A product estimate for Ginzburg–Landau and application to the gradient-flow

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Abstract

We prove a new inequality for the Jacobian (or vorticity) associated to the Ginzburg–Landau energy in any dimension, and give static and dynamical corollaries. We then present a method to prove convergence of gradient-flows of families of energies which Gamma-converge to a limiting energy, which we apply to establish, thanks to the previous dynamical estimate, the limiting dynamical law of a finite number of vortices for the heat-flow of Ginzburg–Landau in dimension 2, with and without magnetic field. To cite this article: E. Sander, S. Serfathy, C. R. Acad. Sci. Paris, Ser. I 336 (2003).

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On s’intéresse à la la limite asymptotique $\varepsilon \to 0$ de l’énergie de Ginzburg–Landau en dimension quelconque : $E_\varepsilon(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2$, où $\Omega$ est un domaine régulier borné de $\mathbb{R}^n$, $n \geq 2$, et $u$ est à valeurs complexes. L’objet essentiel est l’ensemble $u_\varepsilon = 0$ qui converge vers un ensemble de codimension 2 (vortex pour $n = 2$, lignes de vortex pour $n = 3$), ce sont des singularités topologiques lorsque $u_\varepsilon$ a un degré non nul autour d’elles.

Minoration. Pour tout $u$, on introduit la 1-forme $(iu, du) = \sum_{k=1}^n (iu, \partial_k u) dx_k$, où $(\cdot, \cdot)$ désigne le produit scalaire dans $\mathbb{C}$ identifié avec $\mathbb{R}^2$ c.a.d. $(a, b) = \frac{ab+\overline{ab}}{2}$. On introduit ensuite le jacobien $Ju = \frac{1}{2}d(iu, du)$ vu ici comme une
2-forme (il contient l’information de la « vorticité » de $u$). On obtient le résultat suivant qui améliore (et permet de retrouver) les minorations connues :

**Théorème 1** [9]. Soit $u_\varepsilon$ une famille de $H^1(\Omega, \mathbb{C})$ telle que $E_\varepsilon(u_\varepsilon) \leq N_\varepsilon |\log \varepsilon|$, où $N_\varepsilon \leq C |\log \varepsilon|^q$. Quand à extraire, $\frac{J_{u_\varepsilon}}{N_\varepsilon} \rightharpoonup J$ dans $(C_{0,\gamma}^{0,0}(\Omega))'$. $\forall \gamma > 0$, où $J$ est une 2-forme à valeurs mesures bornées, et si $N_\varepsilon$ est indépendent de $\varepsilon$, la limite de $\frac{J_{u_\varepsilon}}{N_\varepsilon}$ est en outre un courant rectifiable de multiplicité entière. De plus, pour tous champs de vecteurs $X$ et $Y$ dans $C^2_0(\Omega)$. $|X \cdot \nabla u_\varepsilon|/\sqrt{N_\varepsilon |\log \varepsilon|}$ et $|Y \cdot \nabla u_\varepsilon|/\sqrt{N_\varepsilon |\log \varepsilon|}$ sont bornés dans $L^2$ et notant $v_X$ et $v_Y$ leurs mesures de défaut de convergence, on a $\|v_X\|^{1/2} \|v_Y\|^{1/2} \geq \int_\Omega J(X, Y)$. On déduit plusieurs nouveaux corollaires dont le suivant s’appliquant au cas dynamique : considérant une famille $u_\varepsilon(x, t)$, on peut écrire $J_{u_\varepsilon} = V_\varepsilon + \mu_\varepsilon$ où $\mu_\varepsilon$ correspond au jacobien pour les variables d’espace, et $V_\varepsilon$ à la partie vitesse du jacobien espace-temps.

**Théorème 2** [9]. Soit $u_\varepsilon(t, x)$ défini sur $[0, T] \times \Omega$ et tel que $\forall t \in [0, T]$. $E_\varepsilon(u_\varepsilon) \leq N_\varepsilon |\log \varepsilon|$ et $\int_{[0,T] \times \Omega} |\partial_t u_\varepsilon|^2 \leq N_\varepsilon |\log \varepsilon|$, il existe $u \in L^\infty([0, T], \mathcal{M}(\Omega))$, $C^{0,1/2}$ en temps pour la norme $(W^{1,\infty}(\Omega))'$, $V \in L^2([0, T], \mathcal{M}(\Omega))$ tels que, après extraction, $\frac{J_{u_\varepsilon}}{N_\varepsilon} \rightharpoonup \mu$ et $\frac{V_{u_\varepsilon}}{N_\varepsilon} \rightharpoonup V$ dans $(C_{0,\gamma}^{0,0}([0, T] \times \Omega))'$. $\forall \gamma > 0$, avec $\partial_t u + dV = 0$. De plus, pour tout $X \in C^\infty([0, T] \times \Omega, \mathbb{R}^p)$, et $f \in C^\infty([0, T]) \times \Omega)$, $v_X$ et $v_Y$ désignant les mesures de défaut de convergence $L^2$ de $|X \cdot \nabla u_\varepsilon|/\sqrt{N_\varepsilon |\log \varepsilon|}$ et $|\partial_t u_\varepsilon|/\sqrt{N_\varepsilon |\log \varepsilon|}$, on a $\|v_X\|^{1/2} \|v_Y\|^{1/2} \geq \int_\Omega J(X, Y)$.

**Gamma-convergence de flots-gradients.** On propose le schéma suivant pour obtenir la convergence de solutions de flots-gradients pour des énergies qui Γ-convergent. $E_\varepsilon$ est une famille de fonctionnelles sur $\mathcal{M}$, qui Γ-converge vers $F$ défini sur $N$ dans le sens que pour tout $u_\varepsilon \in \mathcal{M}$ tel que $E_\varepsilon(u_\varepsilon) \leq C$, quitte à extraire $u_\varepsilon \rightharpoonup^S u \in N$ (le sens $S$ est à préciser), et lim inf $E_\varepsilon(u_\varepsilon) \geq F(u)$. On introduit l’excès d’énergie $D_\varepsilon = E_\varepsilon(u_\varepsilon) - F(u)$, et $D = \limsup \sup \liminf D_\varepsilon$. On suppose que $E_\varepsilon$ et $F$ sont $C^1$ sur $\mathcal{M}$ et $N$, des ouverts d’espaces affines associés à des Banach $B$ et $B'$, sous-espaces des distributions, et que $B$ s’injecte continûment dans un espace de Hilbert $X_\varepsilon$ (resp. $B'$ dans $Y$). Si la différentielle de $E_\varepsilon$, $dE_\varepsilon(u)$ est aussi une forme linéaire continue sur $X_\varepsilon$, on note $\nabla E_\varepsilon(u)$ le vecteur de $X_\varepsilon$ qui la représente (resp. pour $dF$ sur $Y$), et on convient que $\|\nabla E_\varepsilon(u)\|_X = +\infty$ sinon. On définit une solution du flot-gradient de $E_\varepsilon$ sur $[0, T]$ pour la structure $X_\varepsilon$ comme une application $u \in H^1([0, T], X_\varepsilon)$ (donc continue sur $[0, T]$) qui vérifie p.p. en $t \in [0, T]$, $\partial_t u(t) = -\nabla E_\varepsilon(u(t)) \in X_\varepsilon$. On fait de même pour $F$.

**Théorème 3** [10]. Soit $E_\varepsilon$ et $F$ des fonctionnelles telles que $E_\varepsilon$ Γ-converge vers $F$, qui vérifient les hypothèses ci-dessus et les propriétés suivantes : (1) Pour tout $u_\varepsilon$ solution sur $[0, T]$ du flot gradient de $E_\varepsilon$ pour la structure $X_\varepsilon$, quitte à extraire, $u_\varepsilon(t) \rightharpoonup^S u(t) \in N$ tout $t \in [0, T]$, tel que $u \in H^1([0, T], Y)$, et il existe $f \in L^1([0, T])$, tel que pour tout $s \in [0, T]$, lim inf $\int_\Omega f_s \|\partial_t u_\varepsilon(s)\|_{X_\varepsilon}^2$ $d\mathcal{H} \geq \int_\Omega f(t) \|\partial_t u(t)\|_{Y}^2 - f(t) D(t)$ $d\mathcal{H}$. (2) Il existe une fonction $g$ sur $N$, bornée au voisinage de tout point de $N$ telle que pour tout $u_\varepsilon \rightharpoonup^S u$, on a lim inf $\|\nabla E_\varepsilon(u)\|_{X_\varepsilon}^2 \geq \|\nabla F(u)\|_{Y}^2 - g(u) D$.

Alors, si $u_\varepsilon$ est solution sur $[0, T]$ du flot gradient de $E_\varepsilon$ pour la structure $X_\varepsilon$, et si $u_\varepsilon(0) \rightharpoonup^S u(0) \in N$ avec $D(0) = 0$, il existe $0 < T^* \leq T$ tel que pour tout $t \in [0, T^*]$, on ait $u_\varepsilon(t) \rightharpoonup^S u(t) \in N$ et $D(t) = 0$, et $u$ est solution sur $[0, T^*]$ du flot-gradient de $F$ pour la structure $Y$.

**Application à la dynamique de Ginzburg–Landau.** On applique cette méthode (la condition 1 est vérifiée grâce au Théorème 2) pour dériver la dynamique des vortex dans Ginzburg–Landau, retrouvant ainsi le résultat de [7,5]. On obtient aussi dans [9] le résultat analogue, et cette fois nouveau, pour le flot de Ginzburg–Landau avec champ magnétique lorsque le champ appliqué est de l’ordre de $|\log \varepsilon|$ et que le nombre initial de vortex est borné.

1. Introduction

We are interested in the asymptotics as $\varepsilon \to 0$ of the Ginzburg–Landau energy in any dimension:

$$E_\varepsilon(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2,$$

(1)
where \( \Omega \) is a smooth bounded domain of \( \mathbb{R}^n \), with \( n \geq 2 \), and \( u \) is complex-valued. This energy is a simple version of the Ginzburg–Landau energy of superconductivity. The crucial set is the zero-set of \( u \) since \( u \) is complex-valued, it can have a nonzero degree around its zeroes, they are then called topological defects, typically of codimension 2 (point defects called vortices in dimension 2, line vortices in dimension 3). These codimension 2 sets can be clearly extracted at the limit \( \varepsilon \to 0 \). We will be interested on the one hand in lower bounds relating the energy or the variations of \( u \) to the topology of these defects, or vorticity, and on the other hand in the limiting dynamics for these vortex objects under the heat-flow of (1).

2. The lower bounds

Following [6,1], we consider for any \( u \), the 1-form \( j_u = (iu, du) = \sum_{k=1}^{n} (iu, \partial_k u) \, dx_k \), where \( (.,.) \) denotes the scalar product in \( \mathbb{C}^2 \), i.e., \( (a, b) = \frac{ab+ba}{2} \). From \( j_u \), one defines the Jacobian \( J_u = \frac{1}{2} d(j_u) = \frac{1}{2} d(iu, du) \), seen here as a 2-form. Our main result is the following.

**Theorem 2.1** [9]. Let \( u_\varepsilon \) be a family of \( H^1(\Omega, \mathbb{C}) \) such that \( E_\varepsilon(u_\varepsilon) \leq N_\varepsilon |\log \varepsilon| \), with \( N_\varepsilon \leq C |\log \varepsilon|^q \). Up to extraction, \( \frac{J_u}{\varepsilon} \rightharpoonup J \) in \( (C^0_c(\Omega))' \), \( \forall \gamma > 0 \), where \( J \) is a measure-valued 2-form, and if \( N_\varepsilon \) is independent of \( \varepsilon \), the limit of \( \frac{J_u}{\varepsilon} \) is in addition an integer-multiplicity rectifiable current. Moreover, for all vector-fields \( X \) and \( Y \) in \( C^0_c(\Omega) \), \( |X \cdot \nabla u_\varepsilon|/\sqrt{N_\varepsilon |\log \varepsilon|} \) and \( |Y \cdot \nabla u_\varepsilon|/\sqrt{N_\varepsilon |\log \varepsilon|} \) are bounded in \( L^2 \) and if we denote by \( \nu_X \) and \( \nu_Y \) their defect measures, we have

\[
\|\nu_X\|^{1/2} \|\nu_Y\|^{1/2} \geq \int_{\Omega} |J(X, Y)|.
\]

**Proof.** The proof relies on the same ingredients as the other proofs of lower bounds, i.e., on the ball contraction method of [8,3], but the main new idea is to use a deformation of the metric, and thus a construction with growing ellipses instead of balls. \( \square \)

Observe that one can retrieve a bound on the total mass of the Jacobian \( J \) by maximizing (2) over mutually orthogonal vector fields \( X \) and \( Y \) of norm less than 1. There already exist many results bounding below the Ginzburg–Landau energy by the degrees of the vortices or the mass of the Jacobian (see [2,3,8] in dimension 2, [6,1] in higher dimensions...). Our result allows to treat the case of possibly unbounded numbers of vortices, and it yields a sharp inequality, somewhat stronger than the existing ones (which it implies) thanks to its “product”-character. In the case of \( N_\varepsilon \leq C \), i.e., bounded vorticity, one retrieves the results of [1,6]. One can also deduce the following new result.

**Corollary 2.2** [9] \((n = 2)\). If \( E_\varepsilon(u_\varepsilon) \leq C |\log \varepsilon| \) and \( J u_\varepsilon \rightharpoonup J = \pi \sum_{i=1}^k d_i \delta_{a_i} \), then \( \lim_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} (\int_{\Omega} |\partial_\varepsilon u_\varepsilon|^2 \times (\int_{\Omega} |\partial_\varepsilon u_\varepsilon|^2)^{1/2} = \pi \sum_{i=1}^k |d_i| \). Moreover, if \( \frac{1}{2} \int_{\Omega} |\nabla u_\varepsilon|^2 \leq \pi k |\log \varepsilon|(1 + o(1)) \), then \( \forall X, Y \in C^0_c(\Omega, \mathbb{R}^2) \), \( \lim_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \int_{\Omega} (X \cdot \nabla u_\varepsilon, Y \cdot \nabla u_\varepsilon) = \pi \sum_{i=1}^k X(a_i) \cdot Y(a_i) \).

This result asserts that if a vortex of degree \( \pm 1 \) carries only an energy of order \( \pi |\log \varepsilon| \) then the energy of its gradient projected in any direction is the same and equal to half its Dirichlet energy (take for example \( X = Y = e_1 \)).

In the more general case of \( N_\varepsilon \leq C |\log \varepsilon|^q \), we deduce from Theorem 2.1. We also state in [9] a version of this result for the energy with magnetic field.

2.1. Dynamical version

We now consider families \( u_\varepsilon \) which depend both on space and time. We can take in Theorem 2.1 the first coordinate to be time and the remaining \( n \) coordinates to be space. In that framework we have \( j_u = (iu, \partial_t u) \, dt + \).
Theorem 2.3. Let \( u_\varepsilon(t, x) \) be defined over \([0, T] \times \Omega\) (with \( \Omega \subset \mathbb{R}^n \)) and be such that \( \forall t \in [0, T], E_\varepsilon(u_\varepsilon) \leq N_\varepsilon \log \varepsilon \) and \( \int_{[0,T] \times \Omega} |\partial_x u_\varepsilon|^2 \leq N_\varepsilon \log \varepsilon \). \( V_\varepsilon \) and \( M(\Omega) \) such that, after extraction, \( \frac{\varepsilon}{\mu} \to \mu \) and \( \frac{\varepsilon}{\mu} \to V \) in \( (C_0^0([0, T] \times \Omega))^r \), \( \forall \gamma > 0 \), with \( d_\varepsilon \mu + dV = 0 \). Moreover, for all \( X \in C_0^0([0, T] \times \Omega, \mathbb{R}^n) \), and \( f \in C_0^0([0, T]) \), denoting by \( v_X \) and \( v_T \) the defect measures of \( L^2 \) convergence of \( |X \cdot \nabla u_\varepsilon|/\sqrt{N_\varepsilon \log \varepsilon} \) and \( f |\partial_x u_\varepsilon|/\sqrt{N_\varepsilon \log \varepsilon} \), we have \( \|v_X\|_2 \|v_T\|_2 \geq \| f \|_{\Omega \times [0,T]} \) \( V \cdot F \).

In two space dimensions, we retrieve as a corollary Proposition 3 of [4].

3. Gamma-convergence of gradient-flows

Definition 3.1. Let \( E_\varepsilon \) and \( F \) be functionals defined over \( \mathcal{M} \) and \( \mathcal{N} \). We say that \( E_\varepsilon \) \( \Gamma \)-converges to \( F \) in the sense \( S \) if for all \( u_\varepsilon \in \mathcal{M} \) and \( F(u_\varepsilon) \leq C \) there exists a subsequence of \( u_\varepsilon \) (still denoted \( u_\varepsilon \)) and \( u \in \mathcal{N} \) such that \( u_\varepsilon \to^S u \) and \( \liminf_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) \geq F(u) \). The sense \( S \) is to be specified each time, it can be a weak convergence of \( u_\varepsilon \) in a certain norm, it can be a convergence of some function of \( u_\varepsilon \),... Notice that \( u_\varepsilon \) and \( u \) do not necessarily belong to the same space, and that we do not require the usual upper bound for \( \Gamma \)-convergence to be satisfied.

When \( E_\varepsilon \) \( \Gamma \)-converges to \( F \) and \( u_\varepsilon \to^S u \) as in Definition 3.1, we define the energy-excess \( D_\varepsilon = E_\varepsilon(u_\varepsilon) - F(u) \) and \( D = \limsup_{\varepsilon \to 0} D_\varepsilon \). Observe that we must have \( \liminf_{\varepsilon \to 0} D_\varepsilon \geq 0 \).

We assume that \( E_\varepsilon \) and \( F \) are \( C^1 \) on \( \mathcal{M} \) and \( \mathcal{N} \), open subsets of some affine spaces associated to some Banach spaces \( B \) and \( B' \), subspaces of the distributions, and that \( B \) embeds continuously into a Hilbert space \( X_\varepsilon \) (resp. \( B' \) into \( Y \)). If the differential of \( E_\varepsilon \), \( dE_\varepsilon(u) \), is also linear continuous on \( X_\varepsilon \), we denote by \( \nabla E_\varepsilon(u) \) the vector of \( X_\varepsilon \) which represents it (resp. for \( dF \) in \( Y \)), with the convention that \( \nabla E_\varepsilon(u)\|_{\mathcal{M}} = +\infty \) otherwise. We define a solution of the gradient-flow of \( E_\varepsilon \) on \([0, T]\) for the structure \( X_\varepsilon \) as a map \( u \in H^1([0, T], X_\varepsilon) \) (hence continuous in \([0, T]\)) which satisfies \( \partial_t u(t) = -\nabla E_\varepsilon(u(t)) \) in \( X_\varepsilon \) for a.e. \( t \in [0, T] \). And similarly for \( F \).

Theorem 3.2 [10]. Let \( E_\varepsilon \) and \( F \) be functionals such that \( E_\varepsilon \) \( \Gamma \)-converges to \( F \), with the above assumptions, and which satisfy (1) and either (2) or (2') below: (1) (lower bound) For any \( u_\varepsilon \), solution of the gradient flow of \( E_\varepsilon \) in \([0, T]\) for the structure \( X_\varepsilon \), there exists a subsequence such that \( u_\varepsilon(t) \to^S u(t) \in \mathcal{N} \) for \( t \in [0, T] \) with \( u \in H^1([0, T], Y) \), and there exists \( f \in L^1([0, T]) \), such that for every \( s \in [0, T] \),

\[
\liminf_{\varepsilon \to 0} \int_0^s \|\partial_t u_\varepsilon(t)\|_{X_\varepsilon}^2 \, dt \geq \int_0^s (\|\partial_t u\|_{X_\varepsilon}^2 - f(t)D(t)) \, dt.
\]

(3) (2) construction There exists a function \( g \) on \( \mathcal{N} \), bounded on the neighborhood of every \( u \in \mathcal{N} \) such that, given any \( u_\varepsilon \to^S u \) with \( D = \limsup_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) - F(u) \), for any \( v \) defined in a neighborhood of 0 satisfying \( v(0) = u \) and \( \partial_t v(0) = -\nabla F(u) \), there exists \( v_\varepsilon \) such that

\[
\limsup_{\varepsilon \to 0} \|\partial_t v_\varepsilon(0)\|_{X_\varepsilon}^2 \leq \|\partial_t v(0)\|_{X_\varepsilon}^2 + g(u)D,
\]

\[
\liminf_{\varepsilon \to 0} \left( -\frac{d}{dt}|_{t=0} E_\varepsilon(v_\varepsilon) \right) \geq -\frac{d}{dt}|_{t=0} F(v) - g(u)D.
\]

(4)
There exists a function \( g \) on \( \mathcal{N} \), bounded on the neighborhood of every \( u \in \mathcal{N} \) such that for any \( u_\varepsilon \to ^S u \) with \( D = \lim \sup_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) = F(u) \), we have lim inf \( _{\varepsilon \to 0} \| \nabla E_\varepsilon(u_\varepsilon) \|^2_{\mathcal{X}^\varepsilon} \geq \| \nabla F(u) \|^2_T - g(u).D \).

Assume \( u_\varepsilon \) is a solution of the gradient-flow \( \partial_t u_\varepsilon = -\nabla E_\varepsilon(u_\varepsilon) \) on \([0, T]\) (for the structure \( X_\varepsilon \)) such that \( u_\varepsilon(0) \to ^S u(0) \in \mathcal{N} \) and \( D_\varepsilon(0) = o(1) \). Then, there exists \( 0 < T^* \leq T \) such that for \( t \in [0, T^*) \), \( u_\varepsilon(t) \to ^S u(t) \in \mathcal{N} \). \( D_\varepsilon(t) = o(1) \), all the inequalities above are equalities, and \( u \) solves the gradient-flow of \( F \), \( \partial_t u = -\nabla F(u) \) in \([0, T^*)\) for the structure \( Y \). Moreover, if \((2)\) is satisfied, it yields for all \( t_0 \) a \( v_\varepsilon(t) \) defined in a neighborhood of \( 0 \), associated to \( u_\varepsilon(t_0) \to ^S u(t_0) \), and we have for all \( T_1 < T^* \), \( \lim_{\varepsilon \to 0} \int_{t_0}^{T_1} \| \partial_t u_\varepsilon - \partial_t v_\varepsilon(t) \|^2_{\mathcal{X}^\varepsilon} = 0 \).

In order to prove convergence of the gradient-flow in concrete situations, as long as there is a classical flow for the limiting problem, it suffices to prove that \((1)\) and \((2)\), or \((1)\) and \((2)'\) are satisfied. In fact we prove that \((2)'\) implies \((2)\), \((2)\) is thus proposed as a scheme to prove \((2)\) via an explicit construction.

**Idea of the proof.** It relies on the idea that the gradient-flow is the steepest descent. If \((1)\) and \((2)\) are satisfied without the \( D \) term, since \( \partial_t u_\varepsilon = -\nabla E_\varepsilon(u_\varepsilon) \), we have

\[
E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(t)) = \int_0^t -\nabla E_\varepsilon(u_\varepsilon(s)) \cdot \partial_t u_\varepsilon(s) \, ds = \frac{1}{2} \int_0^t \| \nabla E_\varepsilon(u_\varepsilon(s)) \|^2_{\mathcal{X}^\varepsilon} + \frac{1}{2} \| \partial_t u_\varepsilon \|^2_{\mathcal{X}^\varepsilon} \, ds. \tag{5}
\]

But on one hand we start from an initial data such that \( E_\varepsilon(u_\varepsilon(0)) = F(u_\varepsilon(0)) + o(1) \), and by \( F \)-convergence at all time \( t \), \( E_\varepsilon(u_\varepsilon(t)) \to F(u(t)) + o(1) \). Using \((1)\) and \((2)\) on the other hand, we will be able to bound this from below

\[
F(u(0)) - F(u(t)) \geq E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(t)) + o(1) \geq \int_0^t \| \nabla F(u(s)) \|^2_Y + \frac{1}{2} \| \partial_t u \|^2_T \, ds
\]

and thus there must be equality in \((6)\) which implies \( \partial_t u = -\nabla F(u) \) a.e. The case with nonzero energy defect \( D_\varepsilon \) is handled with Gronwall’s lemma, proving that if \( D_\varepsilon \) is initially \( o(1) \), it remains so for all times. \( \square \)

**4. Application to Ginzburg–Landau dynamics**

We apply the method of Theorem 3.2 to derive Ginzburg–Landau dynamics in \( \Omega \subset \mathbb{R}^2 \). Condition \((1)\) of Theorem 3.2 is fulfilled thanks to Theorem 2.4 and condition \((2)\) via a construction. In both cases (with and without magnetic field) the sense of convergence \( S \) that we use is \( u_\varepsilon \to ^S u = (a_1, \ldots, a_n) \) if \( \ast d(\mu_\varepsilon, \mu_\varepsilon) \to 2\pi \sum d_i \delta_{a_i} \).

We assume that the vortices are initially of degree \( \pm 1 \), thus remain such until collision time. The structure we need is \( \| \cdot \|_X = |\log \varepsilon|^{-1/2} \| \cdot \|_{L^2(\Omega)} \) and the structure on \( Y = \mathbb{R}^n \) is \( \pi^{-1/2} \) times the Euclidean norm, while \( \mathcal{N} \) is \( \Omega^n \setminus \bigcup_{i \neq j} \{ a_i = a_j \} \).

In the case without magnetic field, we will recover the following result, in which \( W \) is the renormalized energy as defined in [2]. Well-prepared means that the energy of \( u_\varepsilon(0) \) is initially equal to its minimum (up to \( o(1) \)) for \((a_1, \ldots, a_n)\) given.

**Theorem 4.1.** Let \( u_\varepsilon \) be a family of solutions of

\[
\frac{1}{|\log \varepsilon|} \partial_t u = \Delta u + \frac{1}{\varepsilon^2} (1 - |u|^2) \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial \Omega \quad \text{or} \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial \Omega.
\]

We assume that \( u_\varepsilon(0) \) is such that \( \ast d(\mu_\varepsilon, \mu_\varepsilon) \to 2\pi \sum_{i=1}^n d_i \delta_{a_i(0)} \) with \( d_i = \pm 1 \), and that \( u_\varepsilon \) is well-prepared. Then, there exists a time \( T^* \) such that, for all \( t \in [0, T^*) \), \( \ast d(\mu_\varepsilon, \mu_\varepsilon) \to 2\pi \sum d_i \delta_{a_i(t)} \), with \( a_i(t) \) all distinct, and...
then \( \frac{d u}{d \varepsilon} = -\frac{1}{\varepsilon} \partial_t W(a_1(t), \ldots, a_n(t)) \). Moreover, for all \( B_i(t) \) disjoint open balls centered at \( a_i(t) \), \( 1_{B_i(t)} \) denoting the characteristic function of \( B_i(t) \), we have for all \( T < T^* \), \( \frac{1}{\log(1/\varepsilon)} \int_{\Omega \times (0, T]} \partial_t u - \sum_i 1_{B_i(t)} \frac{d u}{d \varepsilon} \cdot \nabla u |^2 \to 0 \) as \( \varepsilon \to 0 \).

We thus recover the same type of result as in [7,5], i.e., the convergence to the flow of the limiting energy, up to collision time, plus the extra estimate (14).

We also derive the dynamics for the full Ginzburg–Landau energy of superconductivity with magnetic field

\[
J_\varepsilon(u, A) = 1/2 \int_\Omega |\nabla_A u|^2 + |\text{curl} A - h_{\text{ex}}|^2 + \frac{1}{2\varepsilon^2} (1 - |\nabla A|^2)^2, \tag{7}
\]

where the additional unknown is the magnetic potential \( A : \Omega \to \mathbb{R}^2 \) with \( \nabla A = \nabla - iA \), and \( h_{\text{ex}} \) is a parameter: the intensity of the applied field. We consider the case of \( h_{\text{ex}} = \lambda |\log \varepsilon| \) with \( 0 < \lambda < \infty \) and the heat flow for Ginzburg–Landau equations as proposed by Gorkov–Eliashberg (see [11]).

The limit of this dynamics as \( \varepsilon \to 0 \) has been only rigorously established in the case \( h_{\text{ex}} = O(1) \) (for which \( \lambda = 0 \) in [11]). Our result is thus new. Let \( \xi_0 \) be the solution of \(-\Delta \xi_0 + \xi_0 + 1 = 0 \) in \( \Omega \), and \( \xi_0 = 0 \) on \( \partial \Omega \) and let \( J_0 = \frac{1}{2} \int_\Omega (|\nabla \xi_0|^2 + |\Delta \xi_0 - 1|^2) \). With the use of Theorems 2.4 and 3.2, we are able to deduce:

**Theorem 4.2** [10]. Let \( (u, A) \) be a family of solutions of the heat-flow equations given in [11]. We assume that \((u_\varepsilon, A_\varepsilon)(0)\) is such that \( \text{curl}((iu_\varepsilon, \nabla u_\varepsilon - iA_\varepsilon, A_\varepsilon) \to 2\pi \sum_{i=1}^n d_i \delta_{a_i(0)} \), with \( d_i = \pm 1 \), and that \((u_\varepsilon(0), A_\varepsilon(0))\) is well-prepared in the sense that \( J_\varepsilon(u, A) \leq h_{\text{ex}}^2 J_0 + \pi n |\log \varepsilon| + 2\pi h_{\text{ex}} \sum_i d_i \xi_0(a_i(0)) + o(|\log \varepsilon|) \). Then, there exists a time \( T^* > 0 \) such that, for all \( t \in [0, T^*) \), \( \text{curl}((iu, \nabla u, A) \to 2\pi \sum_{i=1}^n d_i \delta_{a_i(t)} \) with \( \forall i, \frac{d a_i}{d t} = -d_i \lambda \nabla \xi_0(a_i(t)) \).

**References**