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Differential Geometry

Growth tightness of negatively curved manifolds
Croissance forte des variétés à courbure négative

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Abstract

We show that any closed negatively curved manifold X is growth tight: this means that its universal covering $\widetilde{X}$ has an exponential growth rate $\omega(\widetilde{X})$ which is strictly greater than the exponential growth rate $\omega(\overline{X})$ of any other normal covering $\overline{X}$. Moreover, we give an explicit formula which estimates the difference between $\omega(\widetilde{X})$ and $\omega(\overline{X})$ in terms of the systole of $\overline{X}$ and of some geometric parameters of the base manifold X. Then, we describe some applications to systoles and periodic geodesics. ***To cite this article: A. Sambusetti, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

On montre que toute variété fermée X de courbure négative est à croissance forte : cela signifie que le revêtement universel $\widetilde{X}$ a un taux de croissance exponentielle $\omega(\widetilde{X})$ strictement supérieur à celui de n'importe quel autre revêtement normal $\overline{X}$ de X. Plus précisément, on donne une formule estimant explicitement la différence entre ces taux de croissance, $\omega(\widetilde{X})$ et $\omega(\overline{X})$, en termes de la systole de $\overline{X}$ et d'autres simples paramètres géométriques de la variété de base X. On en déduit ensuite une inégalité systolique et une application aux géodésiques périodiques. ***Pour citer cet article : A. Sambusetti, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

La *fonction de croissance* d'une variété riemannienne complète X est la fonction $\beta_X(R)$ donnée par le volume des boules de rayon R de X, centrées en un point fixé x_0. Le *taux de croissance exponentielle* de X est défini comme la limite $\omega(X) = \limsup_{R\to\infty} R^{-1}\log\beta_X(R)$ (cette limite est indépendante du choix de x_0). Le taux de croissance exponentielle du revêtement universel riemannien $\widetilde{X}$ d'une variété X est usuellement appelé *entropie* (volumique) de X et noté $\mathrm{Ent}(X)$.

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Le théorème qui suit exprime que *le revêtement universel d'une variété de courbure négative est le revêtement normal de taux de croissance exponentielle maximal* (on remarque que cette caractérisation est en général fausse pour les variétés riemanniennes quelconques, même de courbure négative ou nulle). Plus précisément, on donne une formule estimant explicitement la différence entre le taux de croissance $\omega(\widetilde{X})$ et celui de n'importe quel autre revêtement normal de X, en termes de sa systole et d'autres paramètres géométriques simples de la variété de base X :

Théorème 0.1. *Soit X une variété riemannienne fermée de dimension n telle que $-k \leqslant k(X) \leqslant -1$, et notons $v = \mathrm{vol}(X)$ et $d = \mathrm{diam}(X)$. Soit $\widetilde{X}$ le revêtement riemannien universel de X et $\overline{X}$ un autre revêtement riemannien normal de X. On a alors* :

$$\omega(\widetilde{X}) \geqslant \omega(\overline{X}) + \frac{\log(1 + \varepsilon_n v k^{n/2}\, \mathrm{e}^{-5(n-1)\sqrt{k}\,[\mathrm{syst}(\overline{X})+26d+33]})}{6[\mathrm{syst}(\overline{X}) + 26d + 33]},$$

où $\varepsilon_n = (n-1)2^{n-3}/\mathrm{vol}(S^{n-1})$ et $\mathrm{syst}(\overline{X})$ est la systole *du revêtement $\overline{X}$, c'est-à-dire la longueur de la plus petite géodésique non contractile de $\overline{X}$.*

Une conséquence directe du Théorème 0.1 est l'inégalité systolique suivante :

Corollaire 0.2. *Soit $\overline{X}$ une variété de dimension n et de courbure négative, admettant un groupe d'isométries Γ cocompact qui agit librement et de façon proprement discontinue. Supposons que la courbure de $\overline{X}$ soit normalisée de sorte que $-k \leqslant k(\overline{X}) \leqslant -1$, et notons $v = \mathrm{vol}(\Gamma \setminus \overline{X})$, $d = \mathrm{diam}(\Gamma \setminus \overline{X})$. On a alors*

$$\mathrm{syst}(\overline{X}) \geqslant \frac{1}{12(n-1)\sqrt{k}} \log\left[\frac{\varepsilon_n\, k^{n/2}\, v}{\omega(\widetilde{X}) - \omega(\overline{X})}\right] - 26d - 33.$$

(Bien sûr, cette formule n'est significative que dans le cas où $\omega(\overline{X})$ est suffisamment proche de $\omega(\widetilde{X})$.)

Ce resultat peut être réformulé comme une contrainte sur la longueur minimale d'une géodésique périodique γ représentant une classe d'homotopie libre fixée dans une variété de courbure négative, en termes d'un invariant asymptotique associé à $[\gamma]$:

Corollaire 0.3. *Soit X une variété de dimension n telle que $-k \leqslant k(X) \leqslant -1$. Pour toute géodésique périodique γ de X, on a*

$$\ell(\gamma) \geqslant \frac{1}{12(n-1)\sqrt{k}} \log\left[\frac{\varepsilon_n\, k^{n/2}\,\mathrm{vol}(X)}{\mathrm{Ent}(X) - \mathrm{Ent}([\gamma])}\right] - 26\,\mathrm{diam}(X) - 33,$$

où $\mathrm{Ent}([\gamma])$ est le taux de croissance exponentielle du revêtement riemannien de X associé au sous-groupe distingué de $\pi_1(X)$ engendré par γ.

1. Statement of the main result

The *growth function* of a complete Riemannian manifold X is the function $\beta_X(R)$ given by the volume of balls of radius R of X centered at some point x_0. Correspondingly, the exponential growth rate of X is defined as the limit

$$\omega(X) = \limsup_{R\to\infty} R^{-1} \cdot \log \beta_X(R)$$

(which is independent of the choice of the base point x_0). The exponential growth rate of the Riemannian universal covering $\widetilde{X}$ of a manifold X is known as the (volume) *entropy* of X, and it is usually denoted by $\mathrm{Ent}(X)$.

We shall say that a Riemannian manifold X is growth tight if $\omega(\widetilde{X}) > \omega(\overline{X})$ for *any* normal covering $\overline{X}$ different from the universal covering $\widetilde{X}$. The notion of "growth tightness" was originally introduced in [2] for finitely generated groups, relatively to word metrics: more generally [3], a discrete group (G, d) endowed with a left-invariant distance[2] is *growth tight* if $\omega(G, d) > \omega(G/N, d/N)$ for every nontrivial normal subgroup $N \lhd G$ (where $\omega(G, d)$ is the exponential growth rate of G with respect to d, and d/N is the distance induced on the quotient group G/N). This means that G is characterized, among all of its quotients, by an asymptotic invariant. Arzhantseva and Lysenok [1] recently proved that every Gromov hyperbolic group without finite normal subgroups (in particular, the fundamental group of any closed, negatively curved manifold) is growth tight *with respect to word metrics*, thus answering to a conjecture of Grigorchuk and de la Harpe. It is a natural problem to know if this behavior extends to Riemannian metrics: saying that a manifold X is growth tight is precisely equivalent to say that its fundamental group $\pi_1(X)$ is growth tight *with respect to the geometric distance* induced by the Riemannian length. Actually, if $\overline{X} = N \setminus \widetilde{X}$ for some nontrivial normal subgroup $N \lhd \pi_1(X)$, it is well known that $\omega(\widetilde{X})$ and $\omega(\overline{X})$ coincide with the exponential growth rates of the automorphism groups $\mathrm{Aut}(\widetilde{X}) \cong \pi_1(X)$ and $\mathrm{Aut}(\overline{X}) \cong \pi_1(X)/N$.

Notice that, generally, a Riemannian manifold X with fundamental group of exponential growth is not growth tight, even if X is non-positively curved. For instance, consider the Riemannian product $X = \Sigma_1 \times \Sigma_g$ of a flat torus with any closed hyperbolic surface: then, X admits a normal covering $\overline{X} = \Sigma_1 \times \widetilde{\Sigma}_g$ which has the same exponential growth as its universal covering $\widetilde{X}$, since the fiber of the projection $\widetilde{X} \to \overline{X}$ has polynomial growth in $\widetilde{X}$ (being an orbit of the abelian group $\pi_1(\Sigma_1)$).

The main significance of Theorem 1.1 below is that this property holds in negative curvature: in other words, *the universal covering of a closed negatively curved manifold is characterized as the normal covering of maximal exponential growth rate* (cp. [4]). In this sense, this theorem is the geometric analogue of Arzhantseva and Lyonosok's result. Moreover, our theorem gives an explicit measure of the gap $\omega(\widetilde{X}) - \omega(\overline{X})$ in terms of elementary geometric parameters of the base manifold X and of the *systole* of the intermediate covering $\overline{X}$, that is the length of the smallest noncontractible closed geodesic of $\overline{X}$.

Theorem 1.1. *Let X be a closed, negatively curved manifold of dimension n with sectional curvature normalized so that $-k \leqslant k(X) \leqslant -1$, and assume that $v = \mathrm{vol}(X)$, $d = \mathrm{diam}(X)$. Let $\widetilde{X}$ and $\overline{X}$ be respectively the Riemannian universal covering and any other Riemannian normal covering of X. Then*:

$$\omega(\widetilde{X}) \geqslant \omega(\overline{X}) + \frac{\log(1 + \varepsilon_n v k^{n/2}\, \mathrm{e}^{-5(n-1)\sqrt{k}\,[\mathrm{syst}(\overline{X}) + 26d + 33]})}{6[\mathrm{syst}(\overline{X}) + 26d + 33]}, \tag{1}$$

where $\varepsilon_n = (n-1)2^{n-3}/\mathrm{vol}(S^{n-1})$.

2. Applications

A systolic inequality is a (non-tautological) constraint on the systole of a Riemannian manifold in terms of other geometric parameters; formula (1) can be precisely interpreted in this sense. Let a base, closed, negatively curved manifold X being fixed: if $\overline{X}$ is a normal covering of X, then the closer $\omega(\overline{X})$ is to $\omega(\widetilde{X})$, the bigger must be the systole of $\overline{X}$. It is somewhat surprising that an invariant of the geometry at infinity forces a condition of finite type on the topology. More precisely, as an immediate corollary of Theorem 1.1 we obtain the following estimate of the systole of an open, negatively curved covering $\overline{X}$:

Corollary 2.1. *Let $\overline{X}$ be a n-dimensional negatively curved manifold admitting a cocompact group Γ of isometries acting freely and properly discontinuously. Assume that $-k \leqslant k(\overline{X}) \leqslant -1$, $v = \mathrm{vol}(\Gamma \setminus \overline{X})$ and $d = \mathrm{diam}(\Gamma \setminus \overline{X})$. Then*:

$$\mathrm{syst}(\overline{X}) \geqslant \frac{1}{12(n-1)\sqrt{k}} \log\left[\frac{\varepsilon_n k^{n/2} v}{\omega(\widetilde{X}) - \omega(\overline{X})}\right] - 26d - 33. \tag{2}$$

[2] One has to assume that (G, d) has the property that the balls of finite radius are finite sets, in order that $\omega(G, d)$ makes sense.

Clearly, this formula is significant only when $\omega(\overline{X})$ is sufficiently close to $\omega(\widetilde{X})$. Also notice that, by Theorem 1.1, one always has $\omega(\overline{X}) \neq \omega(\widetilde{X})$ unless $\overline{X}$ is simply connected (that is $\overline{X} = \widetilde{X}$ and $\mathrm{syst}(\overline{X}) = \infty$ by definition).

We give now another interpretation of (1), in terms of periodic geodesics. A pretty natural question is: given a free homotopy class of paths $[\gamma]$ in a closed Riemannian manifold X, can one find a lower bound on the length of the smallest periodic geodesic in this class (in terms, of course, of geometric invariants associated to $[\gamma]$)? The above corollary may be reformulated by saying that the length of any periodic geodesic γ on a closed negatively curved manifold X must be greater than a function depending on some geometric parameters of X and on an asymptotic invariant attached to the homotopy class $[\gamma]$:

Corollary 2.2. *Let X be a closed n-dimensional manifold with $-k \leqslant k(X) \leqslant -1$. For any periodic geodesic γ of X we have*

$$\ell(\gamma) \geqslant \frac{1}{12(n-1)\sqrt{k}} \log\left[\frac{\varepsilon_n\, k^{n/2}\, \mathrm{vol}(X)}{\mathrm{Ent}(X) - \mathrm{Ent}([\gamma])}\right] - 26\,\mathrm{diam}(X) - 33, \tag{3}$$

where $\mathrm{Ent}([\gamma])$ *is the exponential growth rate of the Riemannian covering associated with the normal subgroup of $\pi_1(X)$ generated by γ.*

Notice that $\mathrm{Ent}([\gamma])$ only depends on the free homotopy class of γ and that, by (3), as $\mathrm{Ent}([\gamma])$ grows, the length of the smallest periodic geodesic in this class tends to infinity.

3. Sketch of proof of Theorem 1.1

The *first step* of the proof is the discretization of the problem. Let $\bar\eta$ be a closed geodesic realizing the systole of $\overline{X}$, which projects into a periodic geodesic η of X, and choose preimages $\tilde{x}_0$, $\bar{x}_0$, respectively in $\widetilde{X}$ and $\overline{X}$, of a point x_0 belonging to η. Let N be the (nontrivial) normal subgroup of $G = \pi_1(X, x_0)$ corresponding to the covering $\overline{X}$; then, one clearly has $\eta \in N$, and the automorphisms groups of the coverings $\widetilde{X}$ and $\overline{X}$ can be identified with G and with $\Gamma = G/N$. These groups are respectively endowed with the norms $\|\cdot\|_G$ and $\|\cdot\|_\Gamma$ induced by the Riemannian length: as explained at the end of Section 1, the exponential growth rates of $\widetilde{X}$ and $\overline{X}$ are equal to the exponential growth rates of the groups G and Γ, with respect to these norms. We shall therefore prove inequality (1) with $\omega(G)$ and $\omega(\Gamma)$ in place of $\omega(\widetilde{X})$ and $\omega(\overline{X})$.

The *second step* is to consider the group $\ell^1(\Gamma)$ of "infinite" sequences (γ_i) of elements of Γ with finite ℓ^1-norm, that is such that $\sum_i \|\gamma_i\|_\Gamma < \infty$. Notice that, as Γ is discrete, this means that $\gamma_i = 1$ for all $i \gg 0$; we call *depth* of the sequence (γ_i) the greatest i such that $\gamma_i \neq 1$, and we denote it by $\partial(\gamma_i)$.

Then, for any $c > 0$, we define $\ell^1_c(\Gamma)$ to be the same space as above, but endowed with the norm

$$\big\|(\gamma_i)\big\|_c = \sum_i \|\gamma_i\|_\Gamma + c \cdot \partial(\gamma_i).$$

This norm induces a left-invariant distance on $\ell^1_c(\Gamma)$ which makes of it a discrete metric space; so, it makes sense to consider its asymptotic growth rate, and an explicit computation shows that

$$\omega\big(\ell^1_c(\Gamma)\big) \geqslant \omega(\Gamma) + \varepsilon(c; d, k) \tag{4}$$

for a positive, explicit function $\varepsilon(c; d, k)$ (which tends to zero when $c \to +\infty$).

The *third step* is to show that one can embed $\ell^1_c(\Gamma)$ into G, for $c \gg 0$, by means of a *contracting immersion* j: that is, an injective, Lipschitz map between metric spaces, of Lipschitz constant 1. The map j is constructed inductively on the depth of sequences as follows. Given an element $\gamma \in \Gamma \setminus \{1\}$ (that is, a sequence of $\ell^1_c(\Gamma)$ of depth 1), we choose a *minimal* representative $g \in G$ for γ (that is, such that $\|g\|_G = \|\gamma\|_\Gamma$) and we set $j(\gamma) = g$. For a general sequence (γ_i) of depth m we then set

$$j(\gamma_1, \ldots, \gamma_m) = j(\gamma_2, \ldots, \gamma_m)\eta^{\sigma_1} j(\gamma_1)^{\tau_1}$$

for some well-chosen exponents $\sigma_i, \tau_i \in \{\pm 1\}$. Then, the map j will certainly contract norms if $c \geqslant \|\eta\|_G$, as

$$\big\| j\big((\gamma_i)\big)\big\|_G \leqslant \sum_i \|g_i\|_G + \|\eta\|_G \cdot \partial(\gamma_i) \leqslant \big\|(\gamma_i)\big\|_c.$$

The nontrivial point is to show that the map j is injective *up to choosing opportunely the exponents* σ_i, τ_i. The reason of the exponents ± 1 is that we need that, for each i, the norm of $j(\gamma_1, \ldots, \gamma_i)$ is *almost* equal to the sum of $\|j(\gamma_1)\|_G$ and $\|j(\gamma_2, \ldots, \gamma_i)\|_G$. If this condition is satisfied, then one can exhibit a geometric algorithm permitting to reconstruct the whole sequence $(\gamma_1, \ldots, \gamma_m)$ looking at the geodesic of $\widetilde{X}$ with endpoints $\tilde{x}_0$ and $j(\gamma_1, \ldots, \gamma_m) \cdot \tilde{x}_0$.

Conclusion: as the R-balls of $\ell^1_c(\Gamma)$ can be injected in the R-balls of G, by (4) we obtain: $\omega(G) \geqslant \omega(\ell^1_c(\Gamma)) \geqslant \omega(\Gamma) + \varepsilon(c; d, k)$, for $c = \mathrm{syst}(\overline{X})$.

Actually, here we have slightly simplified the proof: in fact, in order that the algorithm really works, we need to apply step 2 and 3 to a maximal δ-separated set Γ_δ of Γ, and not directly to Γ (where δ is a function of c and d). But the growth function of Γ_δ can be estimated in terms of the growth function of Γ and of the parameters δ and $v = \mathrm{vol}(X)$; then, we can show that formula (4) still holds for Γ_δ instead of Γ, for a function $\varepsilon = \varepsilon(c; d, k, \delta, v)$. This leads therefore to inequality (1).

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