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More on the duality conjecture for entropy numbers

Sur la conjecture de la dualité pour les nombres d'entropie

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Abstract

We verify, up to a logarithmic factor, the duality conjecture for entropy numbers in the case where one of the bodies is an ellipsoid. ***To cite this article: S. Artstein et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Nous démontrons, à un facteur logarithmique près, la conjecture concernant la dualité de nombres d'entropie dans le cas où l'un de deux corps est un ellipsoïde. ***Pour citer cet article : S. Artstein et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

For two convex bodies K and T in $\mathbb{R}^n$, the covering number of K by T, denoted $N(K,T)$, is defined as the minimal number of translates of T needed to cover K. An old problem, going back to Pietsch ([9], p. 38) and usually referred to as the "duality conjecture for entropy numbers", can be stated in terms of covering numbers in the following way. (Below and in what follows all logarithms are to the base 2.)

Conjecture 1 (Duality Conjecture). *There exist two numerical constants a and b, such that for any $n \in \mathbb{N}$ and any two symmetric convex bodies K and T in $\mathbb{R}^n$ one has*

$$\log N\big(T^\circ, aK^\circ\big) \leqslant b \log N(K,T),$$

where A° denotes the polar body of A.

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We refer the reader to the recent paper [7] and the book [11] for background information on the problem and further references.

In this Note we concentrate only on the case in which either K or T is an ellipsoid. [If, as is more customary, we use the language of compact operators between Banach spaces and their entropy numbers rather than that of convex bodies, this corresponds to either the domain or the range of the operator being a Hilbert space.] By affine invariance of the problem, we may and shall assume that the ellipsoid in question is the Euclidean unit ball $D \subset \mathbb{R}^n$.

For a bounded set $A \subset \mathbb{R}^n$, we set

$$M^*(A) := \int_{S^{n-1}} \sup_{x \in A} \langle x, u \rangle \, \mathrm{d}\sigma(u),$$

where σ is the normalized Lebesgue measure on the sphere. Our goal is to complement the following result obtained recently in [7] (see Proposition 4.1 there) going in the direction of Conjecture 1.

Theorem 2 [7,8]. *There exists a universal constant $C > 0$, such that for any $n \in \mathbb{N}$ and any convex 0-symmetric body $K \subset \mathbb{R}^n$ we have*

$$\log N\big(D, C\gamma K^\circ\big) \leqslant 2 \log N(K, D), \tag{1}$$

where $\gamma := \max\{1, M^*(K \cap D)(\log N(K, D)/n)^{-1/2}\}$.

Moreover, it was shown in [7] that one can reduce estimating covering numbers in (1) to the special case in which the body not only is covered by $N = N(K, D)$ balls, but is also the convex hull of (at most) N points. It was conjectured there that for this class of bodies the quantity γ is in fact bounded by a universal constant, which of course would then imply "half" of the Duality Conjecture in our setting, i.e., $T = D$ in Conjecture 1 (observe that the roles of K and D in Theorem 2 are not exchangeable). The conjecture concerning bounded ness of γ was discussed in [7] at length, and equivalent geometrical formulations were proposed. Although the conjecture has not been settled yet, it was proved up to a logarithmic factor. More precisely, we have (see [7], Theorem 9.3)

Proposition 3. *There is a universal constant C_0 such that if $n \in \mathbb{N}$ and $S \subset \mathbb{R}^n$ is any finite set, then setting $K = \operatorname{conv} S$ and $k = \max\{\log |S|, \log N(K, D)\}$ we have*

$$M^*(K \cap D) \leqslant C_0 \sqrt{k/n}\,(1 + \log k)^3. \tag{2}$$

Together with Theorem 2 and the remark following it, this easily implies a one-sided duality result with a factor – playing the role of a from Conjecture 1 – which is not a numerical constant but instead equals $\mathrm{O}((\log k)^3) = \mathrm{O}((\log\log N(K, D))^3)$; see [7], Theorem 9.4.

In this Note we wish to show how the validity of the conjecture from [7], i.e., the uniform boundedness of parameters γ for the class of sets described in the paragraph following Theorem 2, implies also the other half of the duality conjecture in our setting (i.e., when, in the notation of Conjecture 1, $K = D$) and that partial results such as Proposition 3 give duality-type estimates in that reverse inequality. We prove the following result complementing Theorem 2.

Theorem 4. *There exists an absolute constant $c > 0$ such that if $n \in \mathbb{N}$, $K \subset \mathbb{R}^n$ is a 0-symmetric convex body, and γ has the same meaning as in Theorem 2, then*

$$\log N(K, D) \leqslant 3 \log N\big(D, c\gamma^{-1} K^\circ\big). \tag{3}$$

Let now $\alpha > 0$ and assume that, in the above context, $\log N(K, D) \geqslant \alpha n$. Since we have "for free" that $M^*(K \cap D) \leqslant 1$, Theorems 2 and 4 imply a two sided duality result

$$\frac{1}{2} \log N\big(D, a(\alpha) K^\circ\big) \leqslant \log N(K, D) \leqslant 3 \log N\big(D, a(\alpha)^{-1} K^\circ\big), \tag{4}$$

where $a(\alpha) = \mathrm{O}(\alpha^{-1/2})$. This completes the following result by König and Milman [4], which implies duality (for two *general* symmetric convex bodies) in the case where the logarithm of a covering number is *sufficiently large* when compared to n: if K and T are two 0-symmetric convex bodies in $\mathbb{R}^n$, then

$$\frac{1}{C'} \leqslant \left(\frac{N(K,T)}{N(T^\circ, K^\circ)}\right)^{1/n} \leqslant C', \tag{5}$$

where C' is a universal constant. (In fact the bodies need not be symmetric; it suffices to require that their centroids are 0, see [6].) We shall use this result in the sequel. On the other hand, (4) should be compared with [10], where M-ellipsoids are used to achieve results in the same direction. Our proof, despite using simpler tools, provides slightly better results than [10] specialized to our setting.

Proof of Theorem 4. Denote $N := N(K, D)$. Our goal is to prove that $N(D, c\gamma^{-1}K^\circ) \geqslant N^{1/3}$, where $\gamma = \max\{1, \sqrt{n/\log N}\, M^*(K \cap D)\}$. Let $C_1 = c^{-1}/2$ (our choice of c will be specified later). Assume first that $N(K, C_1\gamma D) \leqslant N^{2/3}$. A result of Tomczak-Jaegermann [12] (see [2], Proposition 2, for a more explicit variant) stating that $N(K, D) \leqslant N(K, \theta D) N(D, \frac{1}{2\theta} K^\circ)$ for all $\theta > 0$ implies then that

$$N\big(D, (2C_1\gamma)^{-1}K^\circ\big) \geqslant N^{1/3},$$

which yields (3).

It remains to handle the case $N(K, C_1\gamma D) > N^{2/3}$. This implies that there exists a $C_1\gamma$-separated (in the Euclidean metric) subset S of K of cardinality $> N^{2/3}$. We set $k_1 := \lfloor c_1 \log N \rfloor$, with $c_1 = (3 \log C')^{-1}$, where C' comes from (5). [This will ensure that when we use (5) in dimension k_1, we will get a meaningful result; notice also that we may assume that $k_1 < n$, as otherwise $\log N \geqslant c_1^{-1} n$ and the answer follows easily from (5).] Now consider a random orthogonal projection P_E of K onto a k_1-dimensional subspace E, distributed uniformly with respect to the Haar measure on the corresponding Grassmanian. It is well known that under such projections a distance between two points is typically multiplied by a factor close to $\sqrt{k_1/n}$. Moreover, from the results described in [1], which are an isomorphic extension of the Johnson–Lindenstrauss lemma [3], we know that if $S \subset \mathbb{R}^n$ is a finite set, then – with probability close to 1 – *all* the distances between elements of S are multiplied by factors $\geqslant c_2\sqrt{k_1/n}$, where $c_2 > 0$ depends only on (an upper bound for) the ratio $\log|S|/k_1$; in particular under our assumptions $c_2 \in (0, 1]$ is universal. It follows that, with probability close to 1, the image $P_E S$ of the $C_1\gamma$-separated set S is $c_2 C_1 \sqrt{k_1/n}\,\gamma$-separated. Noting that $P_E S \subset P_E K \subset E$, we deduce that

$$N\big(P_E K, C_2\sqrt{k_1/n}\,\gamma D_E\big) > N^{2/3},$$

where D_E denotes the unit ball in E and $C_2 = c_2 C_1 = c_2 c^{-1}/2$. However, we are now in a position to use (5) in dimension k_1, which (by our choice of k_1) implies

$$N\big(D_E, C_2\sqrt{k_1/n}\,\gamma K^\circ \cap E\big) > N^{1/3}. \tag{6}$$

All that is left is to return to the original dimension. For this we use a Dvoretzky theorem-type assertion, which claims that a random projection of $K \cap D$ onto dimension k_1 is, with probability close to 1, contained in the Euclidean ball of radius $C_3 \max\{M^*(K \cap D), \sqrt{k_1/n}\}$, where $C_3 \geqslant 1$ is a universal constant (for reference see [5] or Fact 3.1 in [7]). Dualizing this (and using the definitions of γ and k_1) we obtain

$$C_4\sqrt{k_1/n}\,\gamma\, \mathrm{conv}(K^\circ \cup D) \cap E \supset D_E$$

(with $C_4 = C_3 \max\{1, \sqrt{2/c_1}\}$), which combined with (6) yields

$$N\big(C_4\, \mathrm{conv}(K^\circ \cup D) \cap E, C_2 K^\circ \cap E\big) > N^{1/3}.$$

We now choose the original c in the statement of the theorem: $c = (C_4)^{-1} c_2/8$. This means precisely that $C_2 = 4C_4$. Then

$$\begin{aligned} N^{1/3} &< N\big(\operatorname{conv}(K^\circ \cup D) \cap E, 4K^\circ \cap E\big) \\ &\leqslant N\big(\operatorname{conv}(K^\circ \cup D) \cap E, 2K^\circ\big) \\ &\leqslant N\big(K^\circ + D, 2K^\circ\big) \\ &\leqslant N\big(D, K^\circ\big), \end{aligned}$$

where for the second inequality we use the fact that, for every $x \in \mathbb{R}^n$, the set $(x + 2K^\circ) \cap E$ is contained in a translate of $4K^\circ \cap E$. The proof is now complete. Notice that in the second case the parameter γ does not enter into the final formulae and we obtain duality with absolute constants.

As suggested earlier, we can combine Theorem 4 with Proposition 3 to show the following

Corollary 5. *There exists an absolute constant $c' > 0$ such that if K is a 0-symmetric convex body in $\mathbb{R}^n$ and if $\log N(K, D) =: k$, then*

$$\log N(K, D) \leqslant 3 \log N\big(D, c'(1 + \log k)^{-3} K^\circ\big). \tag{7}$$

Proof. Since $N(K, D) = 2^k$, there exists a 1-separated set $S = \{x_1, \ldots, x_{2^k}\} \subset K$. Denote $K_1 = \operatorname{conv} S$, then of course

$$N(2K_1, D) = N\left(K_1, \frac{1}{2}D\right) \geqslant 2^k.$$

The body K_1 satisfies the hypotheses of Proposition 3 and so

$$M^*(2K_1 \cap D) \leqslant 2M^*(K_1 \cap D) \leqslant 2C_0\sqrt{k/n}\,(1 + \log k)^3.$$

Using Theorem 4 (observe that γ is at most $\max\{1, 2C_0(1 + \log k)^3\}$) we see that

$$\log N(2K_1, D) \leqslant 3 \log N\big(D, c'(1 + \log k)^{-3} K_1^\circ\big).$$

Combining the estimates and taking into account that $K^\circ \subset K_1^\circ$ we obtain (7).

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