Mathematical Analysis

Uncertainty principle and $L^p–L^q$-sufficient pairs on noncompact real symmetric spaces

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Abstract

We consider a real semi-simple Lie group $G$ with finite center and a maximal compact sub-group $K$ of $G$. Let $G = K \exp(\mathfrak{a}_+^\ast)K$ be a Cartan decomposition of $G$. For $x \in G$ denote $\|x\|$ the norm of the $\mathfrak{a}_+$-component of $x$ in the Cartan decomposition of $G$. Let $a > 0$, $b > 0$ and $1 \leq p, q \leq \infty$. In this Note we give necessary and sufficient conditions on $a$, $b$ such that for all $K$-bi-invariant measurable function $f$ on $G$, if $e^{a\|x\|^2} f \in L^p(G)$ and $e^{b\|\lambda\|^2} F(f) \in L^q(\mathfrak{a}_+^\ast)$ then $f = 0$ almost everywhere.

Résumé

Principe d’incertitude et paires $L^p–L^q$-suffisantes sur les espaces symétriques réels non-compacts. On considère un groupe de Lie semi-simple réel $G$ de centre fini et $K$ un sous-groupe compact maximal de $G$. Soit $G = K \exp(\mathfrak{a}_+^\ast)K$ une décomposition de Cartan de $G$. Pour $x \in G$, on note $\|x\|$ la norme de la composante de $x$ dans $\mathfrak{a}_+$. Soient $a > 0$, $b > 0$ et $1 \leq p, q \leq \infty$. Dans cette Note on donne une condition nécessaire et suffisante sur $a$, $b$ telle que pour toute fonction $f$ mesurable et $K$-bi-invariante sur $G$, si $e^{a\|x\|^2} f \in L^p(G)$ et $e^{b\|\lambda\|^2} F(f) \in L^q(\mathfrak{a}_+^\ast)$ alors $f = 0$ presque partout.

1. Introduction

One of the rigorous formalization of the uncertainty principle in the classical Fourier analysis on $\mathbb{R}$, is to study $L^p–L^q$-sufficient pairs of positive functions in the following meaning. A pair of positive functions $(\varphi, \psi)$ is said to be $L^p–L^q$-sufficient, whenever, for all measurable function $f$, the conditions $\varphi^{-1} f \in L^p(\mathbb{R})$ and $\psi^{-1} F(f) \in L^q(\mathbb{R})$ implies that $f = 0$ almost everywhere. When $p = q = \infty$, the pair is simply said to be sufficient as in [10] (p. 128).

This problem has been intensively studied in the literature, in many situations. For example, on $\mathbb{R}$, when $p = q = \infty$ and $(\varphi(x), \psi(\lambda)) = (e^{-ax^2}, e^{-b\lambda^2})$, we obtain the classical Hardy’s theorem, see [9]. Cowling and
Phragmén–Lindelöf type results

Price [5] have proved that the pair \( (e^{-a|x|^2}, e^{-b|x|^2}) \) is \( L^p-L^q \)-sufficient if and only if \( ab \geq 1/4 \) and \( p \) or \( q \) is finite. Analogous of these results have been also studied in [3,7,8,13–15].

Now we consider a real semi-simple Lie group \( G \) with finite center. Let \( G = K \exp(\mathfrak{a}_+^*) K \) be a Cartan decomposition of \( G \). Denote by \( \|z\| \) the norm of the \( \mathfrak{a}_+^* \)-component of \( x \in G \) in the Cartan decomposition. In [12], Narayanan and Ray, have proved that for all \( 1 \leq p \leq \infty \) and for all \( 1 \leq q < \infty \), the pair \( (e^{-a|x|^2}, e^{-b|\xi|^2}) \) defined respectively on \( G \) and \( \mathfrak{a}_+^* \) is \( L^p-L^q \)-sufficient if and only if \( ab \geq 1/4 \) and \( p \) or \( q \) is finite, where \( h_a \) is the heat kernel on \( G \) at time \( a \). We note that this pair gives the correct decay condition to obtain the analogue of the above Cowling and Price result. In the other hand, Sitaram and Sundari in [14], and Cowling, Sitaram and Sundari in [6] have studied the pair \( (e^{-a|x|^2}, e^{-b|\xi|^2}) \) for \( p = q = \infty \).

The aim of this Note is to study the \( L^p-L^q \)-sufficiency of the same pair \( (e^{-a|x|^2}, e^{-b|\xi|^2}) \) for all \( 1 \leq p, q \leq \infty \) and all \( a > 0, b > 0 \).

2. Notations

In this section we introduce some classical notations and results about semi-simple Lie groups. For details we refer to [11].

Let \( G \) be a connected, non compact real semi-simple Lie group with finite center and \( K \) a fixed maximal compact sub-group of \( G \). Take \( \mathfrak{g} = \mathfrak{t} + \mathfrak{p} \) a Cartan decomposition of \( \mathfrak{g} = \text{Lie}(G) \) such that \( \mathfrak{t} = \text{Lie}(K) \). Let \( \mathfrak{a} \) be a maximal abelian subspace of \( \mathfrak{p} \). The associated Killing form defines a scalar product \((\cdot,\cdot)\) on \( \mathfrak{a} \). By duality, we define a scalar product on \( \mathfrak{a}^* \) which can be extended to \( \mathfrak{a}_+^* \) as a hermitian product, denoted also by \((\cdot,\cdot)\). Let \( \|\cdot\| \) be the associated norm. As usual denote by \( W \) the Weyl group and \( \Sigma \) the set of all roots. Let \( \Sigma^+ \) be a fixed set of positive roots, \( \Sigma_0^+ \) the set of positive indivisible roots and \( \mathfrak{a}_+, \mathfrak{a}_+^* \) the corresponding Weyl chambers respectively in \( \mathfrak{a} \) and \( \mathfrak{a}^* \). Let \( \rho = \frac{1}{2} \sum_{\alpha \in \Sigma} m_\alpha \alpha \) be the Weyl group and \( \Sigma \) the set of all roots. We let the Cartan decomposition \( G = K \exp(\mathfrak{a}_+^*) K \). For all \( x \in G \), denote \( |x| = ||x^+|| \) where \( x^+ \) is the \( \mathfrak{a}_+^* \)-component of \( x \) in the above decomposition. For all \( x \in G \), let \( H(x) \) be the unique element in \( a \) such that \( x \in K \exp H(x) N \). The spherical functions on \( G \) are defined by \( \psi_\lambda(x) = \int_K e^{(i\lambda - \rho)(H(xk))} dk, x \in G, \lambda \in \mathfrak{a}_+^* \).

The spherical Fourier transform on \( G \) is defined by \( \mathcal{F}(f)(\lambda) = \int_G f(x) \psi_{-\lambda}(x) dx, \ f \in \mathcal{D}(G) \). Let \( c \) be the Harish–Chandra-function defined on \( \mathfrak{a}^* \). Then the inversion formula is given by \( \mathcal{F}^{-1}(h)(x) = \int_{\mathfrak{a}_+^*} h(\lambda) \psi_\lambda(x)(c(\lambda))^{-1/2} d\lambda, h = \mathcal{F}(f) \).

3. Phragmén–Lindelöf type results

We need some complex analysis results for the proof of the main theorem of this paper.

1. Fix \( G \) a positive measurable function on \([0, \infty[ \). Suppose there exists integer \( k > 0 \) and a reals \( \alpha_0 > 0 \) and \( \varepsilon_0 > 0 \) such that

\[(i) \forall r > 0, \forall x \geq \alpha_0 \gamma(r x) \leq \text{const} \cdot \max(r^k, 1) \gamma(x).
(ii) \forall \sigma > \alpha_0, \ d_\gamma(\sigma) = \int_\sigma^{\sigma+1} \gamma(x) dx \geq \varepsilon_0.

Lemma 3.1. Let \( f \) be an analytic even function on \( \mathbb{C} \). Suppose that for \( 1 \leq q < +\infty \), \( m \in \mathbb{N} \), \( M > 0 \) and a constant \( v > 0 \) we have for all \( z \in \mathbb{C} \)

\[ |f(z)| \leq M (1 + |z|)^m e^{v|z|^2} \quad \text{and} \quad \int_0^{\infty} |f(x)|^q \gamma(x) dx \leq M. \]

Then \( f = 0 \) on \( \mathbb{C} \).
Using the Phragmén–Lindelöf principle (see [10], p. 36) we obtain the following result

**Lemma 3.2.** Let \( f \) be an even analytic function on \( \mathbb{C} \). Suppose that for \( m \in \mathbb{N} \), \( M > 0 \) and \( \nu > 0 \) we have and all \( z \in \mathbb{C} \) and all \( x \in \mathbb{R}^+ \)

\[
|f(z)| \leq M (1 + |z|)^m e^{\nu t} \quad \text{and} \quad |f(x)| \leq M.
\]

Then \( f = \text{const.} \) on \( \mathbb{C} \).

4. The \( L^p-L^q \) version of Hardy’s theorem

We start by the principal theorem of this Note.

**Theorem 4.1.** Let \( 1 \leq p, q \leq \infty \) and \( a > 0, b > 0 \).

If \( 1 \leq p \leq 2 \) then the pair \( (e^{-a|x|^b}, e^{-b|\lambda|^a}) \) is \( L^p-L^q \)-sufficient if and only if \( ab \geq 1/4 \).

If \( 2 \leq p \leq \infty \) and \( ab > 1/4 \), then the pair \( (e^{-a|x|^b}, e^{-b|\lambda|^a}) \) is \( L^p-L^q \)-sufficient.

In Proposition 4.7, we prove that \( ab > 1/4 \) is necessary and sufficient, in the case \( G = \text{SL}(2, \mathbb{C}) \).

The proof of this theorem is a consequence of the following results. For given \( a > 0 \) and \( \mu = (\mu_1, \ldots, \mu_l) \) an \( \mathbb{R}^l \)-invariant and analytic on \( \mathbb{C}^*_+ \). Moreover it satisfies the properties given in the following lemma.

**Lemma 4.2.** Let \( p' \) be the conjugate exponent of \( p \). We have for all \( \lambda = \xi + i\eta \in \mathbb{C}^*_+ \)

\[
\text{if } 1 < p \leq \infty \text{ then } \|\mathcal{F}(f)(\lambda)\| \leq \text{const.} \cdot (1 + \|\eta\|)^d e^{(1/4a)(\xi^2 + \eta^2)^{1/2}};
\]

\[
\text{if } p = 1 \text{ then } \|\mathcal{F}(f)(\lambda)\| \leq \text{const.} \cdot e^{(1/4a)|\eta|^2}.
\]

Let \( \mu_1, \ldots, \mu_l \) be a basis of \( \mathbb{C}^*_+ \) such that \( \mathbb{C}^*_+ = \sum_{i=1}^{l} \mathbb{R}^+_\mu_i \). Let \( A_t = \mu_1 + t_2 \mu_2 + \cdots + t_l \mu_l \) for all \( t = (t_1, \ldots, t_l) \in \mathbb{R}^l_{>0} \). The change of variable \( (x_1, \ldots, x_l) = (x(1, t_2, \ldots, t_l)) \) and Fubini’s theorem gives

**Lemma 4.3.** If \( 1 \leq q < \infty \) and \( e^{b|\lambda|^a} \mathcal{F} f \) is in \( L^q(\mathbb{C}^*_+, |c(\lambda)|^{-2} d\lambda) \) then

\[
\int_0^\infty \left| e^{b|\lambda|^a} \mathcal{F} f(x \Lambda_t) \right|^q |c(x \Lambda_t)|^{-2} x^{l-1} dx < +\infty,
\]

for almost all \( t_2 > 0, \ldots, t_l > 0 \).

**Proposition 4.4.** Let \( 1 \leq p, q \leq \infty \) and \( f \) a \( K \)-bi-invariant measurable function on \( G \) such that

\[
\|e^{a|x|^b} f\|_{L^p(G)} \leq M \quad \text{and} \quad \|e^{b|\lambda|^a} \mathcal{F} f\|_{L^q(\mathbb{C}^*_+, |c(\lambda)|^{-2} d\lambda)} \leq M,
\]

for \( M > 0 \), \( a > 0 \) and \( b > 0 \). If \( ab < 1/4 \) then \( f = 0 \) almost everywhere.

**Proof.** For \( a < a' < 1/4b \) and \( t_2 > 0, \ldots, t_l > 0 \) let \( t = (t_1, \ldots, t_l) \) and \( g_{a',t} : \mathbb{C} \to \mathbb{C} \) be defined by \( g_{a',t}(z) = e^{(1/4a')(|\Lambda_t|^2)} \mathcal{F} f(z \Lambda_t) \) and \( \gamma(x) = |c(x \Lambda_t)|^{-2} x^{l-1} \). For almost \( t_2 > 0, \ldots, t_l > 0 \) Lemma 3.1 or Lemma 3.2 gives that \( g_{a',t} = 0 \) on \( \mathbb{C} \) then \( \mathcal{F} f = 0 \) on \( \mathbb{C}^*_+ \). Hence \( f = 0 \) almost everywhere.

Using similar proof we obtain the following result
Proposition 4.5. Suppose $1 \leq p \leq 2$ and $1 \leq q \leq \infty$. Let $f$ be a $K$-biinvariant measurable function on $G$ such that

$$\|e^{ia|x|^2} f\|_{L^p(G)} \leq M \quad \text{and} \quad \|e^{b|\lambda|^2} \mathcal{F} f(\lambda)\|_{L^q(a_x^+,|c(\lambda)|^{-2}dx)} \leq M,$$

for $M > 0$, $a > 0$ and $b > 0$. If $ab = 1/4$ then $f = 0$ almost everywhere.

The heat kernel $h_a$ is defined for $a > 0$ and is a positive $K$-bi-invariant $C^\infty$-function on $G$. Using Anker’s estimate [1,2] of $h_a$ we obtain

Proposition 4.6. If $ab < 1/4$ then for all $1 \leq p, q \leq \infty$ and $a < t < 1/4b$, $h_t$ verifies

$$\|e^{ia|x|^2} h_t\|_{L^p(G)} < \infty \quad \text{and} \quad \|e^{b|\lambda|^2} \mathcal{F}(h_t)(\lambda)\|_{L^q(a_x^+,|c(\lambda)|^{-2}dx)} < \infty.$$

Now we consider the group $G = SL(2, \mathbb{C})$ as a real Lie group. We take $K = SU(2)$ and

$$a = \left\{ H_x = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} : x \in \mathbb{R} \right\}.$$

We can identify $a$ to $\mathbb{R}$ in such a way that $\|H_x\| = |x|$. For all real $\lambda$, the map $H_x \mapsto \lambda \cdot x$ gives an identification of $a^*$ with $\mathbb{R}$ such that $\|\lambda\|$ in $a^*$ is $|\lambda|$. Under these conditions we have $\varphi_\lambda(x) = \frac{2\sin 2\lambda x}{\lambda \sinh 2\lambda}$, $|c(\lambda)|^{-2} = \frac{\lambda^2}{4}$ and

$$\delta(x) = \frac{4\sinh^2 2x}{4}.$$

For $a \geq 0$, let $g_a$ be defined on $\mathbb{R}$ by $g_a(\lambda) = e^{-\lambda^2/4} \sin^4(a\lambda)/\lambda^4$.

Proposition 4.7. Let $2 < p \leq \infty$ and $1 \leq q \leq \infty$. For all $0 \leq \alpha < \min(1/4, (p-2)/p)$, the functions $f_\alpha = \mathcal{F}^{-1}(g_a)$ verifies

$$\|e^{ia|x|^2} f_\alpha\|_{L^p(G)} < \infty \quad \text{and} \quad \|e^{b|\lambda|^2} \mathcal{F} f_\alpha\|_{L^q(a_x^+,|c(\lambda)|^{-2}dx)} < \infty.$$

References