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Équations aux dérivées partielles/*Partial Differential Equations*

Uniqueness of the blow-up boundary solution of logistic equations with absorbtion *

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Abstract Let Ω be a smooth bounded domain in $\mathbb{R}^N$. Assume $f \in C^1[0,\infty)$ is a non-negative function such that $f(u)/u$ is increasing on $(0,\infty)$. Let a be a real number and let $b \geqslant 0$, $b \not\equiv 0$ be a continuous function such that $b \equiv 0$ on $\partial\Omega$. We study the logistic equation $\Delta u + au = b(x)f(u)$ in Ω. The special feature of this work is the uniqueness of positive solutions blowing-up on $\partial\Omega$, in a general setting that arises in probability theory. ***To cite this article: F.-C. Cîrstea, V. Rădulescu, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 447–452.***

Unicité de la solution explosant au bord pour équations logistiques avec absorption

Résumé Soit Ω un domaine borné et régulier de $\mathbb{R}^N$. On suppose que $f \in C^1[0,\infty)$ est une fonction non-negative telle que $f(u)/u$ soit strictement croissante sur $(0,+\infty)$. Soit a un réel et $b \geqslant 0$, $b \not\equiv 0$ une fonction continue sur $\overline{\Omega}$. On étudie l'équation logistique $\Delta u + au = b(x)f(u)$ sur Ω. Le but de cette Note est de montrer l'unicité de la solution explosant au bord de Ω dans un contexte général, qui apparaît en théorie des probabilités. ***Pour citer cet article : F.-C. Cîrstea, V. Rădulescu, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 447–452.***

Version française abrégée

Soit $\Omega \subset \mathbb{R}^N$ $(N \geqslant 3)$ un domaine borné et régulier, a un paramètre réel et $b \in C^{0,\mu}(\overline{\Omega})$, $\mu \in (0,1)$, $b \geqslant 0$, $b \not\equiv 0$ dans Ω. On considère l'équation logistique

$$\Delta u + au = b(x)f(u) \quad \text{dans } \Omega, \tag{1}$$

où $f \in C^1[0,\infty)$ satisfait

$$f \geqslant 0 \text{ et } f(u)/u \text{ est strictement croissante sur } (0,+\infty). \tag{A_1}$$

Soit

$$\Omega_0 := \operatorname{int}\{x \in \Omega \colon\ b(x) = 0\}$$

et on suppose que $\partial\Omega_0$ est régulier (éventuellement vide), $\overline{\Omega}_0 \subset \Omega$ et $b > 0$ sur $\Omega \setminus \overline{\Omega}_0$. On désigne par $\lambda_{\infty,1}$ la première valeur propre (avec conditions de Dirichlet) de l'opérateur $(-\Delta)$ dans Ω_0, avec la convention $\lambda_{\infty,1} = +\infty$ si $\Omega_0 = \emptyset$.

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On dit que u est une solution *large* (*explosive*) de (1) si $u \geqslant 0$ dans Ω et $u(x) \to \infty$ si $d(x) := \mathrm{dist}(x, \partial\Omega) \to 0$.

Soit $D > 0$ et $R:[D, \infty) \to (0, +\infty)$ une fonction mesurable. On dit que R a une variation régulière d'indice $\rho \in \mathbb{R}$ (notation : $R \in \mathbb{R}_\rho$) si $\lim_{u\to\infty} R(\xi u)/R(u) = \xi^\rho$, pour chaque $\xi > 0$ (voir [11]).

Soit $\mathcal{K}$ l'ensemble des fonctions $k:(0, \nu) \to (0, +\infty)$ (pour un certain ν), de classe C^1, croissantes, telles que $\lim_{t\to 0^+}(\int_0^t k(s)\,\mathrm{d}s/k(t))^{(i)} := \ell_i$, pour $i = \overline{0, 1}$.

On démontre le résultat suivant.

THÉORÈME 1. – *Supposons que la fonction f satisfait la condition (A_1) et que f' est une fonction à variation régulière d'indice $\rho \neq 0$. De plus, on suppose que le potentiel b vérifie*

$$b(x) = ck^2\big(d(x)\big) + \mathrm{o}\big(k^2\big(d(x)\big)\big) \quad \text{si } d(x) \to 0, \text{ avec } c > 0 \text{ et } k \in \mathcal{K}. \tag{B}$$

Alors, pour chaque $a \in (-\infty, \lambda_{\infty,1})$, l'équation (1) *admet une unique solution explosive u_a. On a, de plus,*

$$\lim_{d(x)\to 0} \frac{u_a(x)}{h(d(x))} = \xi_0, \quad \text{où } \xi_0 = \left(\frac{2 + \ell_1\rho}{c(2+\rho)}\right)^{1/\rho}$$

et la fonction h est définie par

$$\int_{h(t)}^\infty \frac{\mathrm{d}s}{\sqrt{2F(s)}} = \int_0^t k(s)\,\mathrm{d}s, \quad \forall t \in (0, \nu).$$

Let $\Omega \subset \mathbb{R}^N$ $(N \geqslant 3)$ be a smooth bounded domain. Consider the semilinear elliptic equation

$$\Delta u + au = b(x)f(u) \quad \text{in } \Omega, \tag{1}$$

where a is a real parameter and $b \in C^{0,\mu}(\overline{\Omega})$, for some $\mu \in (0, 1)$, such that $b \geqslant 0$, $b \not\equiv 0$ in Ω.

Suppose that $f \in C^1[0, \infty)$ satisfies

$$f \geqslant 0 \text{ and } f(u)/u \text{ is increasing on } (0, \infty). \tag{A_1}$$

In the study of positive solutions for (1), subject to the homogeneous Dirichlet boundary condition, an important role is played by the zero set (see [1])

$$\Omega_0 := \mathrm{int}\big\{x \in \Omega\colon\ b(x) = 0\big\}.$$

We shall assume throughout that Ω_0 is smooth (possibly empty), $\overline{\Omega}_0 \subset \Omega$, and $b > 0$ in $\Omega \setminus \overline{\Omega}_0$.

By a *large* (*explosive*) solution of (1) we mean a solution u of (1) such that $u \geqslant 0$ in Ω and $u(x) \to \infty$ as $d(x) := \mathrm{dist}(x, \partial\Omega) \to 0$. In [3,4] we study the existence of large solutions for (1) and also deduce several existence and unicity results for a related problem. Note that any large solution of (1) is *positive* and it can exists only if the Keller–Osserman condition holds (see [4])

$$\int_1^\infty \frac{\mathrm{d}t}{\sqrt{F(t)}} < \infty, \quad \text{where } F(t) = \int_0^t f(s)\,\mathrm{d}s. \tag{A_2}$$

Let H_∞ define the Dirichlet Laplacian on the set $\Omega_0 \subset \Omega$ as the unique self-adjoint operator associated to the quadratic form $\psi(u) = \int_\Omega |\nabla u|^2\,\mathrm{d}x$ with form domain

$$H_D^1(\Omega_0) = \big\{u \in H_0^1(\Omega)\colon\ u(x) = 0 \text{ for a.e. } x \in \Omega \setminus \Omega_0\big\}.$$

If $\partial\Omega_0$ satisfies an exterior cone condition, then $H_D^1(\Omega_0)$ coincides with $H_0^1(\Omega_0)$ and H_∞ is the classical Laplace operator with Dirichlet condition on $\partial\Omega_0$.

Let $\lambda_{\infty,1}$ be the first Dirichlet eigenvalue of H_∞ in Ω_0. We understand $\lambda_{\infty,1} = +\infty$ if $\Omega_0 = \emptyset$.

The main result in [3] asserts that Eq. (1) has a large solution if and only if $a \in (-\infty, \lambda_{\infty,1})$.

The special feature of this paper is the uniqueness of large solutions of (1) in a general framework for f and b, under the restriction $b \equiv 0$ on $\partial\Omega$, inherited from the logistic equation (see [6]).

We start with

DEFINITION 1 ([11]). – A positive measurable function R defined on $[D,\infty)$, for some $D>0$, is called *regularly varying* (*at infinity*) *with index* $q\in\mathbb{R}$, written $R\in\mathbb{R}_q$, if for all $\xi>0$

$$\lim_{u\to\infty} R(\xi u)/R(u)=\xi^q.$$

When the index of regular variation q is zero, we say that the function is *slowly varying*.

Remark 1. – Any function $R\in\mathbb{R}_q$ can be written in terms of a slowly varying function. Indeed, set $R(u)=u^q L(u)$. From Definition 1 we easily derive that L varies slowly.

The canonical q-varying function is u^q. The functions $\ln(1+u)$, $\ln\ln(\mathrm{e}+u)$, $\exp\{(\ln u)^\alpha\}$, $\alpha\in(0,1)$ vary slowly, as well as any measurable function on $[D,\infty)$ with positive limit at infinity.

In what follows L denotes an arbitrary slowly varying function and $D>0$ a positive number. For details on Properties 1–4 stated below, we refer to Seneta [11] (pp. 7, 18, 53 and 78).

Property 1. – For any $m>0$, $u^m L(u)\to\infty$, $u^{-m}L(u)\to 0$ as $u\to\infty$.

Property 2. – Any positive C^1-function on $[D,\infty)$ satisfying $uL_1'(u)/L_1(u)\to 0$ as $u\to\infty$ is slowly varying. Moreover, if the above limit is $q\in\mathbb{R}$, then $L_1\in\mathbb{R}_q$.

Property 3. – Assume $R:[D,\infty)\to(0,\infty)$ is measurable and Lebesgue integrable on each finite subinterval of $[D,\infty)$. Then R varies regularly if and only if there exists $j\in\mathbb{R}$ such that

$$\lim_{u\to\infty}\frac{u^{j+1}R(u)}{\int_D^u x^j R(x)\,\mathrm{d}x} \tag{2}$$

exists and is a positive number, say a_j+1. In this case, $R\in\mathbb{R}_q$ with $q=a_j-j$.

Property 4 (Karamata Theorem, 1933). – If $R\in\mathbb{R}_q$ is Lebesgue integrable on each finite subinterval of $[D,\infty)$, then the limit defined by (2) is $q+j+1$, for every $j>-q-1$.

LEMMA 1. – *Assume* (A_1) *holds. Then we have the equivalence*

$$\text{(a)}\ f'\in\mathbb{R}_\rho \Longleftrightarrow \text{(b)}\ \lim_{u\to\infty} uf'(u)/f(u):=\vartheta<\infty \Longleftrightarrow \text{(c)}\ \lim_{u\to\infty}(F/f)'(u):=\gamma>0.$$

Remark 2. – Let (a) of Lemma 1 be fulfilled. The following assertions hold

(i) ρ is **non-negative**. Indeed, if $\rho<0$ then Property 1 and Remark 1 would contradict (A_1);
(ii) $\gamma=1/(\rho+2)=1/(\vartheta+1)$ (see the proof of Lemma 1);
(iii) If $\rho\neq 0$, then (A_2) holds (use $\lim_{u\to\infty} f(u)/u^p=\infty$, $\forall p\in(1,1+\rho)$). The converse implication is not necessarily true (take $f(u)=u\ln^4(u+1)$). However, there are cases when $\rho=0$ and (A_2) fails so that (1) has **no** large solutions. This is illustrated by $f(u)=u$ or $f(u)=u\ln(u+1)$.

Inspired by the definition of γ, we denote by $\mathcal{K}$ the set of all positive, increasing C^1-functions k defined on $(0,\nu)$, for some $\nu>0$, which satisfy

$$\lim_{t\to 0^+}\left(\frac{\int_0^t k(s)\,\mathrm{d}s}{k(t)}\right)^{(i)}:=\ell_i,\quad i=\overline{0,1}.$$

It is easy to see that $\ell_0=0$ and $\ell_1\in[0,1]$, for every $k\in\mathcal{K}$. Our next result gives examples of functions $k\in\mathcal{K}$ with $\lim_{t\to 0^+}k(t)=0$, for *every* $\ell_1\in[0,1]$.

LEMMA 2. – *Let* $S\in C^1[D,\infty)$ *be such that* $S'\in\mathbb{R}_q$ *with* $q>-1$. *Hence the following hold*:

(a) *If* $k(t)=\exp\{-S(1/t)\}$, $\forall t\leqslant 1/D$, *then* $k\in\mathcal{K}$ *with* $\ell_1=0$.
(b) *If* $k(t)=1/S(1/t)$, $\forall t\leqslant 1/D$, *then* $k\in\mathcal{K}$ *with* $\ell_1=1/(q+2)\in(0,1)$.
(c) *If* $k(t)=1/\ln S(1/t)$, $\forall t\leqslant 1/D$, *then* $k\in\mathcal{K}$ *with* $\ell_1=1$.

Remark 3. – If $S\in C^1[D,\infty)$, then $S'\in\mathbb{R}_q$ with $q>-1$ if and only if for some $m>0$, $C>0$ and $B>D$ we have $S(u)=Cu^m\exp\{\int_B^u \frac{y(t)}{t}\,\mathrm{d}t\}$, $\forall u\geqslant B$, where $y\in C[B,\infty)$ satisfies $\lim_{u\to\infty}y(u)=0$. In this case, $S'\in\mathbb{R}_q$ with $q=m-1$. This is a consequece of Properties 3 and 4.

Our main result is

THEOREM 1. – *Let* (A_1) *hold and* $f' \in \mathbb{R}_\rho$ *with* $\rho > 0$*. Assume* $b \equiv 0$ *on* $\partial\Omega$ *satisfies*

$$b(x) = ck^2\big(d(x)\big) + \mathrm{o}\big(k^2\big(d(x)\big)\big) \quad \text{as } d(x) \to 0, \text{ for some constant } c > 0 \text{ and } k \in \mathcal{K}. \tag{B}$$

Then, for any $a \in (-\infty, \lambda_{\infty,1})$*, Eq.* (1) *admits a unique large solution* u_a*. Moreover,*

$$\lim_{d(x)\to 0} \frac{u_a(x)}{h(d(x))} = \xi_0, \quad \text{where } \xi_0 = \left(\frac{2 + \ell_1\rho}{c(2+\rho)}\right)^{1/\rho} \tag{3}$$

and h *is defined by*

$$\int_{h(t)}^{\infty} \frac{\mathrm{d}s}{\sqrt{2F(s)}} = \int_0^t k(s)\,\mathrm{d}s, \quad \forall t \in (0, \nu). \tag{4}$$

By Remark 3, the assumption $f' \in \mathbb{R}_\rho$ with $\rho > 0$ holds if and only if there exist $p > 1$ and $B > 0$ such that $f(u) = Cu^p \exp\{\int_B^u \frac{y(t)}{t}\,\mathrm{d}t\}$, for all $u \geqslant B$ (y as before and $p = \rho + 1$). If B is large enough ($y > -\rho$ on $[B, \infty)$), then $f(u)/u$ is increasing on $[B, \infty)$. Thus, to get the whole range of functions f for which our Theorem 1 applies we have only to "paste" a suitable smooth function on $[0, B]$ in accordance with (A_1). A simple way to do this is to define $f(u) = u^{\rho+1} \exp\{\int_0^u \frac{z(t)}{t}\,\mathrm{d}t\}$, for all $u \geqslant 0$, where $z \in C[0, \infty)$ is non-negative such that $\lim_{t\to 0} z(t)/t \in [0, \infty)$ and $\lim_{u\to\infty} z(u) = 0$. Clearly, $f(u) = u^p$, $f(u) = u^p \ln(u+1)$, and $f(u) = u^p \arctan u\, (p > 1)$ fall into this category.

Lemma 2 provides a practical method to find functions k which can be considered in the statement of Theorem 1. Here are some examples: $k(t) = \exp\{-1/t^\alpha\}$, $k(t) = \exp\{-\ln(1 + \frac{1}{t})/t^\alpha\}$, $k(t) = \exp\{-[\arctan(\frac{1}{t})]/t^\alpha\}$, $k(t) = -1/\ln t$, $k(t) = t^\alpha/\ln(1 + \frac{1}{t})$, $k(t) = t^\alpha$, for some $\alpha > 0$.

As we shall see, the uniqueness lies upon the crucial observation (3), which shows that all explosive solutions have the same boundary behaviour. Note that the only case of Theorem 1 studied so far is $f(u) = u^p$ $(p > 1)$ and $k(t) = t^\alpha$ $(\alpha > 0)$ (see [6]). For related results on the uniqueness of explosive solutions (mainly in the cases $b \equiv 1$ and $a = 0$) we refer to [2,5,8,9,12].

Proof of Lemma 1. – From Property 4 and Remark 2(i) we deduce (a) $\Rightarrow$ (b) and $\vartheta = \rho + 1$. Conversely, (b) $\Rightarrow$ (a) follows by Property 3 since $\vartheta \geqslant 1$ cf. (A_1).

(b) $\Rightarrow$ (c) Indeed, $\lim_{u\to\infty} uf(u)/F(u) = 1 + \vartheta$, which yields $\frac{\vartheta}{1+\vartheta} = \lim_{u\to\infty}[1 - (F/f)'(u)] = 1 - \gamma$.

(c) $\Rightarrow$ (b) Choose $s_1 > 0$ such that $(F/f)'(u) \geqslant \frac{\gamma}{2}$, $\forall u \geqslant s_1$. So, $(F/f)(u) \geqslant (u - s_1)\gamma/2 + (F/f)(s_1)$, $\forall u \geqslant s_1$. Passing to the limit $u \to \infty$, we find $\lim_{u\to\infty} F(u)/f(u) = \infty$. Thus, $\lim_{u\to\infty} uf(u)/F(u) = \frac{1}{\gamma}$. Since $1 - \gamma := \lim_{u\to\infty} F(u)f'(u)/f^2(u)$, we obtain $\lim_{u\to\infty} uf'(u)/f(u) = (1-\gamma)/\gamma$. □

Proof of Lemma 2. – Since $\lim_{u\to\infty} uS'(u) = \infty$ (cf. Property 1), from Karamata Theorem we deduce $\lim_{u\to\infty} uS'(u)/S(u) = q + 1 > 0$. Therefore, in any of the cases (a), (b), (c), $\lim_{t\to 0^+} k(t) = 0$ and k is an increasing C^1-function on $(0, \nu)$, for $\nu > 0$ sufficiently small.

(a) It is clear that $\lim_{t\to 0^+} tk'(t)/k(t) \ln k(t) = \lim_{t\to 0^+}(-S'(1/t)/tS(1/t)) = -(q+1)$. By l'Hospital's rule, $\ell_0 = \lim_{t\to 0^+} k(t)/k'(t) = 0$ and $\lim_{t\to 0^+}(\int_0^t k(s)\,\mathrm{d}s) \ln k(t)/tk(t) = -1/(q+1)$. So, $1 - \ell_1 := \lim_{t\to 0^+}(\int_0^t k(s)\,\mathrm{d}s)k'(t)/k^2(t) = 1$.

(b) We see that $\lim_{t\to 0^+} tk'(t)/k(t) = \lim_{t\to 0^+} S'(1/t)/tS(1/t) = q + 1$. By l'Hospital's rule, $\ell_0 = 0$ and $\lim_{t\to 0^+} \int_0^t k(s)\,\mathrm{d}s/tk(t) = 1/(q+2)$. So, $\ell_1 = 1 - \lim_{t\to 0^+} \int_0^t k(s)\,\mathrm{d}s/tk(t) \cdot tk'(t)/k(t) = 1/(q+2)$.

(c) We have $\lim_{t\to 0^+} tk'(t)/k^2(t) = \lim_{t\to 0^+} S'(1/t)/tS(1/t) = q + 1$. By l'Hospital's rule, $\lim_{t\to 0^+} \int_0^t k(s)\,\mathrm{d}s/tk(t) = 1$. Thus, $\ell_0 = 0$ and $\ell_1 = 1 - \lim_{t\to 0^+} \int_0^t k(s)\,\mathrm{d}s/t \cdot tk'(t)/k^2(t) = 1$. □

Proof of Theorem 1. – Fix $a \in (-\infty, \lambda_{\infty,1})$. By [3, Theorem 1], (1) has at least a large solution.

If we prove that (3) holds for an *arbitrary* large solution u_a of (1), then the uniqueness is a consequence of [3, Lemma 3]. Indeed, if u_1 and u_2 are two arbitrary large solutions of (1), then (3) yields

$\lim_{d(x)\to 0^+} u_1(x)/u_2(x) = 1$. Hence, for any $\varepsilon \in (0,1)$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$(1-\varepsilon)u_2(x) \leqslant u_1(x) \leqslant (1+\varepsilon)u_2(x), \quad \forall x \in \Omega \text{ with } 0 < d(x) \leqslant \delta. \tag{5}$$

Choosing eventually a smaller $\delta > 0$, we can assume that $\overline{\Omega}_0 \subset C_\delta$, where $C_\delta := \{x \in \Omega\colon\ d(x) > \delta\}$.

It is clear that u_1 is a positive solution of the boundary value problem

$$\Delta\phi + a\phi = b(x) f(\phi) \quad \text{in } C_\delta, \qquad \phi = u_1 \quad \text{on } \partial C_\delta. \tag{6}$$

By (A_1) and (5), we see that $\phi^- = (1-\varepsilon)u_2$ (resp., $\phi^+ = (1+\varepsilon)u_2$) is a positive sub-solution (resp., super-solution) of (6). By the sub- and super-solutions method, (6) has a positive solution ϕ_1 satisfying $\phi^- \leqslant \phi_1 \leqslant \phi^+$ in C_δ. Since $b > 0$ on $\overline{C}_\delta \setminus \overline{\Omega}_0$, by [3, Lemma 3] we derive that (6) has a *unique* positive solution, i.e., $u_1 \equiv \phi_1$ in C_δ. This yields $(1-\varepsilon)u_2(x) \leqslant u_1(x) \leqslant (1+\varepsilon)u_2(x)$ in C_δ, so that (5) holds in Ω. Passing to the limit $\varepsilon \to 0^+$, we conclude that $u_1 \equiv u_2$.

In order to prove (3) we state some useful properties about h:

(h_1) $h \in C^2(0,\nu)$, $\lim_{t\to 0^+} h(t) = \infty$ (straightforward from (4)).

(h_2) $\lim_{t\to 0^+} h''(t)/k^2(t) f(h(t)\xi) = \frac{1}{\xi^{\rho+1}} \cdot \frac{2+\rho\ell_1}{2+\rho}$, $\forall \xi > 0$ (so, $h'' > 0$ on $(0, 2\delta)$, for $\delta > 0$ small enough).

(h_3) $\lim_{t\to 0^+} h(t)/h''(t) = \lim_{t\to 0^+} h'(t)/h''(t) = 0$.

We check (h_2) for $\xi = 1$ only, since $f \in \mathbb{R}_{\rho+1}$. Clearly, $h'(t) = -k(t)\sqrt{2F(h(t))}$ and

$$h''(t) = k^2(t) f\bigl(h(t)\bigr)\left(1 - 2\frac{k'(t)(\int_0^t k(s)\,\mathrm{d}s)}{k^2(t)}\,\frac{\sqrt{F(h(t))}}{f(h(t))\int_{h(t)}^\infty [F(s)]^{-1/2}\,\mathrm{d}s}\right) \quad \forall t \in (0,\nu). \tag{7}$$

We see that $\lim_{u\to\infty} \sqrt{F(u)}/f(u) = 0$. Thus, from l'Hospital's rule and Lemma 1 we infer that

$$\lim_{u\to\infty} \frac{\sqrt{F(u)}}{f(u)\int_u^\infty [F(s)]^{-1/2}\,\mathrm{d}s} = \frac{1}{2} - \gamma = \frac{\rho}{2(\rho+2)}. \tag{8}$$

Using (7) and (8) we derive (h_2) and also

$$\lim_{t\to 0^+} \frac{h'(t)}{h''(t)} = \frac{-2(2+\rho)}{2+\ell_1\rho} \lim_{t\to 0^+} \frac{\int_0^t k(s)\,\mathrm{d}s}{k(t)} \lim_{u\to\infty} \frac{\sqrt{F(u)}}{f(u)\int_u^\infty [F(s)]^{-1/2}\,\mathrm{d}s} = \frac{-\rho\ell_0}{2+\ell_1\rho} = 0. \tag{9}$$

From (h_1) and (h_2), $\lim_{t\to 0^+} h'(t) = -\infty$. So, l'Hospital's rule and (9) yield $\lim_{t\to 0^+} h(t)/h'(t) = 0$. This and (9) lead to $\lim_{t\to 0^+} h(t)/h''(t) = 0$ which proves (h_3). □

Proof of (3). – Fix $\varepsilon \in (0, c/2)$. Since $b \equiv 0$ on $\partial\Omega$ and (B) holds, we take $\delta > 0$ so that

(i) $d(x)$ is a C^2-function on the set $\{x \in \mathbb{R}^N\colon\ d(x) < 2\delta\}$;

(ii) k^2 is increasing on $(0, 2\delta)$;

(iii) $(c-\varepsilon)k^2(d(x)) < b(x) < (c+\varepsilon)k^2(d(x))$, $\forall x \in \Omega$ with $0 < d(x) < 2\delta$;

(iv) $h''(t) > 0$ $\forall t \in (0, 2\delta)$ (from (h_2)).

Let $\sigma \in (0,\delta)$ be arbitrary. We define $\xi^\pm = [(2+\ell_1\rho)/(c \mp 2\varepsilon)(2+\rho)]^{1/\rho}$ and $v_\sigma^-(x) = h(d(x)+\sigma)\xi^-$, for all x with $d(x)+\sigma < 2\delta$ resp., $v_\sigma^+(x) = h(d(x)-\sigma)\xi^+$, for all x with $\sigma < d(x) < 2\delta$.

Using (i)–(iv), when $\sigma < d(x) < 2\delta$ we obtain (since $|\nabla d(x)| \equiv 1$)

$$\begin{aligned}\Delta v_\sigma^+ + a v_\sigma^+ - b(x) f(v_\sigma^+) \leqslant \xi^+ h''\bigl(d(x)-\sigma\bigr)\biggl(&\frac{h'(d(x)-\sigma)}{h''(d(x)-\sigma)}\Delta d(x) + a\frac{h(d(x)-\sigma)}{h''(d(x)-\sigma)} + 1 \\ &- (c-\varepsilon)\frac{k^2(d(x)-\sigma) f(h(d(x)-\sigma)\xi^+)}{h''(d(x)-\sigma)\xi^+}\biggr).\end{aligned}$$

Similarly, when $d(x)+\sigma < 2\delta$ we find

$$\begin{aligned}\Delta v_\sigma^- + a v_\sigma^- - b(x) f(v_\sigma^-) \geqslant \xi^- h''\bigl(d(x)+\sigma\bigr)\biggl(&\frac{h'(d(x)+\sigma)}{h''(d(x)+\sigma)}\Delta d(x) + a\frac{h(d(x)+\sigma)}{h''(d(x)+\sigma)} + 1 \\ &- (c+\varepsilon)\frac{k^2(d(x)+\sigma) f(h(d(x)+\sigma)\xi^-)}{h''(d(x)+\sigma)\xi^-}\biggr).\end{aligned}$$

Using (h_2) and (h_3) we see that, by diminishing δ, we can assume

$$\Delta v_\sigma^+(x) + a v_\sigma^+(x) - b(x) f\big(v_\sigma^+(x)\big) \leqslant 0 \quad \forall x \text{ with } \sigma < d(x) < 2\delta;$$
$$\Delta v_\sigma^-(x) + a v_\sigma^-(x) - b(x) f\big(v_\sigma^-(x)\big) \geqslant 0 \quad \forall x \text{ with } d(x) + \sigma < 2\delta.$$

Let Ω_1 and Ω_2 be smooth bounded domains such that $\Omega \Subset \Omega_1 \Subset \Omega_2$ and the first Dirichlet eigenvalue of $(-\Delta)$ in the domain $\Omega_1 \setminus \overline{\Omega}$ is greater than a. Let $p \in C^{0,\mu}(\overline{\Omega}_2)$ satisfy $0 < p(x) \leqslant b(x)$ for $x \in \Omega \setminus C_{2\delta}$, $p = 0$ on $\overline{\Omega}_1 \setminus \Omega$ and $p > 0$ on $\Omega_2 \setminus \overline{\Omega}_1$. Denote by w a positive large solution of

$$\Delta w + a w = p(x) f(w) \quad \text{in } \Omega_2 \setminus \overline{C}_{2\delta}.$$

The existence of w is ensured by Theorem 1 in [3].

Suppose that u_a is an arbitrary large solution of (1) and let $v := u_a + w$. Then v satisfies

$$\Delta v + a v - b(x) f(v) \leqslant 0 \quad \text{in } \Omega \setminus \overline{C}_{2\delta}.$$

Since $v_{|\partial\Omega} = \infty > v^-_{\sigma|\partial\Omega}$ and $v_{|\partial C_{2\delta}} = \infty > v^-_{\sigma|\partial C_{2\delta}}$, Lemma 1 in [3] implies

$$u_a + w \geqslant v_\sigma^- \quad \text{on } \Omega \setminus \overline{C}_{2\delta}. \tag{10}$$

Similarly,

$$v_\sigma^+ + w \geqslant u_a \quad \text{on } C_\sigma \setminus \overline{C}_{2\delta}. \tag{11}$$

Letting $\sigma \to 0$ in (10) and (11), we deduce $h(d(x))\xi^+ + 2w \geqslant u_a + w \geqslant h(d(x))\xi^-$, for all $x \in \Omega \setminus \overline{C}_{2\delta}$. Since w is uniformly bounded on $\partial\Omega$, we have

$$\xi^- \leqslant \liminf_{d(x)\to 0} \frac{u_a(x)}{h(d(x))} \leqslant \limsup_{d(x)\to 0} \frac{u_a(x)}{h(d(x))} \leqslant \xi^+.$$

Letting $\varepsilon \to 0^+$ we obtain (3). This concludes the proof of Theorem 1. □

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