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Partial Differential Equations

On a Liouville-type comparison principle for solutions of quasilinear elliptic inequalities ☆

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Abstract

We characterize in terms of monotonicity basic properties of quasilinear elliptic partial differential operators which make it possible to obtain a Liouville-type comparison principle for entire solutions of quasilinear elliptic partial differential inequalities of the form $A(u) + |u|^{q-1}u \leqslant A(v) + |v|^{q-1}v$, which belong only locally to the corresponding Sobolev spaces on $\mathbb{R}^n$, $n \geqslant 2$. We establish that such properties are inherent for a wide class of quasilinear elliptic partial differential operators. Typical examples of such operators are the p-Laplacian and its well-known modifications for $1 < p \leqslant 2$. ***To cite this article: V.V. Kurta, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Sur un principe de comparaison de type Liouville pour des solutions d'inégalités elliptiques quasi-linéaires. On caractérise en terme de monotonie, des propriétés fondamentales d'opérateurs aux dérivées partielles, elliptiques, quasi-linéaires permettant d'établir un principe de comparaison de type Liouville, des solutions faibles d'inégalités aux dérivée partielles, elliptiques, quasi-linéaires de la forme $A(u) + |u|^{q-1}u \leqslant A(v) + |v|^{q-1}v$. Ces solutions appartiennent seulement localement aux espaces de Sobolev correspondant dans $\mathbb{R}^n$, $n \geqslant 2$. On montre que ces propriétés sont valables pour une large classe d'opérateurs aux dérivées partielles elliptiques, quasi-linéaires. Des exemples typiques de tels opérateurs sont le p-laplacien et ses modifications bien connues pour $1 < p \leqslant 2$. ***Pour citer cet article : V.V. Kurta, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

1. Introduction

This work is devoted to the study of a Liouville-type phenomenon for entire solutions of a wide class of quasilinear elliptic partial differential inequalities. It is well known that, in order to obtain and even to formulate Liouville's theorem, for example, for subharmonic functions on $\mathbb{R}^2$, one needs to compare an arbitrary subharmonic function with a constant which is, naturally, a trivial superharmonic function. Due to the linearity of the Laplacian operator one can reformulate this famous result in the form of a Liouville-type comparison principle: *Let* $(u(x), v(x))$ *be an entire solution of the inequality* $\Delta u \leqslant \Delta v$ *on* $\mathbb{R}^2$ *such that* $u(x) \geqslant v(x)$*. Then* $u(x) = v(x)$*, up to a constant, on* $\mathbb{R}^2$.

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This classical field of analysis, well-known as Liouville-type theorems, is again of great interest due to the nonlinearity of the problems considered. However, in a nonlinear situation almost all Liouville-type results obtained have been established by comparing an arbitrary solution of a nonlinear problem with zero, which is a trivial solution of the same or the corresponding nonlinear problem.

The main purpose of this work is to characterize in terms of monotonicity basic properties of quasilinear elliptic partial differential operators which make it possible to obtain a Liouville-type comparison principle for arbitrary solutions of quasilinear elliptic partial differential inequalities of the form

$$A(u) + |u|^{q-1}u \leqslant A(v) + |v|^{q-1}v. \tag{1}$$

Note that such properties are inherent for a wide class of quasilinear elliptic partial differential operators. Typical examples of such operators are the p-Laplacian

$$\Delta_p(w) := \operatorname{div}\big(|\nabla w|^{p-2}\nabla w\big) \tag{2}$$

for $1 < p \leqslant 2$ and its well-known modification, see, e.g., [5, p. 155],

$$\tilde{\Delta}_p(w) := \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial w}{\partial x_i}\right|^{p-2}\frac{\partial w}{\partial x_i}\right) \tag{3}$$

for $n \geqslant 2$ and $1 < p \leqslant 2$.

2. Definitions

Let $A(w)$ be a differential operator given formally by

$$A(w) = \sum_{i=1}^{n} \frac{\mathrm{d}}{\mathrm{d}x_i} A_i(x, \nabla w). \tag{4}$$

Here and in what follows $n \geqslant 2$. Assume that the functions $A_i(x, \xi)$, $i = 1, \ldots, n$, satisfy the Carathéodory conditions on $\mathbb{R}^n \times \mathbb{R}^n$. Namely, they are continuous in ξ at almost all $x \in \mathbb{R}^n$ and measurable in x at all $\xi \in \mathbb{R}^n$.

Definition 2.1. Let $\alpha > 1$ be a given number. An operator $A(w)$, given by (4), is said to be α-monotone if $A_i(x, 0) = 0$, $i = 1, \ldots, n$, at almost all $x \in \mathbb{R}^n$, and there exists a positive constant $\mathcal{K}$ such that

$$0 \leqslant \sum_{i=1}^{n} \big(\xi_i^1 - \xi_i^2\big)\big(A_i\big(x, \xi^1\big) - A_i\big(x, \xi^2\big)\big) \tag{5}$$

and

$$\left(\sum_{i=1}^{n} \big(A_i\big(x, \xi^1\big) - A_i\big(x, \xi^2\big)\big)^2\right)^{\alpha/2} \leqslant \mathcal{K}\left(\sum_{i=1}^{n} \big(\xi_i^1 - \xi_i^2\big)\big(A_i\big(x, \xi^1\big) - A_i\big(x, \xi^2\big)\big)\right)^{\alpha-1} \tag{6}$$

hold at all $\xi^1, \xi^2 \in \mathbb{R}^n$ and almost all $x \in \mathbb{R}^n$.

Note that condition (5) is the well-known monotonicity condition in PDE theory, while condition (6) is the proper α-monotonicity condition for partial differential operators, considered first in [2]; see also [3,4]. Note also that α-monotonicity condition (6) in the case $\xi^2 = 0$ is in turn a special case of the very general growth condition for quasilinear elliptic partial differential operators, considered first in [6].

Now we present algebraic inequalities from which it follows immediately that the p-Laplacian Δ_p and its modification $\tilde{\Delta}_p$ satisfy the α-monotonicity condition for $\alpha = p$ and $1 < p \leqslant 2$.

Lemma 2.2. *Let $1 < \alpha \leqslant 2$, and let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be arbitrary vectors in $\mathbb{R}^n$ of length $|a| = \sqrt{a_1^2 + \cdots + a_n^2}$ and $|b| = \sqrt{b_1^2 + \cdots + b_n^2}$, respectively. Then there exists a positive constant $\mathcal{K}$ such that the* *inequalities*

$$\left(\sum_{i=1}^{n}\bigl(a_i|a|^{\alpha-2}-b_i|b|^{\alpha-2}\bigr)^2\right)^{\alpha/2}\leqslant\mathcal{K}\left(\sum_{i=1}^{n}(a_i-b_i)\bigl(a_i|a|^{\alpha-2}-b_i|b|^{\alpha-2}\bigr)\right)^{\alpha-1}\tag{7}$$

and

$$\left(\sum_{i=1}^{n}\bigl(a_i|a_i|^{\alpha-2}-b_i|b_i|^{\alpha-2}\bigr)^2\right)^{\alpha/2}\leqslant\mathcal{K}\left(\sum_{i=1}^{n}(a_i-b_i)\bigl(a_i|a_i|^{\alpha-2}-b_i|b_i|^{\alpha-2}\bigr)\right)^{\alpha-1}\tag{8}$$

hold.

Remark 1. The statements of Lemma 2.2 were proved in [2]; see also [4].

It is important to note that there exist α-monotone partial differential operators with arbitrary degeneracy. For instance, the weighted p-Laplacian,

$$\bar{\Delta}_p(w):=\operatorname{div}\bigl(a(x)|\nabla w|^{p-2}\nabla w\bigr),\tag{9}$$

see, e.g., [1, p. 55], with any measurable nonnegative uniformly bounded function $a(x)$ on $\mathbb{R}^n$ is α-monotone with $\alpha=p$ for any fixed $1<p\leqslant 2$.

Remark 2. We restrict ourselves here to the study of inequality (1), although all the results formulated below are valid with an arbitrary function $f(x,w)$, instead of $|w|^{q-1}w$, satisfying suitable growth and regularity conditions and being such that

$$\bigl(f(x,v)-f(x,u)\bigr)(v-u)\geqslant c|v-u|^{q+1}\tag{10}$$

for a fixed number $c>0$, almost all $x\in\mathbb{R}^n$, and all $u,v\in\mathbb{R}^1$. Actually, our approach allows us to consider certain nonnegative functions $c(x)$ in place of the constant c.

Definition 2.3. Let $\alpha>1$ and $q>0$ be given numbers, and let the operator $A(w)$ be α-monotone. By an entire solution of inequality (1) we understand a pair of functions $(u(x),v(x))$ on $\mathbb{R}^n$ which belong to the space $W^1_{\alpha,\mathrm{loc}}(\mathbb{R}^n)\cap L_{q,\mathrm{loc}}(\mathbb{R}^n)$ and satisfy the integral inequality

$$\int\limits_{\mathbb{R}^n}\left[\sum_{i=1}^{n}\varphi_{x_i}A_i(x,\nabla u)-|u|^{q-1}u\varphi\right]\mathrm{d}x\geqslant\int\limits_{\mathbb{R}^n}\left[\sum_{i=1}^{n}\varphi_{x_i}A_i(x,\nabla v)-|v|^{q-1}v\varphi\right]\mathrm{d}x\tag{11}$$

for every nonnegative function $\varphi\in C^\infty(\mathbb{R}^n)$ with compact support.

Analogous definitions of solutions of the inequalities

$$-A(u)\geqslant|u|^{q-1}u\tag{12}$$

and

$$-A(v)\leqslant|v|^{q-1}v,\tag{13}$$

which are special cases of inequality (1) for $v=0$ and $u=0$, respectively, can be immediately obtained from Definition 2.3.

Definition 2.4. Let $\alpha>1$ and $q>0$ be given numbers, and let the operator $A(w)$ be α-monotone. By an entire solution of inequality (12) (resp., (13)) we understand a function $w(x)$ on $\mathbb{R}^n$ which belongs to the space $W^1_{\alpha,\mathrm{loc}}(\mathbb{R}^n)\cap L_{q,\mathrm{loc}}(\mathbb{R}^n)$ and satisfies the integral inequality

$$\int_{\mathbb{R}^n} \sum_{i=1}^{n} \varphi_{x_i} A_i(x, \nabla w)\,\mathrm{d}x - \int_{\mathbb{R}^n} |w|^{q-1} w\varphi\,\mathrm{d}x \geqslant 0 \quad (\text{resp., } \leqslant 0) \tag{14}$$

for every nonnegative function $\varphi \in C^{\infty}(\mathbb{R}^n)$ with compact support.

3. Main results

Theorem 3.1. *Let $n \geqslant 2$, and let α and q be given numbers such that $\alpha < n$, $\frac{2n}{n+1} \leqslant \alpha \leqslant 2$ and $1 \leqslant q \leqslant \frac{n(\alpha-1)}{n-\alpha}$. Let the operator $A(w)$ be α-monotone, and let $(u(x), v(x))$ be an entire solution of* (1) *on $\mathbb{R}^n$ such that $u(x) \geqslant v(x)$. Then $u(x) = v(x)$ on $\mathbb{R}^n$.*

Theorem 3.2. *Let $n \geqslant 2$, and let α and q be given numbers such that $\alpha < n$, $1 < \alpha \leqslant 2$, $q \geqslant 1$ and $q > \frac{n(\alpha-1)}{n-\alpha}$. Let the operator $A(w)$ be α-monotone. Then for any given constant $c > 0$ there exists no entire solution $(u(x), v(x))$ of* (1) *on $\mathbb{R}^n$ such that $u(x) \geqslant v(x) + c$.*

Theorem 3.3. *Let $n = 2$, $\alpha = 2$, and let $q > 1$ be a given number. Let the operator $A(w)$ be α-monotone, and let $(u(x), v(x))$ be an entire solution of* (1) *on $\mathbb{R}^n$ such that $u(x) \geqslant v(x)$. Then $u(x) = v(x)$ on $\mathbb{R}^n$.*

Theorem 3.4. *Let $n = 2$, $\alpha = 2$ and $q = 1$. Let the operator $A(w)$ be α-monotone. Then for any given constant $c > 0$ there exists no entire solution $(u(x), v(x))$ of* (1) *on $\mathbb{R}^n$ such that $u(x) \geqslant v(x) + c$.*

Remark 3. To provide more clarity to the understanding of Theorems 3.1–3.4, we note that for $n \geqslant 2$, $n > \alpha > 1$, $q > \frac{(\alpha-1)n}{n-\alpha}$ and a suitable positive constant c, a pair of functions $(u(x), v(x))$ such that

$$u(x) = c\big(1 + |x|^{\alpha/(\alpha-1)}\big)^{(1-\alpha)/(q-\alpha+1)} \quad \text{and} \quad v(x) = 0 \tag{15}$$

is an entire solution of inequality (1) with $A(w) = \Delta_\alpha(w)$ or $A(w) = \tilde{\Delta}_\alpha(w)$, respectively. It is easy to see that there exists no positive constant c such that $u(x) \geqslant v(x) + c$.

Analogous results for solutions of inequalities (12) and (13) can be immediately obtained from Theorems 3.1–3.4. We formulate only two of them.

Theorem 3.5. *Let $n \geqslant 2$, and let α and q be given numbers such that $\alpha < n$, $\frac{2n}{n+1} \leqslant \alpha \leqslant 2$ and $1 \leqslant q \leqslant \frac{n(\alpha-1)}{n-\alpha}$. Let the operator $A(w)$ be α-monotone, and let $u(x)$ and $v(x)$ be entire solutions of inequalities* (12) *and* (13)*, respectively, on $\mathbb{R}^n$ such that $u(x) \geqslant v(x)$. Then $u(x) = v(x)$ on $\mathbb{R}^n$.*

Theorem 3.6. *Let $n \geqslant 2$, and let α and q be given numbers such that $\alpha < n$, $1 < \alpha \leqslant 2$, $q \geqslant 1$ and $q > \frac{n(\alpha-1)}{n-\alpha}$. Let the operator $A(w)$ be α-monotone. Then for any given constant $c > 0$ there exist no entire solutions $u(x)$ and $v(x)$ of inequalities* (12) *and* (13)*, respectively, on $\mathbb{R}^n$ such that $u(x) \geqslant v(x) + c$.*

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