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C^1-generic Pesin's entropy formula

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Abstract The metric entropy of a C^2-diffeomorphism with respect to an invariant smooth measure μ is equal to the average of the sum of the positive Lyapunov exponents of μ. This is the celebrated Pesin's entropy formula, $h_\mu(f) = \int_M \sum_{\lambda_i > 0} \lambda_i$. The C^2 regularity (or $C^{1+\alpha}$) of diffeomorphism is essential to the proof of this equality. We show that at least in the two dimensional case this equality is satisfied for a C^1-generic diffeomorphism and in particular we obtain a set of volume preserving diffeomorphisms strictly larger than those which are $C^{1+\alpha}$ where Pesin's formula holds. ***To cite this article: A. Tahzibi, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 1057–1062.***

La formule d'entropie de Pesin C^1-générique

Résumé L'entropie métrique d'un difféomorphisme C^2, par rapport à une mesure invariante est égale à la moyenne de la somme des exposants de Lyapunov positifs. Ceci est la célèbre formule d'entropie de Pesin. La régularité du difféomorphisme est essentielle pour la preuve de cette égalité. Nous montrons que en dimension deux, cette égalité est satisfaite pour un difféomorphisme C^1-générique et montrons qu'en particulier nous obtenons un ensemble de difféomorphismes conservatifs contenant strictement ceux qui sont $C^{1+\alpha}$, où la formule de Pesin est satisfaite. ***Pour citer cet article : A. Tahzibi, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 1057–1062.***

Version française abrégée

Les exposants de Lyapunov d'une application différentiable de M (une variété compacte) dans M sont définis par le théorème d'Oseledets. Soit μ une mesure de probabilité invariante pour f ; pour presque tout point x il existe des nombres $\lambda_1(x) > \lambda_2(x) > \cdots > \lambda_{k(x)}(x)$ (les exposants) et une unique décomposition $T_xM = E_1(x) \oplus \cdots \oplus E_{k(x)}(x)$ tels que

$$\lim_{n\to\infty} \frac{1}{n} \log\left\| Df_x^n(v)\right\| = \lambda_i(x)$$

pour tout $0 \neq v \in E_i(x)$, $1 \leqslant i \leqslant l$ ($\dim(M) = l$). Les exposants caractéristiques définis comme ci-dessus sont en relation avec l'entropie de f. Par exemple pour une mesure invariante ν et $f \in C^1$, soit

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$\chi(x) := \sum_{\lambda_i>0} \lambda_i$; alors par un résultat de Ruelle :

$$h_\nu(f) \leqslant \int_M \chi \,\mathrm{d}m.$$

Le résultat est en général une inégalité stricte. Mais si m est absolument continue par rapport à la mesure de Lebesgue sur M, et $f \in \mathrm{Diff}_m^{1+\alpha}(M)$, $\alpha > 0$, alors

$$h_m(f) = \int_M \chi \,\mathrm{d}m.$$

En fait, on peut obtenir cette formule pour une plus grande classe de mesures (voir [3]). Mais la régularité de f est une condition nécessaire pour la preuve d'une telle égalité.

Nous allons montrer que si $\dim(M) = 2$, il existe un sous-ensemble générique dans $\mathrm{Diff}_m^1(M)$ où la formule d'entropie de Pesin est satisfaite.

THÉORÈME 0.1. – *Il existe un sous-ensemble générique $\mathcal{G} \in \mathrm{Diff}_m^1(M)$, tel que toute $f \in \mathcal{G}$ satisfait la formule d'entropie de Pesin et $\mathcal{G}$ contient strictement $\bigcup_{\alpha>0} \mathrm{Diff}_m^{1+\alpha}$.*

L'étape clef de la démonstration sera de prouver que les points de continuité des deux fonctions suivantes $h_m(\cdot)$, $L(\cdot)$ forment une partie résiduelle dans la C^1 topologie.

Comme nous considérons des difféomorphismes conservatifs en dimension deux, il existe tout au plus un exposant de Lyapunov positif. Définissons :

– $L(f) = \int_M \lambda_1 \,\mathrm{d}m$ pour $f \in \mathrm{Diff}_m^1(M)$ et
– $h_m(f) =$ l'entropie métrique de f pour $f \in \mathrm{Diff}_m^1(M)$.

Maintenant nous procédons en utilisant la formule d'entropie pour les difféomorphismes dans $\bigcup \mathrm{Diff}_m^{1+\alpha}(M)$. Soit f un point de continuité pour $L(\cdot)$ et $h_m(\cdot)$. Par la densité de $\mathrm{Diff}_m^{1+\alpha}(M)$ dans $\mathrm{Diff}_m^1(M)$ prouvée dans [4], il y a une suite $f_n \in \mathrm{Diff}_m^{1+\alpha}(M)$ telle que f_n converge vers f dans la C^1 topologie. Par la formule de Pesin, $h_m(f_n) = L(f_n)$ et par la continuité en f, $h_m(f) = L(f)$.

The Lyapunov exponents of a diffeomorphism f of a compact manifold M are defined by the Oseledets theorem which states that, for any invariant probability measure μ, for almost all points $x \in M$ there exist numbers $\lambda_1(x) > \lambda_2(x) > \cdots > \lambda_{k(x)}(x)$ (Lyapunov exponents) and a unique splitting $T_xM = E_1(x) \oplus \cdots \oplus E_{k(x)}(x)$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log\left\| Df^n(x)v \right\| = \lambda_i(x)$$

for all $0 \neq v \in E_i(x)$, $1 \leqslant i \leqslant m$. The characteristic exponents defined as above are related to the entropy of f. For example for any invariant measure ν and $f \in C^1$, let $\chi(x) := \sum_{\lambda_i>0} \lambda_i$, then by a result of Ruelle:

$$h_\nu(f) \leqslant \int_M \chi \,\mathrm{d}\nu.$$

An estimation from below in terms of positive Lyapunov exponents is not true for general invariant measures, but if the measure m is absolutely continuous with respect to the Lebesgue measure of M, Pesin's formula states that for m-preserving diffeomorphisms with Hölder continuous derivative, $f \in \mathrm{Diff}_m^{1+\alpha}(M)$

$$h_m(f) = \int_M \chi \,\mathrm{d}m.$$

In fact this entropy formula holds for a larger class of measures [3]. However, the regularity of f is always used to get results of lower bounds for entropy.

We are going to show that if $\dim(M)=2$ then there exists a residual subset in $\mathrm{Diff}^1_m(M)$ such that the diffeomorphisms in this subset satisfy the Pesin's entropy formula.

THEOREM 0.1. – *There exists a* C^1*-residual subset* $\mathcal{G}\subset \mathrm{Diff}^1_m(M)$ *such that any* $f\in\mathcal{G}$ *satisfy the Pesin's entropy formula and* $\mathcal{G}$ *strictly contains* $\bigcup_{\alpha>0}\mathrm{Diff}^{1+\alpha}_m$.

The key idea is to prove that the set of the continuity points of the following two functions, $L(\cdot)$ and $h_m(\cdot)$ is residual in C^1 topology. As we are considering volume preserving diffeomorhisms in dimension two, there exists at most one positive Lyapunov exponent. Define:

- $L(f)=\int_M\sum_{\lambda_i>0}\lambda_i(x)\,\mathrm{d}m$ for $f\in\mathrm{Diff}^1_m(M)$ and
- $h_m(f)=$ the metric entropy of f for any $f\in\mathrm{Diff}^1_m(M)$.

Now we proceed by using the entropy formula for diffeomorphisms in $\bigcup\mathrm{Diff}^{1+\alpha}_m(M)$. Let f be a continuity point for $L(\cdot)$ and $h_m(\cdot)$. By density of $\mathrm{Diff}^{1+\alpha}_m(M)$ in $\mathrm{Diff}^1_m(M)$ proved in [4], there is a sequence $f_n\in\mathrm{Diff}^{1+\alpha}_m(M)$ such that f_n converges to f in C^1 topology. By Pesin's formula, $h_m(f_n)=L(f_n)$ and by continuity at f, $h_m(f)=L(f)$.

1. Continuity points of $L(f)$ and $h_m(f)$

The continuous dependence of Lyapunov exponents on diffeomorphism is an important problem. In fact let $\lambda_1(x,f)\geqslant\lambda_2(x,f)\geqslant\cdots\geqslant\lambda_d(x,f)$ denotes all Lyapunov exponents of f and $\Lambda_i(f)=\int_M\sum_{j=1}^i\lambda_j$ (average of sum of the i-greatest exponents) then it is well known that $f\to\Lambda_i(f)$ is an upper semi-continuous function.

LEMMA 1.1. – *The application* $f\to\Lambda_i(f)$ *is upper semi-continuous for* $f\in\mathrm{Diff}^1_m(M)$.

Proof. – By an standard argument we see that

$$\Lambda_i(f)=\inf_{n\geqslant1}\frac{1}{n}\int_M\log\bigl\|\wedge^i\bigl(Df^n(x)\bigr)\bigr\|\,\mathrm{d}m(x).$$

In fact, to see why the limit is substituted by infimum, observe that the sequence

$$a_n=\int_M\log\bigl\|\wedge^i\bigl(Df^n(x)\bigr)\bigr\|\,\mathrm{d}m(x)$$

is subadditive, i.e., $(a_{n+m}\leqslant a_n+a_m)$ and consequently $\lim a_n/n=\inf(a_n/n)$. Now, as $a_n(f)$ varies continuously with f in C^1 topology and the infimum of continuous functions is upper semi-continuous, the proof of the lemma is complete. □

Let us show that $L(f)$ is an upper semi-continuous function in $\mathrm{Diff}^1_m(M)$ independent of the dimension of M.

LEMMA 1.2. – *Let* $L(x,f)=\sum_{\lambda_i\geqslant0}\lambda_i(x,f)$ *then* $f\to L(f)=\int_M L(x,f)\,\mathrm{d}m(x)$ *is upper semi-continuous.*

Proof. – Observe that the proof of this lemma for two dimensional case is the direct consequence of Lemma 1.1. In fact for any $x\in M$, $\lim_{n\to\infty}\frac{1}{n}\log\|\wedge^p Df^n(x)\|$ exists and is equal to $\lambda_1+\lambda_2(x)+\cdots+\lambda_p(x)$. This functions varies upper semi-continuously with respect to f. From this we claim that for any x the function $f\to\sum_{\lambda_i\geqslant0}\lambda_i(f,x)$ is upper semi-continuous. Because let $f\in\mathrm{Diff}^1_m(M)$ and for $x\in M$, $\lambda_1(x)\geqslant\cdots\geqslant\lambda_p(x)\geqslant0\geqslant\lambda_{p+1}(x)\geqslant\cdots\geqslant\lambda_d(x)$. Take U_ε a neighborhood of f such that for all

$1 \leqslant k \leqslant d$ and any $g \in U_\varepsilon$

$$\sum_{i=1}^{k} \lambda_i(x, g) \leqslant \sum_{i=1}^{k} \lambda_i(x, g) + \varepsilon. \tag{1}$$

This is possible by means of Lemma 1.1. Now take any such g and let $\lambda_1(x) \geqslant \cdots \geqslant \lambda'_p(x) \geqslant 0 \geqslant \lambda_{p'+1} \geqslant \cdots \geqslant \lambda_d(x)$ for some $1 < p' < d$. Using (1) we see that $\sum_{i=1}^{p'} \lambda_i(x, g) \leqslant \sum_{i=1}^{p} \lambda_i(x, f) + \varepsilon$ (consider the three cases $p' < p$, $p = p'$, $p < p'$) and the claim is proved.

Now we prove the lemma. By definition $L(x, g) \leqslant C$ for some uniform C in a neighborhood of f. Define

$$A_n = \left\{ x \in M;\ d(f, g) \leqslant \frac{1}{n} \Rightarrow L(x, g) - L(x, f) \leqslant \frac{\varepsilon}{2} \right\}.$$

As $m(\cup A_n) = 1$ then for some large n we have $m(A_n) \geqslant 1 - \varepsilon/(4C)$. So,

$$\int_M L(x, g) - L(x, f)\,\mathrm{d}m = \int_{A_n} L(x, g) - L(x, f)\,\mathrm{d}m + \int_{A_n^c} L(x, g) - L(x, f)\,\mathrm{d}m$$
$$\leqslant \frac{\varepsilon}{2} + 2C\frac{\varepsilon}{4C} = \varepsilon$$

and the proof of the lemma is complete. □

The upper semi-continuity is the key for the proof of our main theorem, because by a classical theorem in Analysis we know that the continuity points of a semi-continuous function on a Baire space is always a residual subset of the space (see, e.g., [2]).

The upper semi-continuity of $h_m(f)$ for f varying in $\mathrm{Diff}^1_m(M)$ is not known. In fact using Ruelles inequality and Pesin's equality we can show upper semi-continuity of $h_m(f)$ in the C^2 topology. (In this paper all C^2 statements can be replaced by $\mathrm{C}^{1+\alpha}$.) Let $g \in \mathrm{Diff}^2_m(M)$ be near enough to f, by semi continuity of $L(\cdot)$ and Pesin's equality in C^2 topology:

$$h_m(g) \leqslant L(g) \leqslant L(f) + \varepsilon = h_m(f) + \varepsilon.$$

So, we pose the following question:

QUESTION 1.3. – Is it true that $h_m(f)$ is an upper semi-continuous function with C^1 volume preserving diffeomorphisms as its domain?

However, we are able to show that at least in two dimensional case the continuity points of $h_m(f)$ is generic in C^1 topology.

PROPOSITION 1.1. – *The continuity points of the map* h_m: $\mathrm{Diff}^1_m(M) \to \mathbb{R}$ *is a residual set.*

Proof. – We use the result of Bochi [1] which gives a C^1 generic subset $\mathcal{G}' = A \cup Z$ such that any $g \in A$ is Anosov and for $g \in Z$ both Lyapunov exponents vanish almost everywhere. We show that $\mathcal{G}'$ contains a generic subset $\mathcal{G}$ and each diffeomorphism in $\mathcal{G}$ is a continuity point of h_m. Firstly we state the following lemma. □

LEMMA 1.4. – *Any* $f \in Z$ *is a continuity point of* h_m.

Proof. – Let g be near enough to f by Ruelle's inequality and upper semi-continuity of $L(\cdot)$ we get

$$h_m(g) \leqslant L(g) \leqslant L(f) + \varepsilon = \varepsilon.$$ □

Now we prove that $h_m : A \to \mathbb{R}$ is upper semi-continuous. As $A \subset \mathrm{Diff}^1_m(M)$ is open we conclude that the continuity points of $h_m|A$ is generic inside A.

PROPOSITION 1.5. – *h_m restricted to* C^1 *Anosov diffeomorphisms is upper semi-continuous.*

Proof. – To prove the upper semi-continuity of $h_m|A$ we recall the definition of $h_m(f)$. By a theorem of Sinai we know that that if $\mathcal{P}$ is a generating partition then

$$h_m(f) = h_m(f, \mathcal{P}) = \lim_{n\to\infty} \frac{1}{n} H_m\big(\mathcal{P} \vee f^{-1}(\mathcal{P}) \vee \cdots \vee f^{-n+1}(\mathcal{P})\big). \tag{2}$$

If f is Anosov then there is $\varepsilon > 0$ such that any g in a C^1 neighborhood of f is expansive with ε as expansivity constant. By the definition of generating partition any partition with diameter less than ε is generating and so we can choose a unique generating partition for a neighborhood of f. As m is a smooth measure we see that the function

$$f \to \frac{1}{n} H_m\big(\mathcal{P} \vee f^{-1}(\mathcal{P}) \vee \cdots \vee f^{-n+1}(\mathcal{P})\big) = \frac{1}{n} \sum_P m(P) \log\big(m(P)\big)$$

is continuous where the sum is over all elements of $\mathcal{P} \vee f^{-1}(\mathcal{P}) \vee \cdots \vee f^{-n+1}(\mathcal{P})$. The limit in (2) can be replaced by infimum and we know that the infimum of continuous function is upper semi-continuous. □

So, up to now we have proved that there exists a generic subset $A' \subset A$ such that the diffeomorphisms in $\mathcal{G} = A' \cup Z$ are continuity point of h_m. Now we claim that $\mathcal{G}$ is C^1 generic in $\mathrm{Diff}^1_m(M)$. To proof the above claim we show a general fact about generic subsets.

LEMMA 1.6. – *Let $A \cup Z$ be a generic subset of a topological space T where A is an open subset. If $A' \subset A$ is generic inside A then $A' \cup Z$ is also generic in T.*

Proof. – As a countable intersection of generic subsets is also generic, we may suppose that A' is open and dense in A. By hypothesis, $A \cup Z = \bigcap_n C_n$ where C_n are open and dense. So, we have

$$A' \cup Z = A' \cup \left(\bigcap C_n \cap A^c\right) = \bigcap_n (A' \cup C_n) \cap \left(A' \cup A^c\right). \tag{3}$$

First observe that each $A' \cup C_n$ is an open and dense subset and their intersection is generic. To complete the proof it is enough to show that $A' \cup (A^c)^\circ \subseteq A' \cup A^c$ is open and dense. Openness is obvious and denseness is left to reader as an easy exercise of general topology. □

So, as the intersection of generic subsets is again a generic set we conclude that there is generic subset of $\mathrm{Diff}^1_m(M)$ where the diffeomorphisms in this generic subset are the continuity point of both $L(\cdot)$ and $h_m(\cdot)$ and so for this generic subset the Pesin's entropy formula is satisfied.

To finish the proof of Theorem 0.1 we have to show that $\bigcup_{\alpha>0} \mathrm{Diff}^{1+\alpha}_m(M)$ is not a generic subset and so the generic subset of Theorem 0.1 gives us some more diffeomorphisms satisfying Pesin's formula than $\bigcup_{\alpha>0} \mathrm{Diff}^{1+\alpha}_m(M)$.

LEMMA 1.7. – $\mathrm{Diff}^{1+}_n(M) := \bigcup_{\alpha>0} \mathrm{Diff}^{1+\alpha}_m(M)$ *is not generic with* C^1 *topology.*

Proof. – We show that the complement of $\mathrm{Diff}^{1+}_m(M)$ is a generic subset and this implies that $\mathrm{Diff}^{1+}_m(M)$ can not be generic.

As in what follows we are working locally, one may suppose that $M = \mathbb{R}^2$. Let us define

$$\|f\|_\alpha = \operatorname*{Sup}_{x\neq y\in M} \frac{d(Df(x), Df(y))}{d(x, y)^\alpha}$$

and denote

$$H_{n,k} = \left\{ f \in \mathrm{Diff}^1_m(M),\ \|f\|_{1/n} > k \right\}.$$

By the above definition we get $\mathrm{Diff}^1_m(M) \setminus \mathrm{Diff}^{1+}_m(M) = \bigcap_{n,m\in\mathbb{N}} H_{n,m}$. To prove the lemma we claim that for any n, each $H_{n,k}$ is an open dense subset.

1.1. Openness

Let $f \in H_{n,k}$, by definition there exist x, y and $\eta > 0$ such that $d(Df(x), Df(y))/d(x,y)^\alpha > k + \eta$. Take any g, ε-near to f in C^1 topology by the triangular inequality we get:

$$\|g\|_{1/n} > \frac{d(Dg(x), Dg(y))}{d(x,y)^{1/n}} > \frac{d(Df(x), Df(y))}{d(x,y)^{1/n}} - \frac{2\varepsilon}{(d(x,y))^{1/n}}.$$

Taking ε small enough the above inequality shows that $\|g\|_{1/n} > k$ and the openness is proved.

1.2. Density

Let $f \in \mathrm{Diff}^1_m(M)$ we are going to find $g \in \mathrm{Diff}^1_m(M) \setminus \mathrm{Diff}^{1+}_m(M)$ such that g is near enough to f. For this purpose we construct $h \in \mathrm{Diff}^1_m(M) \setminus \mathrm{Diff}^{1+}_m(M)$ near enough to identity and then put $g = h \circ f$.

Considering local charts, it is enough to construct a C^1 volume preserving diffeomorphism $\tilde{I}$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that:

(1) $\tilde{I}$ is C^1 near to identity inside $B(0,\varepsilon)$ for small $\varepsilon > 0$.
(2) $\tilde{I}$ is identity outside the ball $B(0, 2\varepsilon)$.
(3) $\tilde{I} \in \mathrm{Diff}^1_m(\mathbb{R}^2) \setminus \mathrm{Diff}^{1+}_m(\mathbb{R}^2)$.

Let us parameterize $\mathbb{R}^2$ with polar (r,θ) coordinates and $\xi : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 bump function which is equal to one inside the ball $\{r < \varepsilon\}$ and vanishes outside the ball of radius 2ε. Consider the following C^1 but not $\mathrm{C}^{1+\alpha}$ (for any α) real diffeomorphism:

$$\eta(r) = \begin{cases} r + \dfrac{r}{\log 1/r} & \text{if } r > 0, \\ r & \text{if } r \leqslant 0 \end{cases}$$

and define $\tilde{I}(r,\theta) = (r, \theta + \xi(r)\eta(r)\theta_0)$ for small θ_0. The jacobian matrix of $\tilde{I}$ is

$$D\tilde{I} = \begin{pmatrix} 1 & 0 \\ \theta_0(\xi(r)\eta(r))' & 1 \end{pmatrix}$$

and it is obvious that $\tilde{I}$ is volume preserving and taking θ_0 and ε small enough it is near enough to identity in C^1 topology. □

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