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A general Choquet–Deny theorem for nilpotent groups

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Abstract

Let G be a locally compact second countable nilpotent group. Let μ be a probability measure on the Borel sets of G. We prove that any bounded continuous function h on G solution of the convolution equation

$$\forall g \in G, \quad \int_G h(gx)\,\mu(dx) = h(g)$$

verifies $h(gx) = h(g)$ for all $(g, x) \in G \times \operatorname{supp} \mu$.

Résumé

Soient G un groupe nilpotent localement compact à base séparable et μ une mesure de probabilité sur les boréliens de G. Nous montrons que toute fonction continue bornée h sur G solution de l'équation fonctionnelle

$$\forall g \in G, \quad \int_G h(gx)\,\mu(dx) = h(g)$$

admet pour période tous les points du support de μ.

Keywords: Random walk; Harmonic functions

1. Introduction

1.1. Let G be a locally compact second countable (lcsc) group with identity element e. Let μ be a probability measure on the Borel-σ-algebra $\mathcal{B}(G)$ of G. A bounded Borel function h on G is called μ-harmonic if it satisfies the mean value property

$$\forall g \in G, \quad h(g) = \int_G h(gx)\,\mu(dx).$$

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We denote by $\mathcal{H}_c$ the family of bounded continuous μ-harmonic functions. We said that a probability measure μ is *aperiodic* if the closed subgroup of G generated by the support of μ is equal to G. To prove the stated result, we are brought to show that, for all *aperiodic* probability μ, $\mathcal{H}_c$ is reduced to constant functions.

Blackwell [3] has proved the result for a discrete abelian group with a finite number of generators. Later Choquet and Deny [4] have extended it to all abelian groups.

A significant improvement of this result, with a very simple proof, is the following. Let $(S,+)$ be an algebraic abelian semi-group equipped with a σ-algebra $\mathcal{S}$. We assume that the application from $S\times S$ to S which sends (x,y) to $x+y$ is measurable. Let μ be a probability measure on $\mathcal{S}$. Then any bounded μ-harmonic function h satisfies: for all $x\in S$, for μ-almost every $y\in S$, $h(x+y)=h(x)$. (Consider the sequence of functions defined by:

$$u_1(x)=\int_S \big(h(x+y_1)-h(x)\big)^2\,\mu(dy_1)$$

and for all $n\geqslant 2$,

$$u_n(x)=\int_S \big(h(x+y_1+\cdots+y_n)-h(x+y_2+\cdots+y_n)\big)^2\,\mu(dy_1)\ldots\mu(dy_n).$$

One easily sees that the sequence $(u_n)_{n\geqslant 0}$ is increasing (Cauchy–Schwarz theorem) and, for all $x\in S$, the series $\sum_{n\geqslant 0}u_n(x)$ is a telescopic convergent series ($u_n(x)=\int_S h^2(x+y)\mu^n(dy)-\int_S h^2(x+y)\mu^{n-1}(dy)$). Hence the result.)

Dynkin and Malyutov [6] have proved that $\mathcal{H}_c$ is reduced to the constant functions for a discrete nilpotent group with a finite number of generators. Azencott [1] has obtained this result for a class of groups, containing the connected nilpotent groups, when μ is a *spread out* probability (i.e. there exists an integer $n\geqslant 1$ such that the n-fold convolution μ^{*n} of μ by itself is nonsingular with respect to a Haar measure on G). In [15] it is pointed out that, in the work of Azencott, the spread out assumption is only necessary for ensuring that any bounded μ-harmonic function is right uniformly continuous and Azencott's result extends to general μ if we restrict our study to bounded right continuous μ-harmonic functions. Guivarc'h [9] has proved the result for a class of groups, containing nilpotent groups, for which there exists a compact neighborhood V of the identity generating G such that the series $\sum n^a\mu(V^{n+1}-V^n)$ converges for an $a>0$. He also obtained the result for general aperiodic probability μ when G is a nilpotent group of order two. Avez [2] has showed the result for a group with nonexponential growth carrying an aperiodic probability with finite support. Other references on this subject, for other types of groups, are ([5,8,10–13,16]).

In this paper we are proving the result for a nilpotent group and for a *general* aperiodic probability μ.

2. Bounded harmonic function on nilpotent groups

2.1. Theorem. *Let G be a nilpotent locally compact second countable group. Let μ be an aperiodic probability measure on G. Then any continuous bounded μ-harmonic function on G is constant.*

The remaining of this section is devoted to the demonstration of this theorem.

Preliminaries

2.2. Let m be a right Haar measure on G. For all $f\in\mathbb{L}^\infty(G,m)$ and all $\alpha\in\mathbb{L}^1(G,m)$ the function $f^\alpha: g\to\int_G f(x\,g)\,\alpha(x)\,m(dx)$ is *left uniformly continuous* (l.u.c.) on G. That is,

$$\sup_{x\in G}\big|f^\alpha(g\,x)-f^\alpha(x)\big|\leqslant\|f\|_{\mathbb{L}^\infty(G,m)}\big\|\alpha(\cdot\,g^{-1})-\alpha(\cdot)\big\|_{\mathbb{L}^1(G,m)},$$

and $\delta(g) = \|\alpha(\cdot\, g^{-1}) - \alpha(\cdot)\|_{\mathbb{L}^1(G,m)}$ is a continuous function on G satisfying $\delta(e) = 0$. f^α is μ-harmonic when f is. Let $(\alpha_n)_{n\geqslant 0}$ be a sequence of functions on G forming an approximate identity in $\mathbb{L}^1(G,m)$. That is, for all $n \geqslant 0$, α_n is a continuous non-negative function, satisfying $\int_G \alpha_n(x)\, m(dx) = 1$ and with a support contained in a compact neighborhood V_n of e such that $\bigcap_{n\geqslant 0} V_n = \{e\}$. Then, for all continuous and bounded function f on G, f^{α_n} converges uniformly on compact sets to f. We shall denote by $\mathcal{H}_{l.u.c.}$ the family of all l.u.c. μ-harmonic function on G. To show the theorem it is enough to prove that any element of $\mathcal{H}_{l.u.c.}$ is constant.

2.3. We denote by λ the probability measure $\sum_{k\geqslant 0} 2^{-(k+1)}\mu^{*k}$. Any μ-harmonic function is also a λ-harmonic function. The support $\operatorname{supp}\lambda$ of the probability λ is equal to the closure T_μ of the semigroup $\bigcup_{k\in\mathbb{N}}(\operatorname{supp}\mu)^k$. Replacing, if necessary, μ by λ we can suppose that the support of μ is the semigroup T_μ of G, all the convolutions μ^{k*}, $k \geqslant 1$, are equivalent measures and $\mu^{k*}(\{e\}) > 0$.

If H_1 and H_2 are two subgroups of G we denote by $[H_1, H_2]$ the closed subgroup of G generated by the commutators $[x, y] = xyx^{-1}y^{-1}$, $(x, y) \in H_1 \times H_2$. We call r the nilpotence order of G and we denote by

$$G_0 = G \supset G_1 = [G, G] \supset \cdots \supset G_{r+1} = [G, G_r] = \{e\}$$

the lower central series of G. For all $q \in \{1, \ldots, r+1\}$, we call π_q the natural map from G onto the quotient group G/G_q.

2.4. By the right random walk of law μ we mean the sequence of products

$$X_0 = e \quad \text{and} \quad \forall n \geqslant 1,\ X_n = Y_1 \cdots Y_n;$$

where $(Y_n)_{n\geqslant 1}$ is a sequence of i.i.d. G-valued random variables, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, whose common distribution is μ.

Proof of Theorem 2.1

Let $h \in \mathcal{H}_{l.u.c.}$. We denote by H the period group of h; i.e. the closed normal subgroup of G defined by:

$$H = \big\{g \in G\colon \forall (x, y) \in G^2\colon\ h(xgy) = h(xy)\big\}.$$

We reason ad absurdum, assuming that H is not equal to G.

If π is the natural map from G onto G/H, we have $h = \tilde{h} \circ \pi$ for a left uniformly continuous $\pi(\mu)$-harmonic function on G/H. Replacing the triplet (G, μ, h) by $(G/H, \pi(\mu), \tilde{h})$ we can suppose that $H = \{e\}$.

First stage. We will need the following important lemma.

2.5. Lemma. *Let H be a closed normal subgroup of G and π the natural map from G onto G/H. Let μ be a probability measure on the Borel sets of G. The following assertions are equivalent*:

- (i) *There exists a Borel subset W of G^2 with $\mu \otimes \mu\,(W) = 1$ such that for μ-almost every u, (u, u) belongs to W and for all $(u, v) \in W$ verifying $\pi(u) = \pi(v)$, we have $u = v$.*
- (ii) *There exists a Borel map $s : G/H \to G$ such that for μ-almost every u, $u = s \circ \pi(u)$.*
- (iii) *There exists a Borel subset V of G of μ-measure 1 such that for all $(u, v) \in V^2$ verifying $\pi(u) = \pi(v)$ we have $u = v$.*

Proof. The only nonobvious implication is (i) ⇒ (ii).

From hypothesis on W, it follows that there exists a Borel set V with $\mu(V) = 1$ such that for any $u \in V$ there exists a Borel set W_u of μ-measure 1, such that $u \in W_u$, $\{u\} \times W_u \subset W$ and $W_u \times \{u\} \subset W$.

For all $u \in G$ we denote by $\bar{u}$ the element $\pi(u)$ of G/H. We consider a disintegration of μ along the classes modulo H. For all bounded Borel functions f on G and g on G/H, we can write,

(i) $\int_G f(u)g(\bar{u})\mu(du) = \int_G Pf(\bar{u})g(\bar{u})\mu(du)$,

where P is a transition probability from G/H to G (i.e. an application from $G/H \times \mathcal{B}(G)$ to $[0,1]$ satisfying the two following conditions

(ii) for all $\bar{u} \in G/H$, $P(\bar{u}, \cdot)$ is a probability measure on the Borel sets of G,
(iii) for all Borel set B of G, the map $\bar{u} \to P(\bar{u}, B)$ is $\mathcal{B}(G/H)$-measurable.)

We choose a sequence $(\mathcal{P}_n)_{n\geqslant 0}$ of Borel countable partitions of G/H such that: each member of $\mathcal{P}_1$ is bounded; for all $n \in \mathbb{N}$, each member of $\mathcal{P}_n$ is the union of some subfamily of $\mathcal{P}_{n+1}$ and

$$\lim_{n\to+\infty} \sup\{\operatorname{diam}(A):\ A \in \mathcal{P}_n\} = 0.$$

If g is a bounded Borel function on G/H, we know [7, Theorems 2.8.19 and 2.9.8] that for all $\pi(\mu)$-almost every $y \in G/H$, $\forall n \in \mathbb{N}$, $\pi(\mu)(A_{n,y}) > 0$ and

$$\lim_{n\to+\infty}\left(\int_{A_{n,y}} g(x)\pi(\mu)(dx)/\pi(\mu)(A_{n,y})\right) = g(y);$$

where, for all $n \in \mathbb{N}$, $A_{n,y}$ is the member of the partition $\mathcal{P}_n$ containing y. (Remark that the σ-algebras generated by the partitions are increasing. If we know that the union of these σ-algebras generate the Borel σ-algebra, this result is a consequence of the convergence μ-a.e of the martingale $(\mathbb{E}_\mu[f \mid \sigma(\mathcal{P}_n)])_{n\in\mathbb{N}}$ to $\mathbb{E}_\mu[f \mid \mathcal{B}(G)] = f$.)

Let f be a non-negative function of $C_K(G)$, the space of continuous function on G with compact support. We consider the Borel subset U, with $\mu(U) = 1$, defined by:

$$U = \left\{u \in V:\ \forall n \in \mathbb{N},\ \pi(\mu)(A_{n,\bar{u}}) > 0 \text{ and} \lim_{n\to+\infty}\left(\int_{A_{n,\bar{u}}} Pf(x)\pi(\mu)(dx)/\pi(\mu)(A_{n,\bar{u}})\right) = Pf(\bar{u})\right\}.$$

Let $u_0 \in U$ and $\varepsilon > 0$. As any probability measure on a polish topological space is regular [14, Proposition II-7-3] and the Borel subsets $A_{n,\bar{u}_0}$, $n \in \mathbb{N}$, are decreasing, we can find a decreasing sequence $(K_n)_{n\geqslant 0}$ of compact subsets of G such that $K_n \subset \pi^{-1}(A_{n,\bar{u}_0}) \cap W_{u_0}$ and $\mu(K_n) \geqslant (1-\varepsilon)\,\pi(\mu)(A_{n,\bar{u}_0})$. Then we have:

$$\int_G f(u)1_{A_{n,\bar{u}_0}}(\bar{u})\mu(du)/\pi(\mu)(A_{n,\bar{u}_0}) \geqslant (1-\varepsilon)\int_G f(u)1_{K_n}(u)\mu(du)/\mu(K_n).$$

However $\bigcap_{n\in\mathbb{N}} K_n = \{u_0\}$ (if x belongs to $\bigcap_{n\in\mathbb{N}} K_n$, then $\bar{x} = \bar{u}_0$ and $x \in K_0 \subset W_{u_0}$, thus $x = u_0$). For all open balls $B(u_0, r)$ of center u_0 and radius $r > 0$, it follows that $\bigcap_{n\in\mathbb{N}}(K_n \cap B(u_0,r)^c) = \emptyset$ and there exists $p \in \mathbb{N}$ such that $K_p \cap B(u_0,r)^c = \emptyset$. We deduce that, for all $r > 0$,

$$\lim_{n\to+\infty}\int_{B(u_0,r)^c} 1_{K_n}(u)\mu(du)/\mu(K_n) = 0$$

and therefore

$$\lim_{n\to+\infty}\int_G f(u)1_{K_n}(u)\mu(du)/\mu(K_n) = f(u_0).$$

By what precedes, it results that for all $u_0 \in U$, $f(u_0) \leqslant Pf(\bar{u}_0)$. As $\int_G f(u)\mu(du) = \int_G Pf(\bar{u})\mu(du)$ and $C_K(G)$ is separable for the uniform norm, it follows that, there exists a Borel set X with $\mu(X) = 1$ such that, for all $u \in X$: $\forall f \in C_K(G)$, $f(u) = Pf(\bar{u})$ and therefore $P(\bar{u}, \cdot) = \delta_u$. One can assume that X is a countable union of compact subsets; then $\pi(X)$ is a Borel set of G/H. For any $y \in \pi(X)$ denote by $s(y)$ the support of the Dirac measure $P(y, \cdot)$. For all Borel subsets B of G, we have

$$\pi(X) \cap s^{-1}(B) = \pi(X) \cap \{1_B \circ s = 1\} = \pi(X) \cap \{P1_B = 1\}$$

which shows that s is a Borel map from X to G. We extend s to a Borel map from G/H to G.

From equalities (i) above, it follows that on one hand, for $\pi(\mu)$-almost every $y \in G/H$, $y = \pi(s(y))$ and on the other hand $s(\pi(\mu)) = \mu$. Consequently, for μ-almost all $u \in G$, $u = s(\pi(u))$. □

Second stage. By downward induction we prove that for all $q \in \{1, \ldots, r+1\}$ there exists a Borel subset V_q of G with $\mu(V_q) = 1$ such that $V_q V_q^{-1} \cap G_q = \{e\}$.

For $q = r+1$, we can take $V_{r+1} = G$. Let us assume the result true for some $q \in \{2, \ldots, r+1\}$. Consider the right random walk $(X_n)_{n \geqslant 0}$ on G. We denote by $\mathcal{F}_0$ the trivial σ-algebra $\{\emptyset, \Omega\}$ and, for all $n \geqslant 1$, $\mathcal{F}_n$ the σ-algebra generated by the random variables $Y_1, \ldots, Y_n$. For all $g \in G$, the sequence of random variables $(h(gX_n))_{n \geqslant 0}$ is a bounded martingale with respect to the filtration $(\mathcal{F}_n)_{n \geqslant 0}$. Therefore it converges $\mathbb{P}$-a.e. and in norm $\mathbb{L}^s(\Omega, \mathcal{F}, \mathbb{P})$, for all $s \in [1, +\infty[$;

$$\forall n \geqslant 0,\ \forall g \in G, \quad h(g) = \mathbb{E}\big[h(gX_n)\big] = \mathbb{E}\big[\lim_{p \to +\infty} h(gX_p)\big].$$

Moreover we have:

$$\sum_{n \in \mathbb{N}} \int_{T_\mu} \mathbb{E}\big[\big(h(gX_n x) - h(gX_n)\big)^2\big]\mu(dx) \leqslant \sum_{n \in \mathbb{N}} \big(\mathbb{E}\big[h^2(gX_{n+1})\big] - \mathbb{E}\big[h^2(gX_n)\big]\big) \leqslant \|h\|_\infty^2 < +\infty.$$

It follows that for all $g \in G$ and for μ-almost every $x \in G$, the sequences $(h(gX_n(\omega)x))_{n \geqslant 0}$ and $(h(gX_n(\omega)))_{n \geqslant 0}$ converge, for $\mathbb{P}$-a.e. $\omega \in \Omega$, to the same limit.

As h is left uniformly continuous, we deduce that there exists a Borel set U, with $\mu(U) = 1$, such that, for all $(g, x) \in G \times U$, the sequences $(h(gX_n(\omega)x))_{n \geqslant 0}$ and $(h(gX_n(\omega)))_{n \geqslant 0}$ converge, for $\mathbb{P}$-a.e. $\omega \in \Omega$, to the same limit.

Now let us define:

$$W = \big\{(u, v) \in U \times U\colon \text{ for } \mu\text{-almost all } x \in G, uxv(vxu)^{-1} \in V_q V_q^{-1}\big\}.$$

As μ^3 and μ are equivalent measures, the Borel subset W of G^2 has $\mu \otimes \mu$-measure 1. In the same way, for μ-almost all $u \in V_q$, $(u, u) \in W$ and for all $(u, v) \in W$, for μ-almost all t, $(ut, vt) \in W$.

For $(u, v) \in W$ such that $\pi_{q-1}(u) = \pi_{q-1}(v)$, we have $\mathbb{P}$-a.e.,

$$\begin{aligned}\lim_n h(uX_n) &= \lim_n h(uX_n v) = \lim_n h\big((uX_n v)(vX_n u)^{-1} vX_n u\big)\\ &= \lim_n h(vX_n u) = \lim_n h(vX_n),\end{aligned}$$

because $(uX_n v)(vX_n u)^{-1} = [uv^{-1}, vX_n] \in V_q V_q^{-1} \cap G_q = \{e\}$ and therefore

$$h(u) = \mathbb{E}\big[\lim_n h(uX_n)\big] = \mathbb{E}\big[\lim_n h(vX_n)\big] = h(v).$$

From the continuity of h and the last property of W, it follows that

$$\forall (u, v) \in W \text{ such that } \pi_{q-1}(u) = \pi_{q-1}(v),\ \forall t \in T_\mu,\ h(ut) = h(vt).$$

Applying this result to the left translates $h^g : x \to h(gx)$ $(g \in G)$ of h, we obtain

$$\forall g \in G,\ \forall (u, v) \in W \text{ such that } \pi_{q-1}(u) = \pi_{q-1}(v),\ \forall t \in T_\mu,\ h(gut) = h(gvt).$$

Now setting $H=\{x\in G\colon \forall g\in G,\ \forall t\in T_\mu,\ h(gxt)=h(gt)\}$, we obtain a closed subgroup of G. From next Lemma 2.6, it follows that H is a normal subgroup of G contained in the period group of h. Consequently $u^{-1}v$ is a period of h and $u=v$. Then, from Lemma 2.5, we deduce that there exists a Borel subset V_{q-1} of G, with $\mu(V_{q-1})=1$ and $V_{q-1}V_{q-1}^{-1}\cap G_{q-1}=\{e\}$.

Last stage. Finally, for $q=1$, we obtain a Borel subset V_1 with $\mu(V_1)=1$ such that $V_1V_1^{-1}=e$. It follows that $T_\mu T_\mu^{-1}=e$. However $T_\mu T_\mu^{-1}$ is a subgroup [1, Lemme IV 11] dense in G because μ is aperiodic, hence $G=\{e\}$ and we end to a contradiction.

2.6. Lemma. *Let G be a lcsc nilpotent group and H a subgroup of G. For all $t\in G$,*

$$t^{-1}Ht\subset H \quad\Rightarrow\quad t^{-1}Ht=H.$$

Proof. We proceed by induction on the order of nilpotence r of G. If $r=1$ the property is trivial. Assume the property true for a nilpotent group of order $r\geqslant 1$ and let G a nilpotent group of order $r+1$.

We take again the notations of Section 2.3 and note π instead of π_{r+1}. We have $\pi(t)^{-1}\pi(H)\pi(t)\subset\pi(H)$. However $\pi(H)$ is a closed subgroup of the nilpotent group G/G_{r+1} of order r. Therefore $\pi(t)^{-1}\pi(H)\pi(t)=\pi(H)$.

Consequently, if $y\in H$, then there exists $x\in H$ such that $\pi(y)=\pi(t^{-1}xt)$. It follows that $z=y^{-1}t^{-1}xt$ belongs to $G_{r+1}\cap H$ and therefore $y=t^{-1}xtz^{-1}=t^{-1}xz^{-1}t$. Hence the result. □

3. Other result

3.1. Definitions. An element g of G is said *recurrent* if for all neighborhood V of g, $\sum_{n\geqslant 0}1_V(X_n)=+\infty$, $\mathbb{P}$-a.e.. If g is recurrent then, for $\mathbb{P}$-almost all $\omega\in\Omega$, the sequence $(X_n(\omega))_{n\geqslant 0}$ admits g as a closure value. It is well-known that (see for example [17, Chapter 3, §4]):

(i) If one element of G is recurrent then all the elements of G are recurrent. In this case we say that the random walk $(X_n)_{n\geqslant 0}$ on G is *recurrent*. In the contrary case, we say that the random walk $(X_n)_{n\geqslant 0}$ on G is *transient*.
(ii) The random walk $(X_n)_{n\geqslant 0}$ on G is recurrent if and only if $\sum_{n\geqslant 0}\mu^{*n}(V)=+\infty$ for all nonempty open set V.
(iii) The random walk $(X_n)_{n\geqslant 0}$ on G is transient if and only if $\sum_{n\geqslant 0}\mu^{*n}(V)<+\infty$ for all relatively compact open set V.

Let H be a lcsc group on which G acts by automorphisms. For all $g\in G$, we denote by $A(g)$ the automorphism of H associated to g. We call $\bar{A}(G)$ the closure of the subgroup $A(G)=\{A(g)\colon g\in G\}$ in the group $\mathrm{Aut}(H)$ of all automorphisms of H. We shall say that the action of the pair (G,μ) on H is *recurrent* if the random walk $(A(X_n))_{n\geqslant 0}$ is recurrent in $\bar{A}(G)$. In other words, for all $u\in H$, the set of closure values of the sequence $(A(X_n)u)_{n\geqslant 0}$ is the closure of $A(G)u$, $\mathbb{P}$-a.e.

3.2. Theorem. *Suppose that the pair (G,μ) acts, by interior automorphisms, in a recurrent way on the subgroup $H=[G,G]$. Then $\mathcal{H}_c$ is reduce to constants.*

Proof. It is enough to prove the result for $h\in\mathcal{H}_{l.u.c.}$. We denote by π the natural map from G to G/H. We consider the Borel set U of second stage of the proof of Theorem 2.1. For all $(g,x)\in G\times U$, the sequences $(h(gX_n(\omega)x))_{n\geqslant 0}$ and $(h(gX_n(\omega)))_{n\geqslant 0}$ converge, for $\mathbb{P}$-a.e. $\omega\in\Omega$, to the same limit. For $u,v\in U$ such that $\pi(u)=\pi(v)$, we have, $\mathbb{P}$-a.e.,

$$\lim_n h(uX_n) = \lim_n h(uX_nv) = \lim_n h\big(uX_nvu^{-1}u\big)$$
$$= \lim_n h\big((uX_nvu^{-1}X_n^{-1}u^{-1})uX_nu\big) = \lim_n h(vX_nu) = \lim_n h(vX_n).$$

(Remark that $(uX_nvu^{-1}X_n^{-1}u^{-1})uX_nu$ and vX_n are two elements of G such that

$$\big|h\big((uX_nvu^{-1}X_n^{-1}u^{-1})uX_nu\big) - h(vX_nu)\big| \leqslant \|h\|_\infty \delta\big((uX_nvu^{-1}X_n^{-1}u^{-1})uv^{-1}\big)$$

that converges to zero along a subsequence.)

It follows that for all $(u, v) \in U^2$ verifying $\pi(u) = \pi(v)$,

$$h(u) = \mathbb{E}\big[\lim_n h(uX_n)\big] = \mathbb{E}\big[\lim_n h(vX_n)\big] = h(v).$$

We deduce that there exists a measurable function $\tilde{h}$ on G/H such that $\forall u \in U, h(u) = \widetilde{h}(\pi(u))$ and

$$\forall u \in \pi(U), \quad \tilde{h}(u) = \int_{G/H} \tilde{h}(ug)\, \pi(\mu)(dg).$$

As G/H is an abelian group, by taking again the arguments quoted in the introduction to obtain a generalization of the Choquet–Deny theorem in abelian semi-groups, we obtain, for all $n \geqslant 1$,

$$u_n = \int_{G/H} \big(\tilde{h}(y_1 + y_2 + \cdots + y_n) - \tilde{h}(y_2 + \cdots + y_n)\big)^2\, \mu(dy_1) \ldots \mu(dy_n) = 0.$$

It follows that $\tilde{h}(\cdot) = \tilde{h}(e)$, μ-a.e., then $h(\cdot) = h(e)$, μ-a.e. and finally $\forall u \in T_\mu, h(u) = h(e)$. The application of this equality to the left translates h^g: $x \to h(gx)$ $(g \in G)$ of h, gives us the required result. □

3.3. Example. $G = \mathbb{R} \times \mathbb{R}^d$ with product $(a, x)(b, y) = (a + b, x + e^a y)$. Denote by π the projection on $\mathbb{R}$. A pair (G, μ) acts on $\mathbb{R}^d$, in a recurrent way, if and only if the random walk $\pi(X_n) = \pi(Y_1) + \cdots + \pi(Y_n)$ is recurrent. This assumption is satisfied when $\mathbb{E}[|\pi(Y_1)|] < +\infty$ and $\mathbb{E}[\pi(Y_1)] = 0$. If the random variables $\pi(Y_1)$ and $-\pi(Y_1)$ have the same distribution, this assumption is satisfied [8] if $\lim_{t\to+\infty} t\mathbb{P}[\{\pi(X_1) > t\}] = 0$.

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