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COMPTES RENDUS MATHEMATIQUE

Partial Differential Equations

A product estimate for Ginzburg–Landau and application to the gradient-flow

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Abstract

We prove a new inequality for the Jacobian (or vorticity) associated to the Ginzburg–Landau energy in any dimension, and give static and dynamical corollaries. We then present a method to prove convergence of gradient-flows of families of energies which Gamma-converge to a limiting energy, which we apply to establish, thanks to the previous dynamical estimate, the limiting dynamical law of a finite number of vortices for the heat-flow of Ginzburg–Landau in dimension 2, with and without magnetic field. ***To cite this article: E. Sandier, S. Serfaty, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Une estimée-produit pour Ginzburg–Landau, et application au flot-gradient. Nous prouvons une nouvelle inégalite sur le jacobien (ou vorticité) associé à l'énergie de Ginzburg–Landau en dimension quelconque, et en donnons des corollaires statiques et dynamiques. Nous présentons ensuite une méthode pour prouver la convergence de flots-gradient associés à une famille d'énergies qui Gamma-convergent vers une énergie limite, que nous appliquons pour établir à l'aide de l'estimée dynamique précédemment obtenue, la loi limite de la dynamique d'un nombre fini de vortex pour le flot (de la chaleur) de Ginzburg–Landau en dimension 2 avec et sans champ magnétique. ***Pour citer cet article : E. Sandier, S. Serfaty, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

On s'intéresse à la la limite asymptotique $\varepsilon \to 0$ de l'énergie de Ginzburg–Landau en dimension quelconque : $E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2$, où Ω est un domaine régulier borné de $\mathbb{R}^n$, $n \geqslant 2$, et u est à valeurs complexes. L'objet essentiel est l'ensemble $u_\varepsilon = 0$ qui converge vers un ensemble de codimension 2 (vortex pour $n = 2$, lignes de vortex pour $n = 3$), ce sont des singularités topologiques lorsque u_ε a un degré non nul autour d'elles.

Minoration. Pour tout u, on introduit la 1-forme $(\mathrm{i}u, \mathrm{d}u) = \sum_{k=1}^n (\mathrm{i}u, \partial_k u)\, \mathrm{d}x_k$, où $(\cdot,\cdot)$ désigne le produit scalaire dans $\mathbb{C}$ identifié avec $\mathbb{R}^2$ c.a.d. $(a, b) = \frac{\bar{a}b + a\bar{b}}{2}$. On introduit ensuite le jacobien $Ju = \frac{1}{2}d(\mathrm{i}u, \mathrm{d}u)$ vu ici comme une

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2-forme (il contient l'information de la « vorticité » de u). On obtient le résultat suivant qui améliore (et permet de retrouver) les minorations connues :

Théorème 1 [9]. *Soit u_ε une famille de $H^1(\Omega,\mathbb{C})$ telle que $E_\varepsilon(u_\varepsilon)\leqslant N_\varepsilon|\log\varepsilon|$, où $N_\varepsilon\leqslant C|\log\varepsilon|^q$. Quitte à extraire, $\frac{Ju_\varepsilon}{N_\varepsilon}\rightharpoonup J$ dans $(C_c^{0,\gamma}(\Omega))'$, $\forall\gamma>0$, où J est une 2-forme à valeurs mesures bornées, et si N_ε est indépendent de ε, la limite de $\frac{Ju_\varepsilon}{\pi}$ est en outre un courant rectifiable de multiplicité entière. De plus, pour tous champs de vecteurs X et Y dans $C_c^0(\Omega)$, $|X\cdot\nabla u_\varepsilon|/\sqrt{N_\varepsilon|\log\varepsilon|}$ et $|Y\cdot\nabla u_\varepsilon|/\sqrt{N_\varepsilon|\log\varepsilon|}$ sont bornés dans L^2 et notant ν_X et ν_Y leurs mesures de défaut de convergence, on a $\|\nu_X\|^{1/2}\|\nu_Y\|^{1/2}\geqslant|\int_\Omega J(X,Y)|$.*

On déduit plusieurs nouveaux corollaires dont le suivant s'appliquant au cas dynamique : considérant une famille $u_\varepsilon(x,t)$, on peut écrire $Ju_\varepsilon=V_\varepsilon+\mu_\varepsilon$ où μ_ε correspond au jacobien pour les variables d'espace, et V_ε à la partie vitesse du jacobien espace-temps.

Théorème 2 [9]. *Soit $u_\varepsilon(t,x)$ défini sur $[0,T]\times\Omega$ et tel que $\forall t\in[0,T]$, $E_\varepsilon(u_\varepsilon)\leqslant N_\varepsilon|\log\varepsilon|$ et $\int_{[0,T]\times\Omega}|\partial_t u_\varepsilon|^2\leqslant N_\varepsilon|\log\varepsilon|$, il existe $\mu\in L^\infty([0,T],\mathcal{M}(\Omega))$, $C^{0,1/2}$ en temps pour la norme $(W^{1,\infty}(\Omega))'$, et $V\in L^2([0,T],\mathcal{M}(\Omega))$ tels que, après extraction, $\frac{\mu_\varepsilon}{N_\varepsilon}\rightharpoonup\mu$ et $\frac{V_\varepsilon}{N_\varepsilon}\rightharpoonup V$ dans $(C_c^{0,\gamma}([0,T]\times\Omega))'$, $\gamma>0$, avec $\mathrm{d}_t\mu+\mathrm{d}V=0$. De plus, pour tout $X\in C_c^\infty([0,T]\times\Omega,\mathbb{R}^n)$, et $f\in C_c^\infty([0,T])\times\Omega)$, ν_X et ν_T désignant les mesures de défaut de convergence L^2 de $|X\cdot\nabla u_\varepsilon|/\sqrt{N_\varepsilon|\log\varepsilon|}$ et $f|\partial_t u_\varepsilon|/\sqrt{N_\varepsilon|\log\varepsilon|}$, on a $\|\nu_X\|^{1/2}\|\nu_T\|^{1/2}\geqslant|\int_{\Omega\times[0,T]}V\cdot fX|$.*

Gamma-convergence de flots-gradient. On propose le schéma suivant pour obtenir la convergence de solutions de flots-gradient pour des énergies qui Γ-convergent. E_ε est une famille de fonctionelles sur $\mathcal{M}$, qui Γ-converge vers F définie sur $\mathcal{N}$ dans le sens que pour tout $u_\varepsilon\in\mathcal{M}$ tel que $E_\varepsilon(u_\varepsilon)\leqslant C$, quitte à extraire $u_\varepsilon\rightharpoonup^S u\in\mathcal{N}$ (le sens S est à préciser), et $\liminf_{\varepsilon\to0}E_\varepsilon(u_\varepsilon)\geqslant F(u)$. On introduit l'excès d'énergie $D_\varepsilon=E_\varepsilon(u_\varepsilon)-F(u)$, et $D=\limsup_{\varepsilon\to0}D_\varepsilon$. On suppose que E_ε et F sont C^1 sur $\mathcal{M}$ et $\mathcal{N}$, des ouverts d'espaces affines associés à des Banach B et B', sous-espaces des distributions, et que B s'injecte continûment dans un espace de Hilbert X_ε (resp. B' dans Y). Si la différentielle de E_ε, $\mathrm{d}E_\varepsilon(u)$ est aussi une forme linéaire continue sur X_ε, on note $\nabla E_\varepsilon(u)$ le vecteur de X_ε qui la représente (resp. pour $\mathrm{d}F$ sur Y), et on convient que $\|\nabla E_\varepsilon(u)\|_{X_\varepsilon}=+\infty$ sinon. On définit une solution du flot-gradient de E_ε sur $[0,T]$ pour la structure X_ε comme une application $u\in H^1(]0,T[,X_\varepsilon)$ (donc continue sur $[0,T]$) qui vérifie p.p. en $t\in[0,T]$, $\partial_t u(t)=-\nabla E_\varepsilon(u(t))\in X_\varepsilon$. On fait de même pour F.

Théorème 3 [10]. *Soit E_ε et F des fonctionnelles telles que E_ε Γ-converge vers F, qui vérifient les hypothèses ci-dessus et les propriétes suivantes* : (1) *Pour tout u_ε solution sur $[0,T]$ du flot gradient de E_ε pour la structure X_ε, quitte à extraire, $u_\varepsilon(t)\rightharpoonup^S u(t)\in\mathcal{N}$ pour tout $t\in[0,T]$, tel que $u\in H^1(]0,T[,Y)$, et il existe $f\in L^1([0,T])$, tel que pour tout $s\in[0,T]$, $\liminf_{\varepsilon\to0}\int_0^s\|\partial_t u_\varepsilon(t)\|_{X_\varepsilon}^2\,\mathrm{d}t\geqslant\int_0^s(\|\partial_t u\|_Y^2-f(t)D(t))\,\mathrm{d}t$.* (2) *Il existe une fonction g sur $\mathcal{N}$, bornée au voisinage de tout point de $\mathcal{N}$ telle que pour tout $u_\varepsilon\rightharpoonup^S u$, on a $\liminf_{\varepsilon\to0}\|\nabla E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon}^2\geqslant\|\nabla F(u)\|_Y^2-g(u)D$.*

Alors, si u_ε est solution sur $[0,T]$ du flot gradient de E_ε pour la structure X_ε, et si $u_\varepsilon(0)\rightharpoonup^S u(0)\in\mathcal{N}$ avec $D(0)=0$, il existe $0<T^*\leqslant T$ tel que pour tout $t\in[0,T^*[$ on ait $u_\varepsilon(t)\rightharpoonup^S u(t)\in\mathcal{N}$ et $D(t)=0$, et u est solution sur $[0,T^*[$ du flot-gradient de F pour la structure Y.

Application à la dynamique de Ginzburg–Landau. On applique cette méthode (la condition 1 est vérifiée grâce au Théorème 2) pour dériver la dynamique des vortex dans Ginzburg–Landau, retrouvant ainsi le résultat de [7,5]. On obtient aussi dans [9] le résultat analogue, et cette fois nouveau, pour le flot de Ginzburg–Landau avec champ magnétique lorsque le champ appliqué est de l'ordre de $|\log\varepsilon|$ et que le nombre initial de vortex est borné.

1. Introduction

We are interested in the asymptotics as $\varepsilon\to0$ of the Ginzburg–Landau energy in any dimension:

$$E_\varepsilon(u)=\frac{1}{2}\int_\Omega|\nabla u|^2+\frac{1}{2\varepsilon^2}\left(1-|u|^2\right)^2, \tag{1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^n$, with $n \geqslant 2$, and u is complex-valued. This energy is a simple version of the Ginzburg–Landau energy of superconductivity. The crucial set is the zero-set of u: since u is complex-valued, it can have a nonzero degree around its zeroes, they are then called *topological defects*, typically of codimension 2 (point defects called *vortices* in dimension 2, *line vortices* in dimension 3). These codimension 2 sets can be clearly extracted at the limit $\varepsilon \to 0$. We will be interested on the one hand in lower bounds relating the energy or the variations of u to the topology of these defects, or vorticity, and on the other hand in the limiting dynamics for these vortex objects under the heat-flow of (1).

2. The lower bounds

Following [6,1], we consider for any u, the 1-form $ju = (\mathrm{i}u, \mathrm{d}u) = \sum_{k=1}^n (\mathrm{i}u, \partial_k u)\,\mathrm{d}x_k$, where $(\cdot,\cdot)$ denotes the scalar product in $\mathbb{C}$ identified with $\mathbb{R}^2$, i.e., $(a,b) = \frac{\bar{a}b + a\bar{b}}{2}$. From ju, one defines the Jacobian $Ju = \frac{1}{2}d(ju) = \frac{1}{2}d(\mathrm{i}u, \mathrm{d}u)$, seen here as a 2-form. Our main result is the following.

Theorem 2.1 [9]. *Let u_ε be a family of $H^1(\Omega, \mathbb{C})$ such that $E_\varepsilon(u_\varepsilon) \leqslant N_\varepsilon|\log\varepsilon|$, with $N_\varepsilon \leqslant C|\log\varepsilon|^q$. Up to extraction, $\frac{Ju_\varepsilon}{N_\varepsilon} \rightharpoonup J$ in $(C_c^{0,\gamma}(\Omega))'$, $\forall\gamma > 0$, where J is a measure-valued 2-form, and if N_ε is independent of ε, the limit of $\frac{Ju_\varepsilon}{\pi}$ is in addition an integer-multiplicity rectifiable current. Moreover, for all vector-fields X and Y in $C_c^0(\Omega)$, $|X \cdot \nabla u_\varepsilon|/\sqrt{N_\varepsilon|\log\varepsilon|}$ and $|Y \cdot \nabla u_\varepsilon|/\sqrt{N_\varepsilon|\log\varepsilon|}$ are bounded in L^2 and if we denote by ν_X and ν_Y their defect measures, we have*

$$\|\nu_X\|^{1/2}\|\nu_Y\|^{1/2} \geqslant \left|\int_\Omega J(X,Y)\right|. \tag{2}$$

Proof. The proof relies on the same ingredients as the other proofs of lower bounds, i.e., on the ball contruction method of [8,3], but the main new idea is to use a deformation of the metric, and thus a construction with *growing ellipses* instead of balls. □

Observe that one can retrieve a bound on the total mass of the Jacobian J by maximizing (2) over mutually orthogonal vector fields X and Y of norm less than 1. There already exist many results bounding below the Ginzburg–Landau energy by the degrees of the vortices or the mass of the Jacobian (see [2,3,8] in dimension 2, [6,1] in higher dimensions...). Our result allows to treat the case of possibly unbounded numbers of vortices, and it yields a sharp inequality, somewhat stronger than the existing ones (which it implies) thanks to its "product"-character. In the case of $N_\varepsilon \leqslant C$, i.e., bounded vorticity, one retrieves the results of [1,6]. One can also deduce the following new result.

Corollary 2.2 [9] $(n = 2)$. *If $E_\varepsilon(u_\varepsilon) \leqslant C|\log\varepsilon|$ and $Ju_\varepsilon \rightharpoonup J = \pi\sum_{i=1}^k d_i\delta_{a_i}$, then $\liminf_{\varepsilon\to 0} \frac{1}{|\log\varepsilon|}(\int_\Omega |\partial_1 u_\varepsilon|^2 \times \int_\Omega |\partial_2 u_\varepsilon|^2)^{1/2} \geqslant \pi\sum_{i=1}^k |d_i|$. Moreover, if $\frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 \leqslant \pi k|\log\varepsilon|(1 + \mathrm{o}(1))$, then $\forall X, Y \in C_c^0(\Omega, \mathbb{R}^2)$, $\lim_{\varepsilon\to 0} \frac{1}{|\log\varepsilon|}\int_\Omega (X \cdot \nabla u_\varepsilon, Y \cdot \nabla u_\varepsilon) = \pi\sum_{i=1}^k X(a_i) \cdot Y(a_i)$.*

This result asserts that if a vortex of degree ± 1 carries only an energy of order $\pi|\log\varepsilon|$ then the energy of its gradient projected in any direction is the same and equal to half its Dirichlet energy (take for example $X = Y = e_1$).

In the more general case of $N_\varepsilon \leqslant C|\log\varepsilon|^q$, we deduce from Theorem 2.1. We also state in [9] a version of this result for the energy with magnetic field.

2.1. *Dynamical version*

We now consider families u_ε which depend both on space and time. We can take in Theorem 2.1 the first coordinate to be time and the remaining n coordinates to be space. In that framework we have $ju = (\mathrm{i}u, \partial_t u)\,\mathrm{d}t +$

$(\mathrm{i}u, \mathrm{d}_{sp}u)$, where $\mathrm{d}_{sp}u$ denotes the differential with respect to the space coordinates only. We can split the total Jacobian Ju between time and space coordinates and write $Ju = V + \mu$, where $\mu = \frac{1}{2}\mathrm{d}_{sp}(\mathrm{i}u, \mathrm{d}_{sp}u)$ is the space-only Jacobian (i.e., the vorticity), and $V = \sum_{i=1}^{n} V_i\,\mathrm{d}t \wedge \mathrm{d}x_i$, can be identified with a vector-field (it corresponds to the velocity part of the Jacobian). Since the form Ju is closed, i.e., $\mathrm{d}Ju = 0$, we have the transport equation $\mathrm{d}_t\mu + \mathrm{d}V = 0$ where d_t denotes the differential with respect to the time variable only. $\mathcal{M}(\Omega)$ will denote bounded Radon measures on Ω.

Theorem 2.3 [9]. *Let $u_\varepsilon(t,x)$ be defined over $[0,T]\times\Omega$ (with $\Omega \subset \mathbb{R}^n$) and be such that $\forall t \in [0,T]$, $E_\varepsilon(u_\varepsilon) \leqslant N_\varepsilon|\log\varepsilon|$ and $\int_{[0,T]\times\Omega} |\partial_t u_\varepsilon|^2 \leqslant N_\varepsilon|\log\varepsilon|$, V_ε and μ_ε being defined as previously, there exist $\mu \in L^\infty([0,T], \mathcal{M}(\Omega))$, $C^{0,1/2}$ in time for the $(W^{1,\infty}(\Omega))'$-norm, and $V \in L^2([0,T], \mathcal{M}(\Omega))$ such that, after extraction, $\frac{\mu_\varepsilon}{N_\varepsilon} \rightharpoonup \mu$ and $\frac{V_\varepsilon}{N_\varepsilon} \rightharpoonup V$ in $(C_0^{0,\gamma}([0,T]\times\Omega))'$, $\forall\gamma > 0$, with $\mathrm{d}_t\mu + \mathrm{d}V = 0$. Moreover, for all $X \in C_c^0([0,T]\times\Omega, \mathbb{R}^n)$, and $f \in C_c^0([0,T])\times\Omega)$, denoting by ν_X and ν_T the defect measures of L^2 convergence of $|X\cdot\nabla u_\varepsilon|/\sqrt{N_\varepsilon|\log\varepsilon|}$ and $f|\partial_t u_\varepsilon|/\sqrt{N_\varepsilon|\log\varepsilon|}$, we have $\|\nu_X\|^{1/2}\|\nu_T\|^{1/2} \geqslant |\int_{\Omega\times[0,T]} V\cdot fX|$.*

In two space dimensions, we retrieve as a corollary Proposition 3 of [4].

3. Gamma-convergence of gradient-flows

Definition 3.1. Let E_ε and F be functionals defined over $\mathcal{M}$ and $\mathcal{N}$. We say that E_ε Γ-converges to F in the sense S if for all $u_\varepsilon \in \mathcal{M}$ such that $E_\varepsilon(u_\varepsilon) \leqslant C$ there exists a subsequence of u_ε (still denoted u_ε) and $u \in \mathcal{N}$ such that $u_\varepsilon \rightharpoonup^S u$ and $\liminf_{\varepsilon\to0} E_\varepsilon(u_\varepsilon) \geqslant F(u)$. The sense S is to be specified each time, it can be a weak convergence of u_ε in a certain norm, it can be a convergence of some function of u_ε... Notice that u_ε and u do not necessarily belong to the same space, and that we do not require the usual upper bound for Γ-convergence to be satisfied. When E_ε Γ-converges to F and $u_\varepsilon \rightharpoonup^S u$ as in Definition 3.1, we define the energy-excess $D_\varepsilon = E_\varepsilon(u_\varepsilon) - F(u)$ and $D = \limsup_{\varepsilon\to0} D_\varepsilon$. Observe that we must have $\liminf_{\varepsilon\to0} D_\varepsilon \geqslant 0$.

We assume that E_ε and F are C^1 on $\mathcal{M}$ and $\mathcal{N}$, open subsets of some affine spaces associated to some Banach spaces B and B', subspaces of the distributions, and that B embeds continuously into a Hilbert space X_ε (resp. B' into Y). If the differential of E_ε, $\mathrm{d}E_\varepsilon(u)$, is also linear continuous on X_ε, we denote by $\nabla E_\varepsilon(u)$ the vector of X_ε which represents it (resp. for $\mathrm{d}F$ in Y), with the convention that $\|\nabla E_\varepsilon(u)\|_{X_\varepsilon} = +\infty$ otherwise. We define a solution of the gradient-flow of E_ε on $[0,T]$ for the structure X_ε as a map $u \in H^1(]0,T[, X_\varepsilon)$ (hence continuous in $[0,T]$) which satisfies $\partial_t u(t) = -\nabla E_\varepsilon(u(t)) \in X_\varepsilon$ for a.e. $t \in [0,T]$. And similarly for F.

Theorem 3.2 [10]. *Let E_ε and F be functionals such that E_ε Γ-converges to F, with the above assumptions, and which satisfy* (1) *and either* (2) *or* (2′) *below*: (1) (lower bound) *For any u_ε solution of the gradient flow of E_ε in $[0,T]$ for the structure X_ε, there exists a subsequence such that $u_\varepsilon(t) \rightharpoonup^S u(t) \in \mathcal{N}$ for $t \in [0,T]$ with $u \in H^1([0,T], Y)$, and there exists $f \in L^1([0,T])$, such that for every $s \in [0,T]$,*

$$\liminf_{\varepsilon\to0} \int_0^s \|\partial_t u_\varepsilon(t)\|_{X_\varepsilon}^2\,\mathrm{d}t \geqslant \int_0^s \left(\|\partial_t u\|_Y^2 - f(t)D(t)\right)\mathrm{d}t. \tag{3}$$

(2) (construction) *There exists a function g on $\mathcal{N}$, bounded on the neighborhood of every $u \in \mathcal{N}$ such that, given any $u_\varepsilon \rightharpoonup^S u$ with $D = \limsup_{\varepsilon\to0} E_\varepsilon(u_\varepsilon) - F(u)$, for any v defined in a neighborhood of 0 satisfying $v(0) = u$ and $\partial_t v(0) = -\nabla F(u)$, there exists $v_\varepsilon(t)$ such that $v_\varepsilon(0) = u_\varepsilon$ and*

$$\begin{aligned} &\limsup_{\varepsilon\to0}\|\partial_t v_\varepsilon(0)\|_{X_\varepsilon}^2 \leqslant \|\partial_t v(0)\|_Y^2 + g(u)D,\\ &\liminf_{\varepsilon\to0}\left(-\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} E_\varepsilon(v_\varepsilon)\right) \geqslant -\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} F(v) - g(u)D. \end{aligned} \tag{4}$$

(2′) *There exists a function g on $\mathcal{N}$, bounded on the neighborhood of every $u \in \mathcal{N}$ such that for any $u_\varepsilon \rightharpoonup^S u$ with $D = \limsup_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) - F(u)$, we have $\liminf_{\varepsilon\to 0} \|\nabla E_\varepsilon(u_\varepsilon)\|^2_{X_\varepsilon} \geqslant \|\nabla F(u)\|^2_Y - g(u)D$.*

Assume u_ε is a solution of the gradient-flow $\partial_t u_\varepsilon = -\nabla E_\varepsilon(u_\varepsilon)$ on $[0,T]$ (for the structure X_ε) such that $u_\varepsilon(0) \rightharpoonup^S u(0) \in \mathcal{N}$ and $D_\varepsilon(0) = \mathrm{o}(1)$. Then, there exists $0 < T^* \leqslant T$ such that for $t \in [0,T^*)$, $u_\varepsilon(t) \rightharpoonup^S u(t) \in \mathcal{N}$, $D_\varepsilon(t) = \mathrm{o}(1)$, all the inequalities above are equalities, and u solves the gradient-flow of F, $\partial_t u = -\nabla F(u)$ in $[0,T^*)$ for the structure Y. Moreover, if (2′) is satisfied, it yields for all t_0 a $v_\varepsilon^{t_0}(t)$ defined in a neighborhood of 0, associated to $u_\varepsilon(t_0) \rightharpoonup^S u(t_0)$, and we have for all $T_1 < T^*$, $\lim_{\varepsilon\to 0}\int_0^{T_1} \|\partial_t u_\varepsilon - \partial_t v_\varepsilon^t(0)\|^2_{X_\varepsilon} = 0$.

In order to prove convergence of the gradient-flow in concrete situations, *as long as there is a classical flow for the limiting problem*, it suffices to prove that (1) and (2), or (1) and (2′) are satisfied. In fact we prove that (2) implies (2′), (2) is thus proposed as a scheme to prove (2′) via an explicit construction.

Idea of the proof. It relies on the idea that the gradient-flow is the steepest descent. If (1) and (2′) are satisfied without the D term, since $\partial_t u_\varepsilon = -\nabla E_\varepsilon(u_\varepsilon)$, we have

$$E_\varepsilon\big(u_\varepsilon(0)\big) - E_\varepsilon\big(u_\varepsilon(t)\big) = \int_0^t \big(-\nabla E_\varepsilon\big(u_\varepsilon(s)\big), \partial_t u_\varepsilon\big)_{X_\varepsilon}\,\mathrm{d}s = \int_0^t \frac{1}{2}\big\|\nabla E_\varepsilon\big(u_\varepsilon(s)\big)\big\|^2_{X_\varepsilon} + \frac{1}{2}\|\partial_t u_\varepsilon\|^2_{X_\varepsilon}\,\mathrm{d}s. \tag{5}$$

But on one hand we start from an initial data such that $E_\varepsilon(u_\varepsilon(0)) = F(u_\varepsilon(0)) + \mathrm{o}(1)$, and by Γ-convergence at all time t, $E_\varepsilon(u_\varepsilon(t)) \geqslant F(u(t)) + \mathrm{o}(1)$. Using (1) and (2′) on the other hand, we will be able to bound this from below

$$\begin{aligned} F\big(u(0)\big) - F\big(u(t)\big) &\geqslant E_\varepsilon\big(u_\varepsilon(0)\big) - E_\varepsilon\big(u_\varepsilon(t)\big) + \mathrm{o}(1) \geqslant \int_0^t \frac{1}{2}\big\|\nabla F\big(u(s)\big)\big\|^2_Y + \frac{1}{2}\|\partial_t u\|^2_Y\,\mathrm{d}s \\ &\geqslant \int_0^t \big(-\nabla F\big(u(s)\big), \partial_t u\big)_Y\,\mathrm{d}s = F\big(u(0)\big) - F\big(u(t)\big), \end{aligned} \tag{6}$$

and thus there must be equality in (6) which implies $\partial_t u = -\nabla F(u)$ a.e. The case with nonzero energy defect D_ε is handled with Gronwall's lemma, proving that if D_ε is initially $\mathrm{o}(1)$, it remains so for all times. □

4. Application to Ginzburg–Landau dynamics

We apply the method of Theorem 3.2 to derive Ginzburg–Landau dynamics in $\Omega \subset \mathbb{R}^2$. Condition (1) of Theorem 3.2 is fulfilled thanks to Theorem 2.4 and condition (2) via a construction. In both cases (with and without magnetic field) the sense of convergence S that we use is $u_\varepsilon \rightharpoonup^S u = (a_1,\dots,a_n)$ if $\star d(\mathrm{i}u_\varepsilon, \mathrm{d}u_\varepsilon) \rightharpoonup 2\pi\sum_i d_i\delta_{a_i}$. We assume that the vortices are initially of degree ± 1, thus remain such until collision time. The structure we need is $\|\cdot\|_{X_\varepsilon} = |\log\varepsilon|^{-1/2}\|\cdot\|_{L^2(\Omega)}$ and the structure on $Y = \mathbb{R}^n$ is $\pi^{-1/2}$ times the Euclidean norm, while $\mathcal{N}$ is $\Omega^n \setminus \bigcup_{i\neq j}\{a_i = a_j\}$.

In the case without magnetic field, we will recover the following result, in which W is the renormalized energy as defined in [2]. Well-prepared means that the energy of $u_\varepsilon(0)$ is initially equal to its minimum (up to $\mathrm{o}(1)$) for $(a_1,\dots,a_n)$ given.

Theorem 4.1. *Let u_ε be a family of solutions of*

$$\frac{1}{|\log\varepsilon|}\partial_t u = \Delta u + \frac{1}{\varepsilon^2}\big(1 - |u|^2\big) \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial\Omega \quad \text{or} \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

We assume that $u_\varepsilon(0)$ is such that $\star d(\mathrm{i}u_\varepsilon, \mathrm{d}u_\varepsilon) \rightharpoonup 2\pi\sum_{i=1}^n d_i\delta_{a_i(0)}$ with $d_i = \pm 1$, and that u_ε is well-prepared. Then, there exists a time T^* such that, for all $t \in [0,T^*)$, $\star d(\mathrm{i}u_\varepsilon, \mathrm{d}u_\varepsilon) \rightharpoonup 2\pi\sum_i d_i\delta_{a_i(t)}$, with $a_i(t)$ all distinct, and *then* $\frac{\mathrm{d}a_i}{\mathrm{d}t} = -\frac{1}{\pi}\partial_i W(a_1(t),\ldots,a_n(t))$. *Moreover, for all* $B_i(t)$ *disjoint open balls centered at* $a_i(t)$, $\mathbf{1}_{B_i(t)}$ *denoting the characteristic function of* $B_i(t)$, *we have for all* $T < T^*$, $\frac{1}{|\log\varepsilon|}\int_{\Omega\times[0,T]} |\partial_t u_\varepsilon - \sum_i \mathbf{1}_{B_i(t)}\frac{\mathrm{d}a_i}{\mathrm{d}t}\cdot\nabla u_\varepsilon|^2 \to 0$ *as* $\varepsilon\to 0$.

We thus recover the same type of result as in [7,5], i.e., the convergence to the flow of the limiting energy, up to collision time, plus the extra estimate (14).

We also derive the dynamics for the full Ginzburg–Landau energy of superconductivity with magnetic field

$$J_\varepsilon(u,A) = 1/2\int_\Omega |\nabla_A u|^2 + |\mathrm{curl}\,A - h_{\mathrm{ex}}|^2 + \frac{1}{2\varepsilon^2}\big(1-|u|^2\big)^2, \tag{7}$$

where the additional unknown is the magnetic potential $A:\Omega\mapsto\mathbb{R}^2$ with $\nabla_A = \nabla - \mathrm{i}A$, and h_{ex} is a parameter: the intensity of the applied field. We consider the case of $h_{\mathrm{ex}} = \lambda|\log\varepsilon|$ with $0<\lambda<\infty$ and the heat flow for Ginzburg–Landau equations as proposed by Gorkov–Eliashberg (see [11]).

The limit of this dynamics as $\varepsilon\to 0$ has been only rigorously established in the case $h_{\mathrm{ex}} = \mathrm{O}(1)$ (for which $\lambda = 0$) in [11]. Our result is thus new. Let ξ_0 be the solution of $-\Delta\xi_0 + \xi_0 + 1 = 0$ in Ω, and $\xi_0 = 0$ on $\partial\Omega$ and let $J_0 = \frac{1}{2}\int_\Omega |\nabla\xi_0|^2 + |\Delta\xi_0 - 1|^2$. With the use of Theorems 2.4 and 3.2, we are able to deduce:

Theorem 4.2 [10]. *Let* (u,A) *be a family of solutions of the heat-flow equations given in* [11]. *We assume that* $(u_\varepsilon, A_\varepsilon)(0)$ *is such that* $\mathrm{curl}((\mathrm{i}u_\varepsilon, \nabla u_\varepsilon - \mathrm{i}A_\varepsilon u_\varepsilon) + A_\varepsilon) \rightharpoonup 2\pi\sum_{i=1}^n d_i\delta_{a_i(0)}$, *with* $d_i = \pm 1$, *and that* $(u_\varepsilon(0), A_\varepsilon(0))$ *is well-prepared in the sense that* $J_\varepsilon(u,A) \leqslant h_{\mathrm{ex}}^2 J_0 + \pi n|\log\varepsilon| + 2\pi h_{\mathrm{ex}}\sum_i d_i\xi_0(a_i(0)) + \mathrm{o}(|\log\varepsilon|)$. *Then, there exists a time* $T^* > 0$ *such that, for all* $t\in[0,T^*)$, $\mathrm{curl}((\mathrm{i}u_\varepsilon, \nabla_{A_\varepsilon}u_\varepsilon) + A_\varepsilon) \rightharpoonup 2\pi\sum_{i=1}^n d_i\delta_{a_i(t)}$ *with* $\forall i$, $\frac{\mathrm{d}a_i}{\mathrm{d}t} = -d_i\lambda\nabla\xi_0(a_i(t))$.

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