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Differential Geometry

Circle actions and \mathbf{Z}/k -manifolds

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Abstract

We establish an S^1 -equivariant index theorem for Dirac operators on \mathbb{Z}/k -manifolds. As an application, we generalize the Atiyah–Hirzebruch vanishing theorem for S^1 -actions on closed spin manifolds to the case of \mathbb{Z}/k -manifolds. *To cite this article: W. Zhang, C. R. Acad. Sci. Paris, Ser. I* 337 (2003).

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Résumé

Actions du cercle et \mathbb{Z}/k variétés. On établit un théorème d'indice S^1 -équivariant pour les opérateurs de Dirac sur des \mathbb{Z}/k variétés. On donne une application de ce résultat, qui généralise le théorème d'Atiyah–Hirzebruch sur les actions de S^1 aux \mathbb{Z}/k variétés. *Pour citer cet article : W. Zhang, C. R. Acad. Sci. Paris, Ser. I 337 (2003).*

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1. S^1 -actions and the vanishing theorem

Let X be a closed connected smooth spin manifold admitting a non-trivial circle action. A classical theorem of Atiyah and Hirzebruch [1] states that $\hat{A}(X) = 0$, where $\hat{A}(X)$ is the Hirzebruch \hat{A} -genus of X. In this Note we present an extension of the above result to the case of \mathbf{Z}/k -manifolds, which were introduced by Sullivan in his studies of geometric topology. We recall the basic definition for completeness (cf. [6]).

Definition 1.1. A compact connected \mathbb{Z}/k -manifold is a compact manifold X with boundary ∂X , which admits a decomposition $\partial X = \bigcup_{i=1}^{k} (\partial X)_i$ into k disjoint manifolds and k diffeomorphisms $\pi_i : (\partial X)_i \to Y$ to a closed manifold Y.

Let $\pi : \partial X \to Y$ be the induced map. In what follows, we will call an object α (e.g., metrics, connections, etc.) of X a \mathbb{Z}/k -object if there will be a corresponding object β on Y such that $\alpha|_{\partial X} = \pi^*\beta$. We make the assumption that X is \mathbb{Z}/k oriented, \mathbb{Z}/k spin and is of even dimension.

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Let g^{TX} be a \mathbb{Z}/k Riemannian metric of X which is of product structure near ∂X . Let R^{TX} be the curvature of the Levi-Civita connection associated to g^{TX} . Let E be a \mathbb{Z}/k complex vector bundle over X. Let g^E be a \mathbb{Z}/k Hermitian metric on E which is a product metric near ∂X . Let ∇^E be a \mathbb{Z}/k connection on E preserving g^E such that ∇^E is of product structure near ∂X . Let R^E be the curvature of ∇^E . Let $D^E_+ : \Gamma(S_+(TX) \otimes E) \rightarrow$ $\Gamma(S_-(TX) \otimes E)$ be the associated Dirac operator on X and $D^E_{+,\partial X}$ (and then D^E_Y) be its induced Dirac operator on ∂X (and then on Y). Let $\overline{\eta}(D^E_Y)$ be the reduced η -invariant of D^E_Y in the sense of [2]. Then

$$\hat{A}_{(k)}(X,E) = \int_{X} \det^{1/2} \left(\frac{\sqrt{-1}R^{TX}/4\pi}{\sinh(\sqrt{-1}R^{TX}/4\pi)} \right) \operatorname{tr}\left[e^{(\sqrt{-1}/2\pi)R^{E}} \right] - k\bar{\eta} \left(D_{Y}^{E} \right) \mod k\mathbf{Z}$$
(1)

does not depend on (g^{TX}, g^E, ∇^E) and determines a topological invariant in $\mathbb{Z}/k\mathbb{Z}$ (cf. [2] and [6]). Moreover, Freed and Melrose [7] have proved a mod k index theorem, giving $\hat{A}_{(k)}(X, E) \in \mathbb{Z}/k\mathbb{Z}$ a purely topological interpretation. When $E = \mathbb{C}$ is the trivial vector bundle over X, we usually omit the superscript E.

Theorem 1.2. If X admits a nontrivial \mathbb{Z}/k circle action preserving the orientation and the Spin structure on TX, then $\hat{A}_{(k)}(X) = 0$. Moreover, the equivariant mod k index in the sense of Freed and Melrose vanishes.

It turns out that the original method in [1] is difficult to extend to the case of manifolds with boundary to prove Theorem 1.2. Thus we will instead make use of an extension of the method of Witten [10]. Analytic localization techniques developed by Bismut and Lebeau [3, Section 9] and their extensions to manifolds with boundary developed in [5] play important roles in our proof.

2. A mod *k* localization formula for circle actions

We make the assumption that the \mathbb{Z}/k circle action on X lifts to a \mathbb{Z}/k circle action on E. Without loss of generality, we may and we will assume that this \mathbb{Z}/k circle action preserves g^{TX} , g^E and ∇^E . Let $D^E_{+,APS}: \Gamma(S_+(TX) \otimes E) \to \Gamma(S_-(TX) \otimes E)$ be the elliptic operator obtained by imposing the standard Atiyah– Patodi–Singer boundary condition [2] on D^E_+ .

Let *H* be the Killing vector field on *X* generated by the *S*¹ action on *X*. Then $H|_{\partial X} \subset \partial X$ induces a Killing vector field H_Y on *Y*. Let \mathcal{L}_H denote the corresponding Lie derivative acting on $\Gamma(S_{\pm}(TX) \otimes E)$. Then \mathcal{L}_H commutes with $D^E_{\pm,APS}$.

For any $n \in \mathbb{Z}$, let F_{\pm}^n be the eigenspaces of $\Gamma(S_{\pm}(TX) \otimes E)$ with respect to the eigenvalue $2\pi n$ of $\frac{1}{\sqrt{-1}}\mathcal{L}_H$. Let $D_{+,APS}^E(n): F_{\pm}^n \to F_{\pm}^n$ be the restriction of $D_{+,APS}^E$ on F_{\pm}^n . Then $D_{+,APS}^E(n)$ is Fredholm. We denote its index by $\operatorname{ind}(D_{+,APS}^E(n)) \in \mathbb{Z}$.

Let X_H (resp. Y_H) be the zero set of H (resp. H_Y) on X (resp. Y). Then X_H is a \mathbb{Z}/k -manifold and there is a canonical map $\pi_{X_H} : \partial X_H \to Y_H$ induced from π . We fix a connected component $X_{H,\alpha}$ of X_H , and we omit the subscript α if there is no confusion.

We identify the normal bundle to X_H in X to the orthogonal complement of TX_H in $TX|_{X_H}$. Then $TX|_{X_H}$ admits an S^1 -invariant orthogonal decomposition $TX|_{X_H} = N_{m_1} \oplus \cdots \oplus N_{m_l} \oplus TX_H$, where each N_{γ} , $\gamma \in \mathbb{Z}$, is a complex vector bundle on which $g \in S^1 \subset \mathbb{C}$ acts by multiplication by g^{γ} . By using the same notation as in [8, (1.8)], we simply write that $TX|_{X_H} = \bigoplus_{v \neq 0} N_v \oplus TX_H$. Similarly, let $E|_{X_H}$ admits the S^1 -invariant decomposition $E|_{X_H} = \bigoplus_v E_v$.

decomposition $E|_{X_H} = \bigoplus_v E_v$. Let $S(TX_H, (\det N)^{-1})$ be the complex spinor bundle over X_H associated to the canonically induced Spin^c structure on TX_H . It is a \mathbf{Z}/k Hermitian vector bundle and carries a canonically induced \mathbf{Z}/k Hermitian connection.

Recall that by [1, 2.4], one has $\sum_{v} v \dim N_v \equiv 0 \mod 2\mathbb{Z}$. Following [8, (1.15)], set

$$R(q) = q^{1/2\sum_{v} |v| \dim N_{v}} \bigotimes_{v>0} \left(\operatorname{Sym}_{q^{v}}(N_{v}) \otimes \det N_{v} \right) \bigotimes_{v<0} \operatorname{Sym}_{q^{-v}}(\overline{N}_{v}) \otimes \sum_{v} q^{v} E_{v} = \bigoplus_{n} R_{n} q^{n},$$

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$$R'(q) = q^{-1/2\sum_{v}|v|\dim N_{v}} \bigotimes_{v>0} \operatorname{Sym}_{q^{-v}}(\overline{N}_{v}) \bigotimes_{v<0} \left(\operatorname{Sym}_{q^{v}}(N_{v}) \otimes \det N_{v}\right) \otimes \sum_{v} q^{v} E_{v} = \bigoplus_{n} R'_{n} q^{n}.$$

Then each R_n (resp. R'_n) is a \mathbb{Z}/k Hermitian vector bundle over X_H carrying a canonically induced \mathbb{Z}/k Hermitian connection. For any $n \in \mathbb{Z}$, let $D_{X_H,+}^{R_n} : \Gamma(S_+(TX_H, (\det N)^{-1}) \otimes R_n) \to \Gamma(S_-(TX_H, (\det N)^{-1}) \otimes R_n)$ be the canonical twisted Spin^{*c*} Dirac operator on X_H . Let $D_{X_H,+,APS}^{R_n}$ be the corresponding elliptic operator associated to the Atiyah–Patodi–Singer boundary condition [2]. We will use similar notation for R'_n .

Theorem 2.1. For any integer $n \in \mathbb{Z}$, the following identities hold,

$$\operatorname{ind} D_{+,APS}^{E}(n) \equiv \sum_{\alpha} (-1)^{\sum_{0 < v} \dim N_{v}} \operatorname{ind} D_{X_{H,\alpha},+,APS}^{R_{n}} \mod k\mathbf{Z},$$

$$(2)$$

$$\operatorname{ind} D_{+,APS}^{E}(n) \equiv \sum_{\alpha} (-1)^{\sum_{\nu < 0} \dim N_{\nu}} \operatorname{ind} D_{X_{H,\alpha}+,APS}^{R'_{n}} \mod k \mathbb{Z}.$$
(3)

Proof. For any $T \in \mathbf{R}$, following Witten [10], let $D_{T,+}^E: \Gamma(S_+(TX) \otimes E) \to \Gamma(S_-(TX) \otimes E)$ be the Dirac type operator defined by $D_{T,+}^E = D_+^E + \sqrt{-1}Tc(H)$. Let $D_{T,+,APS}^E$ be the corresponding elliptic operator associated to the Atiyah–Patodi–Singer boundary condition [2]. Clearly, $D_{T,+,APS}^E$ also commutes with the S^1 -action. For any integer n, let $D_{T,+,APS}^E(n)$ be the restriction of $D_{T,+,APS}^E$ on F_+^n . Then $D_{T,+,APS}^E(n)$ is still Fredholm. By an easy extension of [5, Theorem 1.2] to the current equivariant and \mathbf{Z}/k situation, one sees that $\operatorname{ind}(D_{T,+,APS}^E(n)) \mod k\mathbf{Z}$ does not depend on $T \in \mathbf{R}$ (compare with [9, Theorem 4.2]). Let $D_{T,+,\partial X}^E: \Gamma((S_+(TX) \otimes E)|_{\partial X}) \to \Gamma((S_+(TX) \otimes E)|_{\partial X})$ be the induced Dirac type operator of $D_{T,+,APS}^E$

Let $D_{T,+,\partial X}^{E}$: $\Gamma((S_{+}(TX) \otimes E)|_{\partial X}) \to \Gamma((S_{+}(TX) \otimes E)|_{\partial X})$ be the induced Dirac type operator of $D_{T,+,\partial X}^{E}$ on ∂X . For any integer *n*, let $D_{T,+,\partial X}^{E}(n)$: $F_{+}^{n}|_{\partial X} \to F_{+}^{n}|_{\partial X}$ be the restriction of $D_{T,+,\partial X}^{E}$ on $F_{+}^{n}|_{\partial X}$. Also, the induced Dirac operators $D_{+,\partial X_{H}}^{R_{n}}$ and $D_{Y_{H}}^{R_{n}}$ can be defined in the same way as in Section 1.

Let $a_n > 0$ be such that $\text{Spec}(D_{Y_H}^{R_n}) \cap [-2a_n, 2a_n] \subseteq \{0\}$. By combining the techniques in [3, Section 9], [4, Section 4b]) and [8, Section 1.2], one can prove the following analogue of [4, Theorem 3.9], stating that there exists $T_1 > 0$ such that for any $T \ge T_1$,

$$\#\left\{\lambda \in \operatorname{Spect}\left(D_{T,+,\partial X}^{E}(n)\right): -a_{n} \leqslant \lambda \leqslant a_{n}\right\} = \dim\left(\ker D_{+,\partial X_{H}}^{R_{n}}\right) = k\dim\left(\ker D_{Y_{H}}^{R_{n}}\right).$$
(4)

If dim(ker $D_{Y_H}^{R_n}$) = 0, then by (4), one sees that when $T \ge T_1$, $D_{T,+,\partial X}^E(n)$ is invertible. Then ind($D_{T,+,APS}^E(n)$) itself does not depend on $T \ge T_1$. Moreover, by combining the techniques in [8, Section 1.2] and [5, Section 3], one can further prove that there exists $T_2 > 0$ such that when $T \ge T_2$,

$$\operatorname{ind}(D_{T,+,APS}^{E}(n)) = \sum_{\alpha} (-1)^{\sum_{0 < v} \dim N_{v}} \operatorname{ind} D_{X_{H,\alpha},+,APS}^{R_{n}}$$
(5)

(compare with [5, (2.13)]). From (5) and the mod k invariance of $ind(D_{T,+,APS}^E(n))$ with respect to $T \in \mathbf{R}$, one gets (2).

In general, dim(ker $D_{Y_H}^{R_n}$) need not be zero, and the eigenvalues of $D_{T,+,\partial X}^E(n)$ lying in $[-a_n, a_n]$ are not easy to control. Thus the above arguments no longer apply directly. Instead, we observe that dim(ker $(D_{Y_H}^{R_n} - a_n)) = 0$, and we use the method in [5] to perturb the Dirac type operators under consideration.

To do this, let $\varepsilon > 0$ be sufficiently small so that g^{TX} , g^E and ∇^E are of product structure on $[0, \varepsilon] \times \partial X \subset X$. Let $f: X \to \mathbf{R}$ be an S^1 -invariant smooth function such that $f \equiv 1$ on $[0, \varepsilon/3] \times \partial X$ and $f \equiv 0$ outside of $[0, 2\varepsilon/3] \times \partial X$. Let r denote the parameter in $[0, \varepsilon]$. Let $D_{XH, -a_n, +}^{R_n}$ be the Dirac type operator acting on $\Gamma(S_+(TX_H, (\det N)^{-1}) \otimes R_n)$ defined by $D_{XH, -a_n, +}^{R_n} = D_{XH, +}^{R_n} - a_n fc(\frac{\partial}{\partial r})$. Let $D_{XH, -a_n, +, APS}^{R_n}$ be the corresponding elliptic operator associated to the Atiyah–Patodi–Singer boundary condition [2]. By an easy extension of [5, Theorem 1.2] (compare with [9, Theorem 4.2]), we see that,

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$$\sum_{\alpha} (-1)^{\sum_{0 < v} \dim N_{v}} \operatorname{ind} D_{X_{H,\alpha}, -a_{n}, +, APS}^{R_{n}} \equiv \sum_{\alpha} (-1)^{\sum_{0 < v} \dim N_{v}} \operatorname{ind} D_{X_{H,\alpha}, +, APS}^{R_{n}} \mod k \mathbf{Z}.$$
(6)

For any $T \in \mathbf{R}$, let $D_{T,-a_n,+}^E: \Gamma(S_+(TX) \otimes E) \to \Gamma(S_-(TX) \otimes E)$ be the Dirac type operator defined by $D_{T,-a_n,+}^E = D_{T,+}^E - a_n f c(\frac{\partial}{\partial r})$. Let $D_{T,-a_n,+,APS}^E$ be the corresponding elliptic operator associated to the Atiyah–Patodi–Singer boundary condition. Let $D_{T,-a_n,+,APS}^E(n)$ be its restriction on F_+^n . Then $D_{T,-a_n,+,APS}^E(n)$ is still Fredholm. By another extension of [5, Theorem 1.2], one has

$$\operatorname{ind} D_{T,-a_n,+,APS}^E(n) \equiv \operatorname{ind} D_{T,+,APS}^E(n) \mod k\mathbf{Z}.$$
(7)

Moreover, since $D_{Y_H}^{R_n} - a_n$, which is the induced Dirac type operator from $D_{X_H, -a_n, +}^{R_n}$ through π_{X_H} , is invertible, by combining the arguments in [8, Section 1.2] with those in [5, Section 3], one deduces that there exists $T_3 > 0$ such that for any $T \ge T_3$, the following analogue of (5) holds,

$$\operatorname{ind} D_{T,-a_n,+,APS}^E(n) = \sum_{\alpha} (-1)^{\sum_{0 < v} \dim N_v} \operatorname{ind} D_{X_{H,\alpha},-a_n,+,APS}^{R_n}.$$
(8)

From (6)–(8) and the mod *k* invariance of $\operatorname{ind}(D_{T,+,APS}^E(n))$ with respect to $T \in \mathbf{R}$, one gets (2). Similarly, by taking $T \to -\infty$, one gets (3). \Box

3. Proof of Theorem 1.2

We apply Theorem 2.1 to the case $E = \mathbf{C}$. First, if $X_H = \emptyset$, by Theorem 2.1, it is obvious that for each $n \in \mathbf{Z}$,

$$\operatorname{ind}(D_{+APS}(n)) \equiv 0 \mod k\mathbf{Z}.$$
(9)

When $X_H \neq \emptyset$, we see that $\sum_{v} |v| \dim N_v > 0$ (i.e., at least one of the N_v 's is nonzero) on each connected component of X_H . Then by (2) and by the definition of the R_n 's, we deduce that for any integer $n \leq 0$, (9) holds. Similarly, by (3) and by the definition of the R'_n 's, one deduces that (9) holds for any integer $n \geq 0$.

In summary, for any $n \in \mathbb{Z}$, (9) holds.

From (1) and (9), by the Atiyah–Patodi–Singer index theorem [2], and using the obvious fact that $ind(D_{+,APS}) = \sum_{n} ind(D_{+,APS}(n))$, one gets $\hat{A}_{(k)}(X) = 0$. \Box

Remark 1. By combining Theorem 2.1 with the arguments in [8, Sections 2–4], one should be able to prove an extension of the Witten rigidity theorem, of which a *K*-theoretic version has been worked out in [8], to \mathbf{Z}/k -manifolds. This, together with some other consequences of Theorem 1.2, will be carried out elsewhere.

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