Lower bounds for the counting function of resonances for a perturbation of a periodic Schrödinger operator by decreasing potential

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Abstract
We are interested here in the counting function of resonances $N(h)$ for a perturbation of a periodic Schrödinger operator $P_0$ by decreasing potential $W(hx)$ ($h \downarrow 0$). We obtain a lower bound for $N(h)$ near some singularities of the density of states measure, associated to the unperturbed Hamiltonian $P_0$. To cite this article: M. Dimassi, M. Mnif, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 1013–1016.
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Des minorations de la fonction de comptage de résonances pour une perturbation d’un opérateur de Schrödinger périodique par un potentiel décroissant

Résumé
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1. Introduction

The purpose of this paper is to give a lower bound for the counting function of resonances for the perturbed periodic Schrödinger operator:

$$P(h) = P_0 + W(hx), \quad P_0 = -\Delta + V(x) \quad (h \downarrow 0).$$

Here $V$ is $C^\infty$, real-valued and $\Gamma$-periodic with respect to a lattice $\Gamma = \bigoplus_{i=1}^m \mathbb{Z} e_i$ in $\mathbb{R}^n$. The potential $W$ is real-valued and satisfies:

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(H1) there exist positive constants $a$ and $C$ such that $W$ extends analytically to $\Gamma(a) := \{ z \in C^n; |\Im(z)| \leq a |\Re(z)| \}$ and

$$|W(z)| \leq C(z)^{-\tilde{n}}, \quad \text{uniformly on } z \in \Gamma(a), \tilde{n} > n.$$ (1)

where $\langle z \rangle = (1 + |z|^2)^{1/2}$. Here $\Re(z)$, $\Im(z)$ denote respectively the real part and the imaginary part of $z$.

For $k \in \mathbb{R}^n$, we define the operator $P_k$ on $L^2(\mathbb{R}^n/\Gamma)$ by:

$$P_k := (D_y + k)^2 + V(y).$$

The Floquet eigenvalues are the eigenvalues $\lambda_1(k) \leq \lambda_2(k) \leq \cdots$ of $P_k$ (enumerated according to their multiplicities). It is well known that [3]:

$$\sigma(P_0) = \sigma_{\text{ac}}(P_0) = \bigcup_{j \geq 1} \Lambda_j, \quad \Lambda_j = \lambda_j(\mathbb{R}^n/\Gamma^*).$$

Here $\Gamma^*$ is the dual lattice corresponding to $\Gamma$.

For $f \in C_0^\infty(\mathbb{R})$, we set

$$\langle \mu, f \rangle = \int [f(W(x)) - f(0)] \, dx,$$ (2)

$$\langle \omega, f \rangle = \sum_{j \geq 1} \int_{E^*} \int_{\mathbb{R}^n} [f(W(x) + \lambda_j(k)) - f(\lambda_j(k))] \, dk \, dx,$$ (3)

where $E^*$ is a fundamental domain of $\mathbb{R}^n/\Gamma^*$.

**Proposition 1.** – The functionals operators $\omega$ and $\mu$ are distributions on $\mathbb{R}$ of order $\leq 1$. Moreover, in $\mathcal{D}'(\mathbb{R})$, we have

$$\omega = d\rho * \mu.$$ (4)

Here

$$\rho(\lambda) := \frac{1}{(2\pi)^n} \sum_{j \geq 1} \int_{E^*} \int_{\mathbb{R}^n} \chi_{\{k \in E^*; \lambda_j(k) \leq \lambda\}} \, dk,$$ (5)

is the density of states measure associated to the unperturbed Hamiltonian $P_0$.

**Proof.** – Applying Taylor’s formula to the r.h.s. of (2), we obtain

$$|\langle \mu, f \rangle| \leq \sup |f'| \int |W(x)| \, dx,$$

which together with (1) implies that $\mu$ is a distribution of order $\leq 1$, with

$$\text{supp} \mu \subset [\inf W(x), \sup W(x)].$$

Consequently, $d\rho * \mu$ is well defined in $\mathcal{D}'(\mathbb{R})$. Using (2), (5) and the definition of the convolution we get easily (4).

When $V = 0$, it was proved by Sjöstrand [4] that if $0 < E \in \text{singsupp}_\mu(\mu)$, then the operator $P(h) = -\Delta + W(hx)$ has at least $C_\Omega r^{-n}$ resonances in any $h$-independent complex neighborhood $\Omega$ of $E$. Here $\text{singsupp}_\mu(\mu)$ denotes the analytic singular support of the distribution $\mu$.

Now let $I$ be an open bounded interval. Assume that for all $\lambda \in I$ the following assumption holds.
(H2) For all $k_0 \in \mathbb{R}^n / \Gamma^*$ with $\lambda_j(k_0) = \lambda$, the eigenvalue $\lambda_j(k_0)$ is simple and $d_\lambda(k_0) \neq 0$.

The case $V \neq 0$ was recently studied by Dimassi and Zerzeri [1]. Under the assumption (H2) they obtained the same lower bound as in [4] near $E \in \text{singsupp}_a(\omega) \cap I$. Surely, in this case $\rho$ is more complicated and $\text{singsupp}_a(\omega)$ will depend on both $\text{singsupp}_a(d\rho)$ and $\text{singsupp}_a(\rho(\lambda))$.

We recall that, when $V = 0$, $\rho(\lambda) = (2\pi)^{-n} \text{vol}(B_{R_0}(0, 1)) \max(\lambda, 0)^{n/2}$. This fact permitted to Sjöstrand to prove that $\text{singsupp}_a(d\rho \ast \mu) = \text{singsupp}_a(\rho(\lambda))$.

In this Note we will use the simple representation of $\omega$ given by Proposition 1 to get a lower bound near some singularities of $\rho(\lambda)$. More precisely we study resonances generated by analytic singularities of $\mu$ near the edge of bands or near some singularities of $\rho$ due to the band crossings.

2. Lower bounds of the counting function near the edges of bands

The following result is a consequence of Morse lemma.

**Lemma 2.** Let $e_0 \in \sigma(P_0)$. We assume that:

(i) If $\lambda_j(k) = e_0$, then $\lambda_j(k)$ is a simple eigenvalue of $P_k$.

(ii) There exist $i_0$ and $k_0$ such that $\lambda_{i_0}(k_0) = e_0$, $\nabla \lambda_{i_0}(k_0) = 0$, $\pm \partial^2 \lambda_{i_0}(k_0) > 0$ and $\nabla \lambda_{i_0}(k) \neq 0 \forall k \in E^*$, $k \neq k_0$.

(iii) For all $k \in \lambda_j^{-1}[e_0]$ and all $i \neq i_0$, $\nabla \lambda_i(k) \neq 0$.

Then there exists an open connected neighborhood $J$ of $e_0$ such that

$$\rho(e) = f(e - e_0) + H((e - e_0) \sqrt{e - e_0}), \quad \forall e \in J,$$

where $f$ and $g_{\pm}$ are $C^\infty$ and $g_{\pm}(0) = 0, \ldots, g_{\pm}^{(n-1)}(0) = 0 \notin (n, g_{\pm}^{(n)}(0) \neq 0$. Here, $+(-)$ corresponds to a local minimum (maximum respectively).

Using (4) and Lemma 2, we obtain:

**Theorem 3.** Let $e_0$ and $J$ be as above, and let $\lambda \in (e_0 + \text{singsupp}_a(\mu))$. We assume that $\lambda$ satisfies (H2) and that $(\lambda - \text{supp}(\mu)) \subset J$. Then for all $h$-independent complex neighborhoods $\Omega$ of $\lambda$, there exist $h_0 = h(\Omega) > 0$ sufficiently small and $C = C(\Omega) > 0$ such that for $h \in [0, h_0]$,

$$\# \{z \in \Omega; z \in \text{Res}(P(h)) \} \geq C h^{-n}.$$

**Remark 4.** The assumption $(\lambda - \text{supp}(\mu)) \subset J$ ensures that, in the study of $d\rho \ast \mu$ near $\lambda$, one only needs the value of $\rho$ in $J$ given by (6). Hence, using (6) and Proposition 1, we show that $\lambda \in \text{singsupp}_a(\omega)$. Therefore, Theorem 3 follows from the result of Dimassi and Zerzeri [1].

3. Lower bounds near singularities due to band crossings

In this subsection we study resonances near singularities of $\rho(\lambda)$ generated by a band crossings. We will only consider the two dimensional case. With similar assumptions, one can treat the case $n \geq 2$.

We assume that $\lambda_j$ is a double eigenvalues $\lambda_{j-1}(k_0) < \lambda_j(k_0) = e_0 = \lambda_{j+1}(k_0) < \lambda_{j+2}(k_0)$ and that for all $k \neq k_0$ such that $\lambda_j(k) = e_0$, $\lambda_j(k)$ is simple and $\nabla \lambda_j(k) \neq 0$.

Since $P_k$ is analytic in $k$, this implies that for $|k - k_0| \leq \delta$ (with $\delta$ small enough), the span $V(k)$, of the eigenvectors of $P_k$ corresponding to eigenvalues in the set $\mathcal{E}; |e - e_0| \leq \delta$ has a basis $\psi_j(x, k), \psi_{j+1}(x, k)$, which is orthonormal and real analytic in $k$. The restriction of $P_k$ to $V(k)$ has the matrix

$$\begin{pmatrix}
a(k) & b(k) \\
b(k) & \beta(k)
\end{pmatrix}.$$
which can be written

\[
\begin{pmatrix}
  a(k) + c(k) & b_1(k) - ib_2(k) \\
  b_1(k) + ib_2(k) & a(k) - c(k)
\end{pmatrix},
\]

where \( a(k) = \alpha(k) + \beta(k)/2 \), \( c(k) = \alpha(k) - \beta(k)/2 \), \( b_1(k) \) and \( b_2(k) \) are real valued. Next, the periodic potential is assumed to have the symmetry \( V(x) = V(-x) \). This symmetry is typical of metals. This symmetry forces \( b(k) \) to be real valued (i.e., \( b_2(k) = 0 \)). Consequently, near \( k_0 \) we have

\[
\lambda_j(k) = a(k) - \sqrt{c_2^2(k) + b_2^2(k)}, \quad \lambda_{j+1}(k) = a(k) + \sqrt{c_2^2(k) + b_2^2(k)}.
\]

We assume that \( \nabla b(k_0), \nabla c(k_0) \) are independent. Since \( n = 2 \), \( (\nabla b(k_0), \nabla c(k_0)) \) is a basis in \( \mathbb{R}^2 \). Set \( \nabla a(k_0) = \alpha_1 \nabla b(k_0) + \alpha_2 \nabla c(k_0) \).

The following result was proved in [2].

**Lemma 5 ([2]).** If \( \alpha_1^2 + \alpha_2^2 < 1 \), then there exist an open connected neighborhood \( J \) of \( e_0 \) and \( C^\infty \) functions \( f \) and \( g \) such that

\[
\rho(e) = f(e) + \left( H(e - e_0) - H(-(e - e_0)) \right) g(e),
\]

with \( g''(e_0) \neq 0 \), \( \forall e \in J \).

**Theorem 6.** Let \( J \) be an open interval in which (7) is valid. Let \( \lambda \in (e_0 + \text{supp}(\mu)) \) be satisfying (H2). We assume that \( (\lambda - \text{supp}(\mu)) \subset J \). Then for all \( h \)-independent complex neighborhoods \( \Omega \) of \( \lambda \), there exist \( h_0 = h(\Omega) > 0 \) sufficiently small and \( C = C(\Omega) > 0 \) such that for \( h \in [0, h_0] \),

\[
\# \{ z \in \Omega; z \in \text{Res}(P(h)) \} \geq C h^{-n}.
\]

**References**


