Examples of wandering domains in $p$-adic polynomial dynamics

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Received 5 July 2002; accepted 20 August 2002

Note presented by Jean-Christophe Yoccoz.

Abstract

For any prime $p > 0$, we construct $p$-adic polynomial functions in $\mathbb{C}_p[z]$ whose Fatou sets have wandering domains. To cite this article: R.L. Benedetto, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 615–620.

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Exemples des domaines errants dans la dynamique polynôme $p$-adique

Résumé


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Version française abrégée

Soit $p > 0$ un nombre premier fixé, soit $\overline{\mathbb{Q}}_p$, une clôture algébrique du corps $\mathbb{Q}_p$ des nombres rationnels $p$-adiques, et soit $\mathbb{C}_p$ le complété de $\overline{\mathbb{Q}}_p$ pour la valeur absolue $p$-adique, notée $| \cdot |$. Pour une fraction rationnelle $\phi(z) \in \mathbb{C}_p(z)$, la dynamique de $\phi$ opérant sur $\mathbb{P}^1(\mathbb{C}_p) = \mathbb{C}_p \cup \{\infty\}$ est analogue à la dynamique des fractions rationnelles complexes sur la sphère de Riemann ; voir [1, 2, 4, 6, 7, 9–11], par exemple. En particulier, on peut définir les ensembles $p$-adiques de Julia, les ensembles de Fatou, et les composantes des ensembles de Fatou, qui se comportent de façon semblable à leurs contre-parties complexes. Bien que quelques résultats partiels suggèrent que l’ensemble de Fatou de $\phi \in \overline{\mathbb{Q}}_p(z)$ ne puisse pas avoir de domaine errant, nous démontrons dans cet article qu’il y a des polynômes dans $\mathbb{C}_p[z]$ avec des domaines errants. Plus précisément, nous démontrons qu’il existe $a \in \mathbb{C}_p$ tel que la fonction $\phi_a$ définie par l’équation (1) a un domaine errant.

Pour $x \in \mathbb{C}_p$ et $r > 0$, on note le disque ouvert $D_r(x) = \{ y \in \mathbb{C}_p : |y-x| < r \}$ et le disque fermé $\overline{D}_r(x) = \{ y \in \mathbb{C}_p : |y-x| \leq r \}$. Nous considérons $a \in \mathbb{C}_p$ tel que $|a| = |p|^{-(p-1)} > 1$. Dans ce cas, $\phi_a$ augmente des distances dans $D_1(1)$ par un facteur de $|a|$ ; voir équation (6). Nous observons avec l’équation (5) que

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S1631-073X(02)02531-1/FLA

615
The main result of this paper implies that the first hypothesis (that the coefficients lie in $\overline{\mathbb{D}}_r(0)$, where $r = |p|^{1-p^{-m}} < 1$, but $\phi_m(\overline{\mathbb{D}}_r(0)) = \overline{D}_1(0)$. De plus, on peut étudier la variation de la famille en employant l’équation (4).

Avec ces outils, nous pouvons construire une valeur $a \in \mathbb{C}_p$ et un point $x \in D_1(0)$ telle que l’orbite $\{\phi_j^a(x)\}_{j \geq 0}$ suit le modèle dans l’équation (7). Dans cette équation, un 0 ou la $j$-ème position indique que $\phi_j^{-1}(x) \in D_1(0)$, et un 1 indique que $\phi_j^{-1}(x) \in D_1(1)$ ; et pour $i \geq 0$, on a $M_i = 2i$ et $m_i = 2p + 2(p - 1)i$. Comme la contraction de $\phi_m$ dans $D_1(0)$ est supérieure à l’expansion de $\phi_{M+1}$ dans $D_1(1)$, on voit que $x$ est contenu dans un disque errant de $\phi_a$.

1. Introduction

Fix a prime number $p > 0$, and let $\mathbb{Q}_p$ denote the field of $p$-adic rationals, formed by completing $\mathbb{Q}$ with respect to the unique absolute value satisfying $|p| = 1/p$. Let $\overline{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$, and let $\mathbb{C}_p$ denote the completion of $\overline{\mathbb{Q}}_p$. The absolute value $|\cdot|$, which extends canonically to $\mathbb{C}_p$, is non-Archimedean, meaning that it satisfies the ultrametric triangle inequality $|x + y| \leq \max\{|x|, |y|\}$. Both $\mathbb{Q}_p$ and $\mathbb{C}_p$ are complete with respect to $|\cdot|$, though $\mathbb{Q}_p$ is not. Note that $\mathbb{Z} \subset \mathbb{Q}_p \subset \mathbb{C}_p$; every $n \in \mathbb{Z}$ satisfies $|n| \leq 1$, with $|p| < 1$. See [5,8,12] for more general background on $p$-adic fields.

Although $\mathbb{Q}_p$ is locally compact, $\overline{\mathbb{Q}}_p$ and $\mathbb{C}_p$ are not. Still, $\mathbb{C}_p$ is algebraically closed and complete, analogous to $\mathbb{C}$; the projective line $\mathbb{P}^1(\mathbb{C}_p) = \mathbb{C}_p \cup \{\infty\}$ is a non-Archimedean version of the Riemann sphere. The dynamics of rational functions $\phi(z) \in \mathbb{C}_p(z)$ acting on $\mathbb{P}^1(\mathbb{C}_p)$ have exhibited many parallels with the existing theory of complex dynamics; see [1,2,4,6,7,9–11], for example. The failure of local compactness, and hence of the Arzelà–Ascoli theorem, means that $p$-adic Fatou and Julia sets should be defined in terms of equicontinuity, rather than normality.

Topologically, $\mathbb{P}^1(\mathbb{C}_p)$ and its subsets are totally disconnected. Nevertheless, the author [1,2] and Rivera-Letelier [9,11] have developed several related definitions of components of $p$-adic Fatou sets which behave as useful analogs of connected components of complex Fatou sets.

The author has proven [11] that if $\phi \in \mathbb{Q}_p(z)$ (acting on the full $\mathbb{P}^1(\mathbb{C}_p)$) has nonempty Julia set $\mathcal{J}$ with no recurrent critical points of order divisible by $p$ in $\mathcal{J}$, then the Fatou set of $\phi$ has no wandering domains. (In fact, the proof in [1] applies equally well to any of the definitions of components even if $\mathcal{J}$ is empty.) The main result of this paper implies that the first hypothesis (that the coefficients lie in $\mathbb{Q}_p$) cannot be removed.

**Theorem 1.1.** – There exists $a \in \mathbb{C}_p$ such that the Julia set $\mathcal{J}$ of

$$
\phi_a(z) = (1 - a)z^{p+1} + az^p
$$

is nonempty; the Fatou set $\mathcal{F}$ of $\phi_a$ has a wandering domain, and all critical points of $\phi_a$ lie in $\mathcal{F}$.

Compared to Sullivan’s complex No Wandering Domains Theorem [13], Theorem 1.1 gives a sharp contrast between non-Archimedean and complex dynamics. Moreover, our result also provides a counterexample disproving Rivera-Letelier’s Conjecture de Non-Errance and his related statement on Structure Conjecturale de l’Ensemble de Fatou in [9, Section 4.3]. However, both of those conjectures may still be true if the hypothesis that all coefficients lie in $\mathbb{Q}_p$ is added; see the conjecture in [1, Section 1].

A generalization of the method of this paper can actually be used to prove the density of parameters for which $\phi_a$ has a wandering domain in the set $\{a \in \mathbb{C}_p : |a| > 1\}$. The argument works for any algebraically closed complete non-Archimedean field with the property that $|p| < 1$. However, in the interest of clarity, we restrict our attention here to announcing the existence of $p$-adic wandering domains, and we leave the generalizations to a forthcoming paper [3].
2. Disks

We will denote the closed disk of radius \( r > 0 \) about a point \( a \in \mathbb{C}_p \) by \( \overline{D}_r(a) \), and the open disk by \( D_r(a) \). We recall some basic properties of non-Archimedean disks. Every disk is both open and closed as a topological set. Any point in a disk \( U \) is a center, but the radius of \( U \) is a well-defined real number, being the same as the diameter of \( U \). If two disks in \( \mathbb{C}_p \) intersect, then one contains the other. If \( f \in \mathbb{C}_p[z] \) is a non-constant polynomial, and if \( U \subset \mathbb{C}_p \) is a disk, then \( f(U) \) is also a disk. If \( a, b \in \mathbb{C}_p, r > 0, \) and \( f \in \mathbb{C}_p[z] \) with \( f(a) = b \), then \( f \) maps \( \overline{D}_r(a) \) bijectively onto \( \overline{D}_r(b) \) if and only if for every \( x \in \overline{D}_r(a) \),

\[
|f(x) - f(a)| = \frac{s}{r} |x - a|.
\]

We also recall Hsia’s criterion \([7]\) for equicontinuity, which is a non-Archimedean analogue of the Montel–Carathéodory theorem. Hsia stated his result for arbitrary meromorphic functions on more general non-Archimedean fields, but for simplicity, we rephrase it for our special case.

**Theorem 2.1 (Hsia).** Let \( F \) be a family of rational functions on a disk \( U \subset \mathbb{C}_p \), and suppose that there are two distinct points \( a_1, a_2 \in \mathbb{P}_1(\mathbb{C}_p) \) such that for all \( f \in F, x \in U \), and \( i = 1, 2 \), we have \( f(x) \neq a_i \). Then \( F \) is an equicontinuous family.

3. The family

We consider the family \( \{\phi_a\} \) defined in equation (1), with \( |a| = |p|^{-(p-1)} > 1 \). For any such \( a, \phi_a \) has a superattracting (hence Fatou) fixed point at \( z = 0 \), and a repelling (hence Julia) fixed point at \( z = 1 \). Furthermore, it is not difficult to see that the filled Julia set \( K \) (that is, the set of points not attracted to \( \infty \)) is completely contained in \( D_1(0) \cup D_1(1) \). The only critical points of \( \phi_a \) besides \( \infty \) lie in \( \overline{D}_1(p)(0) \), which is a bounded open set that maps into itself. Hence, all critical points are Fatou.

Fix \( a \in \mathbb{C}_p \) with \( |a| = |p|^{-(p-1)} \). If \( y_0 \in D_1(1) \) and \( |y_1 - y_0| < 1 \), then it is immediate from the definition of \( \phi_a \) and the ultrametric triangle inequality that

\[
|\phi_a(y_1) - \phi_a(y_0)| = |a| \cdot |y_1 - y_0|.
\]

(2)

If \(|p| < |y_0| < 1 \) and \(|y_1 - y_0| \leq |p|^2 \), then it is only slightly more difficult to show that

\[
|\phi_a(y_1) - \phi_a(y_0)| = |a| \cdot |y_0|^p \cdot |y_1 - y_0|.
\]

(3)

On the other hand, if we fix \( y_0 \in D_1(0) \) and \( y_1 \in D_1(1) \), and if we choose two parameters \( a, b \in \mathbb{C}_p \), then

\[
|\phi_a(y_0) - \phi_a(y_1)| = |y_0|^p \cdot |b - a| \quad \text{and} \quad |\phi_b(y_1) - \phi_a(y_1)| = |y_1 - 1| \cdot |b - a|.
\]

(4)

4. Local mapping properties of \( \phi_a^m \)

Let \( S = |p|^2 \). If \( x \in D_1(0) \) or \( x \in D_1(1) \), then using induction and Eqs. (2) and (3), we can easily prove the following statements concerning the next few iterates of \( x \).

**Lemma 4.1.** Let \( a \in \mathbb{C}_p \) with \( |a| = |p|^{-(p-1)} \). Let \( m \geq 1 \) and \( x \in \mathbb{C}_p \) with \(|x| \leq |p|^{1-p-m} \). Then for all \( 0 \leq i \leq m \),

\[
|\phi_a^i(x)| = |p|^{1-p} |x|^i, \quad \text{and for all } r \in (0, S], \quad \phi_a^i(D_r(x)) \subset \overline{D}_{r \cdot |p|^{1-p}}(\phi_a^i(x)),
\]

where \( e_i = p^{1-m} + p^{2-m} + \cdots + p^{i-m} < 2 \). In particular, if \(|x| = |p|^{1-p-m} \), then \(|\phi_a^m(x)| = 1 \),

\[
\phi_a^m(D_r(x)) \subset \overline{D}_{r \cdot |p|^{m-2}}(\phi_a^m(x)) \quad \text{for all } r \in (0, S],
\]

(5)

and \( \phi^i(x) \in D_1(0) \) for all \( 0 \leq i \leq m - 1 \).
Thus, the iterates of $x$ are all pushed away from 0, but the function $\phi_a^m$ is locally contracting.

On the other hand, all distances within $D_1(1)$ are stretched by a factor of exactly $|a|$, giving us the following simpler statement for that disk.

**Lemma 4.2.** Let $a \in \mathbb{C}_p$ with $|a| = |p|^{-(p-1)}$. Let $M \geq 1$ and $x \in \mathbb{C}_p$ with $|x-1| \leq |a|^{-M}$. Then for all $0 \leq i \leq M$,

$$|\phi_a^i(x) - 1| = |a|^i \cdot |x-1|,$$

and for all $r \in (0, |a|^{-M}]$, 

$$\phi_a^i(D_r(x)) = \overline{D}_{r |a^i|} (\phi_a^i(x)).$$

In particular, if $|x-1| = |a|^{-M}$, then $\phi_a^M(x) - 1 = 1$,

$$\phi_a^M(D_r(x)) = \overline{D}_{r |a^M|} (\phi_a^M(x)) \quad \text{for all } r \in (0, |a|^{-M}],$$

and $\phi(x) \in D_1(1)$ for all $0 \leq i \leq M - 1$.

5. Perturbations

Set the notation $\Phi_n(a, z) = \phi_a^n(z)$. For fixed $x \in \mathbb{C}_p$, $\Phi_n(\cdot, x)$ is a polynomial function of the parameter $a$.

The following lemmas show how the function behaves locally in certain circumstances.

**Lemma 5.1.** Let $a \in \mathbb{C}_p$ with $|a| = |p|^{-(p-1)}$. Let $M \geq 2$, let $n \geq 0$, and let $x \in \mathbb{C}_p$ satisfying $|\phi_a^n(x)| = |p|^{1-p^m}$. Let $A > |p|^{p-1}$ be a real number, let $\varepsilon \in (0, A^{-1}S]$, and suppose that

$$\Phi_n(D_a(x), x) \subseteq \overline{D}_{A \cdot |\phi_a^n(x)|} \quad \text{and} \quad A \leq |p|^{p+1-m}.$$  

Then $\Phi_{n+m}(\cdot, x)$ maps $\overline{D}_e(a)$ bijectively onto $\overline{D}_{e/\ell |a|}(\phi_a^{n+m}(x))$.

Note that the two displayed conditions in Lemma 5.1 say, first, that $A$ is large enough to bound the size of a certain image disk, and second, that $m$ is large enough to make $|p|^{-m+p+1}$ even larger than $A$.

**Lemma 5.2.** Let $a \in \mathbb{C}_p$ with $|a| = |p|^{-(p-1)}$. Let $M \geq 2$, let $n \geq 1$, and let $x \in \mathbb{C}_p$ satisfying $|\phi_a^n(x)| = |p|^{-p^m - 1}$. Let $\varepsilon \in (0, |a|^{-1}M]$. Suppose that $\Phi_n(\cdot, x)$ maps $D_a(x)$ bijectively onto $\overline{D}_{\varepsilon/\ell |a|}(\phi_a^n(x))$.

Then $\Phi_{n+m}(\cdot, x)$ maps $\overline{D}_e(a)$ bijectively onto $\overline{D}_{e/\ell |a|^{M-1}}(\phi_a^{n+m}(x))$.

We sketch the proofs as follows. Pick $b \in \overline{D}_e(a) \setminus \{a\}$, and for every $i \geq 0$, let $\delta_i = |\phi_a^i(x) - \phi_a^j(x)|$.

By the ultrametric triangle inequality, for $i \geq 1$ we have $\delta_i \leq \max\{B_i, C_i\}$, with equality if $B_i \neq C_i$, where

$$B_i = |\phi_b(\phi_a^{i+1}(x)) - \phi_a(\phi_a^{i+1}(x))|, \quad C_i = |\phi_b(\phi_a^{i+1}(x)) - \phi_a(\phi_a^{i+1}(x))|.$$  

Define

$$s_i = |b-a| \cdot \max\{A \cdot |p|^{-\varepsilon i}, |p|^{1-p^{m+i}}\}, \quad \text{and} \quad t_i = |b-a| \cdot |a|^{-1},$$

where $\varepsilon$ is as in the statement of Lemma 4.1.

For Lemma 5.1, we show by induction (using Eqs. (3) and (4) and Lemma 4.1) that $B_i, C_i \leq s_i$ for all $1 \leq i \leq m$. Then we observe that $C_m = B_m = s_m = |b-a|/|a|$, proving that $\delta_m = |b-a|/|a|$, as desired. Similarly, for Lemma 5.2, we show that $B_i < C_i = t_i$, for all $1 \leq i \leq M$. Thus, $\delta_M = t_M = |a|^{M-1} \cdot |b-a|$.

6. Proof of Theorem 1.1

Let $a_0 = p^{-(p-1)} \in \mathbb{C}_p$. For each $i \geq 0$, define $M_i = 2i$ and $m_i = 2p + 2(p-1)i$. Set $r_i = |p|^{1-p^{-m_i}}$ and $\varepsilon_i = |a_0|^{-1-M_i}$. By Lemma 4.1, any $y \in \mathbb{C}_p$ with $|y| = r_0$ satisfies $|\phi_a^{m_0}(y)| = 1$. Because $\phi_a^{m_0}(0) = 0$, it follows that $\phi_a^{m_0}(D_{r_0}(0)) \supset D_1(1)$. In particular, there is some $x \in D_{r_0}(0)$ with $\phi_a^{m_0}(x) = 1$. By the same lemma, we must have $|x| = r_0$. We will find $a \in \overline{D}_1(a_0)$ such that the orbit $\{\phi_a^j(x)\}_{j \geq 0}$ can be described by

$$0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 0, \ldots$$

with $m_0 \subseteq M_1 \subseteq m_1 \subseteq M_2 \subseteq m_2$.  

(7)
where a 0 in the \(j\)-th position in the sequence indicates that \(\phi_a^{j-1}(x) \in D_1(0)\), and a 1 indicates that \(\phi_a^{j-1}(x) \in D_1(1)\).

For \(i \geq 0\), define

\[
  n_i = \sum_{k=1}^{i} (m_{k-1} + M_k) = 2i + pi(i + 1), \quad \text{and} \quad N_i = n_i + m_i = p(i + 1)(i + 2).
\]

That is, \(n_i\) is the number of terms in (7) up to but not including the block of \(m_i\) 0’s, and \(N_i\) is the number of terms up to but not including the block of \(M_{i+1}\) 1’s.

For every \(i \geq 0\), we will find \(a_i \in \overline{D}_{\sigma_{i-1}}(a_{i-1})\) so that for every \(a \in \overline{D}_{\sigma_i}(a_i)\), the orbit \(\{\phi_a^n(x)\}\) follows (7) up to the \(j = N_i\) iterate, with \(\phi_{a_i}^{N_i}(x) = 1\) and such that

\[
  \Phi_{N_i}(\cdot, x) : \overline{D}_{\sigma_i}(a_i) \to \overline{D}_{\sigma_i/|a_i|}(1) \quad \text{is bijective.} \quad (8)
\]

Note that every \(a_i\) will lie in \(\overline{D}_{\rho_{a_0}}(a_0)\), and therefore \(|a_i| = |a_0| = |p|^{-(p-1)}\).

We proceed by induction on \(i\). For \(i = 0\), we already have \(\phi_{a_0}^{N_0}(x) = 1\), and by Lemma 4.1, the orbit \(\{\phi_{a_0}(x)\}\) follows (7) up to the \(N_0 = m_0\) iterate. By Lemma 5.1 (with \(n = n_0 = 0, m = m_0, a = a_0, A = |p|^{(p-1)}, \epsilon = \epsilon_0\)), condition (8) holds. Also, by Lemma 4.1, the orbit \(\{\phi_{a_0}(x)\}\) is correct up to \(j = N_0\) for every \(a \in \overline{D}_{\rho_{a_0}}(a_0)\). Hence, the \(i = 0\) case is already done.

For \(i \geq 1\), assume that we are given \(a_{i-1}\) with the desired properties. Let \(\rho = |a_0|^{1-M_i} \leq \epsilon_{i-1}\); then for every \(a \in \overline{D}_{\sigma_{i-1}}(a_{i-1})\), the orbit \(\{\phi_a^{N_i}(x)\}\) agrees with (7) up to \(j = N_{i-1}\). By Lemma 5.2 (with \(a = a_{i-1}, M = M_i, n = N_{i-1}, \epsilon = \rho\)), there exists \(c_1 \in \mathbb{C}_p\) such that \(|c_1 - a_{i-1}| = \rho\) and

\[
  \Phi_{N_i}(c_1, x) = 0 \quad \text{and} \quad \Phi_{N_i}(\cdot, x) : \overline{D}_{\sigma}(c_1) \to \overline{D}_{n_0}(0) \quad \text{is bijective,} \quad (9)
\]

where \(\sigma = r_i \cdot |a_0|^{1-M_i} \leq \epsilon_{i-1}\). By Lemma 4.2, the orbit \(\{\phi_a^{N_i}(x)\}\) is correct up to \(j = n_i\) for every \(a \in \overline{D}_{\rho_{c_1}}(c_1)\).

Choose \(c_2 \in \overline{D}_{\rho_{c_1}}(c_1)\) so that \(|\Phi_{n_i}(c_2, x)| = r_i\). By Lemma 4.1, \(|\Phi_{N_i}(c_2, x)| = 1\). Furthermore it is clear that \(\Phi_{N_i}(c_1, x) = 0\). Because the polynomial image of a disk is a disk, it follows that \(\Phi_{N_i}(\overline{D}_{\sigma_{i-1}}(c_1), x) \supset \overline{D}_{\gamma_{i-1}}(0)\). We may therefore choose \(a_i \in \overline{D}_{\sigma_{i-1}}(c_1)\) so that \(\Phi_{n_i}(a_i, x) = 1\).

By Eq. (9), the radius of \(\Phi_{n_i}(\overline{D}_{\sigma_{i-1}}(a_i))\) must be \(\epsilon_{i-1} \cdot |a_0|^{M_i-1} = S\). Therefore, by Lemma 5.1 (with \(n = n_{i-1}, m = m_{i-1}, a = a_i, A = |a_i|^{M_i-1}, \epsilon = \epsilon_{i-1}\)), condition (8) holds on \(\overline{D}_{\rho_{a_i}}(a_i)\). By Lemma 4.1, the orbit \(\{\phi_{a_i}^{N_i}(x)\}\) is correct up to \(j = N_i\) for every \(a \in \overline{D}_{\rho_{a_i}}(a_i)\). Our construction of \(a_i\) is complete.

The sequence \(|a_i| \geq 0\) is a Cauchy sequence, because for any \(0 \leq i \leq j\), we have \(|a_i - a_j| \leq \epsilon_i\), and \(\epsilon_i \to 0\). Therefore, the sequence has a limit \(a \in \mathbb{C}_p\), with \(|a - a_0| \leq \epsilon_0\). By construction, \(a \in \overline{D}_{\rho_{a_i}}(a_i)\) for every \(i \geq 0\); hence, the orbit \(\{\phi_{a_i}^{N_i}(x)\}\) follows (7) exactly. In light of Lemmas 4.1 and 4.2, we must have \(|\phi_{a_i}^{N_i}(x)| = |p|^{1-p^{-m_i}}\), and \(|\phi_{a_i}^{N_i}(x) - 1| = |a|^{-M_i+1}\), for any \(i \geq 0\). We only need to verify that \(\phi_a\) has a wandering domain containing \(x\).

Let \(U = \overline{D}_S(x)\); we shall show that \(U\) is contained in a wandering domain of the Fatou set \(\mathcal{F}\) of \(\phi_a\). Every iterate \(U_{ni} = \phi_a^{ni}(U)\) is a disk; we claim that for any \(i \geq 0\), the radius of \(U_{ni}\) is at most \(S = |p|^2\), and the radius of \(U_{N_i}\) is at most \(|a|^{-M_i+1}S\). The claim is easily proven by induction, as follows. For \(i = 0\), \(U_0 = U_0 = U\), and by Eq. (5), \(|U_0|\) has radius at most \(|p|^{m_0-2}S = |a|^{-M_1}S\). For \(i \geq 1\), we assume the radius of \(U_{N_{i-2}}\) is at most \(|a|^{-M_i}S\). By Eq. (6), the radius of \(U_{N_i}\) is at most \(S\); and by Eq. (5), the radius of \(U_{N_i}\) is at most \(|p|^{m_0-2}S = |a|^{-M_{i+1}}S\).

In particular, no \(U_{ni}\) contains the point 1; and because 1 is fixed, it follows that no \(U_{ni}\) contains 1. Clearly, no \(U_{ni}\) contains \(\infty\) either. By Hsia’s theorem, then, the family \(\{\phi_{a_i}\}\) is equicontinuous on \(U\), and therefore \(U \subset \mathcal{F}\).

By any of the definitions of components in [1,2,9,11], the component \(V\) of \(\mathcal{F}\) containing \(U\) must be a disk (see, for example, [2, Theorem 5.4.d]). Again, no iterate of \(V\) can contain 1, and therefore the symbolic
dynamics of any point in \( V \) are also described by Eq. (7). Because those dynamics are not preperiodic, it follows that \( V \) must be wandering.

**Acknowledgements.** The research for this paper was supported by NSF grant DMS-0071541. Many thanks to Bob Devaney, J.-C. Yoccoz, and especially to Juan Rivera-Letelier for their helpful comments and suggestions concerning the exposition of this paper.

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