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A non-linear Riesz respresentation in probabilistic potential theory

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Abstract

For a nice Markov process such as Brownian motion on a bounded domain, we introduce a non-linear potential operator defined in terms of running suprema, and we prove a non-linear Riesz representation of a given function as the sum of a harmonic function and a non-linear potential. The proof involves a family of optimal stopping problems in analogy to the general construction of Bank and El Karoui [Ann. Probab. 32 (1B) (2004) 1030–1067], but here the analysis is carried out in terms of probabilistic potential theory.

Résumé

Pour un processus de Markov régulier, comme le mouvement Brownien dans un domaine borné, nous introduisons un opérateur potentiel non-linéaire défini à l'aide du maximum courant d'un processus et prouvons une représentation de Riesz non linéaire d'une fonction donnée, comme somme d'une fonction harmonique et d'un potentiel non linéaire. La preuve repose sur une famille de problèmes d'arrêt optimal, analogue à la construction qu'on peut trouver dans Bank et El Karoui [Ann. Probab. 32 (1B) (2004) 1030–1067], mais ici l'analyse est faite en termes de théorie du potentiel probabiliste.

MSC: 60J45; 60G40

1. Introduction

In the context of potential theory it is well known that a function u satisfying some strong regularity conditions admits a *Riesz representation*

$$u = Gf + h$$

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as the sum of a harmonic function h and the potential of some function f. The harmonic function is determined by the boundary behavior of u, and the function f can be reconstructed from the induced potential Gf or from u by a differentiation procedure:

$$f = DGf = Du.$$

In probabilistic terms, D can be described as the characteristic operator of the underlying Markov process, and the potential operator G takes the form

$$Gf(x) = E_x\left[\int_0^\zeta f(X_t)\,\mathrm{d}t\right].$$

In view of this probabilistic interpretation, let us now introduce the *non-linear potential operator* $\overline{G}$ defined by

$$\overline{G}f(x) := E_x\left[\int_0^\zeta \sup_{0\leqslant s\leqslant t} f(X_s)\,\mathrm{d}t\right].$$

Our purpose is to study the interplay between the subadditive operator $\overline{G}$ and a corresponding superadditive operator $\underline{D}$ which is a *derivator* in the sense of the non-linear potential theory developed by Dellacherie [7]; see also Mokobodzki [13]. In particular we are going to show that any function u satisfying some very mild regularity conditions admits a *non-linear Riesz decomposition* of the form

$$u = \overline{G}f + h, \tag{1}$$

where h is harmonic, and where f can be reconstructed from the induced non-linear potential $\overline{G}f$ or from u in terms of the derivator $\underline{D}$:

$$f = \underline{D}\,\overline{G}f = \underline{D}u. \tag{2}$$

The non-linear Riesz representation will follow as a corollary from a careful analysis of a family of optimal stopping problems. In potential-theoretic terms, we study the dependence of the functions

$$V_c u = cG1 + R(u - cG1)$$

on the parameter $c \in \mathbb{R}$, where Rv denotes the *réduite* of the function v, i.e., the smallest excessive function larger than v. Each function $V_c u$ is characterized as the smallest function v such that $v \geqslant u$ and $v \succ cG1$ in the strong order defined by the cone of excessive functions. In terms of the non-linear derivator $\underline{D}$, the function $V_c u$ can also be identified as the smallest function v such that $v \geqslant u$ and $\underline{D}v \geqslant c$. We show that the solution of these two equivalent minimization problems is given by the sum of a harmonic function induced by the boundary behavior of u and the non-linear potential

$$\overline{G}(\underline{D}u \vee c)(x) = E_x\left[\int_0^\zeta \sup_{0\leqslant s\leqslant t} \underline{D}u(X_s) \vee c\,\mathrm{d}t\right]. \tag{3}$$

In the limit $c \downarrow -\infty$ we obtain the non-linear Riesz representation (1) of the function u.

As a further consequence of the probabilistic representation (3), we can use the functions $V_c u$ as building blocks for the construction of a concave envelope of the process U defined by $U_t = u(X_t)$. In fact, the process C defined by

$$C_t := V_{\Gamma_t} u(X_t)$$

with

$$\Gamma_t := \sup_{0\leqslant s\leqslant t} \underline{D}u(X_s)$$

dominates the process U, has concave paths up to a martingale, and the derivatives of the concave part only increase at times when $C = U$. Recall that the Snell envelope of U can be viewed as the stochastic version of a decreasing envelope since it has decreasing paths up to a martingale and a point of decrease only occurs at times when $C = U$. In an analogous way, we may view the process C as the stochastic version of a concave envelope.

The crucial idea of studying the dependence of réduites on a parameter goes back to G. Mokobodzki and D. Heath as explained in [12]. It has also appeared in Whittle's construction [15] of Gittins indices for the multi-armed bandit problem. In both versions it has been a source of inspiration for the theory of Gittins indices in continuous time as developed by El Karoui and Karatzas [9–11]. Conversely, the methods developed in that context allow us to give a probabilistic interpretation of the results in [12], and they provide the key to the non-linear Riesz representation (1). Independently, a stochastic representation problem with a similar structure has appeared in the work of Bank [1] and Bank and Riedel [5] on singular optimization problems in intertemporal consumption choice, where it was solved explicitly for a class of Lévy processes. Combining these two developments, Bank and El Karoui [3] solved the representation problem and explored the idea of a concave envelope in a general semimartingale setting; see also [4] for a survey of the general theory and some of its applications.

In this paper, our purpose is to go "back to the roots" and to illustrate these recent developments in the classical setting of probabilistic potential theory. In particular we would like to point out the connection to non-linear potential theory, since we hope that it should be possible to go further in that direction.

2. A non-linear potential operator

Let $(X_t)_{t\geqslant 0}$ denote a strong Markov process with topological state space S and life time ζ, defined on a stochastic base $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, (P_x)_{x\in S})$ which satisfies the usual conditions. We use the notation

$$P_T f(x) := E_x\big[f(X_T);\ T < \zeta\big]$$

for any stopping time T. Let G denote the classical *potential operator* of the process, defined by

$$Gf(x) := \int_0^\infty P_t f(x)\,\mathrm{d}t = E_x\left[\int_0^\zeta f(X_t)\,\mathrm{d}t\right],$$

and recall that the *characteristic operator* D of the process is defined by

$$Du(x) := \lim \frac{u(x) - P_T u(x)}{E_x[T]}$$

for functions u such that the limit exists. Here the limit is taken along decreasing neighborhoods of x, and T denotes the exit time from such a neighborhood.

Let us now consider the subadditive potential operator $\overline{G}$ defined by

$$\overline{G} f(x) := E_x\left[\int_0^\zeta \sup_{0\leqslant s\leqslant t} f(X_s)\,\mathrm{d}t\right]. \tag{4}$$

The corresponding superadditive operator $\underline{D}$ is given by

$$\underline{D}u(x) := \inf \frac{u(x) - P_T u(x)}{E_x[T]}, \tag{5}$$

where the infimum is taken over exit times from open neighborhoods of x. Note that $\underline{D}$ is a "derivator" in the sense of Dellacherie [7], i.e.,

$$u \leqslant v \Longrightarrow \underline{D}u \leqslant \underline{D}v \quad \text{on } \{u = v\}. \tag{6}$$

Our aim is to prove existence and uniqueness of the non-linear Riesz representation (1) in terms of the non-linear operators $\overline{G}$ and $\underline{D}$.

For the ease of exposition, we do not insist on proving our results in the most general setting; this would involve some technical refinements such as the fine topology of the process, in analogy to the general constructions in Bank and El Karoui [3]. Instead, we introduce the following assumptions:

(A1) S is a locally compact metric space, and we denote by $\bar{S} = S \cup \{\Delta\}$ the Alexandrov compactification of S. Functions f on S will also be viewed as functions on $\bar{S}$ with $f(\Delta) := 0$.
(A2) The process $(X_t)_{t \geqslant 0}$ is a Hunt process in the sense of [6] XVI.11 such that $\lim_{t\uparrow\zeta} X_t = \Delta$. In particular it is quasi-leftcontinuous.
(A3) The excessive functions of the process are lower-semicontinuous, and the function g defined by

$$g(x) := E_x[\zeta] = G1(x) \tag{7}$$

is continuous and bounded.

Recall that a measurable function $f \geqslant 0$ on S is *excessive* if

$$P_t f \leqslant f \quad \text{and} \quad \lim_{t\downarrow 0} P_t f = f,$$

and that any excessive function is lower-semicontinuous if, for example, the process has the strong Feller property. In particular, our assumptions are satisfied for Brownian motion on a bounded domain, and in the sequel the reader could simply consider this special case.

Let us now be more precise as to the definition and the properties of the non-linear potential $\overline{G}f$.

Lemma 2.1. *Let f be an upper-semicontinuous function on S. Then the function $u = \overline{G}f$ defined by*

$$u(x) = \overline{G}f(x) := E_x\left[\int_0^\zeta \sup_{0\leqslant u\leqslant s} f(X_u)\,\mathrm{d}s\right],$$

satisfies

$$u \geqslant f \cdot g,$$

and for any $x \in S$ such that $\overline{G}f(x) \in \mathbb{R}$ we have

$$\lim_{t\uparrow\zeta} u(X_t) = 0 \quad P_x\text{-a.s. and in } \mathcal{L}^1(P_x). \tag{8}$$

Moreover, the function $\overline{G}f$ is excessive if $f \geqslant 0$, and it is lower-semicontinuous on $\{f \geqslant c\}$ for any $c \in \mathbb{R}$.

Proof. (1) Since

$$\left\{\sup_{0\leqslant s\leqslant t} f(X_s) \geqslant c,\ t < \zeta\right\} = \{D_c \leqslant t < \zeta\} \in \mathcal{F}_t,$$

where D_c denotes the first entrance time into the closed set $\{f \geqslant c\}$, the process

$$\sup_{0\leqslant s\leqslant t} f(X_s) I_{\{t<\zeta\}} \quad (t \geqslant 0)$$

is adapted. Thus,

$$\overline{G}f(x) = \int_0^\infty E_x\Big[\sup_{0\leqslant s\leqslant t} f(X_s) I_{\{t<\zeta\}}\Big]\,\mathrm{d}t$$

is well defined on $\{f > -\infty\}$ and satisfies

$$\overline{G} f(x) \geqslant f(x) g(x).$$

On $\{f = -\infty\}$ we use the definition

$$\overline{G} f(x) := \lim_{c \downarrow -\infty} E_x\left[\int_0^\zeta \sup_{0 \leqslant u \leqslant s} f(X_u) \vee c \,\mathrm{d}s\right],$$

and by monotone convergence we can write

$$\overline{G} f(x) = E_x\left[\int_0^\zeta \sup_{0 \leqslant s \leqslant t} f(X_s) \,\mathrm{d}t\right]$$

on $\{\overline{G} f > -\infty\}$.

(2) Since

$$P_t u(x) = E_x\left[\int_t^\zeta \sup_{t \leqslant u \leqslant s} f(X_u) \,\mathrm{d}s\right]$$

satisfies $P_t u \leqslant u$ and $\lim_{t \downarrow 0} P_t u = u$ for $f \geqslant 0$, the function $u = \overline{G} f$ is excessive as soon as the function f is non-negative.

(3) If f is bounded from below by $c \in \mathbb{R}$ then $\overline{G} f = \overline{G}(f - c) + cg$ is the sum of an excessive function and of a continuous function, hence lower-semicontinuous due to our assumption (A3).

(4) If $u(x) = \overline{G} f(x)$ is finite then, by dominated convergence,

$$u(X_t) = E_{X_t}\left[\int_0^\zeta \sup_{0 \leqslant u \leqslant s} f(X_u) \,\mathrm{d}s\right] = E_x\left[\int_t^\zeta \sup_{t \leqslant u \leqslant s} f(X_u) \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right]$$

converges to 0, both P_x-a.s. and in $\mathcal{L}^1(P_x)$, as $t \uparrow \zeta$. $\square$

Our aim is to show that, conversely, any function u satisfying some mild regularity conditions admits a unique representation in terms of the non-linear potential operator $\overline{G}$.

3. Uniqueness of the non-linear Riesz representation

In this section, we show that a non-linear Riesz representation of the form (1) is in fact unique.

Definition 3.1. Let us say that a measurable function u on S has *nice boundary behavior* if

$$\exists \lim_{t \uparrow \zeta} u(X_t) \quad P_x\text{-a.s. and in } \mathcal{L}^1(P_x)$$

for any $x \in S$. In this case we define

$$\widetilde{P}_T u(x) := E_x\big[u(X_T); T < \zeta\big] + E_x\Big[\lim_{t \uparrow \zeta} u(X_t); T = \zeta\Big]$$

for any stopping time $T \leqslant \zeta$.

Let $\mathcal{T}(x)$ denote the class of all exit times from relatively compact open neighborhoods of x, and let $\widetilde{\mathcal{T}}(x) \supseteq \mathcal{T}(x)$ denote the class corresponding to all open neighborhoods of x; in particular, we have $\zeta \in \widetilde{\mathcal{T}}(x)$. Recall that a continuous function h on S is *harmonic* if

$$h(x) = P_T h(x) \quad \forall T \in \mathcal{T}(x)$$

and that it is said to be *of class* (D) if, for any $x \in S$, the family $\{h(X_T) \mid T \in \mathcal{T}(x)\}$ is uniformly integrable with respect to P_x. A harmonic function h of class (D) has nice boundary behavior, and it is in fact determined by its boundary behavior:

$$h(x) = E_x\Big[\lim_{t\uparrow\zeta} h(X_t)\Big].$$

Moreover,

$$h(x) = \widetilde{P}_T h(x) \quad \forall T \in \widetilde{\mathcal{T}}(x). \tag{9}$$

Theorem 3.1. *Suppose that u is a real-valued function on S which admits a non-linear Riesz representation*

$$u = \overline{G} f + h, \tag{10}$$

where h is a harmonic function of class (D) and f is upper-semicontinuous. Such a decomposition (10) *is unique. More precisely, u has nice boundary behavior,*

$$h(x) = E_x\Big[\lim_{t\uparrow\zeta} u(X_t)\Big],$$

and

$$f(x) = \underline{D}u(x) := \inf_{T\in\mathcal{T}(x)} \frac{u(x) - P_T u(x)}{E_x[T]} = \inf_{T\in\widetilde{\mathcal{T}}(x)} \frac{u(x) - \widetilde{P}_T u(x)}{E_x[T]} \tag{11}$$

for any $x \in S$. In particular, the function $\underline{D}u$ is upper-semicontinuous. If f is bounded from below by $c \in \mathbb{R}$ then u is lower-semicontinuous and satisfies $u \geqslant cg + h$.

Proof. (1) Since $u(x)$ and $h(x)$ are finite by assumption, we have $\overline{G} f(x) < \infty$ for any $x \in S$. By Lemma 2.1,

$$\lim_{t\uparrow\zeta}\big(u(X_t) - h(X_t)\big) = 0, \quad P_x\text{-a.s. and in } \mathcal{L}^1(P_x).$$

Since h is a harmonic function of class (D), we obtain the existence of $\lim_{t\uparrow\zeta} u(X_t)$ and the identity

$$h(x) = E_x\Big[\lim_{t\uparrow\zeta} h(X_t)\Big] = E_x\Big[\lim_{t\uparrow\zeta} u(X_t)\Big].$$

(2) For any stopping time $T \in \widetilde{\mathcal{T}}(x)$ we have $h(x) = \widetilde{P}_T h(x)$, hence

$$\begin{aligned} u(x) - \widetilde{P}_T u(x) &= \overline{G} f(x) - \widetilde{P}_T \overline{G} f(x) \\ &= E_x\left[\int_0^T \sup_{0\leqslant s\leqslant t} f(X_s)\,\mathrm{d}t\right] + E_x\left[\int_T^\zeta \Big(\sup_{0\leqslant s\leqslant t} f(X_s) - \sup_{T\leqslant s\leqslant t} f(X_s)\Big)\,\mathrm{d}t;\, T < \zeta\right] \\ &\geqslant E_x\left[\int_0^T \sup_{0\leqslant s\leqslant t} f(X_s)\,\mathrm{d}t\right]. \end{aligned} \tag{12}$$

In particular,

$$u(x) - \widetilde{P}_T u(x) \geqslant f(x) E_x[T],$$

and this implies

$$\underline{D}u(x) \geqslant f(x).$$

(3) In order to prove the converse inequality $\underline{D}u(x) \leqslant f(x)$, we fix $\alpha > f(x)$ and define $T_\alpha \in \widetilde{\mathcal{T}}(x)$ as the exit time from the open neighborhood $\{f < \alpha\}$ of x. Since

$$\sup_{T_\alpha \leqslant s \leqslant t} f(X_s) = \sup_{0 \leqslant s \leqslant t} f(X_s)$$

for $t \in [T_\alpha, \zeta)$, the general inequality in (12) becomes an equality for $T := T_\alpha$. Thus

$$u(x) - \widetilde{P}_{T_\alpha} u(x) = E_x\left[\int_0^{T_\alpha} \sup_{0 \leqslant s \leqslant t} f(X_s)\,\mathrm{d}t\right] \leqslant \alpha E_x[T_\alpha],$$

and this implies

$$\inf_{T \in \widetilde{\mathcal{T}}(x)} \frac{u(x) - \widetilde{P}_T u(x)}{E_x[T]} \leqslant f(x). \tag{13}$$

(4) We have to show that the left-hand side of (13) coincides with $\underline{D}u(x)$. Let $(U_n)_{n \geqslant 1}$ be a sequence of relatively compact open sets increasing to S, and denote by S_n the exit time from U_n. For $T \in \widetilde{\mathcal{T}}(x)$, the stopping times $T_n := T \wedge S_n \in \mathcal{T}(x)$ increase to T, and so we have $E_x[T_n] \uparrow E_x[T]$. Moreover, since the process is quasi-left-continuous and u has nice boundary behavior,

$$\lim_n P_{T_n} u(x) = E_x\Big[\lim_n u(X_{T_n})\Big] = E_x\big[u(X_T); T < \zeta\big] + E_x\Big[\lim_{t \uparrow \zeta} u(X_t); T = \zeta\Big] = \widetilde{P}_T u(x).$$

Thus,

$$\frac{u(x) - \widetilde{P}_T u(x)}{E_x[T]} = \lim_n \frac{u(x) - P_{T_n} u(x)}{E_x[T_n]} \geqslant \underline{D}u(x)$$

for any $T \in \widetilde{T}(x)$, and this shows that both expressions for $\underline{D}u(x)$ in (11) coincide. □

4. A family of optimal stopping problems

Let u be a *continuous* function on S. We assume that u is of class (D), that it has nice boundary behavior, and that the function h defined by

$$h(x) := E_x\Big[\lim_{t \uparrow \zeta} u(X_t)\Big]$$

is continuous on S. Our aim is to prove the existence of a non-linear Riesz representation for u. Subtracting the harmonic function h of class (D), we may assume without loss of generality

$$\lim_{t \uparrow \zeta} u(X_t) = 0 \quad P_x\text{-a.s.} \tag{14}$$

for $x \in S$. Note that (14) and our convention $u(\Delta) := 0$ allow us to write

$$\widetilde{P}_T u(x) = E_x\big[u(X_T)\big] = E_x\big[u(X_T); T < \zeta\big]$$

for any stopping time $T \leqslant \zeta$.

As a first step in our construction, we study the following family of optimal stopping problems and the variation of the corresponding *réduites*. For each $c \in \mathbb{R}$ we define

$$V_c u(x) := \sup_{T \in \mathcal{T}(x)} E_x\big[u(X_T) + cT\big]. \tag{15}$$

In other words, $V_c u$ denotes the value function of the optimal stopping problem with parameter c which is defined by the right-hand side of (15). Note that $V_c u(x) < \infty$, due to our assumption (A3). Since $\zeta = T + \zeta \circ \theta_T$ for any stopping time $T \leqslant \zeta$, we have

$$g(x) = E_x[T] + E_x\big[g(X_T); T < \zeta\big], \tag{16}$$

hence

$$V_c u(x) = cg(x) + \sup_{T \in \mathcal{T}(x)} E_x\big[(u - cg)(X_T)\big]. \tag{17}$$

But the value function of the optimal stopping problem introduced on the right-hand side of (17) coincides with the *réduite* Ru_c of the continuous function $u_c := u - cg$, defined as the smallest excessive function v such that $v \geqslant u_c$; see, e.g., [14], Theorem III.1. Moreover, the supremum in (17) is attained by the first entrance time

$$D_c := \inf\{t \geqslant 0 \mid X_t \in A_c\} \leqslant \zeta$$

into the set

$$A_c := \{Ru_c = u_c\} = \{V_c u = u\}.$$

Note that A_c is closed since Ru_c is excessive, hence lower-semicontinuous due to our assumption (A3). These properties of the réduite are well known on various levels of generality; see, e.g., [8], Theorem 2.76. For the convenience of the reader we include a short proof in our present setting.

Lemma 4.1. *The réduite Ru_c of the function u_c is given by*

$$Ru_c(x) = E_x\big[u_c(X_{D_c})\big], \tag{18}$$

and it coincides with the value function of the optimal stopping problem in (17).

Proof. Define $A_{c,n} := \{Ru_c \leqslant (1 + 1/n)u_c\}$ and denote by $D_{c,n}$ the first entrance time into the closed set $A_{c,n}$. By Mokobodzki's theorem as explained in [12] or in [7], No. 16,

$$Ru_c = R\big((Ru_c)I_{A_{c,n}}\big).$$

Applying Hunt's balayage theorem as stated in [6], XIV.97 to the excessive function Ru_c on the right-hand side, we obtain

$$Ru_c(x) = E_x\big[Ru_c(X_{D_{c,n}})\big].$$

Using quasi-left-continuity of the process and our assumptions on u we obtain

$$\lim_n Ru_c(X_{D_{c,n}}) = u_c(X_{D_c}) \quad \text{in} \quad \mathcal{L}^1(P_x),$$

and this implies Eq. (18). In order to identify $Ru_c(x)$ as the value of the optimal stopping problem in (17), note first that the excessive function $v = Ru_c$ satisfies $v \geqslant P_T v \geqslant P_T u$ for any stopping time $T \in \mathcal{T}(x)$. On the other hand, we have just seen that for any $x \notin A_c$ the value $Ru_c(x)$ is approximated by stopping times $D_{c,n} \in \widetilde{\mathcal{T}}(x)$, and this remains true if we replace each $D_{c,n}$ by a suitable $T_n \in \mathcal{T}(x)$. For $x \in A_c$ we can approximate the value $Ru_c(x) = u_c(x)$ by any sequence of stopping times $T_n \in \mathcal{T}(x)$ decreasing to 0. $\square$

Combining the predecing lemma with Eq. (16), we see that the function $V_c u$ is given by

$$V_c u(x) = cg(x) + Ru_c(x) = E_x\big[u(X_{D_c}) + cD_c\big]. \tag{19}$$

In particular, $V_c u$ is lower-semicontinuous since it is the sum of a continuous and an excessive function. Clearly, $V_c u$ dominates the function $u \vee cg$, and it is excessive for $c \geqslant 0$ since the functions Ru_c and g are excessive. This implies $V_c u \geqslant R(u \vee cg)$ for $c \geqslant 0$, but in general there is no equality. The following lemma provides a characterization of $V_c u$ in terms of the *strong order* induced by the cone of excessive functions, i.e.,

$$v \succ w :\Longrightarrow v - w \quad \text{is excessive.}$$

Lemma 4.2. *$V_c u$ is the smallest function v such that $v \geqslant u$ and $v \succ cg$.*

Proof. The function $v := V_c u$ dominates u, and it satisfies $v \succ cg$ since $v - cg = Ru_c$ is excessive. Conversely, consider any function $\tilde v$ such that $\tilde v \geqslant u$ and $\tilde v \succ cg$. Then $\tilde v = cg + w$ for some excessive function w, and w satisfies

$$w = \tilde v - cg \geqslant u - cg = u_c$$

hence $w \geqslant Ru_c$. Thus,

$$\tilde v = cg + w \geqslant cg + Ru_c = V_c u. \qquad \square$$

We are now going to study the dependence of $V_c u(x)$ resp. $Ru_c(x)$ on the parameter $c \in \mathbb{R}$. This idea goes back to G. Mokobodzki and D. Heath, and for the rest of this section we follow their approach, as explained in Heath [12] in the special case where the function u is excessive. The next section will provide a new probabilistic interpretation of the potential theoretic results in [12]. It should be seen as a special case of the general discussion in Bank and El Karoui [3], and it will provide the key to our existence proof for the non-linear Riesz representation of the function u.

Lemma 4.3. *For each $x \in S$, $V_c u(x)$ is increasing and convex in c.*

Proof. Monotonicity is clear from the definition. Convexity follows from the representation (17), since the right-hand side may be viewed as the supremum of functions which are affine in c. $\square$

Since the function $c \mapsto V_c u(x)$ is convex, it is almost everywhere differentiable. We denote by $\partial^+ V_c u(x)$ and $\partial^- V_c u(x)$ the derivatives from the right and the left, respectively, and by $\partial V_c u(x)$ the derivative if it exists. The notation $\partial R_c u(x)$ will be used in the same way. Let us now analyze these derivatives in more detail. Since $V_c u$ is increasing in c and $V_c u \geqslant u$, the sets $A_c = \{V_c u = u\}$ are decreasing in c. Thus, the first entrance times D_c are increasing in c, and we write

$$D_{c-} := \lim_{a \uparrow c} D_a, \qquad D_{c+} := \lim_{b \downarrow c} D_b.$$

We can now describe the derivatives of $c \mapsto V_c u(x)$ and $c \mapsto Ru_c(x)$ in terms of the stopping times D_c.

Lemma 4.4. *For any $c \in \mathbb{R}$,*

$$E_x[D_{c-}] \leqslant \partial^- V_c u(x) \leqslant \partial^+ V_c u(x) \leqslant E_x[D_{c+}], \tag{20}$$

and for almost all $c \in \mathbb{R}$ we have

$$\partial V_c u(x) = E_x[D_c] \tag{21}$$

and

$$\partial Ru_c(x) = E_x[D_c] - g(x) = -E_x\big[g(X_{D_c})\big]. \tag{22}$$

Proof. Applying Eq. (18) with parameter b, we obtain

$$\begin{aligned} V_b u(x) - V_c u(x) &\leqslant (b-c)g(x) + E_x\big[u_b(X_{D_b})\big] - E_x\big[u_c(X_{D_b})\big] \\ &= (b-c)(g - P_{D_b} g)(x) \\ &= (b-c)E_x[D_b] \end{aligned}$$

for $b > c$, and in the same way we obtain

$$V_c u(x) - V_a u(x) \geqslant (c-a)E_x[D_a]$$

for $a < c$. This implies the inequalities in (20). But since $E_x[D_{c-}]$ and $E_x[D_{c+}]$ are the limits from the left and from the right of the increasing function $c \to E_x[D_c]$, we have $E_x[D_{c-}] = E_x[D_c] = E_x[D_{c+}]$ for almost all c. Thus the derivative $\partial V_c u(x)$ exists for almost all c and can be identified with $E_x[D_c]$. □

5. Existence of the non-linear Riesz representation

As in the previous section we assume that u is a continuous function on S of class (D) with boundary behavior (14). In view of Lemma 4.4,

$$V_b u(x) - V_a u(x) = \int_a^b E_x[D_c]\,\mathrm{d}c \tag{23}$$

and

$$Ru_a(x) - Ru_b(x) = \int_a^b E_x\big[g(X_{D_c})\big]\,\mathrm{d}c. \tag{24}$$

In order to analyze these expressions in more detail, we introduce the function γ defined by

$$\gamma(x) := \sup\{c \mid x \in A_c\} \tag{25}$$

and the increasing adapted process

$$\Gamma_t := \sup_{0 \leqslant s \leqslant t} \gamma(X_s) \quad (t \geqslant 0). \tag{26}$$

Note that γ is upper-semicontinuous since the sets A_c are closed, and that the paths of $(\Gamma_t)_{t \geqslant 0}$ are right-continuous.

Proposition 5.1. *For $a < b$,*

$$Ru_a(x) - Ru_b(x) = \overline{G}(\gamma \wedge b)(x) - \overline{G}(\gamma \wedge a)(x).$$

Proof. Due to (24) we have

$$Ru_a(x) - Ru_b(x) = \int_a^b E_x[\zeta - D_c]\,\mathrm{d}c = E_x\left[\int_a^b (\zeta - D_c)\,\mathrm{d}c\right].$$

Now note that

$$\{t < D_c < \zeta\} = \big\{V_c u(X_s) > u_c(X_s)\ \forall s \leqslant t, t < \zeta\big\} = \big\{\gamma(X_s) < c\ \forall s \leqslant t, t < \zeta\big\} = \{\Gamma_t < c, t < \zeta\}. \tag{27}$$

Thus,

$$\int_a^b (\zeta - D_c)\,\mathrm{d}c = \int_a^b \int_0^\zeta I_{\{D_c \leqslant s\}}\,\mathrm{d}s\,\mathrm{d}c = \int_a^b \int_0^\zeta I_{\{\Gamma_s \geqslant c\}}\,\mathrm{d}s\,\mathrm{d}c = \int_0^\zeta (\Gamma_s \wedge b - \Gamma_s \wedge a)\,\mathrm{d}s,$$

and so we get

$$Ru_a(x) - Ru_b(x) = E_x\left[\int_0^\zeta \Gamma_s \wedge b\,\mathrm{d}s\right] - E_x\left[\int_0^\zeta \Gamma_s \wedge a\,\mathrm{d}s\right] = \overline{G}(\gamma \wedge b)(x) - \overline{G}(\gamma \wedge a)(x). \qquad \square$$

Lemma 5.1. $\lim_{c\uparrow\infty} Ru_c(x) = 0$.

Proof. Let $(U_n)_{n\geqslant 1}$ be a sequence of relatively compact open sets U_n increasing to S, and define T_n as the exit time from U_n. Construct an increasing sequence (c_n) such that $0 \leqslant c_n \uparrow \infty$ and $u_{c_n} \leqslant 0$ on $\overline{U}_n$. Then

$$Ru_{c_n}(x) = E_x\left[u_{c_n}(X_{D_{c_n}})\right] \leqslant E_x\left[u_{c_n}(X_{D_{c_n}});\, D_{c_n} > T_n\right] \leqslant E_x\left[u^+(X_{D_{c_n} \vee T_n})\right],$$

and the right-hand side converges to 0 due to our assumption (14) on the boundary behavior of u. $\square$

Theorem 5.1. *For any $a \in \mathbb{R}$, the functions Ru_a and $V_a u$ can be represented as follows*:

$$Ru_a(x) = E_x\left[\int_0^\zeta (\Gamma_t - a)^+\,\mathrm{d}t\right] = \overline{G}(\gamma - a)^+(x), \tag{28}$$

and

$$V_a u(x) = E_x\left[\int_0^\zeta \Gamma_t \vee a\,\mathrm{d}t\right] = \overline{G}(\gamma \vee a)(x). \tag{29}$$

Proof. By Lemma 5.1,

$$Ru_a(x) = \lim_{b\uparrow\infty} E_x\left[\int_0^\zeta (\Gamma_t \wedge b - \Gamma_t \wedge a)\,\mathrm{d}t\right] = E_x\left[\int_0^\zeta (\Gamma_t - \Gamma_t \wedge a)\,\mathrm{d}t\right] = E_x\left[\int_0^\zeta (\Gamma_t - a)^+\,\mathrm{d}t\right]$$

and

$$V_a u(x) = ag(x) + Ru_a(x) = E_x\left[\int_0^\zeta \big((\Gamma_t - a)^+ + a\big)\,\mathrm{d}t\right] = E_x\left[\int_0^\zeta \Gamma_t \vee a\,\mathrm{d}t\right]. \qquad \square$$

The representation (29) allows us to identify the function $V_c u$ as the solution of a minimization problem defined in terms of the non-linear derivator $\underline{D}$.

Corollary 5.2. *Let $\mathcal{V}$ denote the class of all functions v admitting a non-linear Riesz representation. Then $V_c u$ can be characterized as the smallest function $v \in \mathcal{V}$ such that*

(i) $v \geqslant u$;
(ii) $\underline{D}v \geqslant c$.

Proof. The function $v := V_c u$ belongs to $\mathcal{V}$ and satisfies $v \geqslant u$ and $v = \overline{G}(\gamma \vee c)$, hence $\underline{D}v = \gamma \vee c \geqslant c$ due to Theorem 2.1. In order to show that $V_c u$ is minimal, consider a function $\tilde{v} \in \mathcal{V}$ such that $\tilde{v} \geqslant u$ and $\underline{D}\tilde{v} \geqslant c$. Thus, $\tilde{v} = \overline{G}\underline{D}\tilde{v} + \tilde{h}$, where $\tilde{h}$ is a non-negative harmonic function and $\underline{D}\tilde{v} = c + f$ for some lower-semicontinuous function $f \geqslant 0$. Thus,

$$\tilde{v} = \overline{G}c + \overline{G}f + \tilde{h} = c \cdot g + w$$

where $w := \overline{G}f + \tilde{h}$ is excessive, since $w \geqslant 0$ and $\underline{D}w = f \geqslant 0$. This implies $\tilde{v} \succ cg$, and the inequality $\tilde{v} \geqslant v = V_c$ now follows from Lemma 4.2. □

We will now derive the non-linear Riesz representation of the function u from the representation (29) of the functions $V_c u$, combined with the observation that $\gamma = \underline{D}u$.

Theorem 5.2. *Let u be a continuous function of class (D) on S such that*

$$\lim_{t \uparrow \zeta} u(X_t) = 0 \quad P_x\text{-a.s.}$$

Then, for any $x \in S$,

$$u(x) = E_x\left[\int_0^\zeta \sup_{0 \leqslant s \leqslant t} \underline{D}u(X_s)\,\mathrm{d}t\right] = \overline{G}\underline{D}u(x). \tag{30}$$

Proof. (1) For any $c \in \mathbb{R}$ we have $\gamma(x) \geqslant c$ if and only if $V_c u(x) = u(x)$. But this translates into the condition that

$$u(x) - E_x\big[u(X_T)\big] \geqslant c E_x[T]$$

for any stopping time $T \in \mathcal{T}(x)$, which is equivalent to $\underline{D}u(x) \geqslant c$. This shows that the two functions γ and $\underline{D}u$ are in fact identical.

(2) For $c \leqslant \gamma(x)$ we have

$$u(x) = V_c u(x) = E_x\left[\int_0^\zeta \Gamma_t\,\mathrm{d}t\right] = \overline{G}\gamma(x) = \overline{G}\underline{D}u(x),$$

due to (29) and part (1). Thus, the representation (30) holds for any $x \in S$ such that $\gamma(x) > -\infty$.

(3) Suppose that $\gamma(x) = -\infty$. Since

$$-\infty < u(x) \leqslant v(x) := \lim_{c \downarrow -\infty} V_c u(x) = E_x\left[\int_0^\zeta \sup_{0 \leqslant s \leqslant t} \gamma(X_s)\,\mathrm{d}t\right],$$

the entrance time into the set $\{\gamma > -\infty\}$ is P_x-a.s. equal to 0, and this implies

$$\lim_{c \downarrow -\infty} D_c = 0 \quad P_x\text{-a.s.} \tag{31}$$

Comparing Eq. (19) with the representation (29) of $V_c u(x)$, we see that

$$E_x\big[u(X_{D_c}) + cD_c\big] = E_x\left[\int_0^\zeta \Gamma_t \vee c\,\mathrm{d}t\right].$$

This implies

$$E_x\big[u(X_{D_c})\big] = E_x\left[\int_{D_c}^\zeta \Gamma_t\,\mathrm{d}t\right], \tag{32}$$

since

$$\int_0^{D_c} \Gamma_t \vee c \,\mathrm{d}t = cD_c.$$

In view of (31) and since u is of class (D), the left-hand side of (32) converges to $u(x)$, and we can use monotone convergence on the right-hand side to conclude $u(x) = v(x)$, as desired. □

Let us now return to the optimal stopping problem in (15), and let us replace the class $\mathcal{T}(x)$ by the class of all stopping times $T \leqslant \zeta$. The following characterization of the value and of the optimal stopping times in terms of the non-linear operator $\underline{D}$ is a special case of the results developed by Bank [2] in a general semimartingale setting.

Corollary 5.3. *For any $c \in \mathbb{R}$, the optimal stopping problem*

$$\sup_T E_x\big[u(X_T) + cT\big],$$

where the supremum is taken over all stopping times $T \leqslant \zeta$, is solved by the entrance time D_c into the closed set $\{\underline{D}u \geqslant c\}$, and its value coincides with

$$V_c u(x) = \overline{G}(\underline{D}u \vee c)(x).$$

More precisely, a stopping time T is optimal if and only if it satisfies the two conditions

$$D_c \leqslant T \leqslant D_{c+} \tag{33}$$

and

$$\sup_{0 \leqslant t \leqslant T} \underline{D}u(X_t) = \underline{D}u(X_T). \tag{34}$$

Proof. For any stopping time $T \leqslant \zeta$,

$$\begin{aligned} V_c u(x) &= cg(x) + Ru_c(x) \geqslant cg(x) + P_T Ru_c(x) \geqslant cg(x) + P_T u_c(x) = c\big(g(x) - P_T g(x)\big) + P_T u(x) \\ &= E_x\big[u(X_T) + cT\big]. \end{aligned}$$

But Lemma 4.1 shows that the value $V_c u(x)$ is attained by the stopping time $T = D_c$, and so D_c is optimal. On the other hand,

$$E_x\big[u(X_T) + cT\big] = cg(x) + E_x\big[u_c(X_T)\big]$$

for any stopping time $T \leqslant \zeta$, and due to (30) we can write

$$\begin{aligned} E_x\big[u_c(X_T)\big] &= E_x\Bigg[\int_T^{\zeta} \sup_{T \leqslant s \leqslant t} \big(\gamma(X_s) - c\big)\,\mathrm{d}t\Bigg] \leqslant E_x\Bigg[\int_T^{\zeta} \sup_{0 \leqslant s \leqslant t} \big(\gamma(X_s) - c\big)^+\,\mathrm{d}t\Bigg] \\ &\leqslant E_x\Bigg[\int_{D_{c+}}^{\zeta} \sup_{0 \leqslant s \leqslant t} \big(\gamma(X_s) - c\big)^+\,\mathrm{d}t\Bigg] = E_x\big[u_c(X_{D_{c+}})\big] \end{aligned}$$

since $D_{c+} := \lim_{b \downarrow c} D_b = \inf\{t \geqslant 0 \mid \gamma(X_t) > c\}$. Clearly, optimality of T is equivalent to the condition that both inequalities reduce to an equality. For the second inequality this is the case if and only if $T \leqslant D_{c+}$ P_x-a.s. The first inequality becomes an equality if and only if

$$\sup_{T \leqslant s \leqslant t} \big(\gamma(X_s) - c\big) = \sup_{0 \leqslant s \leqslant t} \big(\gamma(X_s) - c\big)^+ \quad \forall t \in [T, \zeta)$$

P_x-a.s. By upper-semicontinuity of γ, this is equivalent to

$$\gamma(X_T) = \sup_{0 \leqslant s \leqslant T} \gamma(X_s) \geqslant c,$$

and thus to the two conditions $T \geqslant D_c$ and (34). □

6. A pathwise concave envelope

Let u be a continuous function on S satisfying the conditions of the last section. If we observe the réduite $Ru = V_0 u$ of the function u along the paths of the Markov process X then we obtain the *Snell envelope* of the process $U := u(X)$. Recall that the Snell envelope is defined as the smallest supermartingale which dominates the process U. As suggested by the trivial case where X is a uniform motion to the right on the unit interval $[0, 1]$, the Snell envelope may be viewed as a stochastic analogue of the decreasing envelope of a function on $[0, 1]$. In a similar way, one can think of introducing a stochastic analogue of the concave envelope. Such an approach was developed by Bank and El Karoui [3] in a general semimartingale context. Let us now illustrate their construction in our present Markovian setting, where it involves the family of functions $V_c u$.

Consider the process C defined by

$$C_t := V_{\Gamma_t} u(X_t) \quad (t \geqslant 0).$$

Clearly, $C \geqslant U$, and C dominates the Snell enveloppe P_x-a.s. if $\gamma(x) \geqslant 0$. For $t \leqslant \zeta$,

$$C_t = E_{X_t}\left[\int_0^\zeta \sup_{0 \leqslant u \leqslant s} \gamma(X_u) \vee \Gamma_t \,\mathrm{d}s\right] = E_x\left[\int_t^\zeta \sup_{t \leqslant u \leqslant s} \gamma(X_u) \vee \Gamma_t \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] = E_x\left[\int_t^\zeta \Gamma_s \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] = M_t - A_t,$$

where the process M defined by

$$M_t := E_x\left[\int_0^\zeta \Gamma_s \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right]$$

is a martingale, and the process A defined by

$$A_t := \int_0^t \Gamma_s \,\mathrm{d}s$$

has convex paths with increasing derivatives Γ_t. Thus, the process C has concave paths up to a martingale, and may thus be viewed as the stochastic analogue of a concave majorant of the process U. Moreover, the process has the following minimality property:

Proposition 6.1. *The process Γ only increases at times where $C_t = U_t$:*

$$\int_0^\zeta (C_t - U_t) \,\mathrm{d}\Gamma_t = 0.$$

Proof. For any time t such that $C_t(\omega) > U_t(\omega)$, the value $c := \Gamma_t(\omega)$ satisfies $V_c u(X_t(\omega)) > u(X_t(\omega))$, hence $\gamma(X_t(\omega)) < c$. Thus, t is not a point of increase for the function $t \mapsto \Gamma_t(\omega)$, and so t does not belong to the support of the corresponding measure $\mathrm{d}\Gamma_t(\omega)$. □

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