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A model of fracture for elliptic problems with flux and solution jumps ☆

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Abstract

A new model of fracture for elliptic problems combining flux and solution jumps as immersed boundary conditions is proposed and proved to be well-posed. An application of this model to the flow in fractured porous media is also proposed including the cases of "impermeable fracture" and "fully permeable fracture" satisfying the so-called "cubic law", as well as intermediate cases. A finite volume scheme on general polygonal meshes is built to solve such problems. Since no unknown is required at the fracture interface, the scheme is as cheap as standard schemes for the same problems without fault. The convergence of the scheme can be proved to the weak solution of the problem. With weak regularity assumptions, we also establish for the discrete H_0^1 and L^2 norms some error estimates in $\mathcal{O}(h)$, where h is the maximum diameter of the control volumes of the mesh. ***To cite this article: Ph. Angot, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Un modèle de fracture pour des problèmes elliptiques avec sauts de flux et de solution. Pour des problèmes elliptiques, un nouveau modèle de fracture combinant des sauts de la solution et du flux comme conditions aux limites immergées est proposé et on montre qu'il est bien posé. On propose également une application de ce modèle à l'écoulement dans des milieux poreux fissurés incluant les cas de « fracture imperméable » et de « fracture totalement perméable » satisfaisant la « loi cubique », ainsi que des cas intermédiaires. Un schéma en volumes finis est construit pour résoudre de tels problèmes sur des maillages polygonaux généraux. Comme aucune inconnue n'est nécessaire sur l'interface de fracture, ce schéma est aussi économique que des schémas standards pour résoudre les mêmes problèmes sans faille. On peut prouver la convergence de ce schéma vers la solution faible du problème. De plus, avec des hypothèses faibles de régularité, on établit pour la norme discrète H_0^1 et pour la norme L^2 des estimations d'erreur en $\mathcal{O}(h)$, où h désigne le pas du maillage, i.e. le diamètre maximum des volumes finis du maillage. ***Pour citer cet article : Ph. Angot, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

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Lorsqu'on s'intéresse par exemple au calcul d'écoulements monophasiques (une seule phase fluide) ou polyphasiques, avec transport massique d'un soluté ou transfert de chaleur, dans des milieux poreux très hétérogènes et fissurés, l'un des problèmes importants est de quantifier l'influence des fractures sur la dispersion de polluants, cf. les revues récentes [6,9]. Ces études trouvent naturellement de nombreuses applications dans des domaines d'intérêt industriel ou environnemental : géologie des aquifères, géothermie des bassins sédimentaires, ingénierie pétrolière, voire même en géophysique (volcanisme, sismique, tectonique)... Les études environnementales concernent en particulier le transport massique par advection–diffusion–réaction et la dispersion de polluants dans des bassins sédimentaires : transport de contaminants chimiques ou d'isotopes radioactifs lors du stockage de déchets... L'objectif de ce travail est de contribuer à la modélisation pertinente et efficace de fractures, vis-à-vis des phénomènes de transfert hydrodynamique ou d'advection–diffusion–réaction.

On propose ici un nouveau modèle de fracture *réduit à l'interface de faille* pour des problèmes elliptiques qui permet, à une échelle macroscopique, aux traces de la solution et du flux normal d'être vraiment *discontinues* à travers cette interface Σ immergée dans le domaine de calcul $\widetilde{\Omega}$. Dans la Section 1, on montre que ce modèle (1)–(5) est bien posé dans un domaine $\Omega \equiv \widetilde{\Omega} \setminus \Sigma$ polygonal, non localement situé d'un seul côté de sa frontière. La formulation faible (6) de ce problème, assortie des hypothèses d'ellipticité (A1)–(A4), satisfait la propriété de coercivité sur $H^1(\Omega)$. Ceci assure, pour ce problème linéaire, l'unicité de solutions a priori. Des résultats techniques, cf. les Lemmes 1.1, 1.2, pour les traces de fonctions de $H^1(\Omega)$ sur la frontière du domaine permettent de montrer le Théorème 1.3 d'existence et d'unicité d'une solution faible du modèle de fracture, qui est continue vis à vis des données. On propose également une application de ce modèle à l'écoulement dans des milieux poreux fissurés couplant sauts de pression et de vitesse normale sur Σ. Le modèle inclut alors les cas de « fracture imperméable » constituant alors des « barrières de perméabilité » [1,2], de « fracture totalement perméable » satisfaisant la « loi cubique » [6,9] pour un écoulement de Hele–Shaw le long de la fracture, ainsi que des cas intermédiaires.

Dans la Section 2, on construit par une généralisation des schémas proposés dans [1–3], une méthode aux volumes finis pour discrétiser le modèle de fracture (1)–(5) sur des maillages généraux polygonaux admissibles, cf. Définition 2.1. Ce schéma VF (14)–(19) est spécialement conçu pour éviter l'introduction d'inconnues supplémentaires sur l'interface Σ, [4]. Il est donc aussi économique que les schémas usuels pour les problèmes elliptiques sans fracture [7]. On peut prouver la convergence forte dans $L^2(\Omega)$ de la solution discrète du schéma vers la solution faible du problème, quand le pas $h_{\mathcal{T}}$ du maillage $\mathcal{T}$ tend zéro, cf. Théorème 2.4, à l'aide d'une généralisation (Lemme 2.3) d'un lemme de traces discrètes et de techniques introduites dans [7]. En outre, avec des hypothèses faibles de régularité, on établit pour la norme discrète H_0^1 et pour la norme L^2 des estimations d'erreur en $\mathcal{O}(h_{\mathcal{T}})$, cf. Théorème 2.5.

1. A well-posed elliptic model of fracture with both flux and solution jumps

Let the domain $\widetilde{\Omega} \subset \mathbb{R}^d$ ($d = 2$ or 3 in practice) be an open bounded polygonal set, $\widetilde{\Gamma} \equiv \partial\widetilde{\Omega}$ being its boundary, which includes a polygonal interface $\Sigma \subset \mathbb{R}^{d-1}$. Let us define the open bounded set Ω such that $\widetilde{\Omega} = \Omega \cup \Sigma$ and its boundary $\Gamma \equiv \partial\Omega = \widetilde{\Gamma} \cup \Sigma$. It is always possible to prolong Σ within a polygonal interface $\widetilde{\Sigma} \supset \Sigma$ which divides the domain $\widetilde{\Omega}$ into two disjointed subdomains Ω^- and Ω^+ such that $\widetilde{\Omega} = \Omega^- \cup \widetilde{\Sigma} \cup \Omega^+$. We denote respectively by χ^- and χ^+ the characteristic functions of Ω^- and Ω^+. Let $\mathbf{n}$ be, either the outward unit normal vector on $\widetilde{\Gamma}$, or the unit normal vector on Σ oriented from Ω^- to Ω^+. For a function ψ in $L^2(\widetilde{\Omega})$, let ψ^+ and ψ^- be the traces of ψ on each side of Σ, $\overline{\psi}|_\Sigma = (\psi^+ + \psi^-)/2$ the arithmetic mean of traces of ψ, and $[\![\psi]\!]_\Sigma = (\psi^+ - \psi^-)$ the jump of traces of ψ on Σ oriented by $\mathbf{n}$.

1.1. A model with jump immersed boundary conditions at the fracture interface

For the data $f \in L^2(\Omega)$, $u_\infty, \phi \in L^2(\widetilde{\Gamma}_N)$, g, h and U given in $L^2(\Sigma)$, we consider the second-order elliptic problem for the real-valued function u defined in Ω and including *fracture immersed boundary conditions* on Σ

which assume jumps of both the normal diffusive flux vector $\boldsymbol{\varphi}(u)\cdot\mathbf{n}\equiv-(\mathbf{a}\cdot\nabla u)\cdot\mathbf{n}$ and the solution u traces through the interface Σ:

$$-\nabla\cdot(\mathbf{a}\cdot\nabla u)+bu=f \quad \text{in } \Omega, \tag{1}$$

$$u=0 \quad \text{on } \widetilde{\Gamma}_D, \qquad \text{meas}(\widetilde{\Gamma}_D)>0 \quad \text{with } \widetilde{\Gamma}\equiv\partial\widetilde{\Omega}=\widetilde{\Gamma}_D\cup\widetilde{\Gamma}_N,\ \widetilde{\Gamma}_D\cap\widetilde{\Gamma}_N=\emptyset, \tag{2}$$

$$-(\mathbf{a}\cdot\nabla u)\cdot\mathbf{n}=\kappa(u-u_\infty)-\phi \quad \text{on } \widetilde{\Gamma}_N, \tag{3}$$

$$[\![(\mathbf{a}\cdot\nabla u)\cdot\mathbf{n}]\!]_\Sigma=\alpha(\bar{u}|_\Sigma-U)-h \quad \text{on } \Sigma, \tag{4}$$

$$\overline{(\mathbf{a}\cdot\nabla u)\cdot\mathbf{n}}|_\Sigma=\beta\,[\![u]\!]_\Sigma-g \quad \text{on } \Sigma, \tag{5}$$

where the symmetric second-order tensor of diffusion $\mathbf{a}\equiv(a_{ij})_{1\leqslant i,j\leqslant d}$, the reaction coefficient b, and κ are measurable and bounded functions verifying classical ellipticity assumptions:

$$\mathbf{a}\in\left(L^\infty(\Omega)\right)^{d^2};\ \exists a_0>0,\ \forall\boldsymbol{\xi}\in\mathbb{R}^d,\quad \mathbf{a}(x)\cdot\boldsymbol{\xi}\cdot\boldsymbol{\xi}\geqslant a_0|\boldsymbol{\xi}|^2 \quad \text{a.e. in } \Omega. \tag{A1}$$

$$b\in L^\infty(\Omega);\ \exists b_0\geqslant 0,\quad b(x)\geqslant b_0 \quad \text{a.e. in } \Omega. \tag{A2}$$

$$\kappa\in L^\infty(\widetilde{\Gamma}_N);\ \exists\kappa_0\geqslant 0,\quad \kappa(x)\geqslant\kappa_0 \quad \text{a.e. on } \widetilde{\Gamma}_N. \tag{A3}$$

The *transfer coefficients* $\alpha,\ \beta$ on Σ satisfy also the following ellipticity assumption:

$$\alpha,\beta\in L^\infty(\Sigma);\ \exists\beta_0\geqslant 0,\quad \alpha(x),\beta(x)\geqslant\beta_0 \quad \text{a.e. on } \Sigma. \tag{A4}$$

When $\alpha,\ h\equiv 0$ in (4), this problem reduces to the model of an insulating fracture with solution jumps, already proposed in [1,2], where the inverse of β can be defined as the *fracture resistance* $\rho=1/\beta$ through the interface Σ. When $\alpha\equiv 0$ and $\beta\equiv 0$ on Σ, the conditions (4), (5) degenerate into the non-homogeneous bi-Neumann boundary condition on Σ: $\overline{\boldsymbol{\varphi}(u)\cdot\mathbf{n}}|_\Sigma=g$ with $[\![\boldsymbol{\varphi}(u)\cdot\mathbf{n}]\!]_\Sigma=h$. The conditions (4), (5) written with ρ instead of β yields the *perfect transmission condition* when $\alpha,h\equiv 0$ and $\rho\equiv 0$ on Σ, see [1,2].

1.2. *Global solvability of the imperfect transmission problem*

We begin by some technical results for the traces of functions of $H^1(\Omega)$ on $\partial\Omega=\widetilde{\Gamma}\cup\Sigma$.

Lemma 1.1 (Traces on $\widetilde{\Gamma}$ and Σ in $H^1(\Omega)$). *If $\widetilde{\Omega}$ is an open bounded polygonal subset of $\mathbb{R}^d$ including a polygonal surface $\Sigma\subset\mathbb{R}^{d-1}$ such that $\widetilde{\Omega}=\Omega\cup\Sigma$ and $\partial\Omega=\partial\widetilde{\Omega}\cup\Sigma$, hereafter called the configuration hypothesis $(\mathcal{H})$, then we can define the following trace applications $\tilde{\gamma},\ \gamma^+,\ \gamma^-$ for all v in $H^1(\Omega)$ by*: $\tilde{\gamma}(v)\equiv v|_{\widetilde{\Gamma}}$, $\gamma^+(v)\equiv v^+|_\Sigma$, $\gamma^-(v)\equiv v^-|_\Sigma$. *They are linear and continuous from $H^1(\Omega)$ into $L^2(\widetilde{\Gamma})$ or $L^2(\Sigma)$, respectively, i.e., $\exists\tilde{c}(\Omega),\ c^+(\Omega),\ c^-(\Omega)>0$ such that* $\forall v\in H^1(\Omega)$:

$$\|v|_{\widetilde{\Gamma}}\|_{L^2(\widetilde{\Gamma})}\leqslant\tilde{c}(\Omega)\|v\|_{H^1(\Omega)},\quad \|v^+|_\Sigma\|_{L^2(\Sigma)}\leqslant c^+(\Omega)\|v\|_{H^1(\Omega)},\quad \|v^-|_\Sigma\|_{L^2(\Sigma)}\leqslant c^-(\Omega)\|v\|_{H^1(\Omega)}.$$

Lemma 1.2 (Average and jump of traces on Σ in $H^1(\Omega)$). *If $(\mathcal{H})$ holds, then $\exists c(\Omega)>0$ such that*:

$$\|\bar{v}|_\Sigma\|_{L^2(\Sigma)}\leqslant c(\Omega)\|v\|_{H^1(\Omega)},\quad \left\|[\![v]\!]_\Sigma\right\|_{L^2(\Sigma)}\leqslant c(\Omega)\|v\|_{H^1(\Omega)},\quad \forall v\in H^1(\Omega).$$

We now define the Hilbert space: $W=\{v\in L^2(\widetilde{\Omega}),\ \text{and } v|_\Omega\in H^1(\Omega);\ \tilde{\gamma}(v)\equiv v|_{\widetilde{\Gamma}}=0 \text{ on } \widetilde{\Gamma}_D\}$, equipped with the usual inner product and associated norm in $H^1(\Omega)$. Then we prove the following proposition by considering the weak formulation of the problem (1)–(5) in Ω as below: *Find $u\in W$ such that* $\forall v\in W$,

$$\int_\Omega(\mathbf{a}\cdot\nabla u)\cdot\nabla v\,\mathrm{d}x+\int_\Omega b\,uv\,\mathrm{d}x+\int_{\widetilde{\Gamma}_N}\kappa\,uv\,\mathrm{d}s+\int_\Sigma\alpha\,\bar{u}|_\Sigma\bar{v}|_\Sigma\,\mathrm{d}s+\int_\Sigma\beta\,[\![u]\!]_\Sigma[\![v]\!]_\Sigma\,\mathrm{d}s$$
$$=\int_\Omega f\,v\,\mathrm{d}x+\int_{\widetilde{\Gamma}_N}(\phi+\kappa\,u_\infty)\,v\,\mathrm{d}s+\int_\Sigma g\,[\![v]\!]_\Sigma\,\mathrm{d}s+\int_\Sigma(h+\alpha\,U)\,\bar{v}|_\Sigma\,\mathrm{d}s. \tag{6}$$

Theorem 1.3 (Existence and uniqueness for the fracture model). *If the assumptions* $(\mathcal{H})$ *and* (A1)–(A4) *hold, the problem* (1)–(5) *with* $g, h, U \in L^2(\Sigma)$, $u_\infty, \phi \in L^2(\widetilde{\Gamma}_N)$ *and* $f \in L^2(\Omega)$ *has a unique weak solution* $u \in W$ *satisfying* (6) *for all* $v \in W$, *such that*: $\exists \alpha_0(\Omega, a_0, b_0, \beta_0, \kappa_0) > 0$,

$$\|u\|_W \leqslant \frac{1}{\alpha_0}\big(\|f\|_{L^2(\Omega)} + \tilde{c}(\Omega)\,\|\phi + \kappa\, u_\infty\|_{L^2(\widetilde{\Gamma}_N)} + c(\Omega)\big(\|g\|_{L^2(\Sigma)} + \|h + \alpha\, U\|_{L^2(\Sigma)}\big)\big).$$

1.3. Application to a model of flow in fractured porous media

The problem (1)–(5) can represent the flow in fractured porous media using the Darcy law, where u stands for the pressure p and $\boldsymbol{\varphi}(u)\cdot\mathbf{n}$ stands for the normal component of the filtration velocity $\mathbf{v}\cdot\mathbf{n}$ as:

$$\nabla\cdot\mathbf{v} = 0 \quad \text{in } \Omega, \tag{7}$$

$$\mathbf{v} = -\frac{1}{\mu}\mathbf{K}\cdot(\nabla p - \varrho\,\mathbf{g}) \quad \text{in } \Omega, \tag{8}$$

$$p = 0 \quad \text{on } \widetilde{\Gamma}_D, \tag{9}$$

$$\mathbf{v}\cdot\mathbf{n} = \kappa(p - p_\infty) + V \quad \text{on } \widetilde{\Gamma}_N, \tag{10}$$

$$[\![\mathbf{v}\cdot\mathbf{n}]\!]_\Sigma = -\alpha\,(\bar{p}|_\Sigma - P) + h, \quad \text{on } \Sigma, \tag{11}$$

$$\overline{\mathbf{v}\cdot\mathbf{n}}|_\Sigma = -\beta[\![p]\!]_\Sigma + g, \quad \text{on } \Sigma, \tag{12}$$

where $\mathbf{K}$ is the permeability tensor of the porous medium, μ and ϱ the dynamic viscosity and the density of the fluid, respectively, and $\mathbf{g}$ the gravity vector. Other mathematical models can be found in [8,10]. The present fracture model allows intermediate cases between the fully permeable fracture and the fully impermeable one. In particular, an "impermeable fracture", already proposed in [1–3], can be characterized by $\alpha \equiv 0$, $g, h \equiv 0$ on Σ and the fracture hydraulic resistance estimated as follows: $\rho(s)|_\Sigma \equiv 1/\beta(s) = (\mu b_f(s))/K_f(s)$, for $s \in [0, S_\Sigma]$, where b_f is the aperture of the "fracture layer", K_f its transversal permeability and s denotes the longitudinal coordinate along the fracture interface Σ. Moreover, a "fully permeable fracture" can be characterized by $\rho \equiv 0$, $g, h \equiv 0$ on Σ and using, for example, the so-called "cubic law" [6,9] for a Hele–Shaw flow along the fracture, it gives for $p(s=0)|_\Sigma = P \equiv 0$: $\alpha(s)|_\Sigma = b_f(s)^2/(12\mu s)$, for $s \in\,]0, S_\Sigma]$.

2. Finite volume approximation on general meshes

For sake of simplicity, we restrict here to homogeneous Dirichlet boundary conditions on $\widetilde{\Gamma}$, i.e., $\widetilde{\Gamma}_D \equiv \widetilde{\Gamma}$. Moreover, we only consider here the case of a scalar coefficient variable in space $a(x)$ satisfying (A1).

Definition 2.1 (*Admissible meshes*). Let $\widetilde{\Omega} = \Omega \cup \Sigma$ be an open bounded polygonal subset of $\mathbb{R}^d$, $d = 2$, or 3 satisfying $(\mathcal{H})$ in Lemma 1.1. An admissible finite volume mesh of Ω, denoted by $\mathcal{T}$, is given by a family of disjointed "control volumes" K, which are open polygonal convex subsets of Ω, a family of edges or sides σ, subsets of $\overline{\Omega}$ contained in hyperplanes of $\mathbb{R}^d$, denoted by $\mathcal{E}$ and a family of points of Ω denoted by $\mathcal{P}$ satisfying the following properties: (i) The closure of the union of all the control volumes is $\overline{\widetilde{\Omega}}$, i.e., $\bigcup_{K\in\mathcal{T}} \overline{K} = \overline{\widetilde{\Omega}}$. (ii) For any $K \in \mathcal{T}$, there exists $\mathcal{E}_K \subset \mathcal{E}$ such that $\partial K = \overline{K} \setminus K = \bigcup_{\sigma\in\mathcal{E}_K} \bar{\sigma}$. Furthermore, $\mathcal{E} = \bigcup_{K\in\mathcal{T}} \mathcal{E}_K$. (iii) For any $(K, L) \in \mathcal{T}^2$ with $K \neq L$, either $\overline{K} \cap \overline{L}$ is $\emptyset$, a vertex, or $\overline{K} \cap \overline{L} = \bar{\sigma}$ for some $\sigma \equiv K|L \in \mathcal{E}$. (iv) The family $\mathcal{P} = (x_K)_{K\in\mathcal{T}}$ is such that $x_K \in \overline{K}$ (for all $K \in \mathcal{T}$) and, if $\sigma = K|L$, it is assumed that $x_K \neq x_L$, and that the straight line $\mathcal{D}_{K,L}$ going through x_K and x_L is orthogonal to $K|L$. (v) There exists $\mathcal{E}_\Sigma \subset \mathcal{E}$ such that $\overline{\Sigma} = \bigcup_{\sigma\in\mathcal{E}_\Sigma} \bar{\sigma}$. (vi) For any $\sigma \in \mathcal{E}$ such that $\sigma \subset \partial\Omega \equiv \partial\widetilde{\Omega} \cup \Sigma$, let K be the control volume such that $\sigma \in \mathcal{E}_K$. If $x_K \notin \sigma$, let $\mathcal{D}_{K,\sigma}$ be the straight line going through x_K and orthogonal to σ, then $y_\sigma = \mathcal{D}_{K,\sigma} \cap \sigma \neq \emptyset$. The mesh size is defined by: $h_{\mathcal{T}} = \sup\{\mathrm{diam}(K),\ K \in \mathcal{T}\}$. For any $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}$, $m(K) = \int_K \mathrm{d}x$ and $m(\sigma) = \int_\sigma \mathrm{d}s$. The set of interior (resp. boundary) edges is denoted by $\mathcal{E}_{\mathrm{int}}$ (resp. $\mathcal{E}_{\mathrm{ext}}$), that is $\mathcal{E}_{\mathrm{int}} = \{\sigma \in \mathcal{E};\ \sigma \not\subset \partial\widetilde{\Omega}\}$ (resp. $\mathcal{E}_{\mathrm{ext}} = \{\sigma \in \mathcal{E};\ \sigma \subset \partial\widetilde{\Omega}\}$). If $\sigma = K|L$, we denote by $d_\sigma > 0$ the Euclidean distance between x_K and x_L and by $d_{K,\sigma}$ the distance from x_K to σ; hence we have $d_\sigma = d_{K,\sigma} + d_{L,\sigma}$. If $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\mathrm{ext}}$, let $d_\sigma = d_{K,\sigma}$.

For each $K \in \mathcal{T}$ or $\sigma \in \mathcal{E}_\Sigma$, define the discrete values of the coefficients by their mean values over K or σ, i.e., for a function ψ in $L^2(\widetilde{\Omega})$ denoting a, b or f and for λ in $L^2(\Sigma)$ denoting α, ρ, g, h or U, we have:

$$\psi_K = \frac{1}{m(K)} \int_K \psi(x)\,\mathrm{d}x, \quad \text{and} \quad \lambda_\sigma = \frac{1}{m(\sigma)} \int_\sigma \lambda(x)\,\mathrm{d}s(x). \tag{13}$$

Lemma 2.2 (Discrete Poincaré inequality). *Let us define $X(\mathcal{T}) \subset L^2(\Omega)$ as the set of functions from Ω to $\mathbb{R}$ which are constant over each control volume of the mesh. For $v \in X(\mathcal{T})$ such that v_K denotes the value taken by v on any control volume $K \in \mathcal{T}$, define the discrete $H_0^1(\Omega)$ norm, with $D_\sigma v = |v_K - v_L|$ for $\sigma = K|L \in \mathcal{E}_{\mathrm{int}}$ and $D_\sigma v = |v_K|$ for $\sigma \in \mathcal{E}_{\mathrm{ext}} \cap \mathcal{E}_K$, by, see* [7]: $\|v\|_{1,\mathcal{T}} = (\sum_{\sigma \in \mathcal{E} \setminus \mathcal{E}_\Sigma} \frac{(D_\sigma v)^2}{d_\sigma} m(\sigma))^{1/2}$, *then* $\|v\|_{L^2(\Omega)} \leqslant \mathrm{diam}(\Omega) \|v\|_{1,\mathcal{T}}$.

2.1. Finite volume method for the fracture model

The finite volume scheme is only written with respect to the discrete unknowns $(u_K)_{K \in \mathcal{T}}$ supposed to be approximations of the values $u(x_K)$ of the exact solution u at the nodal points $x_K \in \overline{K}$ of the mesh.

Our finite volume scheme reads in the following synthetic form [4], as a generalization of the numerical scheme proposed in [3]: for all $K \in \mathcal{T}$,

$$\sum_{\sigma \in \mathcal{E}_K} m(\sigma)\varphi_{K,\sigma} + m(K) b_K u_K = m(K) f_K, \quad \forall K \in \mathcal{T}, \tag{14}$$

$$u_\sigma = 0, \text{ for } \sigma \in \mathcal{E}_{\mathrm{ext}} \cap \mathcal{E}_K, \text{ if } x_K \notin \sigma, \quad \text{or} \quad u_K \equiv u_\sigma = 0, \text{ if } x_K \in \sigma, \tag{15}$$

$$\varphi_{K,\sigma} = -a_K \frac{u_\sigma - u_K}{d_{K,\sigma}}, \quad \text{for } \sigma \in \mathcal{E}_{\mathrm{ext}} \cap \mathcal{E}_K, \text{ if } x_K \notin \sigma, \tag{16}$$

$$\begin{aligned}\varphi_{K,\sigma} = {} & -\tilde{a}_\sigma \left(\frac{u_L - u_K}{d_\sigma}\right) + \tilde{\alpha}_\sigma\, \tilde{\rho}_\sigma \frac{\tilde{a}_\sigma}{2\,d_\sigma}\left(\frac{u_K + u_L}{2}\right) + \tilde{\alpha}_\sigma \frac{\tilde{a}_\sigma\, d_{L,\sigma}}{d_\sigma\, a_L} u_K \\ & - \frac{\tilde{a}_\sigma}{d_\sigma}\left(\frac{\tilde{\rho}_\sigma}{2} + \frac{d_{L,\sigma}}{a_L}\right)\left(\tilde{\alpha}_\sigma\, U_\sigma + \tilde{h}_\sigma\right) + \tilde{\rho}_\sigma \frac{\tilde{a}_\sigma}{d_\sigma}\left(1 + \tilde{\alpha}_\sigma \frac{d_{L,\sigma}}{2\,a_L}\right)\tilde{g}_\sigma, \quad \text{for } \sigma = K|L \in \mathcal{E}_{\mathrm{int}},\end{aligned} \tag{17}$$

$$\tilde{a}_\sigma = \frac{a_\sigma}{1 + \tilde{\rho}_\sigma \frac{a_\sigma}{d_\sigma} + \tilde{\alpha}_\sigma(\tilde{\rho}_\sigma/4 + \lambda_\sigma)}, \quad a_\sigma = \frac{d_\sigma a_K a_L}{d_{L,\sigma} a_K + d_{K,\sigma} a_L}, \quad \text{and} \quad \lambda_\sigma = \frac{d_{K,\sigma} d_{L,\sigma}}{d_{L,\sigma} a_K + d_{K,\sigma} a_L}, \tag{18}$$

$$\tilde{\alpha}_\sigma = \alpha_\sigma,\ \tilde{\rho}_\sigma = \rho_\sigma,\ \tilde{g}_\sigma = g_\sigma,\ \tilde{h}_\sigma = h_\sigma,\ \forall \sigma \in \mathcal{E}_\Sigma \cap \mathcal{E}_K, \quad \text{or} \quad \tilde{\alpha}_\sigma = \tilde{\rho}_\sigma = \tilde{g}_\sigma = \tilde{h}_\sigma = 0,\ \forall \sigma \notin \mathcal{E}_\Sigma. \tag{19}$$

Remark 1 (*Asymptotic coherency*). When $\alpha|_\Sigma \equiv 0$ and $h|_\Sigma \equiv 0$, we get the scheme proposed in [3]; besides, with also $\rho|_\Sigma \equiv 0$, we get the usual cell-centered finite volume scheme for such problems without faults [7].

2.2. Convergence analysis and error estimates

Lemma 2.3 (Discrete trace jumps inequality on Σ). *If* $(\mathcal{H})$ *holds* (*see Lemma* 1.1), *let $\mathcal{T}$ be an admissible mesh* (*see Definition* 2.1), $v \in X(\mathcal{T})$ (*see Lemma* 2.2) *and v_K be the value of v in the finite volume K. For $\sigma = K|L \in \mathcal{E}_{\mathrm{int}}$ and* **n** *oriented from K to L on σ, let $\gamma^-(v)$, $\gamma^+(v)$ be defined by: $\gamma^-(v) = v_K$ and $\gamma^+(v) = v_L$ a.e.* (*for the* $(d-1)$*-Lebesgue measure*) *on σ, and let $\overline{\gamma}(v)$, $[\![\gamma(v)]\!]$ be the arithmetic mean and the oriented jump of these quantities, respectively. Then*: *for all $v \in X(\mathcal{T})$,*

$$\exists c^-(\Omega),\ c^+(\Omega)>0,\quad \big\|\gamma^*(v)\big\|_{L^2(\Sigma)}\leqslant c^*(\Omega)\left(\|v\|_{1,\mathcal{T}}+\|v\|_{L^2(\Omega)}\right),\quad \textit{for } *\in\{-,+\},$$
$$\exists c(\Omega)>0,\quad \big\|[\![\gamma(v)]\!]\big\|_{L^2(\Sigma)}\leqslant c(\Omega)\left(\|v\|_{1,\mathcal{T}}+\|v\|_{L^2(\Omega)}\right),\quad \big\|\overline{\gamma}(v)\big\|_{L^2(\Sigma)}\leqslant c(\Omega)\left(\|v\|_{1,\mathcal{T}}+\|v\|_{L^2(\Omega)}\right).$$

Theorem 2.4 (Convergence of the FV scheme). *If* $(\mathcal{H})$ *and the assumptions* (A1)–(A4) *hold for the problem* (1)–(5) *with* $\widetilde{\Gamma}_D=\widetilde{\Gamma}$, $g,h,U\in L^2(\Sigma)$, $f\in L^2(\Omega)$. *Let* $\mathcal{T}$ *be an admissible mesh in the sense of Definition* 2.1. *Then, there exists a unique discrete solution* $(u_K)_{K\in\mathcal{T}}$ *to the finite volume scheme* (14)–(19) *where the discrete coefficients are calculated by* (13). *Besides, define* $u_{\mathcal{T}}\in X(\mathcal{T})$ *by* $u_{\mathcal{T}}(x)=u_K$ *for a.e.* $x\in K$, *and for all* $K\in\mathcal{T}$. *Then, there exists* $u\in W$, *the unique weak solution of the problem, i.e., satisfying* (6), *such that*: *when* $h_{\mathcal{T}}\to 0$,

$$u_{\mathcal{T}}\to u\quad \text{in } L^2(\Omega),\quad \|u_{\mathcal{T}}\|_{1,\mathcal{T}}\to\left(\int_\Omega\big|\nabla u(x)\big|^2\,\mathrm{d}x\right)^{1/2},$$
$$\gamma^*(u_{\mathcal{T}})\rightharpoonup u^*|_\Sigma\quad \text{in } L^2(\Sigma)\ \text{weakly},\ \text{for } *\in\{-,+\}.$$

Theorem 2.5 (Error estimate of the FV scheme). *If* $(\mathcal{H})$ *and* (A1)–(A4) *hold, let* $u\in W$ *be the unique weak solution of* (1)–(5) *with* $\widetilde{\Gamma}_D=\widetilde{\Gamma}$, $g,h,U\in L^2(\Sigma)$, $f\in L^2(\Omega)$. *Let* $\mathcal{T}$ *be an admissible mesh, see Definition* 2.1, *and* $(u_K)_{K\in\mathcal{T}}$ *be the unique discrete solution of the FV scheme* (14)–(19). *Moreover, assume sufficiently piecewise regular data*: $\forall K\in\mathcal{T}$, $f|_K\in\mathcal{C}^0(\overline{K})$, $b|_K\in\mathcal{C}^1(\overline{K})$ *and* $a|_K$ *in* $\mathcal{C}^1(\overline{K})$, *and* $\forall\sigma\in\mathcal{E}_\Sigma$, $\alpha|_\sigma$, $\rho|_\sigma$ *and* $g|_\sigma$, $h|_\sigma$, $U|_\sigma$ *in* $\mathcal{C}^1(\overline{\sigma})$, *and such that the solution* $u\in H^1(\Omega)$ *satisfies* $u|_K\in\mathcal{C}^2(\overline{K})$, *for any* $K\in\mathcal{T}$. *Define the error* $e_{\mathcal{T}}\in X(\mathcal{T})$ *by* $e_{\mathcal{T}}(x)=u(x_K)-u_K$ *for a.e.* $x\in K$, *and for all* $K\in\mathcal{T}$. *Then,* $\exists C(u,a,b,\alpha,\rho,\Omega)\geqslant 0$, *such that the following error estimates hold for the discrete* H_0^1 *and* L^2 *norms*:

$$\|e_{\mathcal{T}}\|_{1,\mathcal{T}}\leqslant C(u,a,b,\alpha,\rho,\Omega)h_{\mathcal{T}}\quad \text{and}\quad \|e_{\mathcal{T}}\|_{L^2(\Omega)}\leqslant C(u,a,b,\alpha,\rho,\Omega)h_{\mathcal{T}}.$$

Remark 2 (*Weaker regularity assumptions*). With an additional condition of quasi-regularity of the mesh, e.g., [7], error estimates in $\mathcal{O}(h_{\mathcal{T}})$ also hold like above, for weaker regularity assumptions on the solution, i.e., $u\in H^1(\Omega)$ and $u|_K\in H^2(K)$ for any $K\in\mathcal{T}$. It requires the use of Taylor expansions with integral residuals to prove the weak consistency. See [5] for the details.

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