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Algebraic Geometry

Green–Lazarsfeld’s conjecture for generic curves of large gonality

La conjecture de Green–Lazarsfeld pour les courbes génériques
de gonalité élevée

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Abstract

We use Green’s canonical syzygy conjecture for generic curves to prove that the Green–Lazarsfeld gonality conjecture holds for generic curves of genus g , and gonality d , if $g/3 < d < [g/2] + 2$. **To cite this article:** *M. Aprodu, C. Voisin, C. R. Acad. Sci. Paris, Ser. I 336 (2003)*.

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Résumé

Nous utilisons la conjecture de Green sur les syzygies canoniques des courbes génériques pour démontrer la conjecture de la gonalité de Green–Lazarsfeld pour les courbes génériques de genre g et gonalité d , avec $g/3 < d < [g/2] + 2$. **Pour citer cet article :** *M. Aprodu, C. Voisin, C. R. Acad. Sci. Paris, Ser. I 336 (2003)*.

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Version française abrégée

Un théorème de Green et Lazarsfeld (voir l’appendice de [5]) a pour conséquence le résultat suivant :

Théorème 0.1. *Soit L un fibré en droites de degré suffisamment grand au-dessus d’une courbe C de gonalité d . Alors,*

$$K_{h^0(L)-d-1,1}(C, L) \neq 0.$$

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La conjecture de la gonalité prédit que ceci est optimal, c'est à dire :

Conjecture 0.2 (Green et Lazarsfeld, [7]). *Pour toute courbe C de gonalité d et pour tout fibré en droite L de degré suffisamment grand sur C , on a $K_{h^0(L)-d,1}(C, L) = 0$.*

Le but de cette Note est de combiner les résultats principaux de [1,9] et [10] afin de vérifier la conjecture de Green–Lazarsfeld pour les courbes génériques de gonalité assez grande, fixée. Nous démontrons tout d'abord le résultat suivant :

Théorème 0.3. *Soient g et d deux entiers positifs tels que $g/3 + 1 \leq d < [(g + 3)/2]$. Alors, la conjecture de la gonalité est vraie pour les courbes génériques de genre g et de gonalité d .*

Nous remarquons que la gonalité d'une courbe générique de genre g est égale à $[(g + 3)/2]$, donc ce théorème couvre toutes les gonalités assez grandes, sauf la gonalité maximale.

Notre deuxième résultat est le suivant :

Théorème 0.4. *La conjecture de la gonalité est vraie pour les courbes génériques de genre pair.*

Pour démontrer les Théorèmes 0.3 et 0.4 on se sert des courbes nodales sur des surfaces $K3$, comme pour la conjecture de Green générique, voir [9,10]. Un ingrédient nouveau, qui nous permet d'obtenir l'annulation souhaitée, est le fait que le morphisme de normalisation d'une courbe nodale induit une inclusion entre les groupes de cohomologie de Koszul :

Lemme 0.5. *Soient X une courbe nodale, C la normalisée de X et p et q deux points de C qui se projettent tous les deux sur le même noeud x de X . Alors, pour tout $n \geq 1$, on a une inclusion naturelle de $K_{n,1}(C, K_C(p + q))$ dans $K_{n,1}(X, K_X)$.*

1. Introduction

Denoting by $K_{p,q}(X, L)$ the Koszul cohomology with value in a line bundle L (see [5]), Green and Lazarsfeld proved the following (cf. Appendix to [5]):

Theorem 1.1. *Let X be a complex manifold, L_1 , and L_2 be two line bundles on X such that $r_1 := h^0(L_1) - 1 \geq 1$, and $r_2 := h^0(L_2) - 1 \geq 1$. Then $K_{r_1+r_2-1,1}(X, L_1 \otimes L_2) \neq 0$.*

Let now C be a smooth complex smooth projective curve of gonality

$$d := \min\{\deg(L), h^0(L) \geq 2\}.$$

Green–Lazarsfeld's theorem, applied to $L_1 = \mathcal{O}_C(D)$, where $\deg(D) = d$, $h^0(C, \mathcal{O}_C(D)) = 2$, and $L_2 = L - D$ with $\deg(L)$ sufficiently large, implies

$$K_{h^0(L)-d-1,1}(C, L) \neq 0.$$

The gonality conjecture predicts that this is optimal, namely:

Conjecture 1.2 (Green and Lazarsfeld, [7]). *For C a curve of gonality d , and for any line bundle L of sufficiently large degree, we have $K_{h^0(L)-d,1}(C, L) = 0$.*

In spite of the evidence brought by the “ $K_{p,1}$ Theorem” of [5], which solves the hyperelliptic case (among other things), after having formulated the gonality conjecture, Green and Lazarsfeld shown their mistrust of the statement they had just made. Since then, the conjecture has been almost forgotten, and it took a while until some new evidence was discovered (see [3,1]). This delay is probably due to the fact that the conjecture did not count among the mathematical highlights of the last years, as almost all the attention in the theory of syzygies of curves was focused on the more famous Green conjecture.

The aim of this short note is to mix together the main results of [1] on the one hand, and of [9], and [10] on the other hand, in order to verify the Green–Lazarsfeld conjecture for generic curves of large given gonality. The first result we prove is the following:

Theorem 1.3. *For any positive integers g and d such that $g/3 + 1 \leq d < [(g + 3)/2]$, the gonality conjecture is valid for generic curves of genus g and gonality d .*

Note that for generic curves of genus g the gonality equals $[(g + 3)/2]$, and thus Theorem 1.3 covers all possible, not too small, gonalitys, except for the generic gonality.

Our second result is:

Theorem 1.4. *The gonality conjecture is valid for generic curves of even genus.*

In the statements above, the word *generic* should be read in the usual sense. The complex curves of fixed genus g , and gonality d , are parametrised by an irreducible subvariety of the moduli space \mathcal{M}_g , and a *generic curve* is a curve corresponding to a generic point of this variety. Irreducibility follows from the well-known fact that the closure of this subvariety is actually the closure of the image in \mathcal{M}_g of a Hurwitz scheme and then apply [4].

Finally, we mention that all the notation we use in the sequel is standard, and we refer to [5] for basic facts about Koszul cohomology.

2. Proofs of main results

First of all, we recall the following result from [1]:

Theorem 2.1. *If L is a nonspecial line bundle on a curve C , which satisfies $K_{n,1}(C, L) = 0$, for a positive integer n , then, for any effective divisor E of degree e , we have $K_{n+e,1}(C, L + E) = 0$.*

In particular, if $K_{h^0(L)-d,1}(C, L) = 0$ for a nonspecial line bundle L , with $h^0(L) - d > 0$, then $K_{h^0(L')-d,1}(C, L') = 0$ for any L' of sufficiently large degree. By means of the Zariski semi-continuity of graded Betti numbers (see, for example, [2]), for both Theorem 1.3, and Theorem 1.4, it suffices to exhibit one d -gonal curve C of genus g (where $d = g/2 + 1$ for Theorem 1.4), and one nonspecial line bundle L on C satisfying $K_{h^0(L)-d,1}(C, L) = 0$.

Proof of Theorem 1.3. *Step 1.* Construction of C and L .

A suitable choice of such a curve C is provided by the proof of Corollary 1 of [9]. We start with a $K3$ surface S whose Picard group is cyclic, generated by a line bundle \mathcal{L} of self-intersection $\mathcal{L}^2 = 4k - 2$, where $k = g - d + 1$. We denote $\nu = g - 2d + 2 \geq 1$. Under these assumptions, as $\nu \leq k/2$, we know that there exists an irreducible curve $X \in |\mathcal{L}|$, having exactly ν simple nodes as singular points, and no other singularities, and such that its normalization C is of gonality $d = k + 1 - \nu$, see [9].

We set $L = K_C(p + q)$, where p , and q are two distinct points of C that lie over a node x of X .

Step 2. Recall the main result of [9].

Theorem 2.2. *The K3 surface S being as above, we have $K_{k,1}(S, \mathcal{L}) = 0$.*

It follows directly from this result, from the adjunction formula, and from the hyperplane section theorem for Koszul cohomology (see [5]) that $K_{k,1}(X, K_X) = 0$. Since $k = h^0(C, L) - d$, the proof of our theorem is concluded by the following lemma.

Lemma 2.3. *Let X be a nodal curve, $C \xrightarrow{f} X$ be the normalization of X , and $p, q \in C$ be two distinct points lying over the same node x of X . Then, for any $n \geq 1$, we have a natural injective map $K_{n,1}(C, K_C(p + q)) \subset K_{n,1}(X, K_X)$.*

Proof of Lemma 2.3. Firstly, we remark that there is a natural inclusion of spaces of sections $H^0(C, K_C(p + q)) \subset H^0(X, K_X)$. Indeed, the two spaces are both contained in the space of meromorphic differentials on C . Thus $H^0(X, K_X)$ identifies to the meromorphic differentials on C having poles of multiplicity one over the nodes, and regular outside these points, and whose sums of residues over the nodes vanish. The inclusion above is then a direct consequence of the Residue Theorem.

This inclusion yields the following injection between the Koszul complexes of $K_C(p + q)$ and K_X

$$\begin{array}{ccccccc}
 0 \rightarrow & \bigwedge^{n+1} H^0(C, K_C(p + q)) & \rightarrow & \bigwedge^n H^0(C, K_C(p + q)) \otimes H^0(C, K_C(p + q)) & \rightarrow & \dots & \\
 & \downarrow & & \downarrow & & & \\
 0 \rightarrow & \bigwedge^{n+1} H^0(X, K_X) & \rightarrow & \bigwedge^n H^0(X, K_X) \otimes H^0(X, K_X) & \rightarrow & \dots &
 \end{array}$$

To conclude that this induces an injection on the degree 1 cohomology groups, we use the existence of the retraction-homotopy (up to a coefficient of $(n + 1)!$) given by the wedge product :

$$\bigwedge^n H^0(C, K_C(p + q)) \otimes H^0(C, K_C(p + q)) \rightarrow \bigwedge^{n+1} H^0(C, K_C(p + q)).$$

Proof of Theorem 1.4. *Step 1. Construction of C and L .*

We make use of the same curves as those used in [10]. Let S be a K3 surface whose Picard group is generated by an ample line bundle \mathcal{L} of self-intersection $\mathcal{L}^2 = 4k$, where $g = 2k$, and by a rational curve Δ such that $\mathcal{L} \cdot \Delta = 2$. We choose X an irreducible nodal curve in the linear system $|\mathcal{L}|$, having exactly one node as singularity. The curve X has arithmetic genus $2k + 1$. We denote by C the normalization of X , p and q the two distinct points of C lying over the node of X , and $L = K_C(p + q)$. Thus C has genus $2k$.

Step 2. Recall the following result from [10]:

Theorem 2.4. *With the notation above, we have $K_{k,1}(S, \mathcal{L}) = 0$.*

Applying the hyperplane section theorem we conclude that $K_{k,1}(X, K_X) = 0$. By means of Lemma 2.3, we obtain $K_{k,1}(C, L) = 0$. Since $k = h^0(L) - (k + 1)$, it follows that the curve C is of maximal gonality $k + 1$, and that the gonality conjecture is valid for C .

3. Final remarks

Remark 1. Using Theorem 2.1 one can see that for any of the curves C considered in the proofs of Theorem 1.3, and Theorem 1.4, the vanishing $K_{h^0(L')-d,1}(C, L') = 0$ predicted by the gonality conjecture holds for any line bundle L' of degree at least $3g$. The same is true for generic d -gonal curves of genus g , where g and d satisfy $g/3 < d < [g/2] + 2$.

Remark 2. The case $g > (d-1)(d-2)$ has already been treated in [1]. Therefore, from the viewpoint of the Green–Lazarsfeld conjecture for generic curves of fixed genus g , and gonality d , for $d \geq 6$, there is a gap remaining for $3d-3 < g \leq (d-1)(d-2)$ that has to be solved differently. The case of maximal gonality $d = k+2$ in the odd genus case $g = 2k+1$, which seems to be the most difficult, is also left over. Nevertheless, we should point out that in all these excepted cases, gonality conjecture is *almost true*, that is, any line bundle L of degree at least $3g-2$ on a generic d -gonal curve C of genus g (for any choice of g and d) satisfies $K_{h^0(L)-d+1,1}(C, L) = 0$. This follows directly from the main results of [8–10], and from Theorem 3 of [1].

Remark 3. An alternative proof of Lemma 2.3 can be obtained in a more algebraic way, by factoring the normalization morphism through $C \xrightarrow{h} Y \xrightarrow{g} X$, where g is the smoothing of the node x . We analyse the three morphisms between the structure sheaves, and we obtain an isomorphism $H^0(X, K_X) \cong H^0(Y, g^*K_X)$, and an inclusion $H^0(C, K_C(p+q)) \subset H^0(Y, g^*K_X)$. Denoting by $W = H^0(C, K_C(p+q))$ inside $H^0(X, K_X)$, we have $K_{n,1}(C, K_C(p+q)) \subset K_{n,1}(X, K_X, W)$. Next, we use the embedding $K_{n,1}(X, K_X, W) \subset K_{n,1}(X, K_X)$, which arises from the spectral sequence of [6], to conclude.

This shows that in Lemma 2.3 the fact of working over the complex numbers is not essential, and the statement is true for nodal curves over any algebraically closed field.

Remark 4. The use of the results of [9,10] in the proofs of our theorems shades a new light on the relationships between Green’s conjecture, and the gonality conjecture, the two statements seemed to be (in a somewhat mysterious way) intimately related to each other (see also [1]). In fact, Lemma 2.3 answers partially to the conjecture made in [1].

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