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Algebraic Geometry

# Counting curves via lattice paths in polygons

## Calcul de courbes holomorphes par chemins en polygones

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### Abstract

This Note presents a formula for the enumerative invariants of arbitrary genus in toric surfaces. The formula computes the number of curves of a given genus through a collection of generic points in the surface. The answer is given in terms of certain lattice paths in the relevant Newton polygon. If the toric surface is  $\mathbb{P}^2$  or  $\mathbb{P}^1 \times \mathbb{P}^1$  then the invariants under consideration coincide with the Gromov–Witten invariants. The formula gives a new count even in these cases, where other computational techniques are available. **To cite this article:** G. Mikhalkin, C. R. Acad. Sci. Paris, Ser. I 336 (2003).

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### Résumé

Cette Note présente une formule pour les invariants énumératifs de genre arbitraire en surfaces toriques. La formule calcule le nombre des courbes de genre donné qui passent par une collection de points génériques sur la surface. Le résultat est donné en fonction de certains chemins dans le polygone de Newton relevant. Si la surface torique est  $\mathbb{P}^2$  ou  $\mathbb{P}^1 \times \mathbb{P}^1$  nos invariants sont les invariants de Gromov–Witten. La formule est nouvelle même dans ces cas, où d'autres techniques de calcul sont disponibles. **Pour citer cet article :** G. Mikhalkin, C. R. Acad. Sci. Paris, Ser. I 336 (2003).

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### Version française abrégée

Rappelons qu'on peut associer à chaque polygone convexe  $\Delta \subset \mathbb{R}^2$  à sommets entiers une surface compacte torique  $\mathcal{C}T_\Delta \supset (\mathbb{C}^*)^2$  et un système linéaire de courbes holomorphes dans cette surface. Par exemple, pour obtenir  $\mathbb{C}P^2$  et les courbes de degré  $d$  on peut prendre un triangle de sommets  $(d, 0)$ ,  $(0, d)$  et  $(0, 0)$ . Pour obtenir  $\mathbb{C}P^1 \times \mathbb{C}P^1$  et les courbes de bi-degré  $(r, s)$  on peut prendre  $\Delta = [0, r] \times [0, s]$ . On voit le polygone  $\Delta$  comme un analogue du degré en  $\mathcal{C}T_\Delta$ . La dimension de  $\mathbb{P}\mathcal{L}$  est  $m = \#(\Delta \cap \mathbb{Z}^2) - 1$ . Une courbe générique du système  $\mathbb{P}\mathcal{L}$  est lisse de genre  $l = \#(\text{Int } \Delta \cap \mathbb{Z}^2)$ . Cette Note concerne les courbes de  $\mathbb{P}\mathcal{L}$  dont le genre est inférieur à  $l$ .

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Pour un entier  $\delta > 0$  on considère la partie  $\Sigma_\delta^\circ \subset \mathbb{P}\mathcal{L}$  formée par les courbes  $\bar{C}$  de genre géométrique  $g(\bar{C}) = l - \delta$  t.q. chaque composante irréductible intersecte  $(\mathbb{C}^*)^2$ . Le genre géométrique  $g(\bar{C})$  est défini comme  $(2 - \chi(\tilde{C}))/2$ , où  $\tilde{C} \rightarrow \bar{C}$  est la normalisation. Remarquons que  $g(\bar{C})$  peut être négatif si  $\bar{C}$  est non-irréductible.

La dimension de  $\Sigma^\circ$  est  $m - \delta$ . On définit  $\Sigma_\delta$  comme l'adhérence de  $\Sigma_\delta^\circ$ . On définit  $N^{\Delta, \delta}$  comme le degré de  $\Sigma_\delta \subset \mathbb{P}\mathcal{L}$ . Le nombre  $N^{\Delta, \delta}$  est un nombre de Severi-type. C'est le nombre de courbes dans  $\mathbb{C}T_\Delta$  de degré  $\Delta$  et de genre  $l - \delta$  sans composantes en dehors de  $(\mathbb{C}^*)^2$ , qui passent par une collection de points  $z_1, \dots, z_{m-\delta} \in \mathbb{C}T_\Delta$  dans une position générale. Si  $\mathbb{C}T_\Delta$  est  $\mathbb{C}P^2$  ou  $\mathbb{C}P^1 \times \mathbb{C}P^1$ , alors  $N^{\Delta, \delta}$  coïncide avec le nombre de Gromov–Witten de genre  $l - \delta$  correspondant. En général  $N^{\Delta, \delta}$  diffère du nombre de Gromov–Witten correspondant par la contribution de courbes dont les composantes sont contenues dans le diviseur exceptionnel ou les points singuliers (isolés) de  $\mathbb{C}T_\Delta$ .

Notre résultat principal consiste en le calcul de  $N^{\Delta, \delta}$  à l'aide de chemins de réseau dans  $\Delta$ . Rappelons qu'un chemin de réseau  $\gamma : [0, n] \rightarrow \Delta$  est un chemin t.q. pour chaque  $k = 1, \dots, n$  la restriction  $\gamma|_{[k-1, k]}$  est affine-linéaire avec  $\gamma(k-1), \gamma(k) \in \mathbb{Z}^2$ . On définit dans le Section 2 la multiplicité d'un chemin qui relie deux sommets  $p$  et  $q$  de  $\Delta$ . Pour calculer  $N^{\Delta, \delta}$  on fixe une application linéaire auxiliaire  $\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}$  qui est injective sur  $\mathbb{Z}^2$ . Soient  $p$  et  $q$  le minimum et le maximum de  $\lambda|_\Delta$ . On dit qu'un chemin  $\gamma$  est  $\lambda$ -croissant si  $\lambda \circ \gamma$  est une fonction croissante.

**Théorème 1.** *Le nombre  $N^{\Delta, \delta}$  est égal au nombre (calculé avec multiplicités) de chemins de réseau  $\lambda$ -croissants  $[0, m - \delta] \rightarrow \Delta$  reliant  $p$  et  $q$ .*

On considère aussi le problème de calcul sur  $\mathbb{R}$ . On fixe une famille de points réels  $z_1, \dots, z_{m-\delta} \in \mathbb{R}T_\Delta$  dans une position générale et on demande combien parmi les  $N^{\Delta, \delta}$  courbes relevantes passant par  $z_1, \dots, z_{m-\delta}$  sont réelles. Dans la Section 4 on définit la multiplicité réelle d'un chemin  $\gamma : [0, n] \rightarrow \Delta$ .

**Théorème 2.** *Pour chaque choix de  $\lambda$  il existe une configuration de  $m - \delta$  points génériques dans  $(\mathbb{R}^*)^2$  t.q. le nombre de courbes réelles parmi les  $N^{\Delta, \delta}$  courbes complexes relevantes est égal au nombre de chemins de réseau  $\lambda$ -croissants  $[0, m - \delta] \rightarrow \Delta$  reliant  $p$  et  $q$  calculés avec multiplicités réelles.*

### 1. Introduction: the numbers $N^{\Delta, \delta}$

Let  $\Delta \subset \mathbb{R}^2$  be a convex polygon with integer vertices. It defines a finite-dimensional linear system  $\mathbb{P}\mathcal{L}$  of curves in  $(\mathbb{C}^*)^2$ , where  $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ . These curves are the zero loci in  $(\mathbb{C}^*)^2$  of the (Laurent) polynomials

$$f(z, w) = \sum_{(j, k) \in \Delta \cap \mathbb{Z}^2} a_{jk} z^j w^k,$$

$a_{jk} \in \mathbb{C}$ . The polynomials  $f$  themselves form the vector space  $\mathcal{L}$ . Recall that the Newton polygon of  $f$  is  $\text{Convexhull}\{(j, k) \mid a_{jk} \neq 0\}$ . Thus  $\mathcal{L}$  contains polynomials whose Newton polygon is contained in  $\Delta$ . Clearly  $\mathbb{P}\mathcal{L}$  is a complex projective space of dimension  $m = \#(\Delta \cap \mathbb{Z}^2) - 1$ . Curves with the Newton polygon  $\Delta$  form an open dense set  $\mathcal{U} \subset \mathbb{P}\mathcal{L}$ . A generic curve in  $\mathcal{L}$  is a smooth curve of genus  $l = \#(\text{Int } \Delta \cap \mathbb{Z}^2)$ .

Let  $C \in \mathbb{P}\mathcal{L}$  be a curve. Even if  $C$  is not irreducible we can define its geometric genus  $g(C) \in \mathbb{Z}$ . Consider the decomposition  $C = C_1 \cup \dots \cup C_n$  into the irreducible components  $C_j$ . We define  $g(C) = \sum_{j=1}^n g(\tilde{C}_j) + 1 - n$ , where  $\tilde{C}_j \rightarrow C_j$  is the normalization. Note that  $g(C)$  can be negative (cf. [1]). If  $C$  is singular then its genus is strictly smaller than  $l$ .

By the Riemann–Roch formula the curves of genus  $l - \delta$  and with the Newton polygon  $\Delta$  form a subvariety  $\Sigma_\delta^\circ \subset \mathcal{U}$  of dimension  $m - \delta$ . Let  $\Sigma_\delta \subset \mathcal{L}$  be the projective closure of  $\Sigma_\delta^\circ$ . We define  $N^{\Delta, \delta}$  to be the degree of the  $(m - \delta)$ -dimensional subvariety  $\Sigma_\delta$  in  $\mathcal{L}$ . The degree is the intersection number with a projective subspace of codimension  $m - \delta$ . Curves from  $\mathcal{L}$  passing through a point  $z \in (\mathbb{C}^*)^2$  form a hyperplane.

The number  $N^{\Delta, \delta}$  has the following enumerative interpretation. Let  $z_1, \dots, z_{m-\delta} \in (\mathbb{C}^*)^2$  be generic points. The number  $N^{\Delta, \delta}$  equals to the number of algebraic curves of Newton polygon  $\Delta$  and genus  $l - \delta$  passing through  $z_1, \dots, z_{m-\delta}$ . Note that  $N^{\Delta, 0} = 1$  for any  $\Delta$ . These numbers get more interesting when  $\delta > 0$ .

**Remark 1.** Another way to look at the same problem is to consider the compactification of the torus  $(\mathbb{C}^*)^2$ . Recall that the polygon  $\Delta$  defines a compact toric surface  $\mathbb{C}T_\Delta$ , see, e.g., [2]. (Some readers may be more familiar with the definition of toric surfaces by fans, in our case the fan is formed by the dual cones at the vertices of  $\Delta$ .) The surface  $\mathbb{C}T_\Delta$  may have isolated singularities that correspond to some vertices of  $\Delta$ .

In addition to a complex structure (which depends only on the dual fan) the polygon  $\Delta$  defines a holomorphic line bundle  $\mathcal{H}$  over  $\mathbb{C}T_\Delta$ . We have a canonical identification  $\Gamma(\mathcal{H}) = \mathcal{L}$ , where  $\Gamma(\mathcal{H})$  is the space of the sections of  $\mathcal{H}$ . The projective space  $\mathbb{P}\mathcal{L}$  can be also considered as the space of effective 1-cycles in  $\mathbb{C}T_\Delta$  Poincaré dual to  $c_1(\mathcal{H})$ . The number  $N^{\Delta, \delta}$  is the number of holomorphic curves  $\bar{C} \subset \mathbb{C}T_\Delta$  such that  $z_1, \dots, z_{m_\delta} \in \bar{C}$ , the homology class  $[\bar{C}]$  is dual to  $c_1(\mathcal{H})$ , the Euler characteristic of the normalization of  $\bar{C}$  is  $2 - 2(l - \delta)$  and no irreducible component of  $\bar{C}$  is contained in  $\mathbb{C}T_\Delta \setminus (\mathbb{C}^*)^2$ .

**Remark 2.** Note that in the set-up of Remark 1 the number  $N^{\Delta, \delta}$  appears related to the Gromov–Witten invariant of  $\mathbb{C}T_\Delta$  (see [3]) corresponding to  $c_1(\mathcal{H})$ . The difference is that the corresponding Gromov–Witten invariant also has a contribution from the curves of genus  $l - \delta$  which have components contained in  $\mathbb{C}T_\Delta \setminus (\mathbb{C}^*)^2$ . This contribution is zero (by the dimension reasons) if  $\mathbb{C}T_\Delta$  is smooth and does not have exceptional divisors. Thus if  $\Delta = \Delta_d = \text{ConvexHull}\{(0, 0), (d, 0), (0, d)\}$  or  $\Delta = [0, r] \times [0, s]$  then the number  $N^{\Delta, \delta}$  is the multicomponent Gromov–Witten invariant of genus  $(l - \delta)$  and degree  $d$  in  $\mathbb{C}P^2$  or of bidegree  $(r, s)$  in  $\mathbb{C}P^1 \times \mathbb{C}P^1$ .

*Special case.* Suppose  $\Delta = \Delta_d$  so that  $\mathbb{C}T_\Delta = \mathbb{C}P^2$ . We have  $m = \frac{d(d+3)}{2}$  and  $l = \frac{(d-1)(d-2)}{2}$ . The number  $N^{\Delta, \delta} = N_{g,d}$  is the number of genus  $g$ , degree  $d$  (not necessarily irreducible) curves passing through  $3d - 1 + g$  generic points in  $\mathbb{C}P^2$ ,  $g = \frac{(d-1)(d-2)}{2} - \delta$ .

The formula  $N^{\Delta, 1} = 3(d - 1)^2$  is well-known as the degree of the discriminant (cf. [2]). An elegant recursive formula for the number of irreducible rational curves (the one-component part of  $N_{0,d}$ ) was found by Kontsevich [3]. An algorithm for computing  $N_{g,d}$  for arbitrary  $g$  is due to Caporaso and Harris [1]. See [7] for computations for some other rational surfaces, in particular, the Hirzebruch surfaces (this corresponds to the case when  $\Delta$  is a trapezoid).

## 2. Lattice paths and their multiplicities

A path  $\gamma : [0, n] \rightarrow \mathbb{R}^2$ ,  $n \in \mathbb{N}$ , is called a *lattice path* if  $\gamma|_{[j-1, j]}$ ,  $j = 1, \dots, n$ , is an affine-linear map and  $\gamma(j) \in \mathbb{Z}^2$ ,  $j \in 0, \dots, n$ . Clearly, a lattice path is determined by its values at the integer points. Let us choose an auxiliary linear map  $\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}$  that is irrational, i.e., such that  $\lambda|_{\mathbb{Z}^2}$  is injective. Let  $p, q \in \Delta$  be the vertices where  $\alpha|_\Delta$  reaches its minimum and maximum respectively. A lattice path is called  *$\lambda$ -increasing* if  $\lambda \circ \gamma$  is increasing.

The points  $p$  and  $q$  divide the boundary  $\partial\Delta$  into two increasing lattice paths  $\alpha^+ : [0, n_+] \rightarrow \partial\Delta$  and  $\alpha^- : [0, n_-] \rightarrow \partial\Delta$ . We have  $\alpha_+(0) = \alpha_-(0) = p$ ,  $\alpha_+(n_+) = \alpha_-(n_-) = q$ ,  $n_+ + n_- = m - l + 3$ . To fix a convention we assume that  $\alpha_+$  goes clockwise around  $\partial\Delta$  while  $\alpha_-$  goes counterclockwise.

Let  $\gamma : [0, n] \rightarrow \Delta \subset \mathbb{R}^2$  be an increasing lattice path such that  $\gamma(0) = p$  and  $\gamma(n) = q$ . The path  $\gamma$  divides  $\Delta$  into two closed regions:  $\Delta_+$  enclosed by  $\gamma$  and  $\alpha_+$  and  $\Delta_-$  enclosed by  $\gamma$  and  $\alpha_-$ . Note that the interiors of  $\Delta_+$  and  $\Delta_-$  do not have to be connected.

We define the positive (resp. negative) multiplicity  $\mu_\pm(\gamma)$  of the path  $\gamma$  inductively. We set  $\mu_\pm(\alpha_\pm) = 1$ . If  $\gamma \neq \alpha_\pm$  then we take  $1 \leq k \leq n - 1$  to be the smallest number such that  $\gamma(k)$  is a vertex of  $\Delta_\pm$  with the angle less than  $\pi$  (so that  $\Delta_\pm$  is locally convex at  $\gamma(k)$ ).

If such  $k$  does not exist we set  $\mu_\pm(\gamma) = 0$ . If  $k$  exist we consider two other increasing lattice paths connecting  $p$  and  $q$   $\gamma' : [0, n - 1] \rightarrow \Delta$  and  $\gamma'' : [0, n] \rightarrow \mathbb{R}^2$ . We define  $\gamma'$  by  $\gamma'(j) = \gamma(j)$  if  $j < k$  and  $\gamma'(j) = \gamma(j + 1)$

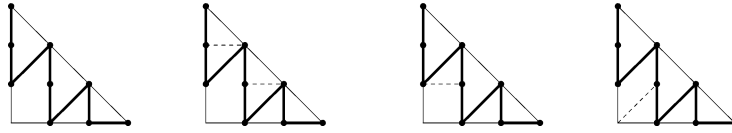


Fig. 1. A path  $\gamma$  with  $\mu_+(\gamma) = 1$  and  $\mu_-(\gamma) = 2$ .

Fig. 1. Un chemin  $\gamma$  avec  $\mu_+(\gamma) = 1$  et  $\mu_-(\gamma) = 2$ .

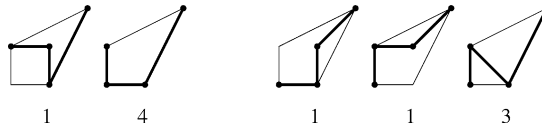


Fig. 2. Computing  $N^{\Delta,1} = 5$  in two different ways.

Fig. 2. Calcul de  $N^{\Delta,1} = 5$  dans deux façons différentes.

if  $j \geq k$ . We define  $\gamma''$  by  $\gamma''(j) = \gamma(j)$  if  $j \neq k$  and  $\gamma''(k) = \gamma(k - 1) + \gamma(k + 1) - \gamma(k) \in \mathbb{Z}^2$ . We set  $\mu_{\pm}(\gamma) = 2 \text{Area}(T)\mu_{\pm}(\gamma') + \mu_{\pm}(\gamma'')$ , where  $T$  is the triangle with the vertices  $\gamma(k - 1)$ ,  $\gamma(k)$  and  $\gamma(k + 1)$ . The multiplicity is always integer since the area of a lattice triangle is half-integer.

Note that it may happen that  $\gamma''(k) \notin \Delta$ . In such case we use a convention  $\mu_{\pm}(\gamma'') = 0$ . We may assume that  $\mu_{\pm}(\gamma')$  and  $\mu_{\pm}(\gamma'')$  is already defined since the area of  $\Delta_{\pm}$  is smaller for the new paths. Note that  $\mu_{\pm} = 0$  if  $n < n_{\pm}$  as the paths  $\gamma'$  and  $\gamma''$  are not longer than  $\gamma$ .

We define the multiplicity of the path  $\gamma$  as the product  $\mu_+(\gamma)\mu_-(\gamma)$ . Note that the multiplicity of a path connecting two vertices of  $\Delta$  does not depend on  $\lambda$ . We only need  $\lambda$  to determine whether a path is increasing.

**Example 1.** Consider the path  $\gamma : [0, 8] \rightarrow \Delta_3$  depicted on the extreme left of Fig. 1. This path is increasing with respect to  $\lambda(x, y) = x - \varepsilon y$ , where  $\varepsilon > 0$  is very small.

Let us compute  $\mu_+(\gamma)$ . We have  $k = 2$  as  $\gamma(2) = (0, 1)$  is a locally convex vertex of  $\Delta_+$ . We have  $\gamma''(2) = (1, 3) \notin \Delta_3$  and thus  $\mu_+(\gamma) = \mu_+(\gamma')$ , since  $\text{Area}(T) = \frac{1}{2}$ . Proceeding further we get  $\mu_+(\gamma) = \mu_+(\gamma') = \dots = \mu_+(\alpha_+) = 1$ .

Let us compute  $\mu_-(\gamma)$ . We have  $k = 3$  as  $\gamma(3) = (1, 2)$  is a locally convex vertex of  $\Delta_-$ . We have  $\gamma''(3) = (0, 0)$  and  $\mu_-(\gamma'') = 1$ . To compute  $\mu_-(\gamma') = 1$  we note that  $\mu_-(\gamma'') = 0$  and  $\mu_-(\gamma') = 1$ . Thus the full multiplicity of  $\gamma$  is 2.

### 3. The formula

In the previous section we fixed an auxiliary linear function  $\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}$  which determines the extremal vertices  $p, q$  of  $\Delta$ .

**Theorem 1.** The number  $N^{\Delta,\delta}$  is equal to the number (counted with multiplicities) of  $\lambda$ -increasing lattice paths  $[0, m - \delta] \rightarrow \Delta$  connecting  $p$  and  $q$ .

This theorem is proved in [5] (to appear). The proof is based on the application of the so-called tropical algebraic geometry (see, e.g., Chapter 9 of [6]). The relation between the classical enumerative problem and the corresponding tropical problem is provided by passing to the “large complex limit” as suggested by Kontsevich (see [4] for these ideas in a more general setting).

Note that an immediate corollary of Theorem 1 is that the sum of the multiplicities of the  $\lambda$ -increasing lattice paths of a fixed length does not depend on the choice of  $\lambda$ .

**Example 2.** Let us compute  $N^{\Delta,1} = 5$  for the polygon  $\Delta$  depicted on Fig. 2 in two different ways. Using  $\lambda(x, y) = -x + \varepsilon y$  for a small  $\varepsilon > 0$  we get the left two paths depicted on Fig. 2. Using  $\lambda(x, y) = x + \varepsilon y$  we

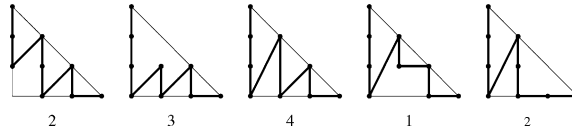


Fig. 3. Computing  $N_{0,3} = 12$ .

Fig. 3. Calcul de  $N_{0,3} = 12$ .

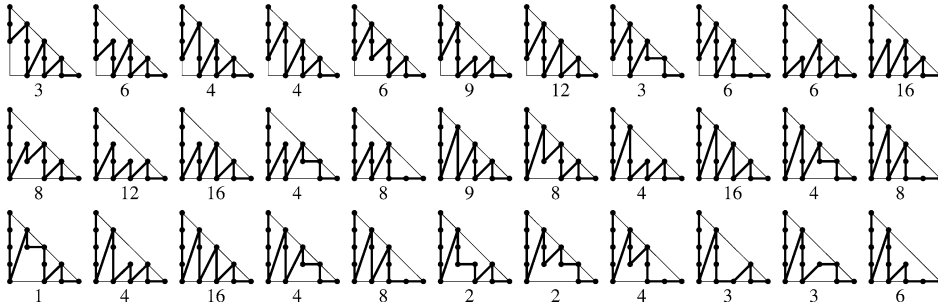


Fig. 4. Computing  $N_{1,4} = 225$ .

Fig. 4. Calcul de  $N_{1,4} = 225$ .

get the three right paths. The corresponding multiplicities are shown under the path. All other  $\lambda$ -increasing paths have zero multiplicity.

In the next two examples we use  $\lambda(x, y) = x - \varepsilon y$  as the auxiliary linear function.

**Example 3.** Fig. 3 shows a computation of the well-known number  $N^{\Delta_{3,1}} = N_{0,3}$ . This is the number of rational cubic curves through 8 generic points in  $\mathbb{C}P^2$ .

**Example 4.** Fig. 4 shows a computation of a less well-known number  $N^{\Delta_{4,2}} = N_{1,4}$ . This is the number of genus 1 quartic curves through 12 generic points in  $\mathbb{C}P^2$ .

#### 4. Real aspects of the count

Suppose that  $z_1, \dots, z_{m-\delta} \in (\mathbb{R}^*)^2 \subset (\mathbb{C}^*)^2$  are generic real points. We may ask how many of the  $N^{\Delta,\delta}$  relevant complex curves are real, i.e., defined over  $\mathbb{R}$ . Note that this number depends on the configuration of real points.

Theorem 1 can be modified to give the relevant count of real curves. In order to do this we need to define the real multiplicity of a lattice path  $\gamma : [0, n] \rightarrow \Delta$  connecting the vertices  $p$  and  $q$ . We introduce the sequence of the pairs of signs  $\sigma_1, \dots, \sigma_n \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$  (this sequence will record the quadrants of the points  $z_j \in (\mathbb{R}^*)^2$ ). The sign  $\sigma_j$  is prescribed to the edge  $\gamma[j-1, j]$ . We make a convention that  $\sigma_j$  and  $\sigma'_j$  are equivalent if  $\sigma_j - \sigma'_j \equiv \gamma(j) - \gamma(j-1) \pmod{2}$ . We set

$$\mu_{\pm}^{\mathbb{R}}(\gamma) = a(T)\mu_{\pm}^{\mathbb{R}}(\gamma') + \mu_{\pm}^{\mathbb{R}}(\gamma''). \tag{1}$$

The definition of the new paths  $\gamma', \gamma''$  and the triangle  $T$  is the same as in Section 2. The sign sequence for  $\gamma''$  is  $\sigma''_j = \sigma_j, j \neq k, k+1, \sigma''_k = \sigma_{k+1}, \sigma''_{k+1} = \sigma_k$ . The sign sequence for  $\gamma'$  is  $\sigma'_j = \sigma_j, j < k, \sigma'_j = \sigma_{j+1}, j > k$ . We define the sign  $\sigma'_k$  and the function  $a(T)$  as follows.

- If all sides of  $T$  are odd we set  $a(T) = 1$  and define the sign  $\sigma'_k$  (up to the equivalence) by the condition that the three equivalence classes of  $\sigma_k, \sigma_{k+1}$  and  $\sigma'_k$  do not share a common element.
- If all sides of  $T$  are even we set  $a(T) = 0$  if  $\sigma_{k-1} \neq \sigma_k$ . In this case we can ignore  $\gamma'$  (and its sequence of signs). We set  $a(T) = 4$  if  $\sigma_k = \sigma_{k+1}$ . In this case we define  $\sigma'_k = \sigma_k = \sigma_{k+1}$ .

- Otherwise we set  $a(T) = 0$  if the equivalence classes of  $\sigma_k$  and  $\sigma_{k+1}$  do not have a common element. We set  $a(T) = 2$  if they do. In the latter case we define the equivalence class of  $\sigma'_k$  by the condition that  $\sigma_k$ ,  $\sigma_{k+1}$  and  $\sigma'_k$  have a common element. There is one exception to this rule. If the even side is  $\gamma(k+1) - \gamma(k-1)$  then there are two choices for  $\sigma'_k$  satisfying the above condition. In this case we replace  $a(T)\mu_{\pm}^{\mathbb{R}}(\gamma')$  in (1) by the sum of the two multiplicities of  $\gamma'$  equipped with the two allowable choices for  $\sigma'_k$  (note that this agrees with  $a(T) = 2$  in this case).

Similar to Section 2 we define  $\mu_{\pm}^{\mathbb{R}}(\alpha_{\pm}) = 1$  and  $\mu^{\mathbb{R}}(\gamma) = \mu_{+}^{\mathbb{R}}(\gamma)\mu_{-}^{\mathbb{R}}(\gamma)$ . As before  $\lambda: \mathbb{R}^2 \rightarrow \mathbb{R}$  is a linear map injective on  $\mathbb{Z}^2$  and  $p$  and  $q$  are the extrema of  $\lambda|_{\Delta}$ .

**Theorem 2.** For any choice of  $\lambda$  and  $\sigma_j$ ,  $j = 1, \dots, m - \delta$ , there exists a configuration of  $m - \delta$  of generic points in the respective quadrants such that the number of real curves among the  $N^{\Delta, \delta}$  relevant complex curves is equal to the number of  $\lambda$ -increasing lattice paths  $\gamma: [0, m - \delta] \rightarrow \Delta$  connecting  $p$  and  $q$  counted with multiplicities  $\mu^{\mathbb{R}}$ .

**Example 5.** Here we use the choice  $\sigma_j = (+, +)$  so all the points  $z_j$  are in the positive quadrant  $(\mathbb{R}_{>0})^2 \subset (\mathbb{R}^*)^2$ . The first count of  $N^{\Delta, 1}$  from Example 2 gives a configuration of 3 real points with 5 real curves. The second count gives a configuration with 3 real curves as the real multiplicity of the last path is 1. Note also that the second path on Fig. 2 changes its real multiplicity if we reverse its direction.

Example 3 gives a configuration of 9 generic points in  $\mathbb{R}P^2$  with all 12 nodal cubics through them real. Example 4 gives a configuration of 12 generic points in  $\mathbb{R}P^2$  with 217 out of the 225 quartics of genus 1 real. The path in the middle of Fig. 4 has multiplicity 9 but real multiplicity 1. A similar computation shows that there exists a configuration of 11 generic points in  $\mathbb{R}P^2$  such that 564 out of the 620 irreducible quartic through them are real.

**Remark 3.** Real nodal curves have three types of nodes: hyperbolic, elliptic and imaginary. Theorem 2 can be refined to count curves with different types of nodes separately. In accordance with [8] let us prescribe a sign  $(-1)^e$  to a real nodal curve, where  $e$  is the number of its elliptic nodes. To compute the corresponding algebraic number of curves we introduce the multiplicity  $v^{\mathbb{R}}$  by replacing (1) with  $v_{\pm}^{\mathbb{R}}(\gamma) = b(T)v_{\pm}^{\mathbb{R}}(\gamma') + v_{\pm}^{\mathbb{R}}(\gamma'')$ . Here we define  $b(T) = 0$  if at least one side of  $T$  is even and  $b(T) = (-1)^{\#(\text{Int } T \cap \mathbb{Z}^2)}$  otherwise. It can be shown with the help of this formula and a combinatorial observation made by Itenberg, Kharlamov and Shustin (to appear) that in the case  $\Delta = \Delta_d$  the algebraic number of irreducible curves counted by  $v^{\mathbb{R}}$  is positive for any genus  $0 \leq g \leq \frac{(d-1)(d-2)}{2}$  if  $\lambda(x, y) = y - \varepsilon x$ .

Note that unlike  $\mu^{\mathbb{R}}$  the multiplicity  $v^{\mathbb{R}}$  does not depend on the quadrant choices  $\sigma_j$ . Furthermore, in [8] Welschinger announced that in the case  $g = 0$  this algebraic number of curves is independent of the configuration of generic real points. Corollary 1.2 of [8] combined with Remark 3 implies the following statement (which answers a question asked, e.g., by Rokhlin and Kharlamov). For any configuration of generic  $3d - 1$  points in  $\mathbb{R}P^2$  there exists a real rational curve of degree  $d$  passing through this configuration.

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