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Partial Differential Equations

Properties of a single vortex solution in a rotating Bose Einstein condensate

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Abstract

In this Note, we study the properties of the line energy for a vortex γ in a Bose Einstein condensate rotating at velocity Ω. The global minimizer is either the vortex free solution or U vortices which exist only for Ω bigger than a critical value. For all values of Ω, we prove the existence of an S type vortex, which is a critical point of the line energy, observed in the experiments. We also prove uniqueness of the minimizer for almost every Ω and a monotonicity property of the curve γ with respect to Ω. The proofs rely on a related isoperimetric problem. ***To cite this article: A. Aftalion, R.L. Jerrard, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Propriétés d'une solution à un vortex pour un condensat de Bose Einstein en rotation. Dans cette Note, nous étudions les propriétés de l'énergie de ligne pour un vortex γ dans un condensat de Bose Einstein en rotation à la vitesse Ω. Nous prouvons que, pour tout Ω, il existe un vortex de type S, qui est un point critique de l'énergie, mais jamais un minimiseur. Le minimiseur global est soit la solution sans vortex soit un vortex en U, qui n'existe que pour Ω plus grand qu'une valeur critique. Nous prouvons également l'unicité des minimiseurs pour presque tout Ω et une propriété de monotonie des courbes γ par rapport à Ω. Les preuves reposent sur un problème de type isopérimétrique. ***Pour citer cet article : A. Aftalion, R.L. Jerrard, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

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Depuis la première réalisation expérimentale de condensats de Bose Einstein atomiques gazeux en 1995 et le prix Nobel en 2001, de nombreuses propriétés de ces systèmes ont été étudiées expérimentalement et théoriquement. Une des questions clés, reliée à la superfluidité est l'existence de vortex. Plusieurs groupes expérimentaux ont obtenu des vortex par la mise en rotation du potentiel de piégeage, notamment le groupe de J. Dalibard à l'ENS à Paris [6,7] et le groupe de W. Ketterle au MIT [1,8]. Le condensat a la forme d'un cigare déterminé par le domaine

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$\mathcal{D}=\{\rho>0\}$, $\rho(\mathbf{r})=\rho_0-(\alpha^2x^2+y^2+\beta^2z^2)$. Les paramètres α and β caractérisent la forme du piège, α étant d'ordre 1, et β petit. Le condensat est mis en rotation suivant l'axe z à la vitesse Ω. Quand la vitesse de rotation est suffisamment grande, des vortex apparaissent. L'une des observations principales est que ces vortex ne sont pas suivant l'axe de rotation mais sont courbés. Le nombre et la forme des vortex dépendent de la vitesse de rotation et de la géometrie du piège.

La modélisation mathématique de ces phénomènes se fait en minimisant la fonctionnelle de Gross–Pitaevskii pour la fonction d'onde du condensat, où il y a un terme qui prend en compte la forme particulière du piège. Dans [3], nous avons fait un développement asymptotique de l'énergie en fonction d'un petit paramètre pour obtenir une énergie approchée qui ne dépend que de la forme et du nombre de vortex. La dérivation rigoureuse se trouve dans [5]. La vérification numérique que le premier vortex pour cette énergie simplifiée est effectivement courbé a été considéré comme une bonne justification par la communauté de physique [8]. Dans [2], nous avons étudié les propriétés de cette énergie approchée en liaison avec les expériences. Très récemment, suite à nos travaux, le groupe de l'ENS a fait de nouvelles expériences qui mettent en évidence la forme des vortex [9]. Le but de cette Note est d'apporter une analyse mathématique rigoureuse des propriétés de la solution à un vortex en liaison avec les expériences et en utilisant l'énergie approchée de [3,2]. Si le vortex est représenté par la ligne γ, alors l'énergie du vortex est donnée par

$$E_\Omega[\gamma]=H-\Omega L=\int_\gamma \rho\,\mathrm{d}l-\Omega\int_\gamma \rho^2\,\mathrm{d}z, \tag{1}$$

le premier terme étant l'énergie cinétique du vortex et le second le moment angulaire dû à la rotation. Le 0 des énergies étant pour la solution sans vortex. Si Ω est tel que $E_\Omega[\gamma]<0$, alors un vortex γ est minimisant. La courbe γ optimale minimisant E_Ω définit la forme du vortex. Cette Note a pour but d'étudier $E_\Omega[\gamma]$ suite aux dernières expériences.

Une première propriété prouvée dans [2] est que la courbe minimisante γ est plane, dans le plan le plus près de l'axe des z, soit le plan yz si $\alpha<1$. Nous pouvons donc nous restreindre au domaine $\mathcal{D}_0:=\{(y,z)\colon\ \rho(y,z)>0\}$, $\rho(y,z)=\rho_0-y^2-\beta^2z^2$ pour $(y,z)\in\mathcal{D}_0$. Le problème dans $\mathcal{D}_0$ peut être écrit en considérant uniquement les vortex de la forme $\gamma=\partial U$, où $U\subset\mathcal{D}_0$ est un ensemble de périmètre fini et ∂U est orientée dans le sens habituel. (Au sens strict γ est la frontière réduite de ∂_*U, voir [4].) Si nous appelons χ la fonction caractéristique de U, alors $L[\gamma]=\int_{\partial U}\rho^2\,\mathrm{d}z=\int_U 2\rho\rho_y\,\mathrm{d}y\,\mathrm{d}z=2\int_{\mathcal{D}_0}\chi\rho\rho_y$ et $H[\gamma]=\int\rho|\nabla\chi|$. On peut donc définir $\mathcal{A}=\{\chi:\mathcal{D}_0\to\{0,1\},\ \chi\in BV\}$ et pour $\chi\in\mathcal{A}$, nous écrivons

$$E_\Omega[\chi]=H[\chi]-\Omega L[\chi]=\int_{\mathcal{D}_0}\rho|\nabla\chi|-2\Omega\int_{\mathcal{D}_0}\chi\rho\rho_y.$$

Alors $E_\Omega[\chi]=E_\Omega[\gamma]$ où $U=\{(y,z)\in\mathcal{D}_0\colon\ \chi(y,z)=1\}$ et $\partial U=\gamma$. Dans la suite de cette Note, nous nous restreignons au plan yz et étudions le problème de minimisation de E_Ω dans $\mathcal{A}$.

Proposition 0.1. (i) *Quel que soit $\Omega\geqslant 0$, il existe un minimiseur de E_Ω dans $\mathcal{A}$. Les minimiseurs sont soit l'état sans vortex $\chi=0$, soit un vortex le long de l'axe des z ($\chi=1$ pour $y<0$), soit ont leur support dans $\{y<0\}$ et ont le vortex à distance finie de l'axe z. Si $\Omega\rho_0<1/2$, il n'existe pas de point critique de l'énergie dans le demi plan $y<0$.*

(ii) *Quel que soit $\Omega\geqslant 0$, il existe un minimiseur de E_Ω dans $\{\chi\in\mathcal{A}\colon\ \chi(y,z)\neq\chi(-y,-z)$ p.p.$\}$. La courbe γ est solution de l'équation d'Euler–Lagrange associée à l'énergie.*

(iii) *Quel que soit ℓ tel que $\mathcal{A}_\ell:=\{\chi\in\mathcal{A}\colon\ L(\chi)=\ell\}$ est non vide, il existe un minimiseur $h(\ell)$ de H dans $\mathcal{A}_\ell$. Dans tous les cas, les minimiseurs sont réguliers dans $\mathcal{D}_0$.*

Dans [2], nous avons prouvé qu'il existe une valeur critique Ω_c, avec $1<\Omega_c\rho_0\leqslant 5/4$, telle que pour $\Omega>\Omega_c$, la solution sans vortex $\chi=0$ ne minimise pas E et pour β petit, un vortex droit selon l'axe z est localement instable. La Proposition 0.1(i) implique donc que sous ces conditions, il existe une solution à vortex courbé, appelé vortex en U. En revanche pour Ω grand, le vortex droit est localement stable.

La minimisation dans le cas (ii) donne des courbes γ qui passent par l'origine et sont appelés vortex en S. Ce ne sont jamais des minima globaux de l'énergie E_Ω mais ils sont observés expérimentalement [9]. Ils existent pour toutes les valeurs de Ω, puisque la solution sans vortex ne vérifie jamais la contrainte. En revanche, si Ω est petit, il n'y a pas de U points critiques de l'énergie. Le théorème suivant justifie l'observation numérique et expérimentale que lorsque on augmente Ω, l'aire entourée par le vortex γ et le bord augmente, si bien que γ se rapproche de l'axe des z.

Théorème 0.2. *Soit $\Omega_1 < \Omega_2$, et pour $i = 1, 2$, soit χ_i le minimiseur de E_{Ω_i} dans $\mathcal{A}$, et soit U_i le support de χ_i. Alors $U_1 \subset U_2$.*

Théorème 0.3. *Pour presque tout $\Omega \geqslant 0$, il existe un unique minimiseir de E_Ω dans $\mathcal{A}$.*

La plupart des preuves s'appuient sur la relation entre le problème de minimisation $E_\Omega = H - \Omega L$ et le problème « isopérimetrique » de minimiser H (la longueur) en fixant L (l'aire) comme dans la Proposition 0.1(iii). C'est un régime où $h(\ell)$ est convexe. On s'attend à ce que $h(\ell)$ change de convexité et cela pourrait permettre de prouver l'existence de points critiques non minimisants en forme de U pour L petit.

Nos arguments dépendent très peu de la forme précise de $\mathcal{D}_0$ et ρ, et s'appliqueraient à des familles beaucoup plus générales. En revanche, la dérivation de l'énergie dépendent très fortement de la forme de ρ.

1. Introduction

Since the first experimental achievement of Bose Einstein condensation in confined alkali gases in 1995 and the Nobel Prize in 2001, many properties of these systems are studied experimentally and theoretically. One of the key issues, related to superfluidity, is the existence of vortices. Several experimental groups have produced vortices by a rotation of the trapping potential, in particular the group of J. Dalibard at the ENS in Paris [6,7] and the group of W. Ketterle at MIT [1,8]. The condensate has a cigar shape determined by the domain $\mathcal{D} = \{\rho > 0\}$, $\rho(\mathbf{r}) = \rho_0 - (\alpha^2 x^2 + y^2 + \beta^2 z^2)$. The parameters α and β characterize the shape of the trap, α being of order 1, and β small. The special shape is due to the confining potential obtained by lasers. The condensate is rotated along the z axis at velocity Ω. When the rotational velocity is sufficiently high, density dips appear in the condensate: these are vortices. One of the major observations is that vortices are not straight along the axis of rotation but bending. The number and shape of vortices depend on the rotational frequency and the geometry of the trap.

The mathematical modelling of these phenomena can be done using the minimization of the Gross–Pitaevskii energy for the wave function of the condensate, with a term taking into account the special geometry of the trap. In [3], we have done an asymptotic expansion of the energy in terms of a small parameter to derive an approximate energy which only depends on the shape and number of vortex lines. A rigorous proof of this derivation is given in [5]. If a single vortex is represented by the line γ then the energy of the vortex is given by

$$E_\Omega[\gamma] = H - \Omega L = \int_\gamma \rho \, \mathrm{d}l - \Omega \int_\gamma \rho^2 \, \mathrm{d}z, \tag{2}$$

the first term being the kinetic energy of the vortex while the second term is the angular momentum due to rotation. The vortex-free solution is set to have 0 energy. If Ω is such that $E_\Omega[\gamma] < 0$, it means that it is favorable to have a vortex. The optimal γ minimizing E_Ω will give the shape of the vortex. We checked numerically in [3] that the first vortex is indeed bending. This was considered as a good justification by the physics community [8]. In [2], we have studied properties of this approximate energy related to the experiments. Very recently and following our works, the ENS group has done extra experiments on vortices to be able to see the bending and study the single vortex configurations [9]. The aim of this Note is to provide further rigorous mathematical analysis of the properties of this single vortex line using the line energy of [3,2], and following the latest experiments.

A first property that we had derived in [2] is that the minimizing γ is planar, in the plane closer to the z axis, that is the yz plane if $\alpha < 1$. In studying minimizers it therefore suffices to consider vortices in the domain $\mathcal{D}_0 := \{(y,z)\colon\ \rho(y,z) > 0\}$, $\rho(y,z) = \rho_0 - y^2 - \beta^2 z^2$ for $(y,z) \in \mathcal{D}_0$. The variational problem in $\mathcal{D}_0$ can be rewritten by considering only vortices of the form $\gamma = \partial U$, where $U \subset \mathcal{D}_0$ is a set of finite perimeter and ∂U is oriented in the standard way, so that Stokes theorem holds. (Strictly speaking γ is the reduced boundary $\partial_* U$, see [4].) If we write χ for the characteristic function of such a set U, then $L[\gamma] = \int_{\partial U} \rho^2 \,\mathrm{d}z = \int_U 2\rho\rho_y \,\mathrm{d}y\,\mathrm{d}z = 2\int_{\mathcal{D}_0} \chi\rho\rho_y$ and $H[\gamma] = \int \rho|\nabla\chi|$. We therefore define $\mathcal{A} = \{\chi : \mathcal{D}_0 \to \{0,1\},\ \chi \in BV\}$ and for $\chi \in \mathcal{A}$ we write

$$E_\Omega[\chi] = H[\chi] - \Omega L[\chi] = \int_{\mathcal{D}_0} \rho|\nabla\chi| - 2\Omega \int_{\mathcal{D}_0} \chi\rho\rho_y.$$

Then $E_\Omega[\chi] = E_\Omega[\gamma]$ when $U = \{(y,z) \in \mathcal{D}_0\colon\ \chi(y,z) = 1\}$ and $\partial U = \gamma$. In the rest of this Note, we restrict our attention to the yz plane and study the problem of minimizing E_Ω in $\mathcal{A}$. A first result is

Proposition 1.1. (i) *For every $\Omega \geqslant 0$, there exists a minimizer of E_Ω in $\mathcal{A}$. Any minimizer is either the vortex free state $\chi = 0$, or has a vortex along the z axis ($\chi = 1$ in $y < 0$) or is supported in the set $\{y < 0\}$ and is bounded away from the z-axis. If $\Omega\rho_0 < 1/2$, there cannot exist a critical point of the energy which lies in the half yz plane $y < 0$.*

(ii) *For every $\Omega \geqslant 0$, there exists a minimizer of E_Ω in $\{\chi \in \mathcal{A}\colon\ \chi(y,z) \neq \chi(-y,-z)$ a.e.$\}$. For any such minimizer, the associated curve γ solves the Euler–Lagrange equations for the line energy.*

(iii) *For every ℓ such that $\mathcal{A}_\ell := \{\chi \in \mathcal{A}\colon\ L(\chi) = \ell\}$ is nonempty, there exists a minimizer $h(\ell)$ of H in $\mathcal{A}_\ell$. In every case, all minimizers are regular in $\mathcal{D}_0$.*

In [2], we have proved that there exists a critical Ω called Ω_c, with $1 < \Omega_c\rho_0 \leqslant 5/4$, such that for $\Omega > \Omega_c$, the vortex-free state $\chi = 0$ does not minimize E and for β small enough, a straight line along the z axis is locally unstable. Proposition 1.1(i) thus implies that under these conditions there exists a nontrivial and non straight minimizing vortex. This minimizer is seen in experiments and is called a U vortex.

The minimization in (ii) gives rise to curves γ that pass through the origin and are called S vortices. They are never global minimizers of E_Ω but are observed experimentally [9]. They exist whatever the value of Ω, since the vortex-free solution never satisfies the constraint. On the other hand, if Ω is small, there are no U vortices that are critical points of the energy. The following theorem explains a fact seen both in experiments and in numerical simulations, where it is observed that as Ω increases, the area in the yz plane enclosed by the vortex curve increases, so that the curves get closer to the z axis.

Theorem 1.2. *Fix $\Omega_1 < \Omega_2$, and for $i = 1, 2$, let χ_i minimize E_{Ω_i} in $\mathcal{A}$, and let U_i be the support of χ_i. Then $U_1 \subset U_2$.*

Theorem 1.3. *For almost every $\Omega \geqslant 0$, there is a unique minimizer of E_Ω in $\mathcal{A}$.*

Many of our proofs are based on the relationship between the problem of minimizing $E_\Omega = H - \Omega L$ and the "isoperimetric" problem of minimizing H (a weighted arclength) while fixing L (a weighted area), as in Proposition 1.1(iii). This is a regime where the isoperimetric function $h(\ell)$ is convex. But $h(\ell)$ is expected to change convexity and this could be used to prove for small L, the existence of nonminimizing, critical points of E_Ω with U shape.

Our arguments depend very little on the specific geometry of $\mathcal{D}_0$ and ρ, and with small modifications would apply quite generally to families of isoperimetric-type problems. On the other hand, the derivation and shape of the line energy strongly rely on the special function ρ.

2. Proofs

Proof of Proposition 1.1. The existence of minimizers mostly follows from standard facts about BV functions: $E_\Omega[\chi]$ is bounded below for χ in $\mathcal{A}$ and taking a minimizing sequence, we can pass to the limit and obtain convergence to a minimizer. In [5], it is shown that any such minimizer is a local minimizer (in a suitable sense) of the full 3-dimensional line energy.

Note that if $\gamma = (y(t), z(t))$ is a curve in $\mathcal{D}_0$ and if $\tilde{\gamma} = (-|y(t)|, z(t))$, then $E_\Omega[\gamma] = E_\Omega[\tilde{\gamma}]$. So we may assume that $y(t) \leqslant 0$ for all t for γ minimizing E. By regularity, it follows that if $y(t_0) = 0$ at some t_0, then $y'(t_0) = 0$. Then the Euler–Lagrange equations imply that $y(t) = 0$ for all t, and hence that γ is the straight vortex. If this does not hold, then regularity implies that γ is bounded away from the z-axis.

Suppose that γ is a vortex, parametrized by $\gamma(t) = (y(t), z(t))$ where y, z are smooth functions on an interval (a, b). We are going to construct a perturbation along which the energy gradient has a sign when $\Omega\rho_0 < 1/2\}$. For $s > 0$ define $\gamma_s(t) = (y(t)_s, z(t))$, $y_s(t) = \min(y(t) + s, (\rho_0 - \beta^2 z^2(t))^{1/2})$, and let $I := \{t \in (a, b)\colon\ \rho(\gamma(t)) > 0\}$. We compute

$$\frac{\mathrm{d}}{\mathrm{d}s} E[\gamma_s]\big|_{s=0} = \int\limits_{t \in I} -2y(t)\left(\sqrt{\dot{y}^2 + \dot{z}^2} - 2\Omega\rho\dot{z}\right)\mathrm{d}t.$$

If γ stays in $y < 0$ and $\Omega\rho_0 < 1/2$, this is always positive thus γ_s cannot be a critical point of the energy.

Existence of minimizers in cases (ii) and (iii) follows by exactly the same arguments, once one observes that the constraints are preserved by L^1 convergence. In case (ii), the curve γ associated to a minimizer χ must pass through the origin. It is easily seen that γ solve the Euler–Lagange equations away from the origin, and at the origin the Euler–Lagrange equations are satisfied if and only if the curvature vanishes, which must occur due to symmetry. Regularity follows from standard theory, see for example [4]. □

Our remaining results require some preliminary definitions and lemmas.

Let $\bar{\ell} := \max\{L[\chi]\colon\ \chi \in \mathcal{A}\}$. For $0 \leqslant \ell \leqslant \bar{\ell}$, define $h(\ell) = \min\{H[\chi]\colon\ \chi \in \mathcal{A},\ L[\chi] = \ell\}$. Proposition 1.1 guarantees that the minimum is attained. Let $\partial h(\ell)$ denote the subgradient of h at ℓ: $\partial h(\ell) = \{\Omega > 0\colon\ h(\ell') \geqslant h(\ell) + \Omega(\ell' - \ell)$ for all $0 \leqslant \ell' \leqslant \bar{\ell}\}$ and let h_c denote the convex envelope of h on the interval $[0, \bar{\ell}]$, that is $h_c(\ell) = \sup\{u(\ell)\colon\ u \leqslant h,\ u$ is convex on $[0, \bar{\ell}]\}$. Finally define $\Sigma = \{\ell \in [0, \bar{\ell}]\colon\ \partial h(\ell)$ is nonempety$\} = \{\ell \in [0, \bar{\ell}]\colon\ h(\ell) = h_c(\ell))\}$. Note that $\Omega \in \partial h(\ell)$ if and only if $\ell \in \Sigma$ and $\Omega \in \partial h_c(\ell)$. Theorem 2 of [2] implies that h is convex for ℓ close to $\bar{\ell}$ if β is sufficiently small, which ensures that Σ is nonempty. On the other hand, Theorem 4 of [2] shows that $h(\ell) \geqslant c\ell^{2/3}$. Thus we expect that h is concave near $\ell = 0$. We prove

Lemma 2.1. *If $\chi \in \mathcal{A}$ minimizes $E_\Omega = H - \Omega L$, then $H[\chi] = h(L[\chi])$. Also, $L[\chi] \in \Sigma$, and $\Omega \in \partial h(L[\chi])$. Conversely, for any $\ell \in \Sigma$ and $\Omega \in \partial h(\ell)$, if $\chi \in \mathcal{A}_\ell$ satisfies $H[\chi] = h(\ell)$, then χ minimizes E_Ω in $\mathcal{A}$.*

To prove the first assertions, fix χ minimizing E_Ω, and let $\ell = L[\chi]$. For any $\tilde{\chi}$ such that $L(\tilde{\chi}) = \ell$, $H[\tilde{\chi}] = E_\Omega[\tilde{\chi}] + \Omega L[\tilde{\chi}] \geqslant E_\Omega[\chi] + \Omega L[\chi] = H[\chi]$, which proves that χ minimizes H in $\mathcal{A}_\ell$, i.e., that $H[\chi] = h(\ell)$.

To prove that $\ell \in \Sigma$, fix any $\ell' \neq \ell$ and find χ' such that $L(\chi') = \ell'$, $H(\chi') = h(\ell')$. Then $h(\ell') - \Omega\ell' = H[\chi'] - \Omega L[\chi'] = E_\Omega[\chi'] \geqslant E_\Omega[\chi] = h(\ell) - \Omega\ell$. Rearranging this gives $h(\ell') \geqslant h(\ell) + \Omega(\ell' - \ell)$, and so $\ell \in \Sigma$ and $\Omega \in \partial h(\ell)$ as claimed.

To prove the other assertions, fix $\ell \in \Sigma$ and $\chi \in \mathcal{A}$ such that $L[\chi] = \ell$, $H[\chi] = h(\ell)$, and fix $\Omega \in \partial h(\ell)$. For any $\chi' \in \mathcal{A}$, let $\ell' = L[\chi']$. Then $E_\Omega[\chi'] = H[\chi'] - \Omega\ell' \geqslant h(\ell') - \Omega\ell' \geqslant h(\ell) - \Omega\ell = E_\Omega[\chi]$.

Lemma 2.2. *If $\chi_1, \chi_2 \in \mathcal{A}$, then for $\chi_* = \chi_1\chi_2$ and $\chi^* = \chi_1 + \chi_2 - \chi_1\chi_2$,*

$$L[\chi_1] + L[\chi_2] = L[\chi_*] + L[\chi^*], \qquad H[\chi_1] + H[\chi_2] \geqslant H[\chi_*] + H[\chi^*].$$

Note that if χ_i is the characteristic function of U_i for $i = 1, 2$, then χ_* and χ^* are the characteristic function of $U_1 \cap U_2$ and $U_1 \cup U_2$ respectively.

The first conclusion is obvious. The second follows from noting that $|\nabla\chi_1| + |\nabla\chi_2| \geqslant \left|\nabla(\chi_1 + \chi_2)\right| = \left|\nabla(\chi_* + \chi_*)\right| = |\nabla\chi_*| + |\nabla\chi^*|$ as measures. The last equality is a consequence of the fact that $U_* := \operatorname{supp}\chi_*$ is a subset of $U^* := \operatorname{supp}\chi^*$. Thus if $\partial U_* \cap \partial U^*$ is a set of positive one-dimensional measure, then their outer unit normals must be parallel (rather than antiparallel) along this set. Hence there can be no cancellation.

Proof of Theorem 1.2. Fix $\Omega_1 < \Omega_2$ and let χ_i be a minimizer of E_{Ω_i}, $i = 1, 2$. Let $\ell_i = L[\chi_i]$ for $i = 1, 2$. Define χ_* and χ^* as in Lemma 2.2 and let $\ell_* = L[\chi_*]$, $\ell^* = L[\chi^*]$. From Proposition 1.1 we know that minimizers are contained in the region where the integrand in L is positive, and it follows that $\ell_* \leqslant \ell_1, \ell_2 \leqslant \ell^*$. Moreover, to prove the theorem it suffices to show that $\ell^* = \ell_2$, as this will prove that $\chi_2 = \chi^*$, or in other words that $U_1 \cup U_2 = U_2$, for $U_i = \operatorname{supp}\chi_i$.

To do this, write $h_* := H[\chi_*]$ and note that $h_* \geqslant h(\ell_*) \geqslant h_1 + \Omega_1(\ell_* - \ell_1)$. Similarly $h^* = H(\chi^*) \geqslant h_2 + \Omega_2(\ell^* - \ell_2)$. Lemma 2.2 implies that $h_* + h^* \leqslant h_1 + h_2$ and that $\Omega_1(\ell^* - \ell_1) = \Omega_1(\ell_2 - \ell^*)$, and so by adding the two equations and rearranging we find that $0 \geqslant (\Omega_2 - \Omega_1)(\ell^* - \ell_2)$. Since $\Omega_2 > \Omega_1$ and $\ell^* \geqslant \ell_2$, we deduce that $\ell_* = \ell_2$ as required. □

Proof of Theorem 1.3. First we claim that the set $M := \{\Omega > 0\colon\ \Omega \in \partial h(\ell^*) \cap \partial h(\ell_*)$ for some $\ell_* < \ell^*\}$ is at most countable. Indeed, if $\Omega \in M$, then also $\Omega \in \partial h_c(\ell_*) \cap \partial h_c(\ell^*)$. And because h_c is convex, it follows that, in the interval $\ell_* < \ell < \ell^*$, h_c is affine with slope Ω. Clearly, there can be at most countably many values Ω with this property, proving the claim.

Now suppose that Ω is a value such that there are distinct minimizers $\chi_1 \neq \chi_2$ of E_Ω. Define χ_* and χ^* as in Lemma 2.2. In view of Lemma 2.2, $E_\Omega[\chi_*] + E_\Omega[\chi^*] \leqslant E_\Omega[\chi_1] + E_\Omega[\chi_2]$, and so it follows that χ_*, χ^* are also minimizers. Because (by Proposition 1.1) χ_1 and χ_2 are supported in the set $\{y < 0\}$, the form of ρ implies that $\ell_* := L(\chi_*) < L(\chi^*) =: \ell^*$. Then Lemma 2.1 implies that Ω belongs to the countable set M defined above. This proves uniqueness of minimizers away from a set of measure zero. □

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