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On rigid displacements and their relation to the infinitesimal rigid displacement lemma in shell theory

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Abstract

Let ω be an open connected subset of $\mathbb{R}^2$ and let $\boldsymbol{\theta}$ be an immersion from ω into $\mathbb{R}^3$. It is established that the set formed by all rigid displacements of the surface $\boldsymbol{\theta}(\omega)$ is a submanifold of dimension 6 and of class $\mathcal{C}^\infty$ of the space $\boldsymbol{H}^1(\omega)$. It is shown that the infinitesimal rigid displacements of the same surface $\boldsymbol{\theta}(\omega)$ span the tangent space at the origin to this submanifold. ***To cite this article: P.G. Ciarlet, C. Mardare, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Déplacements rigides et leur relation au lemme du mouvement rigide infinitésimal en théorie des coques. Soit ω un ouvert connexe de $\mathbb{R}^2$ et $\boldsymbol{\theta}$ une immersion de ω dans $\mathbb{R}^3$. On établit que l'ensemble formé par les déplacements rigides de la surface $\boldsymbol{\theta}(\omega)$ est une sous-variété de dimension 6 et de classe $\mathcal{C}^\infty$ de l'espace $\boldsymbol{H}^1(\omega)$. On montre aussi que les déplacements rigides infinitésimaux de la même surface $\boldsymbol{\theta}(\omega)$ engendrent le plan tangent à l'origine à cette sous-variété. ***Pour citer cet article : P.G. Ciarlet, C. Mardare, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***
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Version française abrégée

Les notations sont définies dans la version anglaise.

Le *lemme du déplacement rigide infinitésimal sur une surface*, qui joue un rôle important dans l'analyse du modèle linéaire de Koiter [12], et plus généralement des coques linéairement élastiques (cf. [6, Chapitre 2]), s'énonce ainsi : soit ω un ouvert connexe de $\mathbb{R}^2$, soit $\boldsymbol{\theta}$ une immersion suffisamment régulière de ω dans un espace euclidien tri-dimensionnel $\mathbb{E}^3$, et soit $\tilde{\boldsymbol{\eta}} \in \boldsymbol{H}^1(\omega)$ un champ de vecteurs vérifiant

$$\gamma_{\alpha\beta}(\tilde{\boldsymbol{\eta}}) = 0 \quad \text{p.p. dans } \omega \quad \text{et} \quad \rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}}) = 0 \quad \text{dans } H^{-1}(\omega),$$

où

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$$\gamma_{\alpha\beta}(\tilde{\boldsymbol{\eta}}) = \frac{1}{2}(\partial_\alpha \tilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\beta + \partial_\beta \tilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\alpha),$$
$$\rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}}) = \partial_{\alpha\beta}(\tilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_3) - \partial_\alpha \tilde{\boldsymbol{\eta}} \cdot \partial_\beta \boldsymbol{a}_3 - \partial_\beta(\tilde{\boldsymbol{\eta}} \cdot \partial_\alpha \boldsymbol{a}_3) - \Gamma^\sigma_{\alpha\beta} \partial_\sigma \tilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_3,$$

les vecteurs $\boldsymbol{a}_\alpha = \partial_\alpha \boldsymbol{\theta}$ sont tangents à la surface $\boldsymbol{\theta}(\omega)$, le vecteur unitaire $\boldsymbol{a}_3 = \frac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2}{|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|}$ est normal à $\boldsymbol{\theta}(\omega)$, et les fonctions $\Gamma^\sigma_{\alpha\beta}$ sont les symboles de Christoffel. Alors il existe des vecteurs $\boldsymbol{c} \in \mathbb{E}^3$ et $\boldsymbol{d} \in \mathbb{E}^3$ tels que

$$\tilde{\boldsymbol{\eta}}(y) = \boldsymbol{c} + \boldsymbol{d} \wedge \boldsymbol{\theta}(y) \quad \text{pour presque tout } y \in \omega.$$

Une première démonstration de ce lemme a été donnée dans [4, Théorème 5.1-1] pour des champs de vecteurs $\tilde{\boldsymbol{\eta}} \in \boldsymbol{H}^1(\omega)$ tels que $\tilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_3 \in H^2(\omega)$, sous les hypothèses suivantes : l'ouvert ω est borné, sa frontière est lipschitzienne, et $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega})$; on en trouve une démonstration plus simple dans [6, Théorème 2.6-2]. Diverses généralisations, correspondant à des hypothèses de régularité nettement moins restrictives sur l'application $\boldsymbol{\theta}$, ont été obtenues ; cf. [3] et [5].

En théorie des coques, la surface $\boldsymbol{\theta}(\omega) \subset \mathbb{E}^3$ est la *configuration de référence* de la surface moyenne d'une coque élastique et le champ $\tilde{\boldsymbol{\eta}}$ est un *champ de déplacements* de la surface $\boldsymbol{\theta}(\omega)$. Les fonctions $\gamma_{\alpha\beta}(\tilde{\boldsymbol{\eta}})$ et $\rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}})$ sont les composantes covariantes du *tenseur linéarisé de changement de métrique* et du *tenseur linéarisé de changement de courbure* associés au champ $\tilde{\boldsymbol{\eta}}$. Un champ de déplacement de la forme ci-dessus $\tilde{\boldsymbol{\eta}} = \boldsymbol{c} + \boldsymbol{d} \wedge \boldsymbol{\theta}$ est appelé un *déplacement rigide infinitésimal* de la surface $\boldsymbol{\theta}(\omega)$.

L'objet de cette Note est d'établir que le lemme du déplacement rigide infinitésimal sur une surface n'est autre que la version linéarisée (dans un sens précisé au Théorème 4.1) du *théorème de rigidité* bien connu de la théorie des surfaces, une fois celui-ci convenablement étendu à l'espace de Sobolev $\boldsymbol{H}^1(\omega)$.

Cette extension (Théorème 2.1) est établie à partir d'une extension, récemment établie dans [8] et rappelée dans le Théorème 1.1, du théorème de rigidité pour un ouvert tri-dimensionnel.

On établit ensuite (Théorème 3.1 et son corollaire) que l'ensemble $\boldsymbol{M}_{\rm rig}$ formé par les *déplacements rigides* (au sens du Théorème 2.1) de la surface $\boldsymbol{\theta}(\omega)$ est une *sous-variété de dimension* 6 *et de classe* $\mathcal{C}^\infty$ *de l'espace* $\boldsymbol{H}^1(\omega)$.

On montre enfin (Théorème 4.1) que l'espace vectoriel engendré par les déplacements rigides infinitésimaux de la surface $\boldsymbol{\theta}(\omega)$ n'est autre que *l'espace tangent à l'origine à la variété* $\boldsymbol{M}_{\rm rig}$. Ce dernier résultat est démontré à partir d'une généralisation, également établie dans [8] et rappelée dans le Théorème 1.2, du lemme du déplacement rigide infinitésimal en coordonnées curvilignes sur un ouvert tri-dimensionnel.

Les énoncés des théorèmes mentionnés ci-dessus se trouvent dans la version anglaise. On trouvera les démonstrations complètes des Théorèmes 2.1, 3.1, et 4.1 dans [9].

1. Preliminaries

Complete proofs of Theorems 2.1, 3.1, and 4.1 are found in [9].

All spaces, matrices, etc., considered are real. The notations $\mathbb{M}^3$, $\mathbb{O}^3$, $\mathbb{O}^3_+$, and $\mathbb{A}^3$ respectively designate the sets of all square matrices of order 3, of all orthogonal matrices of order 3, of all matrices $\boldsymbol{Q} \in \mathbb{O}^3$ with $\det \boldsymbol{Q} = 1$, and of all antisymmetric matrices of order 3.

Latin indices range over the set $\{1, 2, 3\}$ except when they are used for indexing sequences, and the summation convention with respect to repeated indices is used in conjunction with this rule.

The notation $\mathbb{E}^3$ designates a three-dimensional Euclidean space and $\boldsymbol{a} \cdot \boldsymbol{b}$, $\boldsymbol{a} \wedge \boldsymbol{b}$, and $|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$ respectively designate the Euclidean inner product and the exterior product of $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{E}^3$, and the Euclidean norm of $\boldsymbol{a} \in \mathbb{E}^3$.

Let Ω be an open subset of $\mathbb{R}^3$, let x_i denote the coordinates of a point $x \in \mathbb{R}^3$, and let $\partial_i := \partial/\partial x_i$. Let $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \mathbb{E}^3)$ be an *immersion*. The *metric tensor field* $(g_{ij}) \in \mathcal{C}^0(\Omega; \mathbb{M}^3)$ of the set $\boldsymbol{\Theta}(\Omega)$ is then defined by means of its covariant components

$$g_{ij}(x) := \boldsymbol{g}_i(x) \cdot \boldsymbol{g}_j(x), \quad x \in \Omega,$$

where $\boldsymbol{g}_i(x) := \partial_i \boldsymbol{\Theta}(x)$. The following result, established in [8], is an extension of the classical *rigidity theorem for an open set*, to mappings in the Sobolev space $\boldsymbol{H}^1(\Omega)$ (this theorem is usually established for mappings in the class $\boldsymbol{\mathcal{C}}^1(\Omega)$; see, e.g., [7, Theorem 3]). Note that this extension itself relies on a crucial extension of the classical Liouville theorem, originally due to Reshetnyak [13] and recently given a particularly concise and elegant proof by Friesecke, James and Müller [11]. The notation $\boldsymbol{\nabla\Theta}(x)$ designates the matrix whose columns are the vectors $\boldsymbol{g}_i(x)$, $x \in \Omega$.

Theorem 1.1. *Let Ω be a connected open subset of $\mathbb{R}^3$ and let $\boldsymbol{\Theta} \in \boldsymbol{\mathcal{C}}^1(\Omega) := \mathcal{C}^1(\Omega; \mathbb{E}^3)$ be a mapping that satisfies* $\det \boldsymbol{\nabla\Theta} > 0$ *in Ω. Assume that there exists a vector field $\widetilde{\boldsymbol{\Theta}} \in \boldsymbol{H}^1(\Omega) := H^1(\Omega; \mathbb{E}^3)$ that satisfies*

$$\det \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}} > 0 \quad \text{a.e. in } \Omega \quad \text{and} \quad \tilde{g}_{ij} = g_{ij} \quad \text{a.e. in } \Omega$$

(with self-explanatory notations). Then there exist a vector $\boldsymbol{c} \in \mathbb{E}^3$ and a matrix $\boldsymbol{Q} \in \mathbb{O}^3_+$ such that

$$\widetilde{\boldsymbol{\Theta}}(x) = \boldsymbol{c} + \boldsymbol{Q}\boldsymbol{\Theta}(x) \quad \text{for almost all } x \in \Omega.$$

The covariant components of the *linearized change of metric tensor* associated with a displacement field $\tilde{\boldsymbol{v}}$ of the set $\boldsymbol{\Theta}(\Omega)$ are defined by

$$e_{i\|j}(\tilde{\boldsymbol{v}}) = \frac{1}{2}(\partial_i \tilde{\boldsymbol{v}} \cdot \boldsymbol{g}_j + \partial_j \tilde{\boldsymbol{v}} \cdot \boldsymbol{g}_i), \quad \text{where } \boldsymbol{g}_i := \partial_i \boldsymbol{\Theta}.$$

A displacement field $\tilde{\boldsymbol{v}} \in \boldsymbol{H}^1(\Omega)$ that satisfies $e_{i\|j}(\tilde{\boldsymbol{v}}) = 0$ a.e. in Ω is called an *infinitesimal rigid displacement*. The next theorem, due to [8], is an extension of the *infinitesimal rigid displacement lemma in curvilinear coordinates* found in [6, Theorem 1.7-3].

Theorem 1.2. *Let Ω be a connected open subset of $\mathbb{R}^3$ and let $\boldsymbol{\Theta} \in \boldsymbol{\mathcal{C}}^1(\Omega) \cap \boldsymbol{H}^1(\Omega)$ be a mapping that satisfies* $\det \boldsymbol{\nabla\Theta} > 0$ *in Ω. Then a vector field $\tilde{\boldsymbol{v}} \in \boldsymbol{H}^1(\Omega)$ satisfies $e_{i\|j}(\tilde{\boldsymbol{v}}) = 0$ a.e. in Ω if and only if there exist a vector $\boldsymbol{c} \in \mathbb{E}^3$ and a matrix $\boldsymbol{A} \in \mathbb{A}^3$ such that*

$$\tilde{\boldsymbol{v}}(x) = \boldsymbol{c} + \boldsymbol{A}\boldsymbol{\Theta}(x) \quad \text{for almost all } x \in \Omega.$$

2. The rigidity theorem on a surface and its extension to Sobolev spaces

Greek indices range over the set $\{1, 2\}$ and the summation convention for Latin indices also applies to these. Let ω be an open subset of $\mathbb{R}^2$, let y_α denote the coordinates of a point $y \in \mathbb{R}^2$, and let $\partial_\alpha := \partial/\partial y_\alpha$ and $\partial_{\alpha\beta} := \partial^2/\partial y_\alpha \partial y_\beta$.

Let $\boldsymbol{\theta} \in \boldsymbol{\mathcal{C}}^1(\omega) := \mathcal{C}^1(\omega; \mathbb{E}^3)$ be an *immersion*. The *first fundamental form* of the *surface* $\boldsymbol{\theta}(\omega) \subset \mathbb{E}^3$ is defined by means of its covariant components

$$a_{\alpha\beta}(y) := \boldsymbol{a}_\alpha(y) \cdot \boldsymbol{a}_\beta(y), \quad y \in \omega,$$

where $\boldsymbol{a}_\alpha(y) := \partial_\alpha \boldsymbol{\theta}(y)$. Let $\boldsymbol{a}_3(y) := \frac{\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)}{|\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)|}$, $y \in \omega$. If $\boldsymbol{a}_3 \in \boldsymbol{\mathcal{C}}^1(\omega)$, the *second fundamental form* of the surface is defined by means of its covariant components

$$b_{\alpha\beta}(y) := -\boldsymbol{a}_\alpha(y) \cdot \partial_\beta \boldsymbol{a}_3(y), \quad y \in \omega.$$

The classical *rigidity theorem on a surface* (cf., e.g., [7, Theorem 6]) asserts that, if two immersions $\tilde{\boldsymbol{\theta}} \in \boldsymbol{\mathcal{C}}^2(\omega) := \mathcal{C}^2(\omega; \mathbb{E}^3)$ and $\boldsymbol{\theta} \in \boldsymbol{\mathcal{C}}^2(\omega)$ have the same first and second fundamental forms, i.e., if $\tilde{a}_{\alpha\beta} = a_{\alpha\beta}$ and $\tilde{b}_{\alpha\beta} = b_{\alpha\beta}$ in ω (with self-explanatory notations) and ω is connected, then there exist a vector $\boldsymbol{c} \in \mathbb{E}^3$ and a matrix $\boldsymbol{Q} \in \mathbb{O}^3_+$ such that

$$\tilde{\boldsymbol{\theta}}(y) = \boldsymbol{c} + \boldsymbol{Q}\boldsymbol{\theta}(y) \quad \text{for all } y \in \omega.$$

The following result shows that *a similar result holds under the assumptions that $\tilde{\boldsymbol{\theta}} \in \boldsymbol{H}^1(\omega) := H^1(\omega; \mathbb{E}^3)$ and* $\tilde{\boldsymbol{a}}_3 := \frac{\tilde{\boldsymbol{a}}_1 \wedge \tilde{\boldsymbol{a}}_2}{|\tilde{\boldsymbol{a}}_1 \wedge \tilde{\boldsymbol{a}}_2|} \in \boldsymbol{H}^1(\omega)$ (again with self-explanatory notations). As shown below, the vector field $\tilde{\boldsymbol{a}}_3$, which is not necessarily well defined a.e. in ω for an arbitrary mapping $\tilde{\boldsymbol{\theta}} \in \boldsymbol{H}^1(\omega)$, is nevertheless well defined a.e. in ω for those mappings $\tilde{\boldsymbol{\theta}}$ that satisfy the assumptions of the next theorem.

Theorem 2.1. *Let ω be a connected open subset of $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \boldsymbol{\mathcal{C}}^1(\omega)$ be an immersion that satisfies $\boldsymbol{a}_3 \in \boldsymbol{\mathcal{C}}^1(\omega)$. Assume that there exists a vector field $\tilde{\boldsymbol{\theta}} \in \boldsymbol{H}^1(\omega)$ that satisfies*

$$\tilde{a}_{\alpha\beta} = a_{\alpha\beta} \quad \text{a.e. in } \omega, \quad \tilde{\boldsymbol{a}}_3 \in \boldsymbol{H}^1(\omega), \quad \text{and} \quad \tilde{b}_{\alpha\beta} = b_{\alpha\beta} \quad \text{a.e. in } \omega.$$

Then there exist a vector $\boldsymbol{c} \in \mathbb{E}^3$ and a matrix $\boldsymbol{Q} \in \mathbb{O}^3_+$ such that

$$\tilde{\boldsymbol{\theta}}(y) = \boldsymbol{c} + \boldsymbol{Q}\boldsymbol{\theta}(y) \quad \text{for almost all } y \in \omega.$$

Sketch of proof. (i) To begin with, several *technical preliminaries* are established: *first*, we observe that the relations $\tilde{a}_{\alpha\beta} = a_{\alpha\beta}$ a.e. in ω and the assumption that $\boldsymbol{\theta} \in \boldsymbol{\mathcal{C}}^1(\omega)$ is an immersion together imply that

$$|\tilde{\boldsymbol{a}}_1 \wedge \tilde{\boldsymbol{a}}_2| = \sqrt{\det(\tilde{a}_{\alpha\beta})} = \sqrt{\det(a_{\alpha\beta})} > 0 \quad \text{a.e. in } \omega,$$

so that *the vector field $\tilde{\boldsymbol{a}}_3$, and thus the functions $\tilde{b}_{\alpha\beta}$, are well defined a.e. in ω*. *Second*, we claim that

$$b_{\alpha\beta} = b_{\beta\alpha} \quad \text{in } \omega \quad \text{and} \quad \tilde{b}_{\alpha\beta} = \tilde{b}_{\beta\alpha} \quad \text{a.e. in } \omega,$$

i.e., that $\boldsymbol{a}_\alpha \cdot \partial_\beta \boldsymbol{a}_3 = \boldsymbol{a}_\beta \cdot \partial_\alpha \boldsymbol{a}_3$ in ω and $\tilde{\boldsymbol{a}}_\alpha \cdot \partial_\beta \tilde{\boldsymbol{a}}_3 = \tilde{\boldsymbol{a}}_\beta \cdot \partial_\alpha \tilde{\boldsymbol{a}}_3$ a.e. in ω. To prove this, we note that either the assumptions $\boldsymbol{\theta} \in \boldsymbol{\mathcal{C}}^1(\omega)$ and $\boldsymbol{a}_3 \in \boldsymbol{\mathcal{C}}^1(\omega)$, or the assumptions $\boldsymbol{\theta} \in \boldsymbol{H}^1(\omega)$ and $\boldsymbol{a}_3 \in \boldsymbol{H}^1(\omega)$, imply that $\boldsymbol{a}_\alpha \cdot \partial_\beta \boldsymbol{a}_3 = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{a}_3 \in L^1_{\text{loc}}(\omega)$, hence that $\partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{a}_3 \in \mathcal{D}'(\omega)$. It is then shown, after some intermediate computations relying on distribution theory, that the expression ${}_{\mathcal{D}'(\omega)}\langle \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{a}_3, \varphi \rangle_{\mathcal{D}(\omega)}$ is symmetric with respect to α and β, for any $\varphi \in \mathcal{D}(\omega)$. Hence $\partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{a}_3 = \partial_\beta \boldsymbol{\theta} \cdot \partial_\alpha \boldsymbol{a}_3$ in $L^1_{\text{loc}}(\omega)$. *Third*, let

$$\tilde{c}_{\alpha\beta} := \partial_\alpha \tilde{\boldsymbol{a}}_3 \cdot \partial_\beta \tilde{\boldsymbol{a}}_3 \quad \text{and} \quad c_{\alpha\beta} := \partial_\alpha \boldsymbol{a}_3 \cdot \partial_\beta \boldsymbol{a}_3.$$

Then we claim that $\tilde{c}_{\alpha\beta} = c_{\alpha\beta}$ a.e. in ω. To see this, we note that the matrix fields $(\tilde{a}^{\alpha\beta}) := (\tilde{a}_{\alpha\beta})^{-1}$ and $(a^{\alpha\beta}) := (a_{\alpha\beta})^{-1}$ are well defined and equal a.e. in ω since $\boldsymbol{\theta}$ is an immersion and $\tilde{a}_{\alpha\beta} = a_{\alpha\beta}$ a.e. in ω. The formula of Weingarten can thus be applied a.e. in ω, showing that $\tilde{c}_{\alpha\beta} = \tilde{a}^{\sigma\tau}\tilde{b}_{\sigma\alpha}\tilde{b}_{\tau\beta}$ a.e. in ω. The assertion then follows from the assumptions $\tilde{b}_{\alpha\beta} = b_{\alpha\beta}$ a.e. in ω.

(ii) *Starting from the set ω and the mapping $\boldsymbol{\theta}$* (as given in the statement of Theorem 2.1), *we next construct a set Ω and a mapping $\boldsymbol{\Theta}$ that satisfy the assumptions of Theorem* 1.1. More precisely, let

$$\boldsymbol{\Theta}(y, x_3) := \boldsymbol{\theta}(y) + x_3 \boldsymbol{a}_3(y) \quad \text{for all} \quad (y, x_3) \in \omega \times \mathbb{R},$$

and let ω_n, $n \geqslant 0$, be open subsets of $\mathbb{R}^2$ such that $\overline{\omega}_n$ is a compact subset of ω and $\omega = \bigcup_{n \geqslant 0} \omega_n$. Then it is easily seen that, for each $n \geqslant 0$, there exists $0 < \varepsilon_n \leqslant 1$ such that *the mapping $\boldsymbol{\Theta} \in \boldsymbol{\mathcal{C}}^1(\Omega)$ satisfies* $\det \nabla \boldsymbol{\Theta} > 0$ *over the connected open set*

$$\Omega := \bigcup_{n \geqslant 0} \big(\omega_n \times]-\varepsilon_n, \varepsilon_n[\big) \subset \mathbb{R}^3.$$

Note that the covariant components $g_{ij} \in \mathcal{C}^0(\Omega)$ of the metric tensor field associated with the mapping $\boldsymbol{\Theta}$ are then given by (the symmetries $b_{\alpha\beta} = b_{\beta\alpha}$ established in (i) are used here)

$$g_{\alpha\beta} = a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 c_{\alpha\beta}, \quad g_{\alpha 3} = 0, \quad g_{33} = 1.$$

(iii) Starting with the mapping $\tilde{\boldsymbol{\theta}}$ (as given in the statement of Theorem 2.1), we *construct a mapping $\widetilde{\boldsymbol{\Theta}}$ that satisfies the assumptions of Theorem* 1.1. To this end, we define a mapping $\widetilde{\boldsymbol{\Theta}} : \Omega \to \mathbb{E}^3$ by letting

$$\widetilde{\boldsymbol{\Theta}}(y, x_3) := \tilde{\boldsymbol{\theta}}(y) + x_3 \tilde{\boldsymbol{a}}_3(y) \quad \text{for all } (y, x_3) \in \Omega,$$

where the set Ω is defined as in (ii). Hence $\widetilde{\boldsymbol{\Theta}} \in \boldsymbol{H}^1(\Omega)$, since $\Omega \subset \omega \times]-1,1[$. Besides, $\det \nabla\widetilde{\boldsymbol{\Theta}} = \det \nabla\boldsymbol{\Theta}$ a.e. in Ω since the functions $\tilde{a}^{\beta\sigma}\tilde{b}_{\alpha\sigma}$, which are well defined a.e. in ω, are equal a.e. in ω to the functions $a^{\beta\sigma}b_{\alpha\sigma}$. Likewise, the components $\tilde{g}_{ij} \in L^1(\Omega)$ of the metric tensor field associated with the mapping $\widetilde{\boldsymbol{\Theta}}$ satisfy $\tilde{g}_{ij} = g_{ij}$ a.e. in Ω since $\tilde{a}_{\alpha\beta} = a_{\alpha\beta}$ and $\tilde{b}_{\alpha\beta} = b_{\alpha\beta}$ a.e. in ω by assumption and $\tilde{c}_{\alpha\beta} = c_{\alpha\beta}$ a.e. in ω by part (i).

(iv) By Theorem 1.1, there exist a vector $\boldsymbol{c} \in \mathbb{E}^3$ and a matrix $\boldsymbol{Q} \in \mathbb{O}^3_+$ such that

$$\tilde{\boldsymbol{\theta}}(y) + x_3\tilde{\boldsymbol{a}}_3(y) = \boldsymbol{c} + \boldsymbol{Q}\big(\boldsymbol{\theta}(y) + x_3\boldsymbol{a}_3(y)\big) \quad \text{for almost all } (y, x_3) \in \Omega.$$

Differentiating with respect to x_3 in this equality between functions in $\boldsymbol{H}^1(\Omega)$ shows that $\tilde{\boldsymbol{a}}_3(y) = \boldsymbol{Q}\boldsymbol{a}_3(y)$ for almost all $y \in \omega$. Hence $\tilde{\boldsymbol{\theta}}(y) = \boldsymbol{c} + \boldsymbol{Q}\boldsymbol{\theta}(y)$ for almost all $y \in \omega$. □

3. The submanifold of rigid displacements on a surface

All the results needed below about *submanifolds in infinite-dimensional Banach spaces* are found in [1]. The *tangent space* at a point m of a submanifold $\mathcal{M}$ of a Banach space X is denoted $T_m\mathcal{M}$. If $f : X \to Y$ is a Fréchet-differentiable mapping into a Banach space Y, the *tangent map* at m is denoted $T_m f$.

The next theorem shows that the set $\boldsymbol{M}$ formed by all the mappings $\tilde{\boldsymbol{\theta}} \in \boldsymbol{H}^1(\omega)$ that satisfy the assumptions of the rigidity theorem on a surface (Theorem 2.1) is a *finite-dimensional submanifold of the space* $\boldsymbol{H}^1(\omega)$. The *tangent space to* $\boldsymbol{M}$ *at* $\boldsymbol{\theta}$ is also identified. Another equally important characterization of the same tangent space, involving this time the linearized change of metric and linearized change of curvature tensors, will be given in Theorem 4.1.

Theorem 3.1. *Let ω be a connected open subset of $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \boldsymbol{\mathcal{C}}^1(\omega) \cap \boldsymbol{H}^1(\omega)$ be an immersion that satisfies $\boldsymbol{a}_3 \in \boldsymbol{\mathcal{C}}^1(\omega) \cap \boldsymbol{H}^1(\omega)$. Then the set*

$$\boldsymbol{M} := \big\{\tilde{\boldsymbol{\theta}} \in \boldsymbol{H}^1(\omega);\ \tilde{a}_{\alpha\beta} = a_{\alpha\beta} \text{ a.e. in } \omega,\ \tilde{\boldsymbol{a}}_3 \in \boldsymbol{H}^1(\omega),\ \tilde{b}_{\alpha\beta} = b_{\alpha\beta} \text{ a.e. in } \omega\big\}$$

is a submanifold of class $\mathcal{C}^\infty$ and of dimension 6 *of the space $\boldsymbol{H}^1(\omega)$. Its tangent space at $\boldsymbol{\theta}$ is given by*

$$T_{\boldsymbol{\theta}}\boldsymbol{M} = \big\{\tilde{\boldsymbol{\eta}} \in \boldsymbol{H}^1(\omega);\ \exists \boldsymbol{c} \in \mathbb{E}^3,\ \exists \boldsymbol{A} \in \mathbb{A}^3,\ \tilde{\boldsymbol{\eta}} = \boldsymbol{c} + \boldsymbol{A}\boldsymbol{\theta} \text{ a.e. in } \omega\big\}.$$

Sketch of proof. (i) Define the linear mapping $\boldsymbol{f} : (\boldsymbol{c}, \boldsymbol{F}) \in \mathbb{E}^3 \times \mathbb{M}^3 \to \boldsymbol{f}(\boldsymbol{c}, \boldsymbol{F}) = \boldsymbol{c} + \boldsymbol{F}\boldsymbol{\theta} \in \boldsymbol{H}^1(\omega)$. By the rigidity theorem (Theorem 2.1), the above set $\boldsymbol{M}$ may be equivalently defined as $\boldsymbol{M} = \boldsymbol{f}(\mathbb{E}^3 \times \mathbb{O}^3_+)$. If the mapping $\boldsymbol{f} : \mathbb{E}^3 \times \mathbb{M}^3 \to \boldsymbol{H}^1(\omega)$ is *injective,* in which case $\boldsymbol{f}$ is a $\mathcal{C}^\infty$-diffeomorphism from $\mathbb{E}^3 \times \mathbb{M}^3$ onto $\boldsymbol{f}(\mathbb{E}^3 \times \mathbb{M}^3)$, the proof that $\boldsymbol{M} = \boldsymbol{f}(\mathbb{E}^3 \times \mathbb{O}^3_+)$ is a submanifold of $\boldsymbol{H}^1(\omega)$ is simple. Since submanifolds of class $\mathcal{C}^\infty$ are preserved by $\mathcal{C}^\infty$-diffeomorphisms, $\boldsymbol{M}$ is a submanifold of class $\mathcal{C}^\infty$ and of dimension 6 of $\boldsymbol{f}(\mathbb{E}^3 \times \mathbb{M}^3)$. As a closed subspace of the Hilbert space $\boldsymbol{H}^1(\omega)$, the image $\boldsymbol{f}(\mathbb{E}^3 \times \mathbb{M}^3)$ has a closed complement, i.e., $\boldsymbol{f}(\mathbb{E}^3 \times \mathbb{M}^3)$ is "split" in $\boldsymbol{H}^1(\omega)$. The set $\boldsymbol{M}$ is thus also a submanifold of class $\mathcal{C}^\infty$ and of dimension 6 of $\boldsymbol{H}^1(\omega)$ (this conclusion follows from the definition of a submanifold; see [1, Definition 3.2.1]).

If the mapping $\boldsymbol{f} : \mathbb{E}^3 \times \mathbb{M}^3 \to \boldsymbol{H}^1(\omega)$ is *not injective*, some care has to be exercised. More specifically, one needs to prove that the restriction $\boldsymbol{f}^\sharp$ of the mapping $\boldsymbol{f}$ to the set $\mathbb{E}^3 \times \mathbb{O}^3_+$ is an *embedding*, in the sense that the following two properties are satisfied: first, for each $(\boldsymbol{c}, \boldsymbol{Q}) \in \mathbb{E}^3 \times \mathbb{O}^3_+$, the tangent map $T_{(\boldsymbol{c},\boldsymbol{Q})}\boldsymbol{f}$ is injective, with a closed range having a closed complement in $\boldsymbol{H}^1(\omega)$. Second, the restriction $\boldsymbol{f}^\sharp$ of the mapping $\boldsymbol{f}$ to the submanifold $\mathbb{E}^3 \times \mathbb{O}^3_+$ is a homeomorphism, hence a $\mathcal{C}^\infty$-diffeomorphism since $\boldsymbol{f}$ is linear, from $\mathbb{E}^3 \times \mathbb{O}^3_+$ onto the image $\boldsymbol{f}(\mathbb{E}^3 \times \mathbb{O}^3_+)$ equipped with the relative topology induced by that of $\boldsymbol{H}^1(\omega)$.

Once these two properties are established (by means of arguments that are essentially technical, but too lengthy to be sketched here), it can be concluded that the set $\boldsymbol{M} = \boldsymbol{f}^\sharp(\mathbb{E}^3 \times \mathbb{O}^3_+)$ *is a submanifold of dimension* 6 *of* $\boldsymbol{H}^1(\omega)$, since $\mathbb{E}^3 \times \mathbb{O}^3_+$ is a submanifold of dimension 6 of $\mathbb{E}^3 \times \mathbb{M}^3$ (see [1, Section 3.5]). Since the manifolds $\mathbb{E}^3 \times \mathbb{O}^3_+$ and $\boldsymbol{H}^1(\omega)$ are of class $\mathcal{C}^\infty$ and the mapping $\boldsymbol{f}^\sharp$ is of class $\mathcal{C}^\infty$, *the submanifold* $\boldsymbol{M}$ *is also of class* $\mathcal{C}^\infty$.

(ii) Since $\boldsymbol{f}$ is linear and $T_{\boldsymbol{I}}\mathbb{O}^3_+ = \mathbb{A}^3$, the tangent space to $\boldsymbol{M}$ at $\boldsymbol{\theta}$ is given by

$$T_{\boldsymbol{\theta}}\boldsymbol{M} = T_{\boldsymbol{f}(\boldsymbol{0},\boldsymbol{I})}\boldsymbol{f}\big(\mathbb{E}^3 \times \mathbb{O}^3_+\big) = \boldsymbol{f}\big(T_{(\boldsymbol{0},\boldsymbol{I})}\big(\mathbb{E}^3 \times \mathbb{O}^3_+\big)\big) = \boldsymbol{f}\big(\mathbb{E}^3 \times \mathbb{A}^3\big)$$
$$= \big\{\tilde{\boldsymbol{\eta}} \in \boldsymbol{H}^1(\omega);\ \exists \boldsymbol{c} \in \mathbb{E}^3,\ \exists \boldsymbol{A} \in \mathbb{A}^3,\ \tilde{\boldsymbol{\eta}} = \boldsymbol{c} + \boldsymbol{A}\boldsymbol{\theta} \text{ a.e. in } \omega\big\}. \qquad \square$$

Remark. Interestingly, one can establish that *the mapping* $\boldsymbol{f} : \mathbb{E}^3 \times \mathbb{M}^3 \to \boldsymbol{H}^1(\omega)$ *is injective if and only if the surface* $\boldsymbol{\theta}(\omega)$ *is not contained in a plane*.

In shell theory, the surface $\boldsymbol{\theta}(\omega)$ is the *reference configuration* of the middle surface of an elastic shell (under the additional assumption that the immersion $\boldsymbol{\theta}$ is injective, but this assumption is irrelevant for our present purposes). Then, for each $\tilde{\boldsymbol{\theta}} \in \boldsymbol{H}^1(\omega)$, the surface $\tilde{\boldsymbol{\theta}}(\omega)$ is a *deformed configuration* of the middle surface and the field $\tilde{\boldsymbol{\eta}} \in \boldsymbol{H}^1(\omega)$ defined by $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta} + \tilde{\boldsymbol{\eta}}$ is a *displacement field* of the reference configuration $\boldsymbol{\theta}(\omega)$. If in particular $\tilde{\boldsymbol{\theta}} \in \boldsymbol{M}$, the field $\tilde{\boldsymbol{\eta}}$ defined in this fashion is called a *rigid displacement*, and the subset $\boldsymbol{M}_{\mathrm{rig}}$ of $\boldsymbol{H}^1(\omega)$ defined by

$$\boldsymbol{M} = \boldsymbol{\theta} + \boldsymbol{M}_{\mathrm{rig}}$$

is called the *manifold of rigid displacements*. We now recast Theorem 3.1 in terms of the manifold $\boldsymbol{M}_{\mathrm{rig}}$.

Corollary to Theorem 3.1. *Let ω be a connected open subset of $\mathbb{R}^2$, and let $\boldsymbol{\theta} \in \boldsymbol{\mathcal{C}}^1(\omega) \cap \boldsymbol{H}^1(\omega)$ be an immersion that satisfies $\boldsymbol{a}_3 \in \boldsymbol{\mathcal{C}}^1(\omega) \cap \boldsymbol{H}^1(\omega)$. Then the manifold of rigid displacements of the surface $\boldsymbol{\theta}(\omega)$, viz.,*

$$\boldsymbol{M}_{\mathrm{rig}} := \big\{\tilde{\boldsymbol{\eta}} \in \boldsymbol{H}^1(\omega);\ \tilde{a}_{\alpha\beta} = a_{\alpha\beta} \text{ a.e. in } \omega,\ \tilde{\boldsymbol{a}}_3 \in \boldsymbol{H}^1(\omega),\ \tilde{b}_{\alpha\beta} = b_{\alpha\beta} \text{ a.e. in } \omega\big\},$$

is a submanifold of class $\mathcal{C}^\infty$ and of dimension 6 *of the space $\boldsymbol{H}^1(\omega)$ and its tangent space at* $\mathbf{0}$ *is given by*

$$T_{\boldsymbol{0}}\boldsymbol{M}_{\mathrm{rig}} = T_{\boldsymbol{\theta}}\boldsymbol{M} = \big\{\tilde{\boldsymbol{\eta}} \in \boldsymbol{H}^1(\omega);\ \exists \boldsymbol{c} \in \mathbb{E}^3,\ \exists \boldsymbol{A} \in \mathbb{A}^3,\ \tilde{\boldsymbol{\eta}} = \boldsymbol{c} + \boldsymbol{A}\boldsymbol{\theta} \text{ a.e. in } \omega\big\}.$$

4. The infinitesimal rigid displacement lemma on a surface revisited

The covariant components of the *linearized change of metric tensor* and *linearized change of curvature tensor* associated with a smooth enough displacement field $\tilde{\boldsymbol{\eta}}$ of the surface $\boldsymbol{\theta}(\omega)$ are defined by $\gamma_{\alpha\beta}(\tilde{\boldsymbol{\eta}}) := \frac{1}{2}[\tilde{a}_{\alpha\beta} - a_{\alpha\beta}]^{\mathrm{lin}}$ and $\rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}}) := [\tilde{b}_{\alpha\beta} - b_{\alpha\beta}]^{\mathrm{lin}}$, where $a_{\alpha\beta}$ and $\tilde{a}_{\alpha\beta}$, and $b_{\alpha\beta}$ and $\tilde{b}_{\alpha\beta}$, respectively designate the covariant components of the first, and second, fundamental forms of the surfaces $\boldsymbol{\theta}(\omega)$ and $\tilde{\boldsymbol{\theta}}(\omega)$ where $\tilde{\boldsymbol{\theta}} := \boldsymbol{\theta} + \tilde{\boldsymbol{\eta}}$, and $[\cdot]^{\mathrm{lin}}$ denotes the linear part with respect to $\tilde{\boldsymbol{\eta}}$ in the expression $[\cdot]$. A formal computation immediately gives

$$\gamma_{\alpha\beta}(\tilde{\boldsymbol{\eta}}) = \frac{1}{2}(\partial_\alpha \tilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\beta + \partial_\beta \tilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\alpha), \quad \text{where } \boldsymbol{a}_\alpha := \partial_\alpha \boldsymbol{\theta}.$$

This expression thus shows that $\gamma_{\alpha\beta}(\tilde{\boldsymbol{\eta}}) \in L^2_{\mathrm{loc}}(\omega)$ if $\tilde{\boldsymbol{\eta}} \in \boldsymbol{H}^1(\omega)$ and $\boldsymbol{\theta} \in \boldsymbol{\mathcal{C}}^1(\omega)$.

Another formal, but substantially less immediate, computation shows that (see, e.g., [6, Theorem 2.5-1])

$$\rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}}) = \partial_{\alpha\beta}(\tilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_3) - \partial_\alpha \tilde{\boldsymbol{\eta}} \cdot \partial_\beta \boldsymbol{a}_3 - \partial_\beta(\tilde{\boldsymbol{\eta}} \cdot \partial_\alpha \boldsymbol{a}_3) - \Gamma^\sigma_{\alpha\beta} \partial_\sigma \tilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_3,$$

where the functions $\Gamma^\sigma_{\alpha\beta} := a^{\sigma\tau} \boldsymbol{a}_\tau \cdot \partial_\alpha \boldsymbol{a}_\beta$ are the Christoffel symbols of the surface $\boldsymbol{\theta}(\omega)$. This expression thus shows that $\rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}}) \in H^{-1}(\omega)$ if $\tilde{\boldsymbol{\eta}} \in \boldsymbol{H}^1(\omega)$ and $\boldsymbol{\theta} \in \boldsymbol{\mathcal{C}}^2(\omega)$ and $\boldsymbol{a}_3 \in \boldsymbol{\mathcal{C}}^2(\omega)$.

Under these assumptions on the mapping $\boldsymbol{\theta}$ and the field $\boldsymbol{a}_3$, a displacement field $\tilde{\boldsymbol{\eta}} \in \boldsymbol{H}^1(\omega)$ that satisfies $\gamma_{\alpha\beta}(\tilde{\boldsymbol{\eta}}) = 0$ a.e. in ω and $\rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}}) = 0$ in $H^{-1}(\omega)$ is called an *infinitesimal rigid displacement* of the surface $\boldsymbol{\theta}(\omega)$. Accordingly, the *infinitesimal rigid displacement lemma on a surface* consists in identifying the vector space $\boldsymbol{V}^{\mathrm{lin}}_{\mathrm{rig}}$

formed by such displacements. This is the object of the next theorem, which shows that the space $\boldsymbol{V}^{\rm lin}_{\rm rig}$ has a *remarkably simple interpretation* in terms of the manifold $\boldsymbol{M}_{\rm rig}$ of rigid displacements introduced at the end of Section 3.

Theorem 4.1. *Let ω be a connected open subset of $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^2(\omega) \cap \boldsymbol{H}^1(\omega)$ be an immersion that satisfies $\boldsymbol{a}_3 \in \mathcal{C}^2(\omega) \cap \boldsymbol{H}^1(\omega)$. Then the space of infinitesimal rigid displacements of the surface $\boldsymbol{\theta}(\omega)$, viz.,*

$$\boldsymbol{V}^{\rm lin}_{\rm rig} := \big\{\tilde{\boldsymbol{\eta}} \in \boldsymbol{H}^1(\omega);\ \gamma_{\alpha\beta}(\tilde{\boldsymbol{\eta}}) = 0 \text{ a.e. in } \omega \text{ and } \rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}}) = 0 \text{ in } H^{-1}(\omega)\big\},$$

is given by

$$\boldsymbol{V}^{\rm lin}_{\rm rig} = T_{\boldsymbol{0}}\boldsymbol{M}_{\rm rig},$$

where the tangent space $T_0\boldsymbol{M}_{\rm rig}$ has been identified in the Corollary to Theorem 3.1.

Sketch of proof. The proof is reminiscent of that used in [2] or [10] for establishing the Korn inequality on a surface as a consequence of its three-dimensional counterpart in curvilinear coordinates.

(i) Starting from the set ω and the mapping $\boldsymbol{\theta}$, we begin by constructing a set Ω and a mapping $\boldsymbol{\Theta}$ as in parts (ii) and (iii) of the proof of Theorem 2.1, with the additional requirements that the open sets ω_n be connected and that they satisfy $\omega_n \subset \omega_{n+1}$ for any $n \geqslant 0$.

(ii) Given any displacement field $\tilde{\boldsymbol{\eta}} \in \boldsymbol{V}^{\rm lin}_{\rm rig}$, let

$$\tilde{\boldsymbol{v}}(y, x_3) := \tilde{\boldsymbol{\eta}}(y) - x_3\big(\big\{\partial_\alpha(\tilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_3) - \tilde{\boldsymbol{\eta}} \cdot \partial_\alpha \boldsymbol{a}_3\big\}\boldsymbol{a}^\alpha\big)(y)$$

for almost all $(y, x_3) \in \Omega$, where Ω is defined as in part (i) and $\boldsymbol{a}^\alpha := a^{\alpha\beta}\boldsymbol{a}_\beta$. The vector field $\tilde{\boldsymbol{v}}$ defined in this fashion is such that $\tilde{\boldsymbol{v}} \in \boldsymbol{H}^1(\Omega_n)$ for all $n \geqslant 0$, where $\Omega_n = \omega_n \times]-\varepsilon_n, \varepsilon_n[$.

A careful computation then shows that, for any $n \geqslant 0$, the covariant components $e_{i\|j}(\tilde{\boldsymbol{v}}) \in L^2(\Omega_n)$ of the linearized change of metric tensor (see Section 1) associated with the above displacement field $\tilde{\boldsymbol{v}}$ vanish a.e. in Ω_n.

Theorem 1.2 can thus be applied, showing that, for each $n \geqslant 0$, there exist a vector $\boldsymbol{c}_n \in \mathbb{E}^3$ and a matrix $\boldsymbol{A}_n \in \mathbb{A}^3$ such that $\tilde{\boldsymbol{v}}(x) = \boldsymbol{c}_n + \boldsymbol{A}_n\boldsymbol{\Theta}(x)$ for almost all $x \in \Omega_n$. This last relation in turn implies that $\tilde{\boldsymbol{\eta}}(y) = \boldsymbol{c}_n + \boldsymbol{A}_n\boldsymbol{\theta}(y)$ for almost all $y \in \omega_n$. That the vectors $\boldsymbol{c}_n$ and $\boldsymbol{A}_n$ are in fact independent of $n \geqslant 0$ is then a consequence of the inclusions $\omega_n \subset \omega_{n+1}$, $n \geqslant 0$. □

By Theorem 4.1, *the infinitesimal rigid displacements of the surface $\boldsymbol{\theta}(\omega)$ thus span the tangent space at the origin to the manifold formed by the rigid displacements of $\boldsymbol{\theta}(\omega)$.*

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