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C. R. Acad. Sci. Paris, Ser. I 336 (2003) 937–940
COMPTES RENDUS MATHEMATIQUE

Topology

Thin presentation of knots in lens spaces and $\mathbb{R}P^3$-conjecture

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Received 23 January 2003; accepted after revision 17 April 2003

Presented by Étienne Ghys

Abstract

This Note concerns knots in a lens space L that produce S^3 by Dehn surgery. We introduce the thin presentation of knots in L, with respect to a standard spine. We prove that among such knots, those having a thin presentation with only maxima, are 0-bridge or 1-bridge braids in L. In the case $L = \mathbb{R}P^3$, we deduce that minimally braided knots in $\mathbb{R}P^3$ cannot yield S^3 by Dehn surgery. ***To cite this article: A. Deruelle, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Présentation mince des nœuds dans les espaces lenticulaires. Cette Note concerne les nœuds d'un espace lenticulaire L qui produisent S^3 par chirurgies de Dehn. Nous introduisons ici une présentation mince des nœuds de L, par rapport à une épine standard. Nous prouvons alors que parmi ces nœuds, ceux qui possèdent une présentation mince n'ayant que des maxima sont des 0 ou 1-tresses. Dans le cas où $L = \mathbb{R}P^3$, nous déduisons que les nœuds minimalement tressés de $\mathbb{R}P^3$ ne peuvent produire S^3 par chirurgie de Dehn. ***Pour citer cet article : A. Deruelle, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

1. Introduction

This work is about Dehn surgeries on knots in S^3, and what kind of 3-manifolds can arise from such surgeries. In particular, we are interested in the Dehn surgeries that yield lens spaces. We study knots in lens spaces yielding S^3, trying to characterize such knots by introducing their thin presentation.

Let $X = L - \operatorname{Int} N(K)$ be the exterior of the knot K in a lens space L. If α is a slope on the boundary of X and V_α is a solid torus, then the closed 3-manifold obtained by *α-Dehn surgery* on the knot K is defined to be $X(\alpha) = X \cup V_\alpha$, where α bounds a meridional disk in V_α. The core of V_α becomes a knot K_α in $X(\alpha)$, called the *core of the surgery*. Note that the exterior of the knot K_α in $X(\alpha)$, $X(\alpha) - \operatorname{Int} N(K_\alpha)$, is also homeomorphic to X.

For V a Heegaard solid torus of L, a 1-bridge braid in L is a union of a non-essential arc in V and a simple arc on ∂V [7]. A 0-bridge braid is a torus knot on ∂V.

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1631-073X/03/$ – see front matter
doi:10.1016/S1631-073X(03)00218-8

We know that a non-trivial knot in S^3 cannot produce $S^1 \times S^2$ nor S^3 by Dehn surgery [6,11]. In the following, L is assumed to be a lens space (neither S^3 nor $S^1 \times S^2$). Let K be a knot in L that produces S^3 by α-Dehn surgery and let K_α be the core of the surgery in S^3. Then, following Berge [1] and Gordon [9], we conjecture that K is a 0-bridge or a 1-bridge braid in L. The main result of this Note is given in the following theorem.

Theorem 1.1. *Let K be a knot in L yielding S^3 by Dehn surgery. If there exists a standard spine Σ such that a thin presentation of K with respect to Σ has only maxima, then K is a 0-bridge or 1-bridge braid in L. Furthermore, L cannot be $\mathbb{R}P^3$.*

Gordon has conjectured that *Dehn surgery on a non-trivial knot in S^3 cannot yield a lens space of order less than six* [9]. The first lens space in terms of order is $L(2,1) = \mathbb{R}P^3$. So, as a consequence of the Gordon Conjecture, we have the $\mathbb{R}P^3$-conjecture that states [12]: *$\mathbb{R}P^3$ cannot be obtained by Dehn surgery on a non-trivial knot in S^3.* Let a *minimally braided knot* be a knot in $\mathbb{R}P^3$ having a thin presentation with only maxima. Then Theorem 1.1 implies the following corollary.

Corollary 1.2. *A minimally braided knot in $\mathbb{R}P^3$ cannot yield S^3 by Dehn surgery.*

For the details of the proofs of the above results, we refer to [3,4] for the case $L = \mathbb{R}P^3$ in Theorem 1.1 and Corollary 1.2. And in the case where $L \neq \mathbb{R}P^3$ in Theorem 1.1, we refer to [5].

Let us now describe in a few words the sections of this Note and give a sketch of the proof of Theorem 1.1.

We consider a *thin presentation* of K_α, as defined by Gabai [6]. In a similar way, we define here a *thin presentation* of knots in a lens space and an associated *tight presentation*, after fixing a spine Σ in L. This is the goal of Section 2. Let us consider K in such a thin presentation in L. Then S^3 and L each admits a foliation by spheres corresponding to the thin presentations of K_α and K, respectively. In the following, we denote by $\widehat{S}_\alpha$ and $\widehat{S}_\beta$, the level 2-spheres in the S^3 and L-foliations, respectively. Let $S_\alpha = \widehat{S}_\alpha \cap X$ and $S_\beta = \widehat{S}_\beta \cap X$ denote the corresponding level surfaces in X.

Let S_α and S_β be such planar surfaces in X. Then we may assume that S_α and S_β are transverse and in general position. So, we define a pair of intersection planar graphs (G_α, G_β) in the usual way [10]. A *trivial loop* in a planar graph is the boundary of a 1-sided disk-face. By [11,13], G_β cannot represent all types. So, if G_α and G_β contain no trivial loop, by [11, Proposition 2.0.1], G_α must contain a *Scharlemann cycle* [15].

Let us remark that if G_α (resp. G_β) contains a trivial loop, then we say that S_β (resp. S_α) is *High* or *Low* with respect to S_α (resp. S_β) or for short is $\mathcal{H}$ or $\mathcal{L}$, according to the side where the disk-face is, in S_β (resp. S_α), with respect to S_α (resp. S_β) [6,11]. Furthermore, we define here another characteristic: if G_α contains a Scharlemann cycle then we say that S_β is a *Carrier* with respect to S_α, or just $\mathcal{C}$ for short [3–5]. If S_α (resp. S_β) is none of these, we say it is $\mathcal{N}$.

In Section 3, we assume that K has only maxima in its thin (and so its tight) presentation in L. We then produce two one-parameter families of level surfaces, coming from the S^3-foliation and the L-foliation, respectively.

We introduce in Section 4 the graph of singularities associated to the intersection of these two families of level surfaces; for Cerf Theory, we refer to [2]. We then find two planar surfaces S_α in S^3 and S_β in L, and prove that $|K \cap \Sigma| = 1$. Studying the configuration of K we deduce that it is a 0-bridge or a 1-bridge braid (see [3] for the details). In the case $L = \mathbb{R}P^3$, K is then trivial or bounds a $\mathcal{M}$öbius strip, so we have a contradiction [8].

2. Thin presentations

For K_α in S^3, the thin presentation is due to Gabai [6]. For K in $L = L(p,q)$, we remark that L is a closed 3-ball B with an equivalence relation $\sim$ on its boundary which is a $\frac{2\pi q}{p}$-rotation on ∂B. So, $L = B/\sim$ and $\partial B/\sim = \Sigma$ is defined to be a *standard spine* of L [14]. The identified equator of B becomes the singular 1-complex of Σ; this complex $\mathcal{C}$ is the *axis* of the standard spine Σ. Note that if $L = \mathbb{R}P^3$ then $\Sigma = \mathbb{R}P^2$ is a projective plane and $\mathcal{C}$ is not singular.

Now let ∞ be an interior point of B. Then $L - (\Sigma \cup \{\infty\}) \cong S^2 \times \mathbb{R}_+^*$ defines a foliation by 2-spheres. So, we put the knot K in transverse position with respect to this L-foliation. As in S^3, we define the complexity of K to be the sum of the geometric intersections of K with the generic level 2-spheres of the foliation. A *thin presentation* of K is realized for a minimal complexity.

Let us remark that if $L \neq \mathbb{R}P^3$, then K does not intersect the axis of Σ, because of the openness of the property for a knot to be in transverse position with respect to a foliation (for the Morse property see [2,11]). But, for a given thin presentation of K, one can define what we call a *tight presentation*, minimizing $|K \cap \Sigma|$ by allowing intersection with $\mathcal{C}$; this corresponds to first maxima cancelling, so tightening the knot, with possible intersections between K and $\mathcal{C}$.

3. One-parameter families of spheres

From now on, we suppose that K is in a thin presentation (in L) with only maxima; this is the corresponding tight presentation. Let K_α be also in a thin presentation (in S^3).

Let $\{\widehat{S}_\mu\}_{\mu \in [0,1]}$ denote a family of level 2-spheres in the tight presentation of K, between Σ and the first maximum. Let $\{\widehat{S}_\lambda\}_{\lambda \in [0,1]}$ denote a family of spheres in the thin presentation of K_α between a consecutive minimum and maximum. Such a family is called a *middle slab* [11]. For convenience, we fix the index notations $\lambda \in [0,1]$ for S^3 and $\mu \in [0,1]$ for L.

Lemma 3.1. (i) *A surface S_λ or S_μ is one and only one of $\mathcal{H}$, $\mathcal{L}$ or $\mathcal{C}$.* (ii) $\forall \lambda \in [0,1]$ $\exists \mu \in [0,1]$ *such that S_μ is $\mathcal{N}$ with respect to S_λ.*

We deduce the previous result by studying the thin presentation of K_α and that of K. If one supposes the contrary, then we can minimize the complexity. For details in the case $L = \mathbb{R}P^3$, we refer to [4, Lemmas 2.2 and 2.3] and for the general case see [5, Lemmas 4.4 and 4.7].

4. The graph of singularities

We now study, using Cerf Theory, the intersection of the two corresponding families of punctured spheres, $\{S_\lambda\}_{\lambda \in [0,1]}$ and $\{S_\mu\}_{\mu \in [0,1]}$ embedded in $X = L - \operatorname{Int} N(K)$. Without loss of generality, we suppose that $S_{\lambda=0}$ is $\mathcal{H}$, $S_{\lambda=1}$ is $\mathcal{L}$, $S_{\mu=0}$ is $\mathcal{C}$ and $S_{\mu=1}$ is $\mathcal{L}$. This is what we call the extremal conditions. About transversality arguments, due to Cerf, we refer to [2, Chapter 2].

We then obtain a *"Cerf graph of singularities"* Γ. A point in Γ is a couple of parameters $(\lambda, \mu) \in [0,1]^2$ for which the corresponding surfaces S_λ and S_μ are tangent.

A point in $\Gamma^c = [0,1]^2 - \Gamma$, the exterior of the graph, corresponds to transverse surfaces. Note that for all (λ, μ) in the same connected component of Γ^c, all the S_λ's have the same characteristic $\mathcal{H}$, $\mathcal{L}$, $\mathcal{C}$ or $\mathcal{N}$ with respect to S_μ; and similarly for the S_μ's with respect to S_λ. The characteristics $\mathcal{H}$, $\mathcal{L}$, $\mathcal{C}$ and $\mathcal{N}$ are locally constant in Γ^c. So, we associate to each component of Γ^c, two characteristics from the set $\{\mathcal{H}, \mathcal{L}, \mathcal{C}, \mathcal{N}\}$: one with respect to λ and the other with respect to μ. From Lemma 3(i), we then have the following result [5, Lemma 4.6].

Lemma 4.1. *For a fixed λ, the connected components of Γ^c on the same vertical line all have the same characteristic $\mathcal{H}$, $\mathcal{L}$ or $\mathcal{C}$, except that some can be $\mathcal{N}$.*

For a fixed μ, the connected components of Γ^c on the same horizontal line all have the same characteristic $\mathcal{H}$, $\mathcal{L}$ or $\mathcal{C}$, except that some can be $\mathcal{N}$.

By Gordon and Luecke [11], we may suppose that there does not exist a pair (λ, μ) in Γ^c such that S_μ is $\mathcal{N}$ with respect to S_λ and S_λ is $\mathcal{N}$ with respect to S_μ. Therefore, in a single connected component of Γ^c, we cannot have both characteristics of S_λ and S_μ being $\mathcal{N}$.

Let $t = \sup\{\mu \in [0,1] \mid S_\mu \text{ is } \mathcal{C}\}$. By the extremal conditions, we have $t \in]0,1[$. An index-1 point in Γ is a critical point of Γ that corresponds locally to the crossing of two straight lines. Lemmas 3(ii) and 4 then imply that the corresponding point (s,t) in Γ is an index-1 point of the graph Γ. This means that $S_{\lambda=s}$ and $S_{\mu=t}$ are tangent in two different points. And furthermore, each of $S_{\lambda=s-\varepsilon} \cap S_{\mu=t+\varepsilon}$, $S_{\lambda=s+\varepsilon} \cap S_{\mu=t+\varepsilon}$, $S_{\lambda=s+\varepsilon} \cap S_{\mu=t-\varepsilon}$ or $S_{\lambda=s-\varepsilon} \cap S_{\mu=t-\varepsilon}$, is a single tangency point, for small enough $\varepsilon > 0$.

Finally, we study the configuration of the planar surfaces in a neighbourhood of (s,t). We then deduce that K intersects at most two times the spheres $\widehat{S_\mu}$ in the L-foliation and so exactly once the spine Σ. This proves that $|K \cap \Sigma| = 1$.

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