Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Ann. I. H. Poincaré – PR 41 (2005) 1083–1099
ANNALES DE L'INSTITUT HENRI POINCARÉ PROBABILITÉS ET STATISTIQUES
www.elsevier.com/locate/anihpb

Eigenvalues of Hermite and Laguerre ensembles: large beta asymptotics

Ioana Dumitriu [a,*], Alan Edelman [b]

[a] *Department of Mathematics, U.C. Berkeley, Berkeley, CA, USA*
[b] *Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA, USA*

Received 25 June 2004; received in revised form 21 October 2004; accepted 19 November 2004

Available online 24 June 2005

Abstract

In this paper we examine the zero and first order eigenvalue fluctuations for the β-Hermite and β-Laguerre ensembles, using tridiagonal matrix models, in the limit as $\beta \to \infty$. We prove that the fluctuations are described by multivariate Gaussians of covariance $\mathrm{O}(1/\beta)$, centered at the roots of a corresponding Hermite (Laguerre) polynomial. The covariance matrix itself is expressed as combinations of Hermite or Laguerre polynomials respectively.

We show that the approximations are of real value even for small β; we can use them to approximate the true functions even for the traditional $\beta = 1, 2, 4$ values.

Résumé

Dans cet article on examine les fluctutations d'ordre zéro et du premier ordre pour les valeurs propres des ensembles β-Hermite et β-Laguerre, en utilisant les modèles de matrices tridiagonales, dans la limite $\beta \to \infty$. Nous prouvons que les fluctuations suivent des distributions gaussiennes multivariées de covariances $\mathrm{O}(1/\beta)$, centrées sur les zéros des polynômes correspondants. Les matrices de covariances elles mêmes s'expriment en termes de polynômes d'Hermite ou de Laguerre.

Nous montrons que les approximations sont très bonnes, même pour les petites valeurs de β. On peut les utiliser même pour les valeurs traditionnelles de $\beta : 1, 2, 3, 4$.

* Corresponding author.
E-mail addresses: dumitriu@math.berkeley.edu (I. Dumitriu), edelman@math.mit.edu (A. Edelman).

0246-0203/$ – see front matter
doi:10.1016/j.anihpb.2004.11.002

1. Introduction

This paper provides insight into the shape of random matrix laws such as the finite semi-circle law, the finite quarter-circle law and its generalization. We investigate, in a completely rigorous and mathematical fashion, the zero and first order fluctuations for the β-Hermite and β-Laguerre ensembles at large β.

We begin with a simple example. Suppose A is a random $k \times k$ complex matrix with real and imaginary parts all i.i.d. standard normals. Let $S = (A + A^H)/2$ be the Hermitian part of A. The matrix S has a distribution commonly known as the Gaussian Unitary Ensemble; this matrix distribution and the joint distribution of its (real) eigenvalues have been well studied. For a good reference on the subject, see Mehta [10].

We draw below histograms of normalized eigenvalues taken from this distribution, the known theoretical distribution (see [10, page 93]), and the semicircle limit corresponding to $k \to \infty$. For the histograms, we have chosen 40 000 samples from the GUE with $k = 4$ and $k = 6$.

Notice, for k finite, the k "bumps" in the distribution that wiggle above and below the semi-circle. A natural question to many engineers, physicists, mathematicians, and other scientists who have seen these pictures is whether they can be well approximated by the sum of k appropriately chosen Gaussians. (Of course when $k = 1$, this is exactly true.) The answer, as proved in this paper, is yes. We give a sum of Gaussians approximation that is asymptotically correct for the $\beta \to \infty$ limit but useful even for small values of β.

For those well versed in random matrix theory, the GUE is the $\beta = 2$ case of a Hermite matrix ensemble [10]. Had we started with A real (quaternion), we would have the Gaussian Orthogonal (Symplectic) Ensemble corresponding to $\beta = 1$ ($\beta = 4$).

The joint eigenvalue density $f_\beta^H(\lambda_1, \ldots, \lambda_k)$ defined on $\mathbb{R}^k$ for the k eigenvalues for an arbitrary $\beta > 0$ is given in the formula below.

$$f_\beta^H(\lambda_1, \ldots, \lambda_k) = (2\pi)^{-k/2} \prod_{j=1}^{k} \frac{\Gamma(1+\beta/2)}{\Gamma(1+j\beta/2)} \prod_{1 \leqslant i<j \leqslant k} |\lambda_i - \lambda_j|^\beta \, \mathrm{e}^{-\sum_{i=1}^k \lambda_i^2/2}. \tag{1}$$

Note the "repulsion" factor $\Delta(\Lambda) \equiv \Delta(\lambda_1, \ldots, \lambda_k) \equiv \prod_{1 \leqslant i<j \leqslant k} |\lambda_i - \lambda_j|$.

Similarly, for the $k \times k$ Laguerre ensembles of statistics (Wishart matrix theory), the joint eigenvalues density $F_{\beta,a}^L$ is defined on $[0, \infty)^k$ for arbitrary β and parameter $a > (k-1)\beta/2$ (for the $k \times n$ Wishart ensembles of $\beta = 1, 2, 4$, $a = n\beta/2$). Once again note the repulsion factor $\Delta(\Lambda)$:

$$f_{\beta,a}^L(\lambda_1, \ldots, \lambda_k) = c_{\beta,a}^L \prod_{1 \leqslant i<j \leqslant k} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^{k} \lambda_i^{a-(k-1)\beta/2-1} \, \mathrm{e}^{-\sum_{i=1}^k \lambda_i/2}, \tag{2}$$

where

$$c_{\beta,a} = 2^{-ka} \prod_{j=1}^{k} \frac{\Gamma(1+\beta/2)}{\Gamma(1+j\beta/2)\Gamma(a-(k-j)\beta/2)}.$$

In [5], we have found (real) tridiagonal matrix models whose eigenvalue distributions are given by (1) and (2); we depict the distributions in Table 1. Note that the variables have either standard normal distribution or a χ distribution (sometimes scaled by $\sqrt{2}$).

For generating efficiently eigenvalues for the β-ensemble distributions, we recommend using the tridiagonal/bidiagonal model above.

The marginal density of a single eigenvalue (known as the level density) can be computed exactly as a function of x in the case of the Hermite ensembles for β an even integer (with the help of a formula found by Baker and Forrester [2]), using the MOPS (*Multivariate Orthogonal Polynomials symbolically*) software [6]. For general β no closed-form formula is known, but it is likely that computational (approximation, if not symbolical) techniques are not far out of reach.

Table 1
Tridiagonal matrix models for the β-Hermite and β-Laguerre ensembles with any $\beta > 0$

Hermite matrix $k \in \mathbb{N}$	$A_\beta \sim \frac{1}{\sqrt{2}} \begin{pmatrix} N(0,2) & \chi_{(k-1)\beta} & & & \\ \chi_{(k-1)\beta} & N(0,2) & \chi_{(k-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & N(0,2) & \chi_\beta \\ & & & \chi_\beta & N(0,2) \end{pmatrix}$
Laguerre matrix $k \in \mathbb{N}$ $a \in \mathbb{R}$ $a > \frac{\beta}{2}(k-1)$	$L_\beta = B_\beta B_\beta^T$, where $B_\beta \sim \begin{pmatrix} \chi_{2a} & & & \\ \chi_{\beta(k-1)} & \chi_{2a-\beta} & & \\ & \ddots & \ddots & \\ & & \chi_\beta & \chi_{2a-\beta(k-1)} \end{pmatrix}$

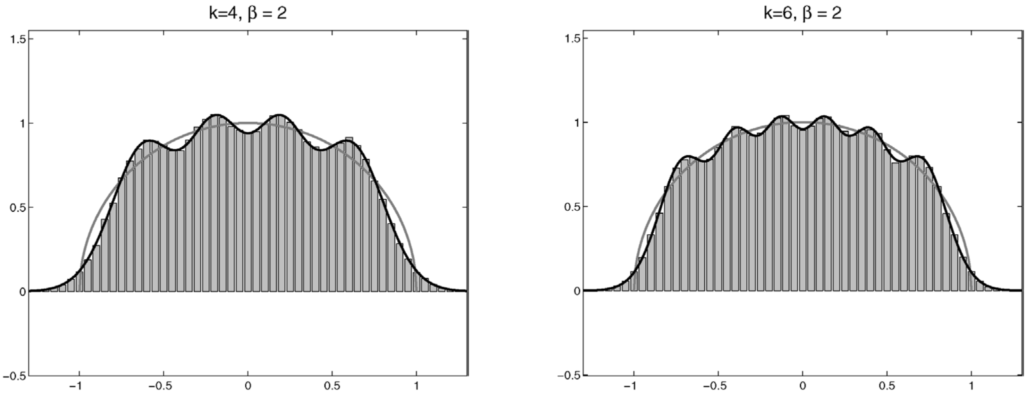

Fig. 1. Histograms of eigenvalues, finite ($k = 4, 6$) exact level densities, and semicircle law ($k \to \infty$) for matrices of the GUE ensemble.

For fixed k and general β one finds that for β getting larger, the bumps of Fig. 1 get "bumpier". To be precise, for the Hermite ensembles, we prove here that at $\beta = \infty$ the bumps become delta functions at the roots of the kth Hermite polynomial, while for β large, the bumps behave like Gaussians centered at these roots with variance $\mathrm{O}(1/\beta)$.

The model of β as an inverse temperature is apparent from (1). As β goes to 0, the strength of the repulsion factor $\Delta(\Lambda)$ decreases until annihilation; the interdependence among eigenvalues disappears, and the randomness increases (each eigenvalue behaves like an independent Gaussian). In the frozen state ($\beta = \infty$), we can imagine the k eigenvalues fixed at the roots of the Hermite polynomial. Warming the system a little (β very large but not infinite) gives the particles a little energy, and the eigenvalues have Gaussian distribution to first order around the Hermite polynomial roots.

Similarly, in the Laguerre case, at $\beta = 0$, the eigenvalues become i.i.d. variables with distribution χ^2_{2a}. As β grows the eigenvalues have Gaussian distribution to first order around the Laguerre polynomial roots, while at $\beta = \infty$ we reach the freezing point when the eigenvalues are fixed at those roots.

In the following, we use first order eigenvalue perturbation theory and the tridiagonal ensembles in [5] to rigorously investigate this phenomenon mathematically obtaining precisely the asymptotic variance along with the mean.

These results draw a parallel to the Tracy–Widom laws [13,14] for the $\beta = 1, 2, 4$-Hermite ensembles, later extended to $\beta = 1, 2$-Laguerre ensembles by Johansson [8] and Johnstone [9].

The Tracy–Widom laws compute the fluctuation in the distribution of the largest eigenvalue of a β-Hermite ensemble with $\beta = 1, 2, 4$, as $k \to \infty$, and obtain it in terms of the solution to a Painleve differential equation. From the semicircle law, we know that as $n \to \infty$, regardless of β, the largest eigenvalue (scaled by $\sqrt{2k\beta}$) goes to 1. From Theorem 3.1, constrained to $i = 1$, with the help of [11] and [1, page 450], we obtain Corollary 3.4, which gives an intuition of how the $\beta = 1, 2, 4$ Tracy–Widom distributions evolve towards a normal distribution at $\beta = \infty$ (as we state in Remark 3.5, we strongly believe that the limits in Corollary 3.4 are interchangeable).

The theoretical results of Section 5 are similar to the "Central Limit Theorems", i.e. the computation of the global fluctuations from the semicircle and semicircle-type laws done by Johansson in [7] for Hermite-like ensembles of any β and by Silverstein and Bai [12] for a class of Laguerre-like ensembles with real or complex entries ($\beta = 1, 2$). Roughly said, the eigenvalues can be thought of as fluctuating (like Gaussians) around the roots of the corresponding orthogonal polynomial as β grows large; if one lets n grow large, too, the global eigenvalue fluctuation becomes a Gaussian process. The larger $1/\beta$, the "warmer" it gets, and the larger the "vibration". The larger β, the "cooler" it gets, and the eigenvalues "freeze" into place.

At the end of Section 5 we perform computational experiments to see how good the β large approximation is even for relatively small k and β.

Before we delve into the main part of this paper, we thought it appropriate to mention one more possible connection. Similar to the β-Hermite ensemble, we have the circular ensembles defined by the joint eigenvalue $e^{i\theta_j}$ (with $\theta_j \in [0, 1]^n$) density proportional to

$$f_\beta \propto \prod_{1 \leqslant j < l \leqslant k} \left| e^{i\theta_j} - e^{i\theta_l} \right|^\beta.$$

The $\beta = 2$ circular ensemble is also known as the Haar measure on the unitary group U_n. The eigenvalues of U_n appear to be almost uniformly distributed on the unit circle (see the experiment with $k = 100$ in Diaconis' paper [4]). For any fixed k, as $\beta \to \infty$, the eigenvalues freeze into place uniformly at the kth roots of unity. We believe that the same Gaussian phenomenon will hold, and the fluctuation of eigenvalue i will behave like a normal centered at the ith root of the unity, with variance depending on $1/\beta$.

2. Eigenvalue perturbation and χ asymptotics

In this section we present two lemmas we need in the proofs of our main results (Theorems 3.1 and 4.1).

The first lemma involves perturbation theory; for a good reference on Perturbation theory and a more general form of the result below, see Demmel's book [3, Section 4.3].

Lemma 2.1. *Let A and B be $n \times n$ symmetric matrices, and let $\epsilon > 0$. Assume A has all distinct eigenvalues. Let $M = A + \epsilon B + o(\epsilon)$, where by $o(\epsilon)$ we mean a matrix in which every entry goes to 0 faster than ϵ. Let $\lambda_i(X)$ denote the ith eigenvalue of X, for $1 \leqslant i \leqslant n$. Finally, let Q be an eigenvector matrix for A. Then*

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \big(\lambda_i(M) - \lambda_i(A) \big) = Q(:, i)^T B Q(:, i),$$

where, following MATLAB notation, $Q(:, i)$ represents the ith column of Q.

Remark 2.2. Equivalently, for every $1 \leqslant i \leqslant n$,

$$\lambda_i(M) = \lambda_i(A) + \epsilon Q(:, i)^T B Q(:, i) + o(\epsilon).$$

The second result is an approximation lemma for the χ_r distribution as r grows large.

Lemma 2.3. *Let $r > 0$, and let X be a variable with distribution χ_r. Then as $r \to \infty$ the p.d.f. of $X - \sqrt{r}$ converges uniformly* on any fixed interval *to the p.d.f. of a normal distribution of mean* 0 *and variance* 1/2*.*

Proof. We prove this lemma by looking at the density function of χ_r when $r \to \infty$. Recall that the p.d.f. of a variable with χ_r distribution is

$$f_r(x) = \frac{2^{1-r/2}}{\Gamma(r/2)} x^{r-1}\,\mathrm{e}^{-x^2/2}.$$

Using the Stirling approximation formula

$$\Gamma(z) \sim z^{z-1/2}\,\mathrm{e}^{-z}\sqrt{2\pi}\left(1 + \frac{1}{12z} + \mathrm{O}\left(\frac{1}{z^2}\right)\right), \tag{3}$$

for r large, we obtain

$$E[X] = \sqrt{2}\frac{\Gamma((r+1)/2)}{\Gamma(r/2)} = \sqrt{r}\,\big(1 + \mathrm{O}\big(r^{-1}\big)\big).$$

Let $Y := X - \sqrt{r}$, the p.d.f. of Y is

$$f(t) = \frac{2^{1-r/2}}{\Gamma(r/2)}(t + \sqrt{r}\,)^{r-1}\,\mathrm{e}^{-(t+\sqrt{r}\,)^2/2}.$$

We examine this p.d.f. in a "small" neighborhood of 0, such that $t = \mathrm{o}(r^{1/2})$. With the help of the Stirling approximation (3), we obtain

$$f(t) = \frac{1}{\sqrt{\pi}}\left(1 + \frac{t}{\sqrt{r}}\right)^{r-1}\mathrm{e}^{-t^2/2-\sqrt{r}t}\big(1 + \mathrm{O}\big(r^{-1}\big)\big),$$

and so

$$f(t) = \frac{1}{\sqrt{\pi}}\,\mathrm{e}^{-t^2}\left(1 + \mathrm{O}\left(\frac{t}{\sqrt{r}}\right)\right). \tag{4}$$

Thus, on any fixed interval, the p.d.f. of Y converges to the p.d.f. of a centered normal of variance $1/2$. □

3. β-Hermite: zero and first-order approximations

Let k be fixed, and let $h_1^{(k)}, \ldots, h_k^{(k)}$ be the roots of the kth univariate Hermite polynomial H_k.

Recall that the Hermite polynomials $H_0(x), H_1(x), \ldots$ are orthonormal with respect to the weight e^{-x^2} on $(-\infty, \infty)$, i.e.

$$\int_{\mathbb{R}} H_i(x)H_j(x)\,\mathrm{e}^{-x^2}\,\mathrm{d}x = \delta_{ij}, \quad \forall i, j \geqslant 0,$$

and $\deg(H_i) = i$ and $[x^i]H_i(x) = 1$, for all $i \geqslant 0$.

Let A_β be a random matrix from the β-Hermite ensemble of size k, scaled by $1/\sqrt{2k\beta}$. For the remainder of this section, we think of β as a parameter.

We state and prove the following theorem.

Theorem 3.1. *Let $\lambda_i(A_\beta)$ be the ith largest eigenvalue of A_β, for any fixed $1 \leqslant i \leqslant k$. Then, as $\beta \to \infty$,*

$$\lambda_i(A_\beta) \to \frac{1}{\sqrt{2k}}h_i^{(k)},$$

and, as $\beta \to \infty$,

$$\sqrt{\beta}\left(\lambda_1(A_\beta) - \frac{1}{\sqrt{2k}}h_1^{(k)}, \lambda_2(A_\beta) - \frac{1}{\sqrt{2k}}h_2^{(k)}, \ldots, \lambda_k(A_\beta) - \frac{1}{\sqrt{2k}}h_k^{(k)}\right) \to \frac{1}{\sqrt{2k}}G$$

where $G \equiv (G_1, G_2, \ldots, G_k)$ is a k-variate Gaussian with covariance matrix

$$\mathrm{Cov}(G_i, G_j) = \frac{\sum_{l=0}^{k-1} H_l^2(h_i^{(k)})H_l^2(h_j^{(k)}) + \sum_{l=0}^{k-2} H_{l+1}(h_i^{(k)})H_l(h_i^{(k)})H_{l+1}(h_j^{(k)})H_l(h_j^{(k)})}{(\sum_{l=0}^{k-1} H_l^2(h_i^{(k)}))(\sum_{l=0}^{k-1} H_l^2(h_j^{(k)}))}.$$

The convergence here is of p.d.f.'s, uniformly on any fixed interval in $\mathbb{R}^k$.

Proof. Let H be the $k \times k$ symmetric tridiagonal matrix

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sqrt{k-1} & & & & \\ \sqrt{k-1} & 0 & \sqrt{k-2} & & & \\ & \sqrt{k-2} & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & \sqrt{1} \\ & & & & \sqrt{1} & 0 \end{pmatrix}. \tag{5}$$

This matrix is the tridiagonal matrix corresponding to the 3-term recurrence for Hermite polynomials (see, for example, [11, pages 105–106]). It is a well-known and easily verified fact that its eigenvalues are the roots of the kth Hermite polynomial $H_k(x)$ (recall that we denoted them by $h_1^{(k)}, \ldots, h_k^{(k)}$), and that the eigenvector corresponding to the ith eigenvalue $h_i^{(k)}$ is

$$v_i = \begin{pmatrix} H_{k-1}(h_i^{(k)}) \\ H_{k-2}(h_i^{(k)}) \\ \vdots \\ H_1(h_i^{(k)}) \\ H_0(h_i^{(k)}) \end{pmatrix}.$$

Lemma 3.2. *Let A_β be as defined in the beginning of this section. Then*

$$\lim_{\beta\to\infty} \sqrt{2k\beta}A_\beta - \sqrt{\beta}H = Z,$$

where Z is a tridiagonal matrix with standard normal variables on the diagonal and normal variables of mean 0 and variance $1/4$ on the subdiagonal. All normal variables in Z are mutually independent, subject only to the symmetry.

Convergence here is a convergence of p.d.f.s, uniformly on any fixed product of intervals.

From now on we use the notation

$$Z = \begin{pmatrix} M_k & N_{k-1} & & & & \\ N_{k-1} & M_{k-1} & N_{k-2} & & & \\ & N_{k-2} & M_{k-2} & & & \\ & & & \ddots & & \\ & & & & M_2 & N_1 \\ & & & & N_1 & M_1 \end{pmatrix}, \tag{6}$$

with Z as above.

Lemma 3.2 follows immediately from Lemma 2.3, since we are dealing with a finite number $(k-1)$ of χ variables on the sub-diagonal of A_β, each converging in p.d.f. to a Gaussian variable, uniformly on any fixed interval.

Hence we have that, entry by entry,

$$A_\beta \sim \frac{1}{\sqrt{2k}} H + \frac{1}{\sqrt{2k\beta}} Z,$$

the p.d.f.'s converging uniformly on any fixed product of intervals as $\beta \to \infty$.

Thus all zero- and first-order properties of A_β are the same as for the random matrix $(1/\sqrt{2k})H + (1/\sqrt{2k\beta})Z$, where Z is as above. In particular, for any $1 \leqslant i \leqslant k$,

$$\lambda_i(A_\beta) \sim \lambda_i\left(\frac{1}{\sqrt{2k}} H + \frac{1}{\sqrt{2k\beta}} Z\right),$$

and with the help of Lemma 2.1, for any $1 \leqslant i \leqslant k$,

$$\lambda_i(A_\beta) \sim \frac{1}{\sqrt{2k}} h_i^{(k)} + \frac{1}{\sqrt{2k\beta}} \frac{v_i^T Z v_i}{v_i^T v_i},$$

with the p.d.f.'s converging uniformly on any fixed interval, as $\beta \to \infty$.

Hence, using the notation (6),

$$\sqrt{\beta}\left(\lambda_i(A_\beta) - \frac{1}{\sqrt{2k}} h_i^{(k)}\right) \sim \frac{1}{\sqrt{2k}} \frac{\sum_{l=0}^{k-1} H_l^2(h_i^{(k)}) M_{l+1} + 2\sum_{l=1}^{k-1} H_l(h_i^{(k)}) H_{l-1}(h_i^{(k)}) N_l}{\sum_{l=0}^{k-1} H_l^2(h_i^{(k)})},$$

with the p.d.f.'s converging uniformly on any fixed interval, as $\beta \to \infty$.

The statement of Theorem 3.1 follows. □

Remark 3.3. There is an alternative way to look at this problem which is reminiscent of what is sometimes known in applied mathematics as the "saddle point" method. The method involves finding the maximum of the potential function $V(\lambda)$ (defined by writing the p.d.f. as $\mathrm{e}^{-V(\lambda)}$), which for this case is

$$V(\lambda) := V(\lambda_1, \ldots, \lambda_k) = -\beta \sum_{1 \leqslant i < j \leqslant n} \log|\lambda_i - \lambda_j| + \sum_{i=1}^{n} \lambda_i^2/2.$$

The fact that the maximum of the potential function is achieved at the Hermite polynomial roots $h_i^{(k)}$ (scaled by $\sqrt{2k\beta}$) has a well known electrostatic interpretation (see [11]).

Once the maximum is found, it is used to approximate (locally, around the maximum point) the potential function by a quadratic function (just as in the univariate case) given by the Hessian matrix $\mathcal{H} = (\partial^2 V(\lambda)/\partial\lambda_i \partial\lambda_j)_{i,j}$, which is the inverse of the covariance matrix we computed in Theorem 3.1. Since $\beta \to \infty$, this should provide zero and first order asymptotics for the eigenvalues, i.e. the equivalent of Theorem 3.1. One could compute $\mathcal{H}$, and manipulate it to show that it matches our covariance matrix; Brian Sutton from MIT has confirmed this by verifying a few small cases (up to $k = 6$).

Letting $k \to \infty$ in Theorem 3.1, we obtain the Corollary below.

Corollary 3.4. *Let A_β be a matrix from the $k \times k$ β-Hermite ensemble, scaled by $1/\sqrt{2k\beta}$, and let $\lambda_1(A_\beta)$ be the largest eigenvalue of A_β. Then*

$$\lim_{k \to \infty} \lim_{\beta \to \infty} k^{-2/3}\big(\lambda_1(A_\beta) - 1\big) \to \frac{a_1}{2} + \sigma^2 G,$$

where $a_1 = -2.33810\ldots$ is the largest root of the Airy Ai *function* (*see* [1])*, and*

$$\sigma^2 = 2\frac{\int_0^\infty \mathtt{Ai}^4(x+a_0)\,\mathrm{d}x}{(\int_0^\infty \mathtt{Ai}^2(x+a_0)\,\mathrm{d}x)^2} \sim 0.41050\ldots.$$

Proof. The corollary follows by using the special properties of the Hermite polynomial roots and the Airy function as in [1] and [11]. We sketch the proof here.

The fact that

$$\frac{h_1^{(k)}}{\sqrt{2k}} \sim 1 + \frac{a_0}{2k^{2/3}}$$

is a special functions result that can be found in [11, pages 131–132]. All we need to prove is that the corresponding eigenvector v_1 is going to a normalized version of the function $Ai(x+a_0)$ with stepsize $1/k^{1/3}$. This we can do as follows: let $D = \frac{1}{2n}H^2$, i.e. D is the pentadiagonal matrix

$$D = \frac{1}{4n}\begin{pmatrix} k-1 & 0 & \sqrt{(k-1)(k-2)} & & & & \\ 0 & 2k-3 & 0 & \sqrt{(k-2)(k-3)} & & & \\ \sqrt{(k-1)(k-2)} & 0 & 2k-5 & 0 & & & \\ 0 & \sqrt{(k-2)(k-3)} & 0 & 2k-7 & & & \\ & & & & \ddots & & \\ & & & & 5 & 0 & \sqrt{6} \\ & & & & 0 & 3 & 0 \\ & & & & \sqrt{6} & 0 & 1 \end{pmatrix}.$$

Note that $\sqrt{(k-i)(k-(i+1))}/k = 1 - i_*/k$ for some $i_* \in [i, i+1]$. As $k \to \infty$, the diagonal of the matrix D is roughly a discretization of the function $\frac{1}{2}(1-\frac{x}{k})$ from 0 to k, with stepsize $1/k$. Similarly, the off-diagonal term can roughly be identified with a discretization of the function $\frac{1}{4}(1-\frac{x}{k})$, once again with stepsize $1/k$, from 0 to k. Since we know that

$$Dv_1 \sim \left(1 + \frac{a_0}{2k^{2/3}}\right)^2 v_1 \sim \left(1 + \frac{a_0}{k^{2/3}}\right) v_1,$$

if follows that v_1 must be a (normalized) discretization with step $1/k^{1/3}$, from 0 to $k^{2/3}$, of a function F_k which solves

$$F_k'' - xF_k = a_0 F_k.$$

Since the equation $f'' - xf = 0$ has 2 independent solutions, Ai and Bi (see [1, page 446]), it follows that $F = (1-c_k)\mathtt{Ai}(x+a_0) + c_k\mathtt{Bi}(x+a_0)$. Due to the interlacing property of the Hermite polynomial eigenvalues, $h_i^{(k)}$ is larger than any root of a polynomial $H_j(x)$ with $j < k$; hence v_1 *has all positive entries*. On the other hand, $\mathtt{Ai}(x+a_0) \geqslant 0$ for $x \geqslant 0$ and $\mathtt{Bi}(a_0) < 0$, while $\mathtt{Ai}(x+a_0) \to 0$ and $\mathtt{Bi}(x+a_0) \to \infty$ as $x \to \infty$. Hence it must be that $c_k \to 0$ as $k \to \infty$ (otherwise v_1 would not have strictly positive entries).

Thus, $v_1/||v_1||_2$ tends to a (normalized to norm 1) discretization with stepsize $1/k^{1/3}$ (from 0 to $k^{2/3}$) of the function $\mathtt{Ai}(x+a_0)$, and the calculations follow. □

Remark 3.5. Note that the limit in Corollary 3.4 is taken first with respect to β, then with respect to k. We believe (and experimental evidence strongly supports this) that the limits are interchangeable.

As a final illustration of Theorem 3.1, we include Fig. 2, where we have meshed the covariance matrix for $k = 20$ and $k = 50$; note that as k increases, the covariance matrix becomes more and more diagonally dominant (at $k = \infty$, the matrix becomes diagonal, as the eigenvalues become independent).

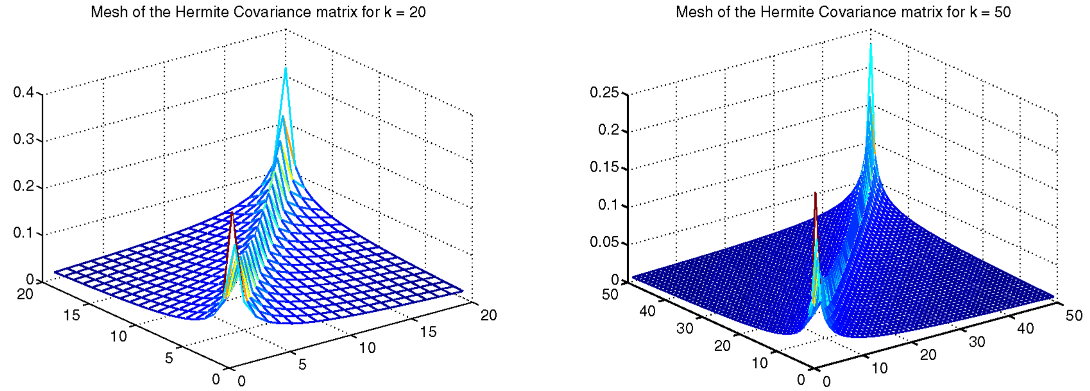

Fig. 2. Meshes of the covariance matrix at $k = 20$ and $k = 50$.

4. β-Laguerre: zero and first-order approximations

Let k be fixed. Given a fixed $\gamma > 0$, let $l_1^{(k)}, \ldots, l_k^{(k)}$ be the roots of the kth Laguerre polynomial of parameter $\gamma - 1$, $L_k^{\gamma-1}$.

Recall that for any $\gamma > -1$, the Laguerre polynomials $L_0^\gamma, L_1^\gamma, \ldots$ are orthonormal with respect to the weight $x^\gamma \mathrm{e}^{-x}$ on $[0, \infty)$:

$$\int\limits_{[0,\infty)} L_i^\gamma(x) L_j^\gamma(x) x^\gamma \mathrm{e}^{-x} \,\mathrm{d}x = \delta_{ij}, \quad \forall i, j \geqslant 0,$$

and $\deg(L_i^\gamma) = i$ and $[x^i] L_i^\gamma(x) = (-1)^i$ for all $i \geqslant 0$.

Let B_β be a random matrix from the β-Laguerre ensemble of size k and parameter a_β, scaled by $1/k\beta$. For the remainder of this section, we think of β as a parameter. Suppose that, as β grows large,

$$\lim_{\beta\to\infty} \frac{a_\beta}{\beta} = \frac{1}{2}(k + \gamma - 1).$$

Note that the requirement $a_\beta > (k-1)\beta/2$ constrains γ to be positive.

Theorem 4.1. *Let $\lambda_i(B_\beta)$ be the ith largest eigenvalue of B_β, for any fixed $1 \leqslant i \leqslant k$. Then, as $\beta \to \infty$,*

$$\lambda_i(B_\beta) \to \frac{1}{k} l_i^{(k)}.$$

Moreover, as $\beta \to \infty$,

$$\sqrt{\beta}\left(\lambda_1(B_\beta) - \frac{1}{k} l_1^{(k)}, \lambda_2(B_\beta) - \frac{1}{k} l_2^{(k)}, \ldots, \lambda_k(B_\beta) - \frac{1}{k} l_k^{(k)}\right) \to \frac{1}{k} G,$$

where $G \equiv (G_1, G_2, \ldots, G_k)$ is a centered k-variate Gaussian of covariance matrix

$$\mathrm{Cov}(G_i, G_j) = 2\frac{(\gamma + k - 1)(L_{k-1}^\gamma(l_i^{(k)}))^2 (L_{k-1}^\gamma(l_j^{(k)}))^2 + A_k(i,j) + B_k(i,j) + C_k(i,j) + D_k(i,j)}{(\sum_{l=0}^{k-1} (L_l^\gamma(l_i^{(k)}))^2)(\sum_{l=0}^{k-1} (L_l^\gamma(l_j^{(k)}))^2)},$$

where

$$A_k(i,j) = \sum_{l=1}^{k-1} (\gamma + 2(k-l) - 1)\big(L_{k-l-1}^\gamma(l_i^{(k)})\big)^2 \big(L_{k-l-1}^\gamma(l_j^{(k)})\big)^2,$$

$$B_k(i,j) = \sum_{l=1}^{k-1} \big(\gamma + 2(k-l)\big) L^{\gamma}_{k-l-1}\big(l_i^{(k)}\big) L^{\gamma}_{k-l-1}\big(l_j^{(k)}\big) L^{\gamma}_{k-l}\big(l_i^{(k)}\big) L^{\gamma}_{k-l}\big(l_j^{(k)}\big),$$

$$C_k(i,j) = \sum_{l=1}^{k-1} \sqrt{\gamma+k-l}\,\sqrt{k-l}\,\Big(\big(L^{\gamma}_{k-l-1}\big(l_i^{(k)}\big)\big)^2 L^{\gamma}_{k-l-1}\big(l_j^{(k)}\big) L^{\gamma}_{k-l}\big(l_j^{(k)}\big) + \big(L^{\gamma}_{k-l-1}\big(l_j^{(k)}\big)\big)^2 L^{\gamma}_{k-l-1}\big(l_i^{(k)}\big) L^{\gamma}_{k-l}\big(l_i^{(k)}\big)\Big), \quad \textit{and}$$

$$D_k(i,j) = \sum_{l=1}^{k-1} \sqrt{\gamma+k-l}\,\sqrt{k-l}\,\Big(\big(L^{\gamma}_{k-l}\big(l_i^{(k)}\big)\big)^2 L^{\gamma}_{k-l-1}\big(l_j^{(k)}\big) L^{\gamma}_{k-l}\big(l_j^{(k)}\big) + \big(L^{\gamma}_{k-l}\big(l_j^{(k)}\big)\big)^2 L^{\gamma}_{k-l-1}\big(l_i^{(k)}\big) L^{\gamma}_{k-l}\big(l_i^{(k)}\big)\Big).$$

The convergence here is of p.d.f.'s, uniformly on any fixed product of intervals as $\beta \to \infty$.

Proof. The proof follows in the footsteps of that of Theorem 3.1.

Let L_γ be the $k \times k$ (symmetric) positive definite matrix

$$L_\gamma = \begin{pmatrix} \gamma+k-1 & \sqrt{\gamma+k-1}\,\sqrt{k-1} & & & & \\ \sqrt{\gamma+k-1}\,\sqrt{k-1} & 2(k-2)+\gamma+1 & \sqrt{\gamma+k-2}\,\sqrt{k-2} & & & \\ & \sqrt{\gamma+k-2}\,\sqrt{k-2} & 2(k-3)+\gamma+1 & & & \\ & & \ddots & & & \\ & & & & \sqrt{\gamma+2}\,\sqrt{2} & \\ & & & \sqrt{\gamma+2}\,\sqrt{2} & 3+\gamma & \sqrt{\gamma+1}\,\sqrt{1} \\ & & & & \sqrt{\gamma+1}\,\sqrt{1} & 1+\gamma \end{pmatrix}. \tag{7}$$

We can write $L_\gamma = B_\gamma B_\gamma^T$, with

$$B_\gamma = \begin{pmatrix} \sqrt{\gamma+k-1} & & & & \\ \sqrt{k-1} & \sqrt{\gamma+k-2} & & & \\ & \ddots & \ddots & & \\ & & \sqrt{2} & \sqrt{\gamma+1} & \\ & & & \sqrt{1} & \sqrt{\gamma} \end{pmatrix}. \tag{8}$$

Using the Laguerre differential recurrence and a 3-term recurrence which relates the Laguerre polynomials of parameter γ and $\gamma-1$ (see, for example, [11, (5.1.13, 5.1.14)]), together with elementary linear algebra, it is easy to check that the matrix L_γ has as eigenvalues the roots of the kth Laguerre polynomial of parameter $\gamma-1$, $L_k^{\gamma-1}(x)$ (recall that we have denoted them by $l_1^{(k)}, \ldots, l_k^{(k)}$), and an eigenvector corresponding to the ith eigenvalue $l_i^{(k)}$ is

$$w_i = \begin{pmatrix} L^{\gamma}_{k-1}(l_i^{(k)}) \\ L^{\gamma}_{k-2}(l_i^{(k)}) \\ \vdots \\ L^{\gamma}_{1}(l_i^{(k)}) \\ L^{\gamma}_{0}(l_i^{(k)}) \end{pmatrix}.$$

We define $\phi_i \equiv w_i/\|w_i\|_2$ to be a length 1 eigenvector corresponding to the ith eigenvalue l_i.

Lemma 4.2. *Let B_β be as in the statement of Theorem* 4.1. *Then*

$$\lim_{\beta\to\infty} k\beta B_\beta - \beta L^\gamma = \frac{1}{\sqrt{2}}\big(B_\gamma Z^T + Z B_\gamma^T\big),$$

with the p.d.f.'s converging uniformly on any fixed product of intervals, as $\beta \to \infty$. Here Z is a lower bidiagonal matrix with standard normal variables on the diagonal and on the subdiagonal. All normal variables in Z are mutually independent, subject only to the symmetry constraint.

We use the notation

$$Z \equiv \begin{pmatrix} M_k & & & & \\ N_{k-1} & M_{k-1} & & & \\ & \ddots & \ddots & & \\ & & N_2 & M_2 & \\ & & & N_1 & M_1 \end{pmatrix}. \tag{9}$$

Once again, the proof for this lemma follows from the construction of the Laguerre matrix as a lower bidiagonal random matrix times its transpose, and from Lemma 2.3.

Just as in the Hermite case, Lemma 4.2 allows us to write that, entry by entry,

$$B_\beta \sim \frac{1}{k}L_\gamma + \frac{1}{k\sqrt{2\beta}}\big(B_\gamma Z^T + Z B_\gamma^T\big),$$

and so

$$\lambda_i(B_\beta) \sim \lambda_i\left(\frac{1}{k}L_\gamma + \frac{1}{k\sqrt{2\beta}}\big(B_\gamma Z^T + Z B_\gamma^T\big)\right),$$

equivalently,

$$\lambda_i(B_\beta) \sim \frac{1}{k}l_i^{(k)} + \frac{1}{k\sqrt{2\beta}}\frac{w_i^T(B_\gamma Z^T + Z B_\gamma^T)w_i}{w_i^T w_i}$$

with the p.d.f.'s converging uniformly on any fixed interval.

Since $w_i^T B_\gamma Z^T w_i = w_i^T Z B_\gamma^T w_i$, as $\beta \to \infty$,

$$\lambda_i(B_\beta) \sim \frac{1}{k}l_i^{(k)} + \frac{\sqrt{2}}{k\sqrt{\beta}}\frac{w_i^T B_\gamma Z^T w_i}{w_i^T w_i} + \mathrm{o}\left(\frac{1}{\sqrt{\beta}}\right)$$

with the p.d.f.'s converging uniformly on any fixed interval.

Thus, using notation (9),

$$\sqrt{\beta}\left(\lambda_i(B_\beta) - \frac{1}{k}l_i^{(k)}\right) \sim \frac{\sqrt{2}}{k}\frac{\sqrt{\gamma}(L_0^\gamma(l_i^{(k)}))^2 + \mathrm{Sum}_1 + \mathrm{Sum}_2}{\sum_{l=0}^{k-1} L_l^\gamma(l_i^{(k)})^2},$$

with

$$\mathrm{Sum}_1 = \sum_{l=1}^{k-1}\big(\sqrt{\gamma+l}\big(L_l^\gamma\big(l_i^{(k)}\big)\big)^2 + \sqrt{l}L_l^\gamma\big(l_i^{(k)}\big)L_{l-1}^\gamma\big(l_i^{(k)}\big)\big)M_{l+1}, \quad \text{and}$$

$$\mathrm{Sum}_2 = \sum_{l=1}^{k-1}\big(\sqrt{\gamma+l}L_l^\gamma\big(l_i^{(k)}\big)L_{l-1}^\gamma\big(l_i^{(k)}\big) + \sqrt{l}\big(L_{l-1}^\gamma\big(l_i^{(k)}\big)\big)^2\big)N_l,$$

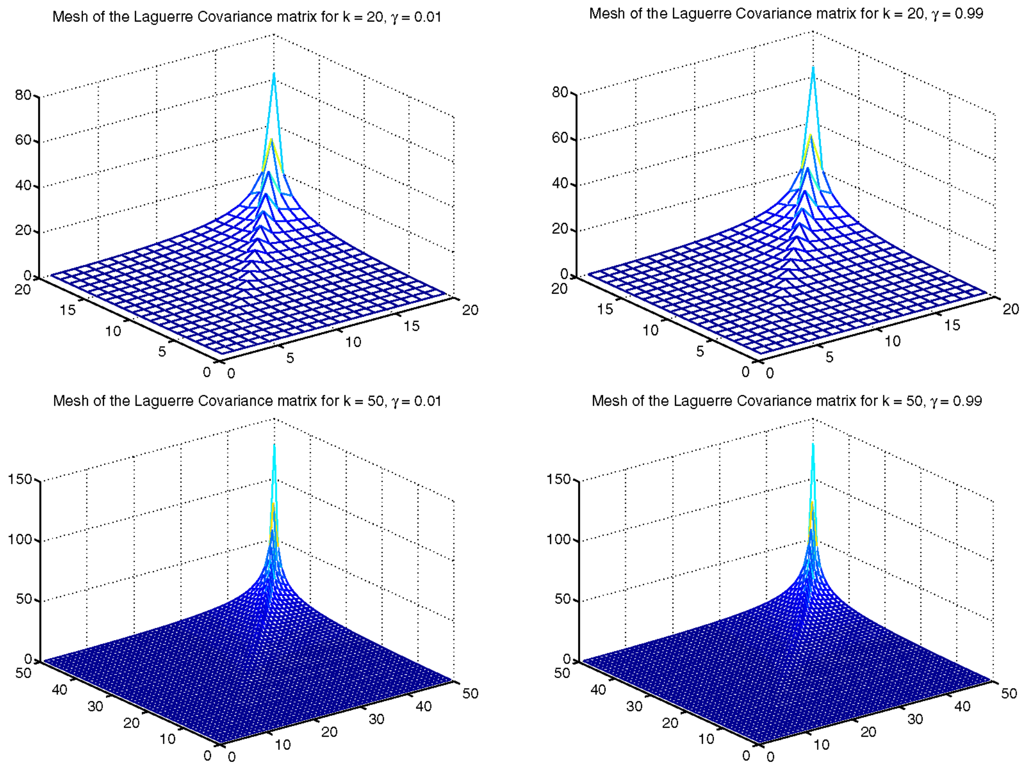

Fig. 3. Meshes of the covariance matrix at $\gamma = 0.01$ and $\gamma = 0.99$, with $k = 20$ and $k = 50$.

with the p.d.f.'s converging uniformly on any fixed product of intervals, as $\beta \to \infty$.

The statement of the theorem follows. □

As in the Hermite case, we include a final illustration of Theorem 4.1 in Fig. 3, where we have meshed the covariance matrix for $k = 20$ and $k = 50$, for both $\gamma = 0.01$ and $\gamma = 0.99$; note that as k increases, the covariance matrix becomes more and more diagonally dominant (at $k = \infty$, the matrix should be diagonal). Also note that since k is relatively large, the plot is almost independent of γ.

5. Applications: level densities

We can compare the large β asymptotics to the theoretical answer for the distribution of a randomly chosen eigenvalue. For large n, this is the well-know semicircle law (for the Hermite ensembles) or equivalent thereof (for Laguerre ensembles), but we are interested in finite n.

We found that even for β small, the approximation can be quite reasonable.

We summarize the large β answer as a sum of Gaussians in Corollaries 5.1 and 5.2.

Corollary 5.1. *Let k be fixed, and $f_{k,\beta}$ be the level density of the scaled (by $1/\sqrt{2k\beta}$) $k \times k$ β-Hermite ensemble. Let $g_{k,\beta}$ be as below:*

$$g_{k,\beta}(x) = \frac{1}{k} \sum_{i=1}^{k} \frac{1}{\sqrt{2\pi}\sigma_i} \mathrm{e}^{-(x-\mu_i)^2/(2\sigma_i^2)},$$

where $\mu_i = h_i^{(k)}/\sqrt{2k}$ *and* $\sigma_i = (1/\sqrt{2k\beta})\sqrt{\mathrm{Var}(G_i)}$*, with* h_i *and* $\mathrm{Var}(G_i)$ *as in Section* 3*. Then for any* x*,*

$$\lim_{\beta\to\infty}\sqrt{\beta}\,\big(f_{k,\beta}(x) - g_{k,\beta}(x)\big) = 0.$$

Corollary 5.2. *Let* k *and* $\gamma > 0$ *be fixed, and* $f_{k,\beta,\gamma}$ *be the level density of the scaled* (*by* $1/(k\beta)$) $k\times k$ β*-Laguerre ensemble of parameter* $a = \frac{\beta}{2}(k-1+\gamma)$*. Let* $g_{k,\beta,\gamma}$ *be as below*:

$$g_{k,\beta,\gamma}(x) = \frac{1}{k}\sum_{i=1}^{k}\frac{1}{\sqrt{2\pi}\,\sigma_i}\,\mathrm{e}^{-(x-\mu_i)^2/(2\sigma_i^2)},$$

where $\mu_i = l_i^{(k)}/k$ *and* $\sigma_i = (1/k\sqrt{\beta})\sqrt{\mathrm{Var}(G_i)}$*, with* $l_i^{(k)}$ *and* $\mathrm{Var}(G_i)$ *as in Section* 4*. Then for any* x*,*

$$\lim_{\beta\to\infty}\sqrt{\beta}\,\big(f_{k,\beta,\gamma}(x) - g_{k,\beta,\gamma}(x)\big) = 0.$$

While these approximations are simple enough (a sum of Gaussians is an easily recognizable shape that is also easy to work with), one may wonder how big β has to be in order for these approximations to become "accurate" (for example, in order to *appear* accurate in a plot, the approximations have to be accurate to about 2–3 digits). We have found that, in either of the two cases, the answer is surprisingly low.

In the following two subsections, we have used only even integer values of β for our plots, because (in addition to $\beta = 1$) those are the only ones for which (to the best of our knowledge) there are exact formulas for the level densities. The plots were obtained with the help of our Maple Library, **MOPs** (*Multivariate Orthogonal Polynomials* (*symbolically*)), which was used for computing the orthogonal and Jack polynomial quantities involved; these were translated into polynomials which were then plotted in **MATLAB**. For a reference on **MOPs** see [6].

5.1. Level density plots: the Hermite case

In the following, we illustrate the accuracy of the sum of Gaussians approximation ($g_{k,\beta}$ from Corollary 5.1) for β relatively small (4 to 10) by plotting it against the true level density ($f_{k,\beta}$ from Corollary 5.1).

Fig. 4 plots the level density and approximation for the 4×4 Hermite case.

In Fig. 4, we let $k = 4$, and gradually increase β (from 4 to 10) to show how the approximation approaches the exact level density. For $\beta = 10$, the dots fall right on the curve.

If we plot the densities for $k = 7$ (as in Fig. 5), for $\beta = 6$ the approximation is already very close to the exact level density.

We can conclude that the approximation works well for low values of β, in the Hermite case.

5.2. Level densities: the Laguerre case

In the Laguerre case, we cut the parameter cube with two different slices, as explained below. For plotting purposes we have considered $k = 4$ in both.

In this story, there are two Laguerre densities: one for the eigenvalue p.d.f., that is, in the Laguerre ensemble density, and a second (different!) one for the Laguerre polynomial corresponding to the limiting level density, as $\beta\to\infty$. We call the first one p and the second one γ, and we hold each of them constant as $\beta\to\infty$, while varying the other one, as depicted in the table below. To further emphasize which of the two parameters, γ or p, we are keeping constant, we have used bold fonts.

Case (a). This case holds γ (and therefore the limiting Laguerre polynomial, whose roots are the limits of the scaled eigenvalues) constant as $\beta\to\infty$.

Note that both the Laguerre ensemble parameter $a = \frac{\beta}{2}(k+\gamma-1)$ and the power $p = \gamma\frac{\beta}{2} - 1$ are increasing functions of β.

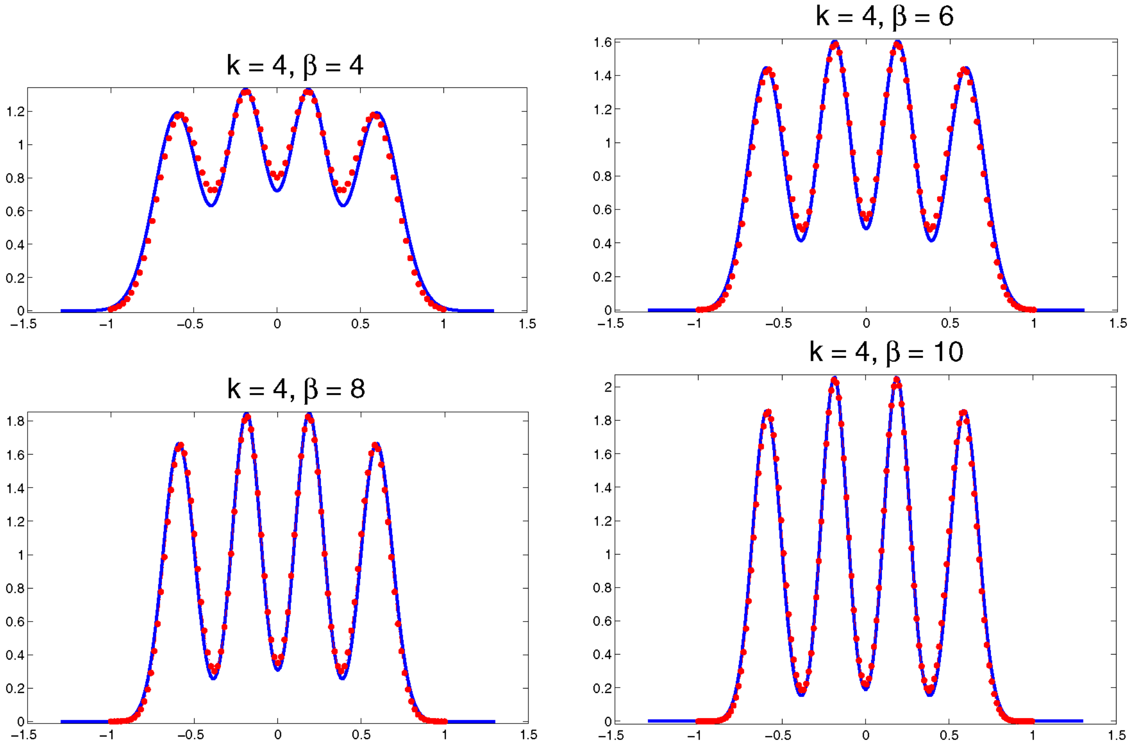

Fig. 4. Hermite case: sum of Gaussians approximation to the level densities (dots) and exact level densities (lines) for $k = 4$, and $\beta = 4, 6, 8, 10$.

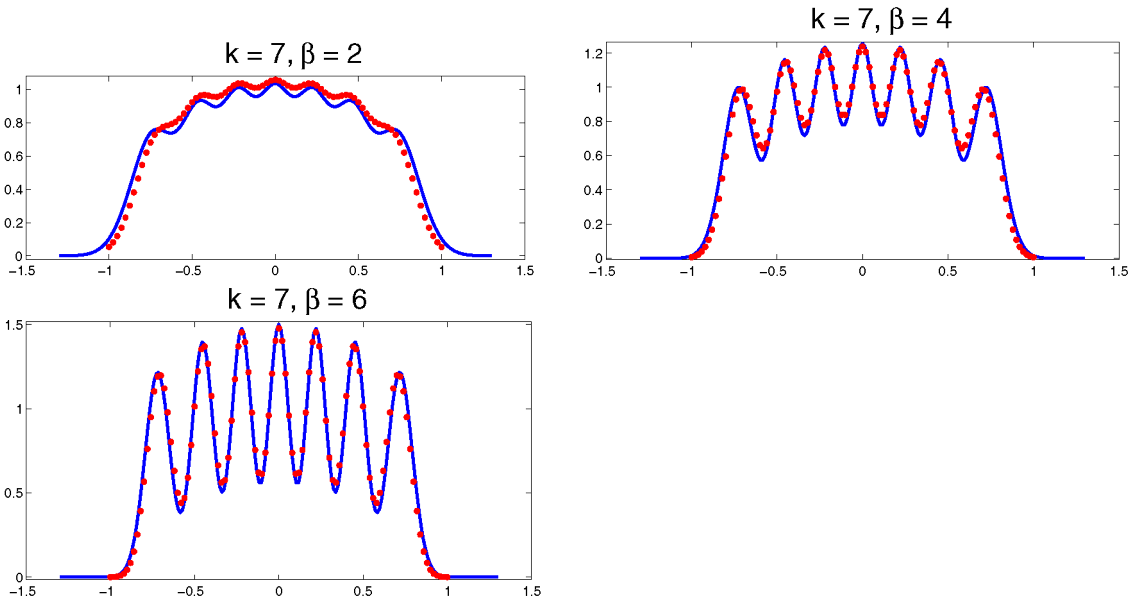

Fig. 5. Hermite case: sum of Gaussians approximation to the level densities (dots) and exact level densities (lines) for $k = 7$, and $\beta = 2, 4, 6$.

Table 2

	Fixed quantities	Variable	Other quantities	Eigenvalue p.d.f.	Limiting Laguerre polynomial
(a)	$k, \boldsymbol{\gamma}$	$\beta \to \infty$	$a = \frac{\beta}{2}(k + \boldsymbol{\gamma} - 1)$ $p = \frac{\beta}{2}\boldsymbol{\gamma} - 1$	$c\lvert\Delta\rvert^{\beta} \prod_{i=1}^{k} \lambda_i^{\frac{\beta}{2}\boldsymbol{\gamma}-1} \mathrm{e}^{-\lambda_i/2}$	$L_k^{\boldsymbol{\gamma}-1}(x)$
(b)	$k, \mathbf{p}$	$\beta \to \infty$	$a = \mathbf{p} + \frac{\beta}{2}(k-1)$ $\gamma = \frac{2}{\beta}(\mathbf{p}+1)$	$c\lvert\Delta\rvert^{\beta} \prod_{i=1}^{k} \lambda_i^{\mathbf{p}} \mathrm{e}^{-\lambda_i/2}$	$L_k^{-1}(x)$

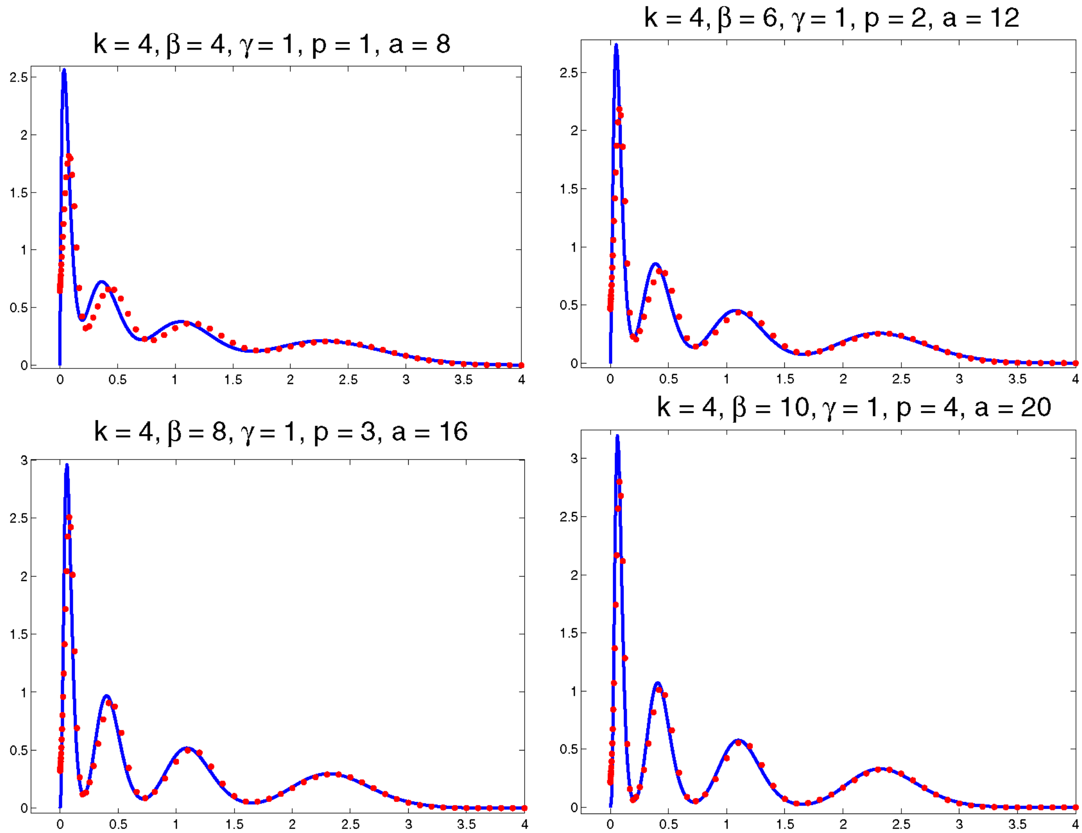

Fig. 6. Laguerre case (a): sum of Gaussians approximation to the level densities (dots) and exact level densities (lines) for $k = 4$, $\gamma = 1$, and $\beta = 4, 6, 8, 10$.

By prescribing γ, in the limit as $\beta \to \infty$ the plot should become a sum of delta functions at the roots of the Laguerre polynomial $L_k^{\gamma-1}(x)$.

In Fig. 6 we take $k = 4$, $\gamma = 1$, $\beta = 4, 6, 8, 10$, and $a = 8, 12, 16, 20$ (equivalently, $p = 1, 2, 3, 4$). Note that the approximation is very good for $\beta = 10$.

Case (b). This case holds the power p constant in the weight $|\Delta(\Lambda)|^{\beta} \prod_{i=1}^{k} \lambda_i^{p} \mathrm{e}^{-\lambda_i/2}$, thereby changing the parameter γ and the Laguerre polynomial. In this second test, as $\beta \to \infty$, $\gamma = \frac{2}{\beta}(p+1) \to 0$.

Thus as $\beta \to \infty$, the plot should become a sum of delta functions at the roots of the polynomial $L_n^{-1}(x)$.

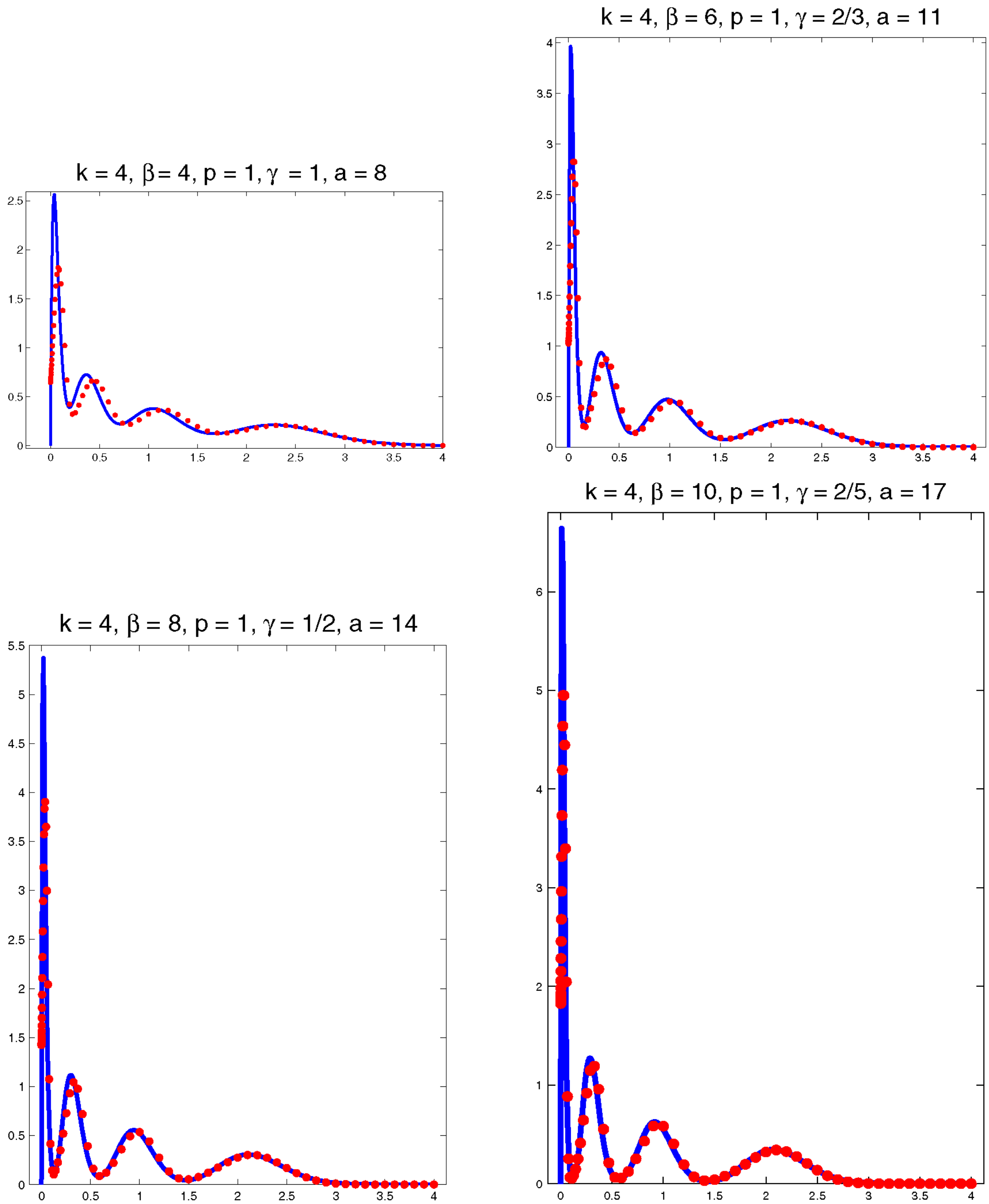

Fig. 7. Laguerre case (b): sum of Gaussians approximation to the level densities (dots) and exact level densities (lines) for $k = 4$, $p = 1$, and $\beta = 4, 6, 8, 10$.

The approximation works, once again, surprisingly well, as demonstrated by Fig. 7, where $n = 4$, $p = 1$, $\beta = 4, 6, 8, 10$, and $\gamma = 1, 2/3, 1/2, 2/5$ (or $a = 8, 11, 14, 17$).

Remark 5.3. Note that case (b), the smallest eigenvalue converges to 0 (which is the smallest root of the Laguerre polynomial $L_4^{-1}(x)$), and the presence of the delta function at 0 in the sum of Gaussians (Fig. 7) is very clearly visible.

Thus we can conclude that in both cases, a good approximation is obtained even for β relatively small.

Acknowledgements

The authors would like to thank the MIT-Singapore Alliance and the National Science Foundation for their support (NSF grant DMS9971591). Ioana Dumitriu's research was supported in part by a Miller Fellowship at U.C. Berkeley. She would like to thank Alexei Borodin for the useful comments and suggestions in presenting the results of this paper. Both authors would like to thank Brian Sutton for helpful discussions and numerical experiments. Last but not least, we would like to thank the referee for comments and suggestions that helped make this paper better.

References

[1] M. Abramowitz, I.A. Stegun (Eds.), Handbook of Mathematical Functions, Dover, New York, 1970.
[2] T. Baker, P. Forrester, The Calogero–Sutherland model and generalized classical polynomials, Commun. Math. Phys. 188 (1997) 175–216.
[3] J. Demmel, Applied Numerical Linear Algebra, SIAM, Philadelphia, PA, 1997.
[4] P. Diaconis, Patterns in eigenvalues: the 70th Josiah Willard Gibbs Lecture, Bull. Amer. Math. Soc. 40 (2003) 155–178.
[5] I. Dumitriu, A. Edelman, Matrix models for beta-ensembles, J. Math. Phys. 43 (2002) 5830–5847.
[6] I. Dumitriu, A. Edelman, G. Shuman, MOPS: Multivariate Orthogonal Polynomials (symbolically), 2004, Preprint found at lanl.arxiv.org/abs/math-ph/0409066.
[7] K. Johansson, On fluctuations of random hermitian matrices, Duke Math. J. 91 (1998) 151–203.
[8] K. Johansson, Shape fluctuations and random matrices, Commun. Math. Phys. 209 (2000) 437–476.
[9] I.M. Johnstone, On the distribution of the largest principal component, Ann. Statist. 29 (2) (2001) 295–327.
[10] M. Lal Mehta, Random Matrices, second ed., Academic Press, Boston, 1991.
[11] G. Szegö, Orthogonal Polynomials, fourth ed., American Mathematical Society, Providence, RI, 1975.
[12] J.W. Silverstein, Z.D. Bai, CLT of linear spectral statistics of large dimensional sample covariance matrices, Preprint, 2003, Ann. Probab., accepted for publication.
[13] C.A. Tracy, H. Widom, On orthogonal and symplectic matrix ensembles, J. Statist. Phys. 92 (1996) 809–835.
[14] C.A. Tracy, H. Widom, The distribution of the largest eigenvalue in the Gaussian ensembles, in: Calogero–Moser–Sutherland Models, in: CRM Ser. in Math. Phys., vol. 4, Springer-Verlag, 2000, pp. 461–472.