# Critical densities for random quotients of hyperbolic groups Densités critiques pour les quotients aléatoires de groupes hyperboliques 

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#### Abstract

We prove that in various natural models of a random quotient of a group, depending on a density parameter, for each hyperbolic group there is some critical density under which a random quotient is still hyperbolic with high probability, whereas above this critical value a random quotient is very probably trivial. We give explicit characterizations of these critical densities for the various models. To cite this article: Y. Ollivier, C. R. Acad. Sci. Paris, Ser. I 336 (2003). © 2003 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS. All rights reserved.


## Résumé

Nous prouvons que pour plusieurs modèles naturels de quotient aléatoire d'un groupe, dépendant d'un paramètre de densité, pour chaque groupe hyperbolique il existe une densité critique sous laquelle un quotient aléatoire reste hyperbolique avec grande probabilité, tandis qu'au-dessus de cette densité le quotient aléatoire est très probablement trivial. Nous donnons des caractérisations explicites de ces densités critiques dans les différents modèles. Pour citer cet article: Y. Ollivier, C. R. Acad. Sci. Paris, Ser. I 336 (2003).
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## 1. Introduction

The study of random groups emerged from an affirmation of Gromov that "almost every group is hyperbolic" (see [9]). A random (discrete) group is a group obtained by a presentation $\left\langle a_{1}, \ldots, a_{m} ; R\right\rangle$ where $R$ is a set of randomly chosen words in the generators $a_{i}^{ \pm 1}$. This depends on a model for the set $R$.

In early studies on random groups (cf. [15] or [4]), $R$ was made up of a fixed number $N$ of randomly uniformly chosen words of prescribed lengths. One can show in this model that the probability to get a hyperbolic group tends to 1 when the lengths of the words tend to infinity.

Later, Gromov introduced (cf. [10]) a finer model of random group, in which the number $N$ of relators is allowed to be much bigger. In this model, a (large) length $\ell$ is chosen, and the set $R$ of relators taken consists in $(2 m-1)^{d \ell}$

[^0]random words uniformly picked from the set of all $(2 m)(2 m-1)^{\ell-1}$ reduced words of length $\ell$ in the generators $a_{i}^{ \pm 1}$. Here $d$ is a number between 0 and 1 called density. The properties of the random group $\left\langle a_{1}, \ldots, a_{m} ; R\right\rangle$ can vary with $d$ and indeed, Gromov states a sharp phase transition theorem between hyperbolicity and triviality.

Theorem 1.1 (Gromov [10]). Fix a density d between 0 and 1 . Choose a length $\ell$ and pick at random a set $R$ of $(2 m-1)^{d \ell}$ uniformly chosen reduced words of length $\ell$ in the letters $a_{1}^{ \pm 1}, \ldots, a_{m}^{ \pm 1}$.

If $d<1 / 2$ then the probability that the presentation $\left\langle a_{1}, \ldots, a_{m} ; R\right\rangle$ defines an infinite hyperbolic group tends to 1 as $\ell \rightarrow \infty$. If $d>1 / 2$ then the probability that the presentation $\left\langle a_{1}, \ldots, a_{m} ; R\right\rangle$ defines the group $\{e\}$ or $\mathbb{Z} / 2 \mathbb{Z}$ tends to 1 as $\ell \rightarrow \infty$.

Following Gromov [10], $d \ell$ should be thought of as a dimension representing the number of "equations" we can impose on a random word so that we still have a reasonable chance to find such a word in a set of $(2 m-1)^{d \ell}$ randomly chosen words. For example, for large $\ell$, in a set of $2^{d \ell}$ randomly chosen words of length $\ell$ in the two letters "a" and "b", there will probably be some word beginning with $d \ell$ letters "a". As another example, in a set of $(2 m-1)^{d \ell}$ randomly chosen words in $a_{i}^{ \pm 1}$, there will probably be two words having the same first $2 d \ell$ letters, but no more. In particular, if $d<1 / 12$ then the set of words will satisfy the small cancellation property $C^{\prime}(1 / 6)$ (see [7] for definitions). But as soon as $d>1 / 12$, we are far from small cancellation, and as $d$ approaches $1 / 2$ we have arbitrarily big cancellation.

Other properties of generic groups have been studied under one or another model, such as small cancellation properties, torsion elements, topology of the boundary, property T , or the fact that most subgroups are free; and more are to come. See for example [4,2,1,20]. In [11], Gromov elaborates on random groups in relation with $C^{\star}$-algebraic conjectures. Generic properties of group can also be studied under a topological rather than statistical approach, see [5].

## 2. Statement of the results

The theorem above states that a random quotient of a free group is hyperbolic. Our theorems state that, more generally, for each hyperbolic group, there is an explicit critical density under which a random quotient is still hyperbolic, whereas above this density the quotient is trivial. The random quotient can be taken with reduced words, geodesic words (which was the same in a free group), or just plain words.

In our theorems we need the well-known notions of cogrowth and gross cogrowth of a group. Cogrowth has been introduced by Grigorchuk in [8] and by Cohen in [6]. For some examples see [3] or [18]. Gross cogrowth is linked to the spectrum of the random walk on the group, introduced by Kesten ([12] and [13]), and extensively studied since then (see, e.g., [19] and the references therein).

Definition 2.1 (Cogrowth, gross cogrowth). Let $G$ be a group generated by elements $a_{i}^{ \pm 1}$ and, for any length $\ell$, let $W^{\ell}$ be the set of words of length $\ell$ in $a_{i}^{ \pm 1}$. The cogrowth of the group $G$ with respect to the generating set $a_{1}, \ldots, a_{m}$ is defined as $\eta=1 / 2$ for a free group, and otherwise

$$
\eta=\lim _{\ell \rightarrow \infty, \ell \text { even }} \frac{1}{\ell} \log _{2 m-1} \#\left\{w \in W^{\ell}, w=e \text { in } G, w \text { reduced }\right\}
$$

The gross cogrowth of the group $G$ with respect to the generating set $a_{1}, \ldots, a_{m}$ is defined as

$$
\theta=\lim _{\ell \rightarrow \infty, \ell \text { even }} \frac{1}{\ell} \log _{2 m} \#\left\{w \in W^{\ell}, w=e \text { in } G\right\}
$$

The limits are well-defined by a superadditivity argument. These quantities depend on the generating set. The definition is restricted to even lengths since in a group presentation containing no relations of odd length (e.g., a free group) there are no words of odd length representing the element $e$.

There are various conventions for cogrowth of a free group (the formula would give $-\infty$ ). The one we use is justified by the following facts: for any non-free group, $\eta$ is greater than $1 / 2$; moreover, there is a formula linking $\eta$ and $\theta$ : for any group (including free groups with our convention), $(2 m)^{\theta}=(2 m-1)^{\eta}+(2 m-1)^{1-\eta}$; and finally this unifies the statement of our theorem.

The quantity $(2 m)^{\theta-1}$ is equal to the spectral radius of the discrete heat (random walk) operator on the Cayley graph of the group. We always have $\eta \geqslant 1 / 2$ and $\theta>1 / 2$. A cogrowth, or gross cogrowth, of 1 is equivalent to amenability.

We now state the theorems. The first two are formal generalizations of Theorem 1.1.
Theorem 2.2 (Random quotient by reduced words). Let $G$ be a non-elementary hyperbolic group generated by the elements $a_{1}, \ldots, a_{m}$. Fix a density $d$ between 0 and 1 . Choose a length $\ell$ and pick at random a set $R$ of $(2 m-1)^{d \ell}$ uniformly chosen reduced words of length $\ell$ in $a_{i}^{ \pm 1}$. Let $\langle R\rangle$ be the normal subgroup generated by $R$. Let $\eta$ be the cogrowth of the group $G$.

If $d<1-\eta$, then, with probability tending to 1 as $\ell \rightarrow \infty$, the quotient $G /\langle R\rangle$ is non-elementary hyperbolic. If $d>1-\eta$, then, with probability tending to 1 as $\ell \rightarrow \infty$, the quotient $G /\langle R\rangle$ is $\{e\}$ or $\mathbb{Z} / 2 \mathbb{Z}$.

Theorem 2.3 (Random quotient by elements of a sphere). Let $G$ be a non-elementary hyperbolic group generated by the elements $a_{1}, \ldots, a_{m}$. Fix a density $d$ between 0 and 1 . Choose a length $\ell$.

Let $S^{\ell}$ be the set of elements of $G$ which are of norm $\ell-1, \ell$ or $\ell+1$ with respect to $a_{1}, \ldots, a_{m}$. Let $N$ be the number of elements of $S^{\ell}$. Pick at random a set $R$ of $N^{d}$ uniformly chosen elements of $S^{\ell}$. Let $\langle R\rangle$ be the normal subgroup generated by $R$.

If $d<1 / 2$, then, with probability tending to 1 as $\ell \rightarrow \infty$, the quotient $G /\langle R\rangle$ is non-elementary hyperbolic. If $d>1 / 2$, then, with probability tending to 1 as $\ell \rightarrow \infty$, the quotient $G /\langle R\rangle$ is $\{e\}$.

Theorem 2.4 (Random quotient by plain words). Let $G$ be a hyperbolic group generated by the elements $a_{1}, \ldots, a_{m}$. Fix a density d between 0 and 1 . Choose a length $\ell$ and pick at random a set $R$ of $(2 m)^{d \ell}$ uniformly chosen words of length $\ell$ in $a_{i}^{ \pm 1}$. Let $\langle R\rangle$ be the normal subgroup generated by $R$.

Let $\theta$ be the gross cogrowth of the group $G$.
If $d<1-\theta$, then, with probability tending to 1 as $\ell \rightarrow \infty$, the quotient $G /\langle R\rangle$ is non-elementary hyperbolic. If $d>1-\theta$, then, with probability tending to 1 as $\ell \rightarrow \infty$, the quotient $G /\langle R\rangle$ is $\{e\}$ or $\mathbb{Z} / 2 \mathbb{Z}$.

Some remarks: at $d=0$, instead of taking only one relator one could take any number of relators growing subexponentially with $\ell$. Also, numbers such as $(2 m)^{d \ell}$ are not necessarily integers: one can take the integer part, but anyway changing the number of relators by a subexponential term in $\ell$ (such as a multiplicative constant) does not affect our results.

The possible occurrence of $\mathbb{Z} / 2 \mathbb{Z}$ above the critical density only reflects the fact if $\ell$ is even and $G$ has no relations of odd length, at least a quotient $\mathbb{Z} / 2 \mathbb{Z}$ remains. The occurrence of such small-scale phenomena is the reason why we take words of length $\ell$ or $\ell+1$ or $\ell-1$ in Theorem 2.3.

Of course, the three theorems are not proven separately: they are a consequence of a more general (and longer to state) theorem about random quotients of hyperbolic groups by words picked under probability measures satisfying some set of axioms (to be published elsewhere). In particular, it is not mandatory that all relations taken have the same length, though this makes the definition of density more involved.

## 3. Sketch of proof

The (rather long, see [14]) proof of these theorems relies on the characterization of hyperbolic groups by isoperimetry of minimal Dehn diagrams (cf. [17]). A Dehn diagram $D$ for a random quotient $G /\langle R\rangle$ can be decomposed into ("new") cells of $R$ and ("old") cells from the presentation of $G$. The old cells form a Dehn diagram $D^{\prime}$ of $G$, and hyperbolicity of $G$ implies that $D^{\prime}$ is narrow. Then, if $\ell$ is taken large enough (depending on
isoperimetric constants of $G$ ), at large scales $D$ looks like a planar graph $D^{\prime \prime}$ whose faces are the new cells of $D$, and whose edges represent thin strips of small old cells between new cells.

To each edge of $D^{\prime \prime}$ between new cells $r_{1}, r_{2} \in R$ can be associated an equality of words $w_{1}=u w_{2} v$ where $w_{1}$ and $w_{2}$ are subwords of $r_{1}$ and $r_{2}$ representing the lateral long sides of the strip, and $u, v$ are short (w.r.t. $\ell$ ) words representing the short sides. The strip is made of old cells, so the equality takes place in the group $G$. The words $w_{1}$ and $w_{2}$ are subwords of randomly chosen words. For the sake of simplicity, consider only the case of plain random words (Theorem 2.4). Then, $w_{1}$ and $w_{2}$ themselves are uniformly chosen random words. The probability that, for two given subwords $w_{1}, w_{2}$ of relators $r_{1}$ and $r_{2}$, there exist short words $u$ and $v$ such that $w_{1}=u w_{2} v$ in $G$ is roughly the same as the probability that $w_{1}=w_{2}$ in $G$ due to the short length of $u$ and $v$. By definition of the gross cogrowth $\theta$, this probability behaves like $(2 m)^{-L(1-\theta)}$ where $L=\left|w_{1}\right|+\left|w_{2}\right|$. Now the boundary length $|\partial D|$ of $D$ is nearly equal to $\ell$ times the number $n$ of new cells in $D$ minus the cumulated length $L_{c}$ of all the subwords of the relators which form the edges of $D^{\prime \prime}$. But we just showed that each edge of length $L$ implies a loss of $(2 m)^{-L(1-\theta)}$ in probability. One can show that these probabilities can be considered as independent. So the probability that such a diagram exists is at most $(2 m)^{-L_{c}(1-\theta)}$. By definition of density we have $(2 m)^{d \ell}$ choices for each of the $n$ new cells of $D$, so the total probability is $(2 m)^{n d \ell-L_{c}(1-\theta)}$. So when $d<1-\theta$, either the boundary length $n \ell-L_{c}$ is big, or the probability $(2 m)^{n d \ell-L_{c}(1-\theta)}$ is small as was to be shown.

This is for a given Dehn diagram $D$. A result of hyperbolic geometry [16] states that it is enough to check the isoperimetric inequality for a finite number of Dehn diagrams, hence the probabilistic evaluation.

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## References

[1] G.N. Arzhantseva, Generic properties of finitely presented groups and Howson's theorem, Comm. Algebra 26 (4) (1998) $3783-3792$.
[2] G.N. Arzhantseva, A.Yu. Ol'shanskii, Generality of the class of groups in which subgroups with a lesser number of generators are free, Mat. Zametki 59 (4) (1996) 489-496. Translation in Math. Notes 59 (3-4) (1996) 350-355.
[3] C. Champetier, Cocroissance des groupes à petite simplification, Bull. London Math. Soc. 25 (5) (1993) 438-444.
[4] C. Champetier, Propriétés statistiques des groupes de présentation finie, J. Adv. Math. 116 (2) (1995) 197-262.
[5] C. Champetier, L'espace des groupes de type fini, Topology 39 (4) (2000) 657-680.
[6] J.M. Cohen, Cogrowth and amenability of discrete groups, J. Funct. Anal. 48 (1982) 301-309.
[7] É. Ghys, P. de la Harpe, Sur les groupes hyperboliques d'après Mikhael Gromov, in: Progr. Math., Vol. 83, Birkhäuser, 1990.
[8] R.I. Grigorchuk, Symmetrical random walks on discrete groups, in: R.L. Dobrushin, Ya.G. Sinai (Eds.), Multicomponent Random Systems, in: Adv. Probab. Related Topics, Vol. 6, Dekker, 1980, pp. 285-325.
[9] M. Gromov, Hyperbolic groups, in: S.M. Gersten (Ed.), Essays in Group Theory, Springer, 1987, pp. 75-265.
[10] M. Gromov, Asymptotic invariants of infinite groups, in: G. Niblo, M. Roller (Eds.), Geometric Group Theory, Cambridge University Press, Cambridge, 1993.
[11] M. Gromov, Random walk in random groups, Preprint IHÉS, 2002.
[12] H. Kesten, Symmetric random walks on groups, Trans. Amer. Math. Soc. 92 (1959) 336-354.
[13] H. Kesten, Full Banach mean values on countable groups, Math. Scand. 7 (1959) 146-156.
[14] Y. Ollivier, Sharp phase transition theorems for hyperbolicity of random groups, 2003, ArXiv document http://www.arxiv.org/abs/ math.GR/0301187.
[15] A.Yu. Ol'shanskii, Almost every group is hyperbolic, Internat. J. Algebra Comput. 2 (1) (1992) 1-17.
[16] P. Papasoglu, An algorithm detecting hyperbolicity, in: G. Baumslag, et al. (Eds.), Geometric and Computational Perspectives on Infinite Groups, in: DIMACS Ser. Discrete Math. Theor. Comput. Sci., Vol. 25, 1996, pp. 193-200.
[17] A. Short, et al., in: É. Ghys, A. Haefliger, A. Verjovsky (Eds.), Group Theory from a Geometrical Viewpoint, World Scientific, 1991.
[18] W. Woess, Cogrowth of groups and simple random walks, Arch. Math. (Basel) 41 (1983) 363-370.
[19] W. Woess, Random Walks on Infinite Graphs and Groups, in: Cambridge Tracts in Math., Vol. 138, Cambridge University Press, 2000.
[20] A. Żuk, Property (T) and Kazhdan constants for discrete groups, Preprint, École normale supérieure de Lyon.


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