]\mathbb{P}_x[t < L_K < \infty]$$
$$= \mathbb{P}_x[T_K = \infty]\mathbb{P}_x[t < L_K < \infty] + \big(16\pi^3\big)^{-1} C(K)^2 \mathbb{P}_x[T_K = \infty] t^{-1} + \mathrm{O}\big(t^{-3/2}\big). \tag{102}$$

Lemma 7. *Let K be a compact and non-polar set in $\mathbb{R}^3$. Then for $t \to \infty$*

$$\int_0^t \mathrm{d}s\, \mathbb{P}_x[s < L_K < t] \int \mu_K(\mathrm{d}y) p(x,y;t-s) = \big(16\pi^3\big)^{-1} C(K)^2 t^{-1} + \mathrm{O}\big(t^{-3/2}\big). \tag{103}$$

Proof. By (19)

$$\int \mu_K(\mathrm{d}y)p(x,y;t-s) \leqslant (4\pi)^{-3/2}C(K)(t-s)^{-3/2}, \tag{104}$$

so that by (104) and (30)

$$\int_0^t \mathrm{d}s\, \mathbb{P}_x[s < L_K < t] \int \mu_K(\mathrm{d}y)p(x,y;t-s) \leqslant \left(32\pi^3\right)^{-1} C(K)^2 t^{-1} \int_0^1 \mathrm{d}s\, s^{-1/2}\left(1+s^{1/2}\right)^{-1}(1-s)^{-1/2}$$
$$= \left(16\pi^3\right)^{-1} C(K)^2 t^{-1}, \tag{105}$$

where the integral with respect to $s \in [0,1]$ is evaluated by the change of variable $s = (\sin\theta)^2$. To prove the lower bound in Lemma 7 we have

$$\int \mu_K(\mathrm{d}y)p(x,y;t-s) \geqslant (4\pi)^{-3/2}C(K)(t-s)^{-3/2} - (4\pi)^{-3/2}C(K)(t-s)^{-3/2}\left(1-\mathrm{e}^{-\frac{(|x|+R)^2}{4(t-s)}}\right). \tag{106}$$

Since

$$\int_0^t \mathrm{d}s\, \mathbb{P}_x[s < L_K < t](t-s)^{-3/2}\left(1-\mathrm{e}^{-\frac{(|x|+R)^2}{4(t-s)}}\right) \leqslant C(K)\int_0^t \mathrm{d}s\,\left(s^{-1/2}-t^{-1/2}\right)(t-s)^{-3/2}\left(1-\mathrm{e}^{-\frac{(|x|+R)^2}{4(t-s)}}\right)$$
$$= 2C(K)t^{-1}\int_0^{\pi/2} \frac{\mathrm{d}\theta}{1+\sin\theta}\left(1-\mathrm{e}^{-\frac{(|x|+R)^2}{4t(\cos\theta)^2}}\right)$$
$$\leqslant 2C(K)t^{-1}\int_0^{\pi/2} \mathrm{d}\theta\,\left(1-\mathrm{e}^{-\frac{(|x|+R)^2}{\theta^2 t}}\right) = \mathrm{O}\left(t^{-3/2}\right), \tag{107}$$

we have that the left-hand side of (103) is bounded from below by

$$(4\pi)^{-3/2}C(K)\int_0^t \mathrm{d}s\, \mathbb{P}_x[s < L_K < t](t-s)^{-3/2} + \mathrm{O}\left(t^{-3/2}\right). \tag{108}$$

Since

$$\mathbb{P}_x[s < L_K < t] \geqslant \int \mu_K(\mathrm{d}y)\int_s^t \mathrm{d}\tau\,(4\pi\tau)^{-3/2}\mathrm{e}^{-\frac{(|x|+R)^2}{4s}}$$
$$= \left(4\pi^{3/2}\right)^{-1}C(K)\left(s^{-1/2}-t^{-1/2}\right)\left(1-\left(1-\mathrm{e}^{-\frac{(|x|+R)^2}{4s}}\right)\right), \tag{109}$$

we have by estimates similar to (107) that (108) is bounded from below by

$$\left(16\pi^3\right)^{-1}C(K)^2t^{-1} - C(K)^2t^{-1}\int_0^1 \mathrm{d}s\,\left(s^{-1/2}-1\right)(1-s)^{-3/2}\left(1-\mathrm{e}^{-\frac{(|x|+R)^2}{4st}}\right) + \mathrm{O}\left(t^{-3/2}\right)$$
$$= \left(16\pi^3\right)^{-1}C(K)^2t^{-1} + \mathrm{O}\left(t^{-3/2}\right). \tag{110}$$

This completes, by (106)–(110), the proof of the lower bound in Lemma 7. □

By Lemma 7 we obtain that the term of order t^{-1} in (102) cancels with the second term in the right-hand side of (101). So the proof of Proposition 2 for $m = 3$ is complete if we can show that

$$\sum_{i=1}^4 A_i = \mathrm{O}\left(t^{-3/2}\right), \quad t \to \infty. \tag{111}$$

However, it turns out that each of the A_i is (for $m = 3$) of order $(\log t)/t^{3/2}$. So in order to obtain (111) we will show that the sum of the coefficients of $\log t/t^{3/2}$ of the A_i's cancel with remainder $\mathrm{O}(t^{-3/2})$. In Proposition 8 we state that $\mathbb{P}_x[s < T_K < t]$ in (64) can be replaced by $\mathbb{P}_x[s < L_K < t]\mathbb{P}_x[T_K = \infty]$ at a cost of $\mathrm{O}(t^{-3/2})$ with similar replacements in (65)–(67) respectively. In Lemma 9 we obtain, using Proposition 8, the desired asymptotic behaviour of each of the A_i. This in turn implies (111) and thereby completing the proof of (111) and of Theorem 1.

Proposition 8. *Let K be a compact, non-polar set in $\mathbb{R}^3$, and let A_i $i = 1, \dots, 4$ be given by* (64)–(67) *respectively. Then for $t \to \infty$*

$$A_1 = \mathbb{P}_x[T_K = \infty] \int_0^t \mathrm{d}s\, \mathbb{P}_x[s < L_K < t]\mathbb{P}_x[t - s < L_K < \infty] \int \mu_K(\mathrm{d}y) p(x, y; t - s) + \mathrm{O}\big(t^{-3/2}\big), \tag{112}$$

$$A_2 = \mathbb{P}_x[T_K = \infty] \int_0^t \mathrm{d}s\, \mathbb{P}_x[s < L_K < \infty]\mathbb{P}_x[s < L_K < t] \int \mu_K(\mathrm{d}y) p(x, y; t - s) + \mathrm{O}\big(t^{-3/2}\big), \tag{113}$$

$$A_3 = \mathbb{P}_x[T_K = \infty] \int_0^t \mathrm{d}s \int_s^t \mathrm{d}\tau\, \mathbb{P}_x[\tau < L_K < t] \int \mu_K(\mathrm{d}z) p(x, z; t - \tau) \times \int \mu_K(\mathrm{d}y) p(x, y; t - s) + \mathrm{O}\big(t^{-3/2}\big), \tag{114}$$

$$A_4 = \mathbb{P}_x[T_K = \infty] \int_0^t \mathrm{d}s \int_0^s \mathrm{d}\tau\, \mathbb{P}_x[\tau < L_K < s] \int \mu_K(\mathrm{d}z) \times \big(p(x, z; t - \tau) - p(x, z; s - \tau)\big) \int \mu_K(\mathrm{d}y) p(x, y; t - s) + \mathrm{O}\big(t^{-3/2}\big). \tag{115}$$

It is convenient to denote the first term in the right-hand sides of (112)–115) respectively by $B_1, \dots, B_4$.

Lemma 9. *Let K be a compact and non-polar set in $\mathbb{R}^3$. Then for $t \to \infty$*

$$B_1 = 2(4\pi)^{-9/2} C(K)^3 \mathbb{P}_x[T_K = \infty] \frac{\log t}{t^{3/2}} + \mathrm{O}\big(t^{-3/2}\big), \tag{116}$$

$$B_2 = 4(4\pi)^{-9/2} C(K)^3 \mathbb{P}_x[T_K = \infty] \frac{\log t}{t^{3/2}} + \mathrm{O}\big(t^{-3/2}\big), \tag{117}$$

$$B_3 = 2(4\pi)^{-9/2} C(K)^3 \mathbb{P}_x[T_K = \infty] \frac{\log t}{t^{3/2}} + \mathrm{O}\big(t^{-3/2}\big), \tag{118}$$

$$B_4 = -8(4\pi)^{-9/2} C(K)^3 \mathbb{P}_x[T_K = \infty] \frac{\log t}{t^{3/2}} + \mathrm{O}\big(t^{-3/2}\big). \tag{119}$$

Proof. By (29), (30) and (104)

$$B_1 \leqslant 4(4\pi)^{-9/2} C(K)^2 \mathbb{P}_x[T_K = \infty] \int_0^t \mathrm{d}s\, \big(s^{-1/2} - t^{-1/2}\big)(t - s)^{-2} \int \mu_K(\mathrm{d}y) \mathrm{e}^{-\frac{|x-y|^2}{4(t-s)}}. \tag{120}$$

On the other hand, by (35)

$$\mathbb{P}_x[t - s < L_K < \infty] \geqslant 2(4\pi)^{-3/2} C(K)(t - s)^{-1/2}\big(1 + \mathrm{e}^{-\frac{(|x|+R)^2}{4(t-s)}} - 1\big). \tag{121}$$

Hence by (109) and (121)

$$B_1 \geqslant 4(4\pi)^{-9/2}C(K)^2\mathbb{P}_x[T_K=\infty]\int_0^t \mathrm{d}s\,\big(s^{-1/2}-t^{-1/2}\big)(t-s)^{-2}$$
$$\times\int \mu_K(\mathrm{d}y)\,\mathrm{e}^{-\frac{|x-y|^2}{4(t-s)}}\Big(1-\big(1-\mathrm{e}^{-\frac{(|x|+R)^2}{4(t-s)}}\big)-\big(1-\mathrm{e}^{-\frac{(|x|+R)^2}{4s}}\big)\Big). \tag{122}$$

Below we will compute the leading asymptotic behaviour of the right-hand side of (120). Substitution of $s=t(\cos\theta)^2$ in (120) yields that the integral equals

$$2t^{-3/2}\int \mu_K(\mathrm{d}y)\int_0^{\pi/2}\mathrm{d}\theta\,(\sin\theta)^{-1}(1+\cos\theta)^{-1}\mathrm{e}^{-\frac{|x-y|^2}{4t(\sin\theta)^2}}. \tag{123}$$

Since

$$(\sin\theta)^{-1}(1+\cos\theta)^{-1}\leqslant(2\theta)^{-1}+4,\quad 0<\theta<\pi/2, \tag{124}$$

we have that the right-hand side of (123) is bounded from above by

$$t^{-3/2}\int \mu_K(\mathrm{d}y)\int_0^{\pi/2}\mathrm{d}\theta\,\theta^{-1}\mathrm{e}^{-\frac{|x-y|^2}{4t\theta^2}}+\mathrm{O}\big(t^{-3/2}\big)=\frac{1}{2}t^{-3/2}\int\mu_K(\mathrm{d}y)\int_{\frac{|x-y|^2}{\pi^2 t}}^{\infty}\mathrm{d}u\,u^{-1}\,\mathrm{e}^{-u}+\mathrm{O}\big(t^{-3/2}\big)$$
$$=\frac{1}{2}t^{-3/2}\int\mu_K(\mathrm{d}y)\log\bigg(\frac{|x-y|^2}{\pi^2 t}\bigg)+\mathrm{O}\big(t^{-3/2}\big). \tag{125}$$

This gives, together with (120), the desired upper bound for the asymptotic behaviour of the right-hand side of (120). The lower bound for the right-hand side of (120) follows similarly, using $(\sin\theta)^{-1}(1+\cos\theta)^{-1}\geqslant(2\theta)^{-1},\ 0<\theta<\pi/2$. Furthermore returning to (122) we have that

$$\int_0^t \mathrm{d}s\,\big(s^{-1/2}-t^{-1/2}\big)(t-s)^{-2}\int\mu_K(\mathrm{d}y)\mathrm{e}^{-\frac{|x-y|^2}{4(t-s)}}\big(1-\mathrm{e}^{-\frac{(|x|+R)^2}{4(t-s)}}\big)$$
$$\leqslant 2t^{-3/2}\int_0^{\pi/2}\mathrm{d}\theta\,(\sin\theta)^{-1}\int\mu_K(\mathrm{d}y)\mathrm{e}^{-\frac{|x-y|^2}{4t(\sin\theta)^2}}\big(1-\mathrm{e}^{-\frac{(|x|+R)^2}{4t(\sin\theta)^2}}\big)$$
$$\leqslant 2t^{-3/2}\int_0^{\pi/2}\mathrm{d}\theta\,(\sin\theta)^{-1}\int\mu_K(\mathrm{d}y)\bigg(\frac{4t(\sin\theta)^2}{|x-y|^2}\bigg)^{1/2}\big(1-\mathrm{e}^{-\frac{(|x|+R)^2}{4t(\sin\theta)^2}}\big)$$
$$\leqslant 16\pi t^{-1}\bigg(\int\mu_K(\mathrm{d}y)\frac{1}{4\pi|x-y|}\bigg)\int_0^{\infty}\mathrm{d}\theta\big(1-\mathrm{e}^{-\frac{\pi^2(|x|+R)^2}{16t\theta^2}}\big)=\mathrm{O}\big(t^{-3/2}\big), \tag{126}$$

and

$$\int_0^t \mathrm{d}s\,\big(s^{-1/2}-t^{-1/2}\big)(t-s)^{-2}\int\mu_K(\mathrm{d}y)\mathrm{e}^{-\frac{|x-y|^2}{4(t-s)}}\big(1-\mathrm{e}^{-\frac{(|x|+R)^2}{4s}}\big)$$
$$\leqslant\int_0^t \mathrm{d}s\,\big(s^{-1/2}-t^{-1/2}\big)(t-s)^{-2}\int\mu_K(\mathrm{d}y)\bigg(\frac{4(t-s)}{|x-y|^2}\bigg)^{1/2}\big(1-\mathrm{e}^{-\frac{(|x|+R)^2}{4s}}\big)$$
$$\leqslant 8\pi t^{-1}\int_0^t \mathrm{d}s\,s^{-1/2}(t-s)^{-1/2}\big(1-\mathrm{e}^{-\frac{(|x|+R)^2}{4s}}\big)$$

$$= 16\pi t^{-1}\int_0^{\pi/2} d\theta\,\bigl(1 - e^{-\frac{(|x|+R)^2}{4t(\sin\theta)^2}}\bigr) = O\bigl(t^{-3/2}\bigr). \tag{127}$$

It follows by (126) and (127) that the two remainders in the right-hand side of (122) contribute each at most $O(t^{-3/2})$. This completes the proof of (116).

To prove (117) we note that by (29), (30) and (104)

$$B_2 \leqslant 2(4\pi)^{-9/2}C(K)^2\mathbb{P}_x[T_K = +\infty]\int_0^t ds\,\bigl(s^{-1/2} - t^{-1/2}\bigr)(t-s)^{-3/2}\int \mu_K(dy)\int_s^\infty d\tau\,\tau^{-3/2}e^{-\frac{|x-y|^2}{4\tau}}$$
$$= 4(4\pi)^{-9/2}C(K)^2\mathbb{P}_x[T_K = \infty]t^{-3/2}\int_0^{\pi/2}\frac{d\theta}{\sin\theta(1+\sin\theta)}\int \mu_K(dy)\int_1^\infty d\tau\,\tau^{-3/2}e^{-\frac{|x-y|^2}{4\tau t(\sin\theta)^2}}. \tag{128}$$

On the other hand

$$B_2 \geqslant 2(4\pi)^{-9/2}C(K)^2\mathbb{P}_x[T_K = \infty]\int_0^t ds\,\bigl(s^{-1/2} - t^{-1/2}\bigr)(t-s)^{-3/2}$$
$$\times \int \mu_K(dy)\int_s^t d\tau\,\tau^{-3/2}e^{-\frac{|x-y|^2}{4\tau}}\bigl(1 - \bigl(1 - e^{-\frac{(|x|+R)^2}{4s}}\bigr) - \bigl(1 - e^{-\frac{(|x|+R)^2}{4(t-s)}}\bigr)\bigr). \tag{129}$$

Below we will compute the leading asymptotic behaviour of the right-hand side of (128). Using the inequality $(\sin\theta)^{-1} \leqslant \theta^{-1} + 4,\ 0 < \theta < \pi/2$, we obtain for (128) the upper bound

$$4(4\pi)^{-9/2}C(K)^2\mathbb{P}_x[T_K = \infty]t^{-3/2}\int_0^{\pi/2} d\theta\,\theta^{-1}\int_1^\infty d\tau\,\tau^{-3/2}\int \mu_K(dy)e^{-\frac{|x-y|^2}{4\tau t\theta^2}} + O\bigl(t^{-3/2}\bigr), \tag{130}$$

and the upper bound follows by a calculation similar to (125). The lower bound for the right-hand side of (128) follows using $(\sin\theta)^{-1}(1+\sin\theta)^{-1} \geqslant \theta^{-1} - 4,\ 0 < \theta < \pi/2$. Furthermore returning to (129) we have a first error term

$$\int_0^t ds\,\bigl(s^{-1/2} - t^{-1/2}\bigr)(t-s)^{-3/2}\bigl(1 - e^{-\frac{(|x|+R)^2}{4s}}\bigr)\int \mu_K(dy)\int_s^t d\tau\,\tau^{-3/2}e^{-\frac{|x-y|^2}{4\tau}}. \tag{131}$$

Since

$$\int \mu_K(dy)\int_s^t d\tau\,\tau^{-3/2}e^{-\frac{|x-y|^2}{4\tau}} \leqslant \int \mu_K(dy)\int_s^\infty d\tau\,\tau^{-3/2}\biggl(\frac{4\tau}{|x-y|^2}\biggr)^{1/4}$$
$$\leqslant 4\sqrt{2}\,s^{-1/4}\int \mu_K(dy)|x-y|^{-1/2}$$
$$\leqslant 4\sqrt{2}s^{-1/4}\biggl(\int \mu_K(dy)|x-y|^{-1}\biggr)^{1/2}C(K)^{1/2}$$
$$\leqslant 8\sqrt{2\pi}\,s^{-1/4}C(K)^{1/2}, \tag{132}$$

we have that (131) is bounded from above by

$$8\sqrt{2\pi}\,C(K)^{1/2}t^{-1}\int_0^t ds\,s^{-3/4}(t-s)^{-1/2}\bigl(1 - e^{-\frac{(|x|+R)^2}{4s}}\bigr)$$

$$\leqslant 16\sqrt{2\pi}C(K)^{1/2}t^{-5/4}\int_0^{\pi/2} \mathrm{d}\theta\,(\sin\theta)^{-1/2}\big(1-\mathrm{e}^{-\frac{(|x|+R)^2}{4t(\sin\theta)^2}}\big)$$

$$\leqslant 16\pi C(K)^{1/2}t^{-5/4}\int_0^{\infty} \mathrm{d}\theta\,\theta^{-1/2}\big(1-\mathrm{e}^{-\frac{(|x|+R)^2}{4t\theta^2}}\big)=\mathrm{O}\big(t^{-3/2}\big). \tag{133}$$

The second error term is bounded by

$$\int_0^t \mathrm{d}s\,\big(s^{-1/2}-t^{-1/2}\big)(t-s)^{-3/2}\big(1-\mathrm{e}^{-\frac{(|x|+R)^2}{4(t-s)}}\big)\int \mu_K(\mathrm{d}y)\int_s^t \mathrm{d}\tau\,\tau^{-3/2}\mathrm{e}^{-\frac{|x-y|^2}{4\tau}}$$

$$\leqslant 8\sqrt{2\pi}\,C(K)^{1/2}t^{-1}\int_0^t \mathrm{d}s\,s^{-3/4}(t-s)^{-1/2}\big(1-\mathrm{e}^{-\frac{(|x|+R)^2}{4(t-s)}}\big)$$

$$\leqslant 8\sqrt{\pi}\,C(K)^{1/2}\big(|x|+R\big)^{1/2}t^{-1}\int_0^t \mathrm{d}s\,s^{-3/4}(t-s)^{-3/4}=\mathrm{O}\big(t^{-3/2}\big), \tag{134}$$

where we have used (132) and the inequality $1-\mathrm{e}^{-\theta}\leqslant\theta^{1/4},\ \theta\geqslant 0$. It follows by (133) and (134) that the two remainders in the right-hand side of (129) contribute each at most $\mathrm{O}(t^{-3/2})$. This completes the proof of (117).

To prove (118) we note that by (30) and (104)

$$B_3\leqslant 2(4\pi)^{-9/2}C(K)^2\mathbb{P}_x[T_K=\infty]\int_0^t \mathrm{d}s\int \mu_K(\mathrm{d}y)\mathrm{e}^{-\frac{|x-y|^2}{4(t-s)}}(t-s)^{-3/2}$$

$$\times\int_s^t \mathrm{d}\tau\,\big(\tau^{-1/2}-t^{-1/2}\big)(t-\tau)^{-3/2}. \tag{135}$$

On the other hand

$$B_3\geqslant 2(4\pi)^{-9/2}C(K)^2\mathbb{P}_x[T_K=\infty]\int_0^t \mathrm{d}s\int \mu_K(\mathrm{d}y)\mathrm{e}^{-\frac{|x-y|^2}{4(t-s)}}(t-s)^{-3/2}\int_s^t \mathrm{d}\tau\,\big(\tau^{-1/2}-t^{-1/2}\big)(t-\tau)^{-3/2}$$

$$\times\Big(1-\big(1-\mathrm{e}^{-\frac{(|x|+R)^2}{4s}}\big)-\big(1-\mathrm{e}^{-\frac{(|x|+R)^2}{4(t-s)}}\big)\Big). \tag{136}$$

Below we will compute the leading asymptotic behaviour of the right-hand side of (135). Firstly, since

$$\int_s^t \mathrm{d}\tau\,\big(\tau^{-1/2}-t^{-1/2}\big)(t-\tau)^{-3/2}=\int_s^t \mathrm{d}\tau\,\tau^{-1/2}t^{-1/2}\big(\tau^{1/2}+t^{1/2}\big)^{-1}(t-\tau)^{-1/2}\geqslant\frac{(t-s)^{1/2}}{t^{3/2}} \tag{137}$$

we have that the right-hand side of (135) is bounded from below by

$$2(4\pi)^{-9/2}C(K)^2\mathbb{P}_x[T_K=\infty]t^{-3/2}\int_0^t \mathrm{d}s\int \mu_K(\mathrm{d}y)(t-s)^{-1}\mathrm{e}^{-\frac{|x-y|^2}{4(t-s)}}$$

$$\geqslant 2(4\pi)^{-9/2}C(K)^3\mathbb{P}_x[T_K=\infty]\frac{\log t}{t^{3/2}}+\mathrm{O}\big(t^{-3/2}\big). \tag{138}$$

Secondly, since

$$\int_s^t \mathrm{d}\tau\,\big(\tau^{-1/2}-t^{-1/2}\big)(t-\tau)^{-3/2}\leqslant\frac{(t-s)^{1/2}}{t^{3/2}}+\frac{2(t-s)^{3/2}}{t^2s^{1/2}} \tag{139}$$

we have that the right-hand side of (135) is bounded from above by

$$2(4\pi)^{-9/2}C(K)^2\mathbb{P}_x[T_K=\infty]t^{-3/2}\int_0^t \mathrm{d}s\int \mu_K(\mathrm{d}y)(t-s)^{-1}\mathrm{e}^{-\frac{|x-y|^2}{4(t-s)}}+\mathrm{O}\big(t^{-3/2}\big)$$
$$\leqslant 2(4\pi)^{-9/2}C(K)^3\mathbb{P}_x[T_K=\infty]\frac{\log t}{t^{3/2}}+\mathrm{O}\big(t^{-3/2}\big). \tag{140}$$

In order to complete the proof of (118) we have to show that the two error terms in the right-hand side of (136) contribute at most $\mathrm{O}(t^{-3/2})$. Since the right-hand side of (139) is bounded from above by $3(t-s)^{1/2}t^{-1}s^{-1/2}$ we have that the first of these error terms is bounded by

$$C(K)^2t^{-1}\int_0^t \mathrm{d}s\int \mu_K(\mathrm{d}y)(t-s)^{-1}s^{-1/2}\mathrm{e}^{-\frac{|x-y|^2}{4(t-s)}}\big(1-\mathrm{e}^{-\frac{(|x|+R)^2}{4s}}\big)$$
$$\leqslant 8\pi C(K)^2t^{-1}\int \mu_K(\mathrm{d}y)(4\pi|x-y|)^{-1}\int_0^t \mathrm{d}s\,(t-s)^{-1/2}s^{-1/2}\big(1-\mathrm{e}^{-\frac{(|x|+R)^2}{4s}}\big)$$
$$\leqslant 16\pi C(K)^2t^{-1}\int_0^{\pi/2} \mathrm{d}\theta\big(1-\mathrm{e}^{-\frac{(|x|+R)^2}{4t(\sin\theta)^2}}\big)=\mathrm{O}\big(t^{-3/2}\big). \tag{141}$$

The upper bound for the second of these error terms follows by a similar calculation. This completes the proof of (118).

To prove (119) we rewrite B_4 as follows.

$$B_4=(4\pi)^{-3/2}C(K)\mathbb{P}_x[T_K=\infty]\int_0^t \mathrm{d}s\int \mu_K(\mathrm{d}y)p(x,y;t-s)$$
$$\times\int_0^s \mathrm{d}\tau\,\big((t-\tau)^{-3/2}-(s-\tau)^{-3/2}\big)\int_\tau^s \mathrm{d}\rho\int \mu_K(\mathrm{d}w)p(x,w;\rho)$$
$$+(4\pi)^{-3/2}\mathbb{P}_x[T_K=\infty]\int_0^t \mathrm{d}s\int \mu_K(\mathrm{d}y)p(x,y;t-s)\int \mu_K(\mathrm{d}z)$$
$$\times\int_0^s \mathrm{d}\tau\,(t-\tau)^{-3/2}\big(\mathrm{e}^{-\frac{|x-z|^2}{4(t-\tau)}}-1\big)\int_\tau^s \mathrm{d}\rho\int \mu_K(\mathrm{d}w)p(x,w;\rho)$$
$$+(4\pi)^{-3/2}\mathbb{P}_x[T_K=\infty]\int_0^t \mathrm{d}s\int \mu_K(\mathrm{d}y)p(x,y;t-s)\int \mu_K(\mathrm{d}z)$$
$$\times\int_0^s \mathrm{d}\tau(s-\tau)^{-3/2}\big(1-\mathrm{e}^{-\frac{|x-z|^2}{4(s-\tau)}}\big)\int_\tau^s \mathrm{d}\rho\int \mu_K(\mathrm{d}w)p(x,w;\rho). \tag{142}$$

We first show that the third term in the right-hand side of (142) is bounded in absolute value by $\mathrm{O}(t^{-3/2})$. Note that

$$\int_\tau^s \mathrm{d}\rho\int \mu_K(\mathrm{d}w)p(x,w;\rho)\leqslant 2(4\pi)^{-3/2}\big(\tau^{-1/2}-s^{-1/2}\big)\int \mu_K(\mathrm{d}w)\mathrm{e}^{-\frac{|x-w|^2}{4s}}$$
$$\leqslant(s-\tau)\tau^{-1/2}s^{-1}\int \mu_K(\mathrm{d}w)\mathrm{e}^{-\frac{|x-w|^2}{4s}}. \tag{143}$$

Hence the absolute value of this third term is bounded by

$$C(K)\int_0^t \mathrm{d}s \int \mu_K(\mathrm{d}y)p(x,y;t-s)\int_0^s \mathrm{d}\tau\,\bigl(1-\mathrm{e}^{-\frac{(|x|+R)^2}{4(s-\tau)}}\bigr)(s-\tau)^{-1/2}\tau^{-1/2}s^{-1}\int \mu_K(\mathrm{d}w)\mathrm{e}^{-\frac{|x-w|^2}{4s}}$$

$$=2C(K)\int_0^t \mathrm{d}s \int \mu_K(\mathrm{d}y)p(x,y;t-s)s^{-1}\int_0^{\pi/2} \mathrm{d}\theta\,\bigl(1-\mathrm{e}^{-\frac{(|x|+R)^2}{4s(\sin\theta)^2}}\bigr)\int \mu_K(\mathrm{d}w)\mathrm{e}^{-\frac{|x-w|^2}{4s}}$$

$$\leqslant (4\pi)^2\bigl(|x|+R\bigr)C(K)\int_0^t \mathrm{d}s \int \mu_K(\mathrm{d}y)p(x,y;t-s)\int \mu_K(\mathrm{d}w)p(x,w;s)=\mathrm{O}\bigl(t^{-3/2}\bigr). \tag{144}$$

Since for $0<\tau<s<t$

$$(t-\tau)^{-3/2}\bigl(1-\mathrm{e}^{-\frac{|x-z|^2}{4(t-\tau)}}\bigr)\leqslant (s-\tau)^{-3/2}\bigl(1-\mathrm{e}^{-\frac{(|x|+R)^2}{4(s-\tau)}}\bigr), \tag{145}$$

we have that the second term in the right-hand side of (142) is also estimated by (144).

It remains to find the asymptotic behaviour of the first term in the right-hand side of (142). By the first inequality in (143) we have that this term is bounded from below by

$$2(4\pi)^{-9/2}C(K)\mathbb{P}_x[T_K=\infty]\int_0^t \mathrm{d}s\int \mu_K(\mathrm{d}y)(t-s)^{-3/2}\mathrm{e}^{-\frac{|x-y|^2}{4(t-s)}}$$

$$\times\int_0^s \mathrm{d}\tau\,\bigl((t-\tau)^{-3/2}-(s-\tau)^{-3/2}\bigr)\bigl(\tau^{-1/2}-s^{-1/2}\bigr)\int \mu_K(\mathrm{d}w)\mathrm{e}^{-\frac{|x-w|^2}{4s}}. \tag{146}$$

A straightforward calculation gives that

$$\int_0^s \mathrm{d}\tau\,\bigl((s-\tau)^{-3/2}-(t-\tau)^{-3/2}\bigr)\bigl(\tau^{-1/2}-s^{-1/2}\bigr)$$

$$=2(t-s)^{3/2}\bigl(t^{1/2}+(t-s)^{1/2}\bigr)^{-1}s^{-1}\bigl[(t-s)^{-1}+\bigl(t+s+(st)^{1/2}\bigr)t^{-1}(t-s)^{-1/2}\bigl(t^{1/2}+s^{1/2}\bigr)^{-1}\bigr]. \tag{147}$$

Hence (146) equals

$$-4(4\pi)^{-9/2}C(K)\mathbb{P}_x[T_K=\infty]\int_0^t \mathrm{d}s\int \mu_K(\mathrm{d}y)\int \mu_K(\mathrm{d}w)\mathrm{e}^{-\frac{|x-y|^2}{4(t-s)}-\frac{|x-w|^2}{4s}}\bigl(t^{1/2}+(t-s)^{1/2}\bigr)^{-1}s^{-1}$$

$$\times\bigl[(t-s)^{-1}+\bigl(t+s+(st)^{1/2}\bigr)t^{-1}(t-s)^{-1/2}\bigl(t^{1/2}+s^{1/2}\bigr)^{-1}\bigr]. \tag{148}$$

The first term in the square brackets of (148) gives the contribution

$$-6(4\pi)^{-9/2}C(K)^3\mathbb{P}_x[T_K=\infty]\frac{\log t}{t^{3/2}}+\mathrm{O}\bigl(t^{-3/2}\bigr), \tag{149}$$

and the second term contributes

$$-2(4\pi)^{-9/2}C(K)^3\mathbb{P}_x[T_K=\infty]\frac{\log t}{t^{3/2}}+\mathrm{O}\bigl(t^{-3/2}\bigr). \tag{150}$$

By (146)–(150) we conclude that the first term in the right-hand side of (142) is bounded from below by the expression in the right-hand side of (119). Since

$$\int_\tau^s \mathrm{d}\rho\int \mu_K(\mathrm{d}w)p(x,w;\rho)\geqslant 2(4\pi)^{-3/2}\bigl(\tau^{-1/2}-s^{-1/2}\bigr)\int \mu_K(\mathrm{d}w)\mathrm{e}^{-\frac{|x-w|^2}{4\tau}} \tag{151}$$

we have, by (143), that the resulting upper bound differs from the lower bound by at most

$$\begin{aligned}
&\int_0^t \mathrm{d}s \int \mu_K(\mathrm{d}y) p(x,y;t-s) \int_0^s \mathrm{d}\tau \left((s-\tau)^{-3/2} - (t-\tau)^{-3/2}\right) \\
&\quad \times C(K)\left(\tau^{-1/2} - s^{-1/2}\right) \int \mu_K(\mathrm{d}w)\left(\mathrm{e}^{-\frac{|x-w|^2}{4s}} - \mathrm{e}^{-\frac{|x-w|^2}{4\tau}}\right) \\
&\quad \leqslant \int_0^t \mathrm{d}s \int \mu_K(\mathrm{d}y) p(x,y;t-s) \int \mu_K(\mathrm{d}w) s^{-1} \mathrm{e}^{-\frac{|x-w|^2}{4s}} \\
&\quad \times C(K) \int_0^s \mathrm{d}\tau\, \tau^{-1/2}(s-\tau)^{-1/2}\left(1 - \mathrm{e}^{-|x-w|^2(\frac{1}{4\tau} - \frac{1}{4s})}\right).
\end{aligned} \tag{152}$$

By substituting $\tau = s(\sin\theta)^2$ we have that

$$\begin{aligned}
\int_0^t \mathrm{d}\tau\, \tau^{-1/2}(s-\tau)^{-1/2}\left(1 - \mathrm{e}^{-|x-w|^2(\frac{1}{4\tau} - \frac{1}{4s})}\right) &\leqslant 2\int_0^{\pi/2} \mathrm{d}\theta \left(1 - \mathrm{e}^{-\frac{|x-w|^2(\cos\theta)^2}{4s(\sin\theta)^2}}\right) \leqslant 2\int_0^{\infty} \mathrm{d}\theta \left(1 - \mathrm{e}^{-\frac{(|x|+R)^2}{s\theta^2}}\right) \\
&\leqslant (4\pi)^{1/2}\left(|x| + R\right) s^{-1/2}.
\end{aligned} \tag{153}$$

Then (152) is bounded from above by

$$(4\pi)^2 C(K)\left(|x| + R\right) \int \mu_K(\mathrm{d}y) \int \mu_K(\mathrm{d}w) \int_0^t \mathrm{d}s\, p(x,w;s) p(x,y;t-s). \tag{154}$$

But (154) has been estimated in (144). This completes the proof of (119), Lemma 9 and Proposition 2 for $m = 3$. □

Finally one can show that, by going through the estimates leading to the proof of Proposition 2, the remainder $\mathrm{O}(t^{-m/2})$ in Theorem 1 is uniform on compact subsets of $\mathbb{R}^m \setminus K$. This completes the proof of Theorem 1.

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