C. R. Acad. Sci. Paris, Ser. I 335 (2002) 485–487

Géométrie algébrique/*Algebraic Geometry*

Conductors of wildly ramified covers, II

Rachel J. Pries

Department of Mathematics, Columbia University, New York, NY 10027, USA

Received 10 June 2002; accepted 20 June 2002

Note presented by Michel Raynaud.

Abstract Consider a wildly ramified G-Galois cover of curves $\phi: Y \to \mathbb{P}^1_k$ branched at only one point over an algebraically closed field k of characteristic p. In this note, I prove using formal patching that all sufficiently large conductors occur for such covers ϕ when the Sylow p-subgroups of G have order p. ***To cite this article: R.J. Pries, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 485–487.***

Conducteurs des revêtements avec ramification sauvage, II

Résumé Soit k un corps algébriquement clos de caractéristique p. Soit $\phi: Y \to \mathbb{P}^1_k$ un revêtement fini galoisien, de groupe G, ramifié seulement au-dessus d'un point (avec ramification sauvage). On montre l'existence d'un revêtement de ce type avec tous conducteurs suffisamment grands quand les p-Sylow de G sont d'ordre p. La démonstration consiste à étudier la géométrie formelle. ***Pour citer cet article : R.J. Pries, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 485–487.***

1. Introduction

Let X_k be a proper smooth k-curve where k is an algebraically closed field of characteristic p. Abhyankar's Conjecture (Raynaud [7] and Harbater [2]) determines which finite groups G occur as a Galois group of a cover $\phi: Y \to X_k$ of smooth connected curves branched at a finite set of points B. An open problem is to determine which filtrations of higher ramification groups can be realized for the inertia groups of such a cover ϕ.

Let S be a chosen Sylow p-subgroup of G. In this note, I restrict to the case that S has order p. Under this assumption, any inertia group of ϕ is of the form $I \simeq \mathbb{Z}/p \rtimes \mu_m$ with $\gcd(p,m)=1$. Furthermore, the filtration of higher ramification groups at a ramification point η is determined by one integer j, namely by the lower jump or conductor; note that $j = \mathrm{val}(g(\pi_\eta) - \pi_\eta) - 1$ where $\mathrm{id} \neq g \in S$ and π_η is a uniformizer at η. There are several necessary conditions on the conductor: $\gcd(p,j)=1$ and the order n' of the prime-to-p part of the center of I equals $\gcd(j,m)$.

When $X_k \simeq \mathbb{P}^1_k$, $B = \{\infty\}$, and $|S| = p$, I prove that all sufficiently large conductors occur for such covers ϕ, Corollary 2.4.

The main idea of the proof is that it is possible to increase the conductor of a given G-Galois cover $\phi: Y \to X_k$ at any wild ramification point. In Section 2, I construct a family of covers so that ϕ is

E-mail address: pries@math.columbia.edu (R.J. Pries).

S1631-073X(02)02492-5/FLA

isomorphic to the normalization of one fibre of the family. The techniques consist of local deformations and formal patching [3]. One can then use another fibre of this family to find another cover of $\mathbb{P}^1_k$ with the same Galois and inertia group but with larger conductor.

For more general results along these lines, see [5].

Suppose $f : Y \to X$ is a morphism of schemes, ξ is a point of X, and $\eta \in f^{-1}(\xi)$. The *germ* $\widehat{X}_\xi$ of X at ξ is defined to be the spectrum of the complete local ring of functions of X at ξ and $\hat{f}_\eta : \widehat{Y}_\eta \to \widehat{X}_\xi$ is the corresponding morphism of germs of curves.

2. Increasing the conductor

In this section, we show that it is possible to increase the conductor at a branch point while preserving the inertia and Galois group. Let $R = k[[t]]$ and $K = k((t))$. Let b be the closed point of $U = \mathrm{Spec}(k[[u]])$. Let $U_R = \mathrm{Spec}(R[[u]])$ and $U_K = U_R \times_R K = \mathrm{Spec}(k[[u,t]][t^{-1}])$.

PROPOSITION 2.1. – *Let $I \simeq \mathbb{Z}/p \rtimes \mu_m$. Suppose there exists an I-Galois cover $\phi : X \to U$ of normal connected germs of curves with conductor j. Then for $i \in \mathbb{N}$ with $\gcd(j + im, p) = 1$, there exists an I-Galois cover $\phi_R : X_R \to U_R$ of irreducible germs of R-curves, whose branch locus consists of only the R-point $b_R = b \times_k R$, such that*:

(1) *The normalization of the special fibre of ϕ_R $(t = 0)$ is isomorphic to ϕ away from b.*

(2) *The generic fibre $\phi_K : X_K \to U_K$ of ϕ_R is an I-Galois cover of normal connected curves whose branch locus consists of only the K-point $b_K = b_R \times_R K$ over which it has inertia I and conductor $j + im$.*

Proof. – After an automorphism A of $k[[u]]$, the equations for $A^*\phi$ are given by: $u_1^m = u$, $x^p - x = u_1^{-j}$. Consider the normal cover $\phi'_R : X'_R \to U_R$ given generically by the equations:

$$u_1^m = u,\ x^p - x = u_1^{-(j+im)}\big(t + u^i\big).$$

The I-Galois action on the variables is given by the same expressions and the cover is irreducible. The curve X'_R is singular only above the point $(u,t) = (0,0)$. The normalization of the special fibre agrees with $A^*\phi$. The cover ϕ'_R is branched only at the R-point $u = 0$ since $u_1 = 0$ is the only pole of the function $u_1^{-(j+im)}(t + u^i)$. Taking the restriction of ϕ_K over $\mathrm{Spec}(K[[u]])$ where $t + u^i$ is a unit, we see that ϕ_K has inertia I and conductor $j + im$ over b_K. Pulling back the cover ϕ'_R by the automorphism $A^{-1} \times_k R$ of $R[[u]]$ changes none of these properties and thus yields the cover ϕ_R. □

We now use Proposition 2.1 to deform a given cover to a family of covers and then specialize to another fibre of the family to get a cover with new ramification data.

THEOREM 2.2. – *Suppose there exists a G-Galois cover $\phi : Y \to X_k$ of smooth connected curves with branch locus B and inertia $I \simeq \mathbb{Z}/p \rtimes \mu_m$ and conductor j above $\xi_1 \in B$. Then for $i \in \mathbb{N}^+$ such that $\gcd(j + im, p) = 1$, there exists a G-Galois cover $\phi' : Y' \to X_k$, where Y' is a smooth connected curve of genus $g_{Y'} = g_Y + |G| i(p-1)/2p$, such that*:

(1) *The branch locus of ϕ' consists only of the k-points ξ for $\xi \in B$. For $\xi \in B$, $\xi \neq \xi_1$, the ramification behavior for ϕ' at ξ is identical to that of ϕ at ξ.*

(2) *The cover ϕ' has inertia I and conductor $j + im$ at ξ_1.*

Proof. – Let $\eta \in \phi^{-1}(\xi_1)$. Applying Proposition 2.1 to the I-Galois cover of germs of curves $\hat{\phi}_\eta : \widehat{Y}_\eta \to \widehat{X}_{\xi_1}$, there is a deformation $\hat{\phi}_R : \widehat{Y}_R \to \widehat{X}_R$ with the desired properties. In particular, $\hat{\phi}_K$ has inertia $I \simeq \mathbb{Z}/p \rtimes \mu_m$ and conductor $j + im$ over $\xi_{1,K}$. Consider the disconnected G-Galois cover $\mathrm{Ind}_I^G(\hat{\phi}_R)$.

The covers ϕ and $\mathrm{Ind}_I^G(\hat{\phi}_R)$ and the isomorphism given by Proposition 2.1 constitute a relative G-Galois thickening problem [3]. The (unique) solution to this thickening problem [3, Theorem 4] yields a G-Galois cover $\phi_R : Y_R \to X_R$. Recall that ϕ_R is isomorphic to $\mathrm{Ind}_I^G(\hat{\phi}_R)$ over $\widehat{X}_R$ and isomorphic to the trivial deformation $\phi_{\mathrm{tr}} : Y_{\mathrm{tr}} \to X_{\mathrm{tr}}$ of ϕ away from ξ_1. Thus Y_R is irreducible since Y is and Y_K is smooth since $Y_{\mathrm{tr},K}$ and $\widehat{Y}_K$ are.

The data for the cover ϕ_R is contained in a subring $\Theta \subset R$ of finite type over k, with $\Theta \neq k$ since the family is non-constant. Since k is algebraically closed, there exist infinitely many k-points of $\mathrm{Spec}(\Theta)$. The closure L of the locus of k-points x of $\mathrm{Spec}(\Theta)$ over which the fibre ϕ_x is not a G-Galois cover of smooth connected curves is closed, [1, Proposition 9.29]. Furthermore, $L \neq \mathrm{Spec}(\Theta)$ since Y_K is smooth and irreducible. Let $\phi' : Y' \to X_k$ be the fibre over a k-point not in L. Note that Y' is smooth and irreducible by definition. The other properties follow immediately from the isomorphisms above. The increase in the genus results from the larger conductor due to the Riemann–Hurwitz formula. □

One can also change the Galois group, the inertia group, or the congruence value of the conductor by deforming covers of germs of semistable curves, see [5].

DEFINITION 2.3. – Let $G(S) \subset G$ be the subgroup generated by all proper quasi-p subgroups G' such that $G' \cap S$ is a Sylow p-subgroup of G'. The group G is *p-pure* if $G(S) \neq G$.

COROLLARY 2.4. – *Let G be a finite p-pure quasi-p group with Sylow p-subgroup S of order $p \neq 2$. Let m_e be the exponent of the normalizer $N_G(S)$ of S in G divided by p. For all $j \in \mathbb{N}$ with $j \geqslant m_e(2 + 1/(p-1))$ and $\gcd(j, p) = 1$, there exists a G-Galois cover $\phi : Y \to \mathbb{P}_k^1$ branched at only one point with inertia group $\mathbb{Z}/p$ and conductor j.*

Proof. – Under these hypotheses, [4, Theorem 3.5] applies. Its conclusion is that for some $I \simeq \mathbb{Z}/p \rtimes \mu_m \subset G$ and some $j \leqslant m(2 + 1/(p-1)$, there exists a G-Galois cover $\phi : Y \to \mathbb{P}_k^1$ of smooth connected curves branched at only one point over which it has inertia group I and conductor j. By Abhyankar's Lemma, a pullback of ϕ has inertia $\mathbb{Z}/p$ and conductor j. The corollary follows from Theorem 2.2. □

For results on groups which are not p-pure, see [5].

References

[1] M. Fried, M. Jarden, Field Arithmetic, Springer-Verlag, Berlin, 1986.
[2] D. Harbater, Abhyankar's conjecture on Galois groups over curves, Invent. Math. 117 (1) (1994) 1–25.
[3] D. Harbater, K. Stevenson, Patching and thickening problems, J. Algebra 212 (1) (1999) 272–304.
[4] R. Pries, Conductors of wildly ramified covers, I, Preprint, 2001.
[5] R. Pries, Conductors of wildly ramified covers, III, Preprint, 2001.
[6] R. Pries, Families of wildly ramified covers of curves, Amer. J. Math., accepted.
[7] M. Raynaud, Revêtements de la droite affine en caractéristique $p > 0$ et conjecture d'Abhyankar, Invent. Math. 116 (1–3) (1994) 425–462.