Numerical Analysis

A cell-centered finite volume scheme on general meshes for the Stokes equations in two space dimensions

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Abstract

This Note presents a new finite volume scheme for the Stokes equations on general non-structured meshes. A convergence result is presented, and an error estimate is given when the solution is regular enough. *To cite this article: R. Eymard, R. Herbin, C. R. Acad. Sci. Paris, Ser. I 337 (2003).*
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Résumé

Un schéma volumes finis centrés par mailles pour la résolution des équations de Stokes sur des maillages 2D généraux.
On présente ici une nouvelle méthode de volumes finis pour la discrétisation des équations de Stokes sur un maillage 2D non structuré. On présente un résultat de convergence, ainsi qu'une estimation d'erreur dans le cas où la solution est suffisamment régulière. *Pour citer cet article : R. Eymard, R. Herbin, C. R. Acad. Sci. Paris, Ser. I 337 (2003).*
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We study the following problems: find an approximation of the weak solution $(u, v, p) \in H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$ to the Stokes equations:

$$
\begin{aligned}
-v\Delta u + p_x &= f \quad \text{in } \Omega, \\
-v\Delta v + p_y &= g \quad \text{in } \Omega, \\
u_x + v_y &= 0 \quad \text{in } \Omega,
\end{aligned}
\tag{1}
$$

with a homogeneous Dirichlet boundary condition on the velocity (u, v), and under the following assumptions on the data:

$$
\Omega \text{ is a polygonal open bounded subset of } \mathbb{R}^2, \quad v \in (0, +\infty), \quad f, g \in L^2(\Omega).
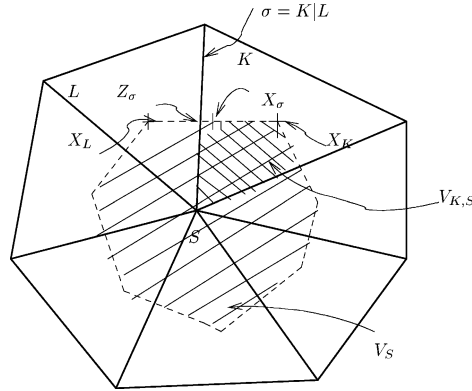\tag{2}
$$

Fig. 1. Example of an admissible triangular discretization.

Under hypotheses (2), (u, v) is a weak solution (see, e.g., [7]) of (1) if

$$
\begin{aligned}
&(u, v) \in E(\Omega) := \left\{ (u', v') \in \left(H_0^1(\Omega) \right)^2, \ \operatorname{div}(u', v') = u'_x + v'_y = 0 \text{ a.e.} \right\}, \\
&\nu \int_\Omega \left(\nabla u(x, y) \cdot \nabla u'(x, y) + \nabla v(x, y) \cdot \nabla v'(x, y) \right) \mathrm{d}x \, \mathrm{d}y \\
&\quad + \int_\Omega \left(f(x, y) u'(x, y) + g(x, y) v'(x, y) \right) \mathrm{d}x \, \mathrm{d}y, \quad \forall (u', v') \in E(\Omega).
\end{aligned}
\tag{3}
$$

In this Note, we present a new finite volume scheme for the discretization of the Stokes equations in two space dimensions. In this scheme, the velocity unknowns are associated to the control volumes and the pressure unknowns to the nodes, as in the scheme introduced in [2,1] (equilateral triangular grid). It differs from the MAC scheme [6,5] (rectangular grids) by the fact that both velocity components are associated to the same cell, and because it may be written on general unstructured meshes, see Fig. 1.

1. The finite volume scheme

Definition 1.1 (Admissible discretization). We consider a finite volume mesh $\mathcal{M}$ of Ω satisfying the usual conditions (see [2]), with the following notations:

$\mathcal{M}$ is a finite family of nonempty open polygonal convex disjoint subsets of Ω such that $\overline{\Omega} = \bigcup_{K \in \mathcal{M}} \overline{K}$.

The set of edges of the mesh $\mathcal{M}$ (resp. of a control volume K) is denoted by $\mathcal{E}$ (resp. $\mathcal{E}_K$).

We assume the existence of a family of points $\mathcal{P} = (X_K)_{K \in \mathcal{M}}$, satisfying the usual orthogonality condition: for any edge $\sigma = K|L \in \mathcal{E}$ separating two control volumes K and L the straight line (X_K, X_L) going through X_K and X_L is orthogonal to $K|L$.

For any $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}_K$, we assume that the orthogonal projection Z_σ of X_K on σ is such that $Z_\sigma \in \sigma$.

Let $\mathcal{V}$ (resp. $\mathcal{V}_K$) be the set of vertices of the mesh (resp. of the control volume K). For any $S \in \mathcal{V}$, we denote by x_S and y_S the coordinates of S and $\mathcal{M}_S = \{ K \in \mathcal{M}, \ S \in \mathcal{V}_K \}$.

The size of the discretization is defined by: $h = \sup\{ \operatorname{diam}(K), \ K \in \mathcal{M} \}$.

For all $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}_K$, we denote by $\mathbf{n}_{K,\sigma}$ the unit vector normal to σ outward to K. We denote by $d_{K,\sigma}$ the Euclidean distance between X_K and σ. We then define $\tau_{K,\sigma} = (\operatorname{meas}(\sigma)) / d_{K,\sigma}$, We shall measure the regularity of the mesh through the measure angle$(\mathcal{M})$ defined as the minimum, for S vertex of K and σ, of of all angles $\alpha_{K,S,\sigma} = \widehat{Z_\sigma X_K S}$ and $\frac{\pi}{2} - \alpha_{K,S,\sigma}$.

The set of interior (resp. boundary) edges is denoted by $\mathcal{E}_{\text{int}}$ (resp. $\mathcal{E}_{\text{ext}}$), that is $\mathcal{E}_{\text{int}} = \{\sigma \in \mathcal{E}; \ \sigma \not\subset \partial\Omega\}$ (resp. $\mathcal{E}_{\text{ext}} = \{\sigma \in \mathcal{E}; \ \sigma \subset \partial\Omega\}$). For any $\sigma \in \mathcal{E}_{\text{int}}, \sigma = K|L$ (resp. $\mathcal{E}_{\text{ext}}, \sigma \in \mathcal{E}_K$), let X_σ be the center point of the line segment $[X_K, X_L]$ (resp. $[X_K, Z_\sigma]$), and x_σ and y_σ its coordinates.

For all $K \in \mathcal{M}$ and all $S \in \mathcal{V}_K$, let σ_1 and σ_2 be the edges of K such that S is the common vertex of σ_1 and σ_2 and such that $(y_{\sigma_1} - y_S)(x_{\sigma_2} - x_S) - (y_{\sigma_2} - y_S)(x_{\sigma_1} - x_S) > 0$. We then define the coefficients

$$A_{K,S} = y_{\sigma_1} - y_{\sigma_2}, \qquad B_{K,S} = x_{\sigma_2} - x_{\sigma_1}. \tag{4}$$

For $K \in \mathcal{M}$ and $S \in \mathcal{V}_K$ let σ_1 and σ_2 be the elements of $\mathcal{E}_K$ such that S is a common vertex to σ_1 and σ_2: we then denote by $V_{K,S}$ the polygonal subset of Ω, whose vertices are $S, Z_{\sigma_1}, X_K, Z_{\sigma_2}$, and we set $V_S = \bigcup_{K \in \mathcal{M}_S} V_{K,S}$.

Definition 1.2. Let Ω be an open bounded polygonal subset of $\mathbb{R}^N$, with $N \in \mathbb{N}_*$. Let $\mathcal{M}$ be an admissible finite volume discretization of Ω in the sense of Definition 1.1. Let $H_{\mathcal{M}}(\Omega) \subset L^2(\Omega)$ be the space of functions which are piecewise constant on each control volume $K \in \mathcal{M}$. For all $u \in H_{\mathcal{M}}(\Omega)$ and for all $K \in \mathcal{M}$, we denote by u_K the constant value of u in K and we define $(u_\sigma)_{\sigma \in \mathcal{E}}$ by:

$$\tau_{K,\sigma}(u_\sigma - u_K) + \tau_{L,\sigma}(u_\sigma - u_L) = 0, \quad \forall \sigma \in \mathcal{E}_{\text{int}}, \ \sigma = K|L, \quad \text{and} \quad u_\sigma = 0, \quad \forall \sigma \in \mathcal{E}_{\text{ext}}. \tag{5}$$

Let $L_{\mathcal{M}}(\Omega)$ be the space of functions which are piecewise constant on the domains V_S, for all $S \in \mathcal{V}$. The discrete divergence operator is defined by: $\text{div}_{\mathcal{M}} : (H_{\mathcal{M}}(\Omega))^2 \to L_{\mathcal{M}}(\Omega)$, by: $\text{div}_{\mathcal{M}}(u, v)(x, y) = \frac{1}{\text{meas}(S)} \sum_{K \in \mathcal{M}_S} \times (A_{K,S} u_K + B_{K,S} v_K)$, for a.e. $(x, y) \in V_S$ and for any $S \in \mathcal{V}$. Let $E_{\mathcal{M}}(\Omega) = \{(u, v) \in H_{\mathcal{M}}(\Omega)^2, \ \text{div}_{\mathcal{M}}(u, v) = 0\}$, and for $(u, u') \in (H_{\mathcal{M}}(\Omega))^2$,

$$[u, u']_{\mathcal{M}} = \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} \tau_{K,\sigma}(u_\sigma - u_K)(u'_\sigma - u'_K). \tag{6}$$

Thanks to the discrete Poincaré inequality (see [2]), we may define the following norm in $H_{\mathcal{M}}(\Omega)$ by: $|u|_{\mathcal{M}} = ([u, u]_{\mathcal{M}})^{1/2}$, and for $(u, v) \in H_{\mathcal{M}}(\Omega)^2$, we define: $|(u, v)|_{\mathcal{M}} = ([u, u]_{\mathcal{M}} + [v, v]_{\mathcal{M}})^{1/2}$.

Under hypotheses (2), let $\mathcal{M}$ be an admissible discretization of Ω in the sense of Definition 1.1. Let $\lambda \in [0, +\infty)$ be given. The finite volume scheme can be written: find $(u_{\mathcal{M}}, v_{\mathcal{M}}, p_{\mathcal{M}})$ such that

$$
\begin{aligned}
&(u_{\mathcal{M}}, v_{\mathcal{M}}, p_{\mathcal{M}}) \in H_{\mathcal{M}}(\Omega) \times H_{\mathcal{M}}(\Omega) \times L_{\mathcal{M}}(\Omega), \\
&\nu\big([u_{\mathcal{M}}, u'_{\mathcal{M}}]_{\mathcal{M}} + [v_{\mathcal{M}}, v'_{\mathcal{M}}]_{\mathcal{M}}\big) - \int_\Omega p_{\mathcal{M}}(x, y) \,\text{div}_{\mathcal{M}}(u'_{\mathcal{M}}, v'_{\mathcal{M}})(x, y) \,dx\,dy \\
&= \int_\Omega \big(f(x, y) u'_{\mathcal{M}}(x, y) + g(x, y) v'_{\mathcal{M}}(x, y)\big) \,dx\,dy, \quad \forall (u'_{\mathcal{M}}, v'_{\mathcal{M}}) \in \big(H_{\mathcal{M}}(\Omega)\big)^2, \\
&\text{div}_{\mathcal{M}}(u_{\mathcal{M}}, v_{\mathcal{M}}) = -\lambda \,\text{size}(\mathcal{M}) p_{\mathcal{M}},
\end{aligned}
\tag{7}
$$

where χ_{V_S} denotes the characteristic function of V_S. In (7), the test functions $(u'_{\mathcal{M}}, v'_{\mathcal{M}})$ are successively taken to be equal to $(\chi_K, 0)$ and $(0, \chi_K)$, for all $K \in \mathcal{M}$, an elimination of the unknowns u_σ and v_σ using (5) yields a linear system of equations with unknowns $(u_K, v_K)_{K \in \mathcal{M}}$ and $(p_S)_{S \in \mathcal{V}}$. If $\lambda \neq 0$, the pressures can then be eliminated using the last equation of (7); thanks to the discrete Poincaré inequality, we get that this system is invertible. In the case $\lambda = 0$, it is still possible to prove the existence and uniqueness of the discrete velocities [3]. Note that if $\lambda \neq 0$, the scheme (7) is a finite volume version of the penalization method which was studied in [4] in the finite element case.

2. Convergence and error estimate

Proposition 2.1. *Under hypotheses* (2)*, let $\mathcal{M}$ be an admissible discretization of Ω in the sense of Definition* 1.1. *Let $\lambda \in (0, +\infty)$ be given. Let $(u, v, p) \in H_{\mathcal{M}}(\Omega) \times H_{\mathcal{M}}(\Omega) \times L_{\mathcal{M}}(\Omega)$ be a solution to* (7)*. Then the following inequalities hold*:

$$\nu |(u, v)|_{\mathcal{M}} \leqslant \operatorname{diam}(\Omega) \|(f, g)\|_{L^2(\Omega)}, \tag{8}$$

and

$$\left(\nu \lambda \operatorname{size}(\mathcal{M})\right)^{1/2} \|p\|_{L^2(\Omega)} \leqslant \operatorname{diam}(\Omega) \|(f, g)\|_{L^2(\Omega)}. \tag{9}$$

An easy consequence of the above estimates is the existence and uniqueness of a solution to (7). We may then state the following convergence result and error estimate, the proof of which may be found in [3].

Proposition 2.2. *Under hypotheses* (2)*, let $\lambda \in (0, +\infty)$ be given and let $(\mathcal{M}^{(n)})_{n \in \mathbb{N}}$ be a sequence of admissible discretization of Ω in the sense of Definition* 1.1*, such that $\lim_{n \to \infty} \operatorname{size}(\mathcal{M}^{(n)}) = 0$ and such that there exists $\alpha > 0$ with $\operatorname{angle}(\mathcal{M}^{(n)}) \geqslant \alpha$, for all $n \in \mathbb{N}$. Let $(u^{(n)}, v^{(n)}, p^{(n)}) \in H_{\mathcal{M}^{(n)}}(\Omega) \times H_{\mathcal{M}^{(n)}}(\Omega) \times L_{\mathcal{M}^{(n)}}(\Omega)$ be the solution to* (7)*. Then the sequence $(u^{(n)}, v^{(n)})_{n \in \mathbb{N}}$ converges in $L^2(\Omega)^2$ to the weak solution (u, v) of the Stokes problem in the sense of* (3).

Proposition 2.3. *Under hypotheses* (2)*, assume that there exists $(\bar{u}, \bar{v}, \bar{p}) \in (H_0^1(\Omega) \cap H^2(\Omega)) \times (H_0^1(\Omega) \cap H^2(\Omega)) \times H^1(\Omega)$ solution of* (1)*. Let $\mathcal{M}$ be an admissible discretization of Ω in the sense of Definition* 1.1*. Let $\lambda \in (0, +\infty)$ be given and $\alpha > 0$ such that $\operatorname{angle}(\mathcal{M}) \geqslant \alpha$. Let $(u, v, p) \in H_{\mathcal{M}}(\Omega) \times H_{\mathcal{M}}(\Omega) \times L_{\mathcal{M}}(\Omega)$ be the solution to* (7)*. Denote by $(\bar{u}^{\mathcal{M}}, \bar{v}^{\mathcal{M}}) \in H_{\mathcal{M}}(\Omega)^2$ the functions respectively defined by $\bar{u}(x_K, y_K)$ and $\bar{v}(x_K, y_K)$ in K, for all $K \in \mathcal{M}$. Then there exists $C > 0$, which only depends on Ω, ν and α, such that the following inequalities hold*:

$$\left|(u - \bar{u}^{\mathcal{M}}, v - \bar{v}^{\mathcal{M}})\right|_{\mathcal{M}} \leqslant C \left(\operatorname{size}(\mathcal{M})\right)^{1/4} \left(\|\bar{u}\|_{H^2(\Omega)}^2 + \|\bar{v}\|_{H^2(\Omega)}^2 + \|\bar{p}\|_{H^1(\Omega)}^2\right)^{1/2} \left(\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\right)^{1/2}, \tag{10}$$

and

$$\left\|(u - \bar{u}, v - \bar{v})\right\|_{L^2(\Omega)} \leqslant C \left(\operatorname{size}(\mathcal{M})\right)^{1/4} \left(\|\bar{u}\|_{H^2(\Omega)}^2 + \|\bar{v}\|_{H^2(\Omega)}^2 + \|\bar{p}\|_{H^1(\Omega)}^2\right)^{1/2} \left(\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\right)^{1/2}. \tag{11}$$

The order of convergence $1/4$ which is obtained in Proposition 2.3 is clearly not optimal: it is a consequence of the fact that the pressures have only been weakly estimated, using the artificial compressibility λ. Indeed, numerical results show that the method is efficient and that the order of convergence is greater than 1.5. Until now, we have only been able to prove the convergence in the case $\lambda \neq 0$, although numerical results suggest that it also holds in the case $\lambda = 0$. This scheme has also been successfully implemented in the case of the nonlinear stationary Navier–Stokes equation, with a centered version and an upstream weighted version. In both cases, convergence theorems may be proven.

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