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Numerical Analysis

An exact Block–Newton algorithm for solving fluid–structure interaction problems

Un algorithme exact de Newton par blocs pour la résolution de problèmes d'interaction fluide–structure

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Abstract

In this Note, we introduce a partitioned Newton based method for solving nonlinear coupled systems arising in the numerical approximation of fluid–structure interaction problems. The originality of this Schur–Newton algorithm lies in the exact Jacobians evaluation involving the fluid–structure linearized subsystems which are here fully developed. ***To cite this article: M.Á. Fernández, M. Moubachir, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Dans cette Note, nous nous intéressons à une méthode à partitions de type Newton pour la résolution de systèmes couplés non-linéaires intervenant dans l'approximation numérique des problèmes d'interaction fluide–structure. Cet algorithme utilise, de manière fondamentale, l'évaluation exacte des jacobiens construits à partir des sous-problèmes fluide–structure linéarisés dont nous fournissons la structure exacte. ***Pour citer cet article : M.Á. Fernández, M. Moubachir, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

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L'utilisation de schémas implicites de discrétisation temporelle pour l'approximation numérique de systèmes couplés fluide-structure permet de garantir la stabilité des solutions associées. Cependant, il est alors nécessaire de résoudre à chaque pas de temps, un système non-linéaire fortement couplé. Une solution est d'utiliser un algorithme

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de type Newton par blocs. Il permet à la fois de découpler ce système en conservant la structuration naturelle en sous-problèmes fluide et structure, ce qui autorise l'utilisation de solveurs dédiés à chacun des sous-systèmes, et en même temps de garantir une convergence rapide vers la solution du problème couplé non-linéaire. Notre contribution réside dans l'obtention de la structure des sous-problèmes linéarisés intervenant dans l'algorithme de Schur–Newton.

Le problème mécanique. Nous considérons un système mécanique occupant le domaine mobile $\Omega(t)$, constitué par un solide déformable $\Omega^{\mathrm{s}}(t)$ entourant un fluide défini dans le domaine $\Omega^{\mathrm{f}}(t)$ (Fig. 1). Le fluide est supposé newtonien visqueux, homogène et en écoulement incompressible. Le solide élastique subit de grandes déformations. On impose à l'interface fluide–structure la continuité cinématique des vitesses (1) et cinétique des efforts (2). Une formulation faible de type ALE est fournie par le système (3).

Discrétisation temporelle. Cette formulation est discrétisée en temps de façon implicite. Ce qui conduit à résoudre à chaque pas de temps le système couplé non-linéaire (6) faisant intervenir les opérateurs fluide et structure (4), (5).

L'algorithme de Schur–Newton. Les algorithmes de type Gauss–Seidel ou Jacobi par blocs s'appuyent sur une itération de type point-fixe dont la convergence est lente. L'algorithme de Newton par blocs est fondé sur une itération de type Newton–Raphson dont les propriétés de convergence sont meilleurs. La contribution principale de cette Note consiste à établir la structure analytique des dérivées des opérateurs (4), (5) par rapport aux variables d'état fluides et structures. Les opérateurs linéarisés (7)–(10) sont obtenus grâce à l'utilisation des techniques de dérivation introduites dans [1]. De plus, nous établissons la structure du système linéarisé générique (11) que l'on doit résoudre lors des différentes étapes de l'algorithme de Schur–Newton en incluant les second membres (12), (13).

1. Introduction

One issue arising in the numerical approximation of coupled fluid–structure systems, is the definition of coupling algorithms based on specific solvers involving efficient discretization for each of the solid and fluid subsystems, that may guarantee accurate and fast convergence of the overall system. This issue is particularly difficult to face when the fluid and the solid densities are of the same order, as it happens in hemodynamics for example, since only implicit schemes can ensure stability of the resulting method. Thus, at each time step, the rule is to solve a coupled non-linear system using efficient methods that may preserve, inside inner loops, the fluid–structure subsystem splitting. This can be achieved using block iteration algorithms. Recent advances in this topic suggest the use of Block Newton based method [3] for a fast convergence towards the solution of the non-linear coupled system. Our contribution lies in the derivation of explicit linearized systems that need to be solved in order to evaluate the different Jacobians involved in the Schur–Newton algorithm.

2. Mechanical problem

We consider a mechanical system occupying a moving domain $\Omega(t)$. It consists of a deformable structure $\Omega^{\mathrm{s}}(t)$ (vessel wall, pipe-line, ...) surrounding a fluid under motion (blood, oil, ...) in the complement $\Omega^{\mathrm{f}}(t)$ of $\Omega^{\mathrm{s}}(t)$ in $\Omega(t)$ (Fig. 1). The problem is to determinate the time evolution of the configuration $\Omega(t)$, as well as the velocity and Cauchy stress tensor within the fluid and the structure. The density, velocity and Cauchy stress tensor in the moving configuration are governed by basic conservation and constitutive laws. To describe the evolution of $\Omega(t)$, we introduce a continuous mapping $x : \Omega_0 \times \mathbb{R}^+ \to \mathbb{R}^3$ which maps any point x_0 of a fixed (reference) configuration Ω_0 into its image $x(x_0, t)$ inside the actual configuration $\Omega(t)$. The choice of both the configuration Ω_0 and the map x may be arbitrary. It is nevertheless simpler to impose that the point $x^{\mathrm{s}}(x_0, t) = x_{|\Omega_0^{\mathrm{s}}}(x_0, t)$ corresponds to the position, at time t, of the material point x_0 inside the solid domain. Conversely, inside the fluid domain, the mapping $x^{\mathrm{f}} = x_{|\Omega_0^{\mathrm{f}}}$ can be any reasonable extension of the material interface deformation: $x^{\mathrm{f}} = \mathrm{Ext}(x^{\mathrm{s}}_{|\Gamma_0^{\mathrm{w}}})$, $\dot{x}^{\mathrm{f}}_{|\Gamma_0^{\mathrm{w}}} = \dot{x}^{\mathrm{s}}_{|\Gamma_0^{\mathrm{w}}}$.

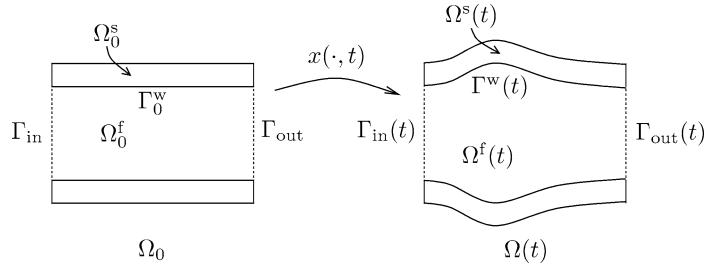

Fig. 1. Geometric configurations.

Fig. 1. Les configurations géométriques.

We deal with a Newtonian viscous, homogeneous fluid under incompressible flow with density ρ and kinetic viscosity μ. Its behavior is described by its velocity u and pressure p. The elastic solid under large displacements is described by its velocity $w^s = \dot{x}^s$ and the stress tensor S (second Piola–Kirchoff tensor). The field S is related to x^s through an appropriate constitutive law. The coupling between the solid and the fluid is realized through standard boundary conditions at the fluid–structure interface, namely, the kinematic continuity of the velocity,

$$u = w^f, \quad \text{on } \Gamma^w(t), \tag{1}$$

and the kinetic continuity of the stress,

$$FSn_0 = J\sigma(u,p)F^{-T}n_0, \quad \text{on} \quad \Gamma_0^w. \tag{2}$$

We end up with the following weak coupled system using an ALE fluid formulation:

$$\begin{aligned}
&\rho\frac{d}{dt}\int_{\Omega^f(t)} u\cdot v\,dx - \rho\int_{\Omega^f(t)} \operatorname{div}\big[u\otimes(u-w^f)\big]\cdot v\,dx + \int_{\Omega^f(t)} \sigma(u,p):\nabla u\,dx - \int_{\Gamma^w(t)} \lambda\cdot v\,da\\
&\quad - \int_{\Gamma_{in}(t)\cup\Gamma_{out}(t)} g(t)\cdot v\,da + \int_{\Omega^f(t)} q\operatorname{div} u\,dx + \int_{\Gamma^w(t)} (u-w^f)\cdot\mu\,da = 0,\\
&\quad \forall (v,q,\mu)\in H^1\big(\Omega^f(t)\big)\times L^2\big(\Omega^f(t)\big)\times H^{-1/2}\big(\Gamma^w(t)\big),\\
&\int_{\Omega_0^s}\rho_0\dot{w}^s\cdot y\,dx + \int_{\Omega_0^s} FS:\nabla y\,dx + \int_{\Gamma_0^w} J\big\|F^{-T}n\big\|\lambda\cdot y\,dx + \int_{\Omega_0}(w-\dot{x})\cdot z\,dx\\
&\quad + \int_{\Omega_0^f}\big(x^f - \operatorname{Ext}\big(x^s_{|\Gamma_0^w}\big)\big)\cdot y\,dx = 0, \quad \forall (y,z)\in H^1(\Omega_0)\times L^2(\Omega_0),
\end{aligned} \tag{3}$$

where g stands for the external forces acting on the fluid (defined on the whole $\mathbb{R}^3$). Moreover, λ stands for the Lagrange multiplier (interface force) associated to the velocity continuity kinematic constraint imposed at the fluid–structure interface in a weak form.

3. Time discretization

The weak coupled formulation (3) is now semi-discretized in time. Numerical experiments shows that only implicit schemes ensure stability and allow to solve effectively problems in which both the fluid and structural densities are of the same order. We use the implicit scheme provided in [2]: it involves an implicit Euler treatment on the fluid domain and a mid-point rule for the structural equation. Introducing the following fluid and solid weak state operators:

$$\begin{aligned}\big\langle \mathcal{F}\big((u,p,\lambda),(x,w)\big),(v,q,\mu)\big\rangle &= \frac{\rho}{\Delta t}\int_{\Omega^{\mathrm{f}}(x)} u\cdot v\,\mathrm{d}x - \frac{\rho}{\Delta t}\int_{\Omega^{\mathrm{f}}(t_n)} u^n\cdot v\,\mathrm{d}x\\ &+ \rho\int_{\Omega^{\mathrm{f}}(x)} \operatorname{div}\big[u\otimes\big(u-w^{\mathrm{f}}\big)\big]\cdot v\,\mathrm{d}x + \int_{\Omega^{\mathrm{f}}(x)} \sigma(u,p):\nabla v\,\mathrm{d}x - \int_{\Gamma^{\mathrm{w}}(x)} \lambda\cdot v\,\mathrm{d}a\\ &- \int_{\Gamma_{\mathrm{in}}(x)\cup\Gamma_{\mathrm{out}}(x)} g(t_{n+1})\cdot v\,\mathrm{d}a + \int_{\Omega^{\mathrm{f}}(x)} q\operatorname{div}u\,\mathrm{d}x + \int_{\Gamma^{\mathrm{w}}(x)} (u-w^{\mathrm{f}})\cdot\mu\,\mathrm{d}a, \end{aligned} \tag{4}$$

$$\begin{aligned}\big\langle \mathcal{S}\big((u,p,\lambda),(x,w)\big),(y,z)\big\rangle &= \frac{2}{\Delta t}\int_{\Omega_0^{\mathrm{s}}} \rho_0\big(w^{\mathrm{s}}-2w^{\mathrm{s},n}+w^{\mathrm{s},n-1/2}\big)\cdot y\,\mathrm{d}x + \int_{\Omega_0^{\mathrm{s}}} FS:\nabla y\,\mathrm{d}x\\ &+ \int_{\Gamma_0^{\mathrm{w}}} J\big\|F^{-\mathrm{T}}n\big\|\lambda\cdot y\,\mathrm{d}x + \int_{\Omega_0^{\mathrm{f}}}\big(x^{\mathrm{f}}-\mathrm{Ext}\big(x^{\mathrm{s}}_{|\Gamma_0^{\mathrm{w}}}\big)\big)\cdot y\,\mathrm{d}x + \int_{\Omega_0^{\mathrm{f}}}\left[w^{\mathrm{f}}-\frac{1}{\Delta t}\big(x^{\mathrm{f}}-x^{\mathrm{f},n}\big)\right]\cdot z\,\mathrm{d}x\\ &+ \int_{\Omega_0^{\mathrm{s}}}\left[w^{\mathrm{s}}-\frac{\Delta t}{2}\left(x^{\mathrm{s}}-2x^{\mathrm{s},n}+x^{\mathrm{s},n-1/2}\right)\right]\cdot z\,\mathrm{d}x, \end{aligned} \tag{5}$$

the semi-discretized in time problem writes: find $((u^{n+1},p^{n+1},\lambda^{n+1}),(x^{n+1},w^{n+1}))$ solution of the following system,

$$\begin{aligned}&\big\langle \mathcal{F}\big(\big(u^{n+1},p^{n+1},\lambda^{n+1}\big),\big(x^{n+1},w^{n+1}\big)\big),(v,q,\mu)\big\rangle = 0,\\ &\big\langle \mathcal{S}\big(\big(u^{n+1},p^{n+1},\lambda^{n+1}\big),\big(x^{n+1},w^{n+1}\big)\big),(y,z)\big\rangle = 0,\end{aligned} \tag{6}$$

for all $(v,q,\mu)\in H^1(\Omega^{\mathrm{f}}(t_{n+1}))\times L^2(\Omega^{\mathrm{f}}(t_{n+1}))\times H^{-1/2}(\gamma(t_{n+1}))$ and $(y,z)\in H^1(\Omega_0)\times L^2(\Omega_0)$.

4. The Schur–Newton method

Standard forms for solving the coupled non-linear problem (6) are Block–Jacobi or Block–Gauss–Seidel iterations. Unfortunately, these methods usually show poor convergence properties. In order to speed up the convergence, we can use Newton–Raphson based methods. These methods require the evaluation of the derivative of the weak operators $(\mathcal{F},\mathcal{S})$ with respect to the state variables. Hence, the Schur–Newton algorithm reads as follows:

1. Choose $(\bar{u}^{\mathrm{f}},\bar{u}^{\mathrm{s}})\in U^{\mathrm{f}}\times U^{\mathrm{s}}$,
2. Do until convergence,
   (a) Solve $D_{u^{\mathrm{f}}}\mathcal{F}(\bar{u}^{\mathrm{f}},\bar{u}^{\mathrm{s}})u_1 = -\mathcal{F}(\bar{u}^{\mathrm{f}},\bar{u}^{\mathrm{s}})$,
   (b) Evaluate the residual $r = \mathcal{S}(\bar{u}^{\mathrm{f}},\bar{u}^{\mathrm{s}}) + D_{u^{\mathrm{f}}}\mathcal{S}(\bar{u}^{\mathrm{f}},\bar{u}^{\mathrm{s}})u_1$,
   (c) Solve $\mathbb{S}(\bar{u}^{\mathrm{f}},\bar{u}^{\mathrm{s}})\delta u^{\mathrm{s}} = -r$, with $\mathbb{S}(\bar{u}^{\mathrm{f}},\bar{u}^{\mathrm{s}})$ the Schur complement operator

$$\mathbb{S} = \big(D_{u^{\mathrm{s}}}\mathcal{S} + D_{u^{\mathrm{f}}}\mathcal{S}\big(D_{u^{\mathrm{f}}}\mathcal{F}\big)^{-1}(-D_{u^{\mathrm{s}}}\mathcal{F})\big)\big(\bar{u}^{\mathrm{f}},\bar{u}^{\mathrm{s}}\big),$$

   (d) Solve $D_{u^{\mathrm{f}}}\mathcal{F}(\bar{u}^{\mathrm{f}},\bar{u}^{\mathrm{s}})u_2 = -D_{u^{\mathrm{s}}}\mathcal{F}(\bar{u}^{\mathrm{f}},\bar{u}^{\mathrm{s}})\delta u^{\mathrm{s}}$,
   (e) Evaluate $\delta u^{\mathrm{f}} = u_1 + u_2$,
   (f) Update rule: $[\bar{u}^{\mathrm{f}},\bar{u}^{\mathrm{s}}]\leftarrow[\bar{u}^{\mathrm{f}}+\delta u^{\mathrm{f}},\bar{u}^{\mathrm{s}}+\delta u^{\mathrm{s}}]$.

Usually steps involving the evaluation of the Jacobians are performed using a finite difference approximation that only requires state operators evaluations [3]. However, the lack of a priori rules for selecting optimal finite difference infinitesimal steps, leads to non-consistent Jacobians and a reduction of the overall convergence speed.

In the next section, we show how to avoid this approximation by establishing the exact expression of the different linearized systems.

4.1. Weak state operators derivatives

Using similar techniques to those developed in [1], we are able to obtain the expressions of the action of the derivatives of weak state operators $(\mathcal{F},\mathcal{S})$ with respect to the state variables $(u^{\mathrm{f}} \overset{\text{def}}{=} ((u,p,\lambda)), u^{\mathrm{s}} \overset{\text{def}}{=} (x,w))$ at point $((\bar u,\bar p,\bar\lambda),(\bar x,\overline{w}))$ in the direction $((\delta u,\delta p,\delta\lambda),(\delta x,\delta w))$:

- Fluid state operator derivatives:

$$\begin{aligned}
&\big\langle D_{(u,p,\lambda)}\mathcal{F}\big((\bar u,\bar p,\bar\lambda),(\bar x,\overline{w})\big)(\delta u,\delta p,\delta\lambda),(v,q,\mu)\big\rangle = \frac{\rho}{\Delta t}\int\limits_{\Omega^{\mathrm{f}}(\bar x)} \delta u\cdot v\,\mathrm{d}x\\
&\quad+\rho\int\limits_{\Omega^{\mathrm{f}}(\bar x)} \operatorname{div}\big[\delta u\otimes\big(\bar u-\overline{w}^{\mathrm{f}}\big)+\bar u\otimes\delta u\big]\cdot v\,\mathrm{d}x+\int\limits_{\Omega^{\mathrm{f}}(\bar x)}\sigma(\delta u,\delta p):\nabla v\,\mathrm{d}x\\
&\quad-\int\limits_{\Gamma^{\mathrm{w}}(\bar x)}\delta\lambda\cdot v\,\mathrm{d}a+\int\limits_{\Omega^{\mathrm{f}}(\bar x)} q\operatorname{div}\delta u\,\mathrm{d}x+\int\limits_{\Gamma^{\mathrm{w}}(\bar x)}\delta u\cdot\mu\,\mathrm{d}a, \qquad (7)\\
&\big\langle D_{(x,w)}\mathcal{F}\big((\bar u,\bar p,\bar\lambda),(\bar x,\overline{w})\big)(\delta x,\delta w),(v,q,\mu)\big\rangle = \frac{\rho}{\Delta t}\int\limits_{\Omega^{\mathrm{f}}(\bar x)}\operatorname{div}\delta x\,\bar u\cdot v\,\mathrm{d}x\\
&\quad+\rho\int\limits_{\Omega^{\mathrm{f}}(\bar x)}\operatorname{div}\big\{\bar u\otimes\big(\bar u-\overline{w}^{\mathrm{f}}\big)\big[\mathrm{I}\operatorname{div}\delta x-(\nabla\delta x)^{\mathrm{T}}\big]\big\}\cdot v\,\mathrm{d}x-\rho\int\limits_{\Omega^{\mathrm{f}}(\bar x)}\operatorname{div}\big(\bar u\otimes\delta w^{\mathrm{f}}\big)\cdot v\,\mathrm{d}x\\
&\quad+\int\limits_{\Omega^{\mathrm{f}}(\bar x)}\sigma(\bar u,\bar p)\big[\mathrm{I}\operatorname{div}\delta x-(\nabla\delta x)^{\mathrm{T}}\big]:\nabla v\,\mathrm{d}x-\int\limits_{\Gamma^{\mathrm{w}}(\bar x)}\big(\eta(\delta x)\cdot n\big)\bar\lambda\cdot v\,\mathrm{d}a\\
&\quad-\int\limits_{\Gamma_{\mathrm{in}}(\bar x)\cup\Gamma_{\mathrm{out}}(\bar x)}\big(\eta(\delta x)\cdot n\big)g(t_{n+1})\cdot v\,\mathrm{d}a-\int\limits_{\Omega^{\mathrm{f}}(\bar x)} q\operatorname{div}\big\{\bar u\big[\mathrm{I}\operatorname{div}\delta x-(\nabla\delta x)^{\mathrm{T}}\big]\big\}\,\mathrm{d}x\\
&\quad+\int\limits_{\Gamma^{\mathrm{w}}(\bar x)}\big[\big(\bar u-\overline{w}^{\mathrm{f}}\big)\big(\eta(\delta x)\cdot n\big)-\delta w^{\mathrm{f}}\big]\cdot\mu\,\mathrm{d}a. \qquad (8)
\end{aligned}$$

- Solid state operator derivatives:

$$\begin{aligned}
&\big\langle D_{(u,p,\lambda)}\mathcal{S}\big((\bar u,\bar p,\bar\lambda),(\bar x,\overline{w})\big)(\delta u,\delta p,\delta\lambda),(y,z)\big\rangle=\int\limits_{\Gamma_0^{w}}\bar J\,\big\|\overline{F}^{-\mathrm{T}}n\big\|\,\delta\lambda\cdot y\,\mathrm{d}x, \qquad (9)\\
&\big\langle D_{(x,w)}\mathcal{S}\big((\bar u,\bar p,\bar\lambda),(\bar x,\overline{w})\big)(\delta x,\delta w),(y,z)\big\rangle=\frac{2}{\Delta t}\int\limits_{\Omega_0^{\mathrm{s}}}\rho_0\delta w^{\mathrm{s}}\cdot y\,\mathrm{d}x\\
&\quad+\int\limits_{\Omega_0^{\mathrm{s}}}\big(\overline{F}\delta S+\nabla\delta x\,\overline{S}\big):\nabla y\,\mathrm{d}x+\int\limits_{\Gamma_0^{\mathrm{w}}}\bar J\,\big\|\overline{F}^{-\mathrm{T}}n\big\|\big(\eta(\delta x)\cdot n\big)\bar\lambda\cdot y\,\mathrm{d}x\\
&\quad+\int\limits_{\Omega_0^{\mathrm{f}}}\big(\delta x^{\mathrm{f}}-\mathrm{Ext}'\big(\bar x^{\mathrm{s}}_{|\Gamma_0^{\mathrm{w}}}\big)\delta x^{\mathrm{f}}_{|\Gamma_0^{\mathrm{w}}}\big)\cdot y\,\mathrm{d}x+\int\limits_{\Omega_0^{\mathrm{s}}}\left(\delta w^{\mathrm{s}}-\frac{\Delta t}{2}\delta x^{\mathrm{s}}\right)\cdot z\,\mathrm{d}x+\int\limits_{\Omega_0^{\mathrm{f}}}\left(\delta w^{\mathrm{f}}-\frac{1}{\Delta t}\delta x^{\mathrm{f}}\right)\cdot z\,\mathrm{d}x. \qquad (10)
\end{aligned}$$

Here, $\eta(\delta x) = [\mathrm{I}\,\mathrm{div}\,\delta x - (\nabla\delta x)^{\mathrm{T}}]n$ stands for the variation at first order of the surface vector $n\,\mathrm{d}a$.

Remark 1. Here, the use of the intrinsic linearized state introduced in [1] does not simplify the Jacobian expressions since the transport lemma is no more valid when the state equations are not satisfied by the reference flow.

4.2. Algorithm substeps in strong formulation

In the Schur–Newton algorithm, step 2(a) can be carried out solving the following strong fluid subproblem:

$$\begin{cases} \dfrac{\rho}{\Delta t}\delta u + \rho\,\mathrm{div}\big[\delta u\otimes\big(\bar u-\overline{w}^{\mathrm{f}}\big)+\bar u\otimes\delta u\big] - 2\mu\,\mathrm{div}\,\varepsilon(\delta u)+\nabla\delta p = l_1, & \text{in } \Omega^{\mathrm{f}}(\bar x),\\ \mathrm{div}\,\delta u = l_2, & \text{in } \Omega^{\mathrm{f}}(\bar x),\\ \delta u = l_3, & \text{on } \Gamma^{\mathrm{w}}(\bar x),\\ \sigma(\delta u,\delta p)n = l_4, & \text{on } \Gamma_{\mathrm{in}}(\bar x)\cup\Gamma_{\mathrm{out}}(\bar x), \end{cases} \tag{11}$$

with

$$\begin{aligned} l_1 &= -\frac{\rho}{\Delta t}\big(\bar u - J^{n,\bar x}u^n\big) - \mathrm{div}\big[\rho\bar u\otimes\big(\bar u-\overline{w}^{\mathrm{f}}\big)-\sigma(\bar u,\bar p)\big], \quad \text{in } \Omega^{\mathrm{f}}(\bar x),\\ l_2 &= -\,\mathrm{div}\,\bar u, \quad \text{in } \Omega^{\mathrm{f}}(\bar x),\\ l_3 &= \overline{w}^{\mathrm{f}} - \bar u, \quad \text{on } \Gamma^{\mathrm{w}}(\bar x),\\ l_4 &= g(t_{n+1}) - \sigma(\bar u,\bar p)n, \quad \text{on } \Gamma_{\mathrm{in}}(\bar x)\cup\Gamma_{\mathrm{out}}(\bar x), \end{aligned} \tag{12}$$

and $\delta\lambda = \sigma(\delta u,\delta p)n + \sigma(\bar u,\bar p) - \bar\lambda$. Here, $J^{n,\bar x} = \det\nabla(x^n\circ\bar x^{-1})$ denotes the Jacobian of the mapping $x^n\circ\bar x^{-1}$ from $\Omega^{\mathrm{f}}(\bar x)$ to $\Omega^{\mathrm{f}}(t_n)$.

Step 2(c) can be carried out using an iterative free matrix method, e.g., GMRES (see [3]). These methods only require matrix vector product operations. Such operations involve the solution of the strong fluid subproblem (11) with:

$$\begin{aligned} l_1 &= -\frac{\rho}{\Delta t}\,\mathrm{div}\,\delta x\bar u - \mathrm{div}\big\{\big[\rho\bar u\otimes\big(\bar u-\overline{w}^{\mathrm{f}}\big)-\sigma(\bar u,\bar p)\big]\big[\mathrm{I}\,\mathrm{div}\,\delta x-(\nabla\delta x)^{\mathrm{T}}\big]\big\} + \mathrm{div}\big(\rho\bar u\otimes\delta w^{\mathrm{f}}\big), \quad \text{in } \Omega^{\mathrm{f}}(\bar x),\\ l_2 &= -\,\mathrm{div}\big\{\bar u\big[\mathrm{I}\,\mathrm{div}\,\delta x-(\nabla\delta x)^{\mathrm{T}}\big]\big\}, \quad \text{in } \Omega^{\mathrm{f}}(\bar x),\\ l_3 &= \delta w^{\mathrm{f}} - \big(\bar u-\overline{w}^{\mathrm{f}}\big)\big(\eta(\delta x)\cdot n\big), \quad \text{on } \Gamma^{\mathrm{w}}(\bar x),\\ l_4 &= g(t_{n+1})\big(\eta(\delta x)\cdot n\big) - \sigma(\bar u,\bar p)\eta(\delta x), \quad \text{on } \Gamma_{\mathrm{in}}(\bar x)\cup\Gamma_{\mathrm{out}}(\bar x), \end{aligned} \tag{13}$$

and $\delta\lambda = \sigma(\delta u,\delta p)n + \sigma(\bar u,\bar p)\eta(\delta x) - (\eta(\delta x)\cdot n)\bar\lambda$. Finally, for step 2(d) we have to solve again (11) provided with data (13).

5. Conclusion

Using Block–Newton based methods for solving nonlinear coupled systems arising in the numerical approximation of fluid–structure interaction problems with time implicit schemes, requires the evaluation of Jacobians associated to each subsystems. In this Note, we have proposed a new strategy consisting in implementing linearized solvers in order to evaluate, in a consistent way, the different jacobians involved in the Block Schur–Newton algorithm. Our contribution lies in the derivation of these linearized systems. Very efficient coupling solvers can be designed with such an approach and this will be the subject of a forthcoming paper.

References

[1] M.A. Fernández, M. Moubachir, Sensitivity analysis for an incompressible aeroelastic system, Math. Models Methods Appl. Sci. 12 (8) (2002) 1109–1130.

[2] P. Le Tallec, J. Mouro, Fluid structure interaction with large structural displacements, Comput. Methods Appl. Mech. Engrg. 190 (24–25) (2001) 3039–3067.

[3] J. Steindorf, H.G. Matthies, Numerical efficiency of different partitioned methods for fluid–structure interaction, Z. Angew. Math. Mech. 80 (2) (2000) 557–558.