Conductors of wildly ramified covers, I

Rachel J. Pries
Department of Mathematics, Columbia University, New York, NY 10027, USA
Received 10 June 2002; accepted 20 June 2002
Note presented by Michel Raynaud.

Abstract
Consider a wildly ramified $G$-Galois cover of curves $\phi: Y \to \mathbb{P}^1_k$ branched at only one point over an algebraically closed field $k$ of characteristic $p$. For any $p$-pure group $G$ whose Sylow $p$-subgroups have order $p$, I show the existence of such a cover with small conductor. The proof uses an analysis of the semi-stable reduction of families of covers. To cite this article: R.J. Pries, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 481–484.

Conducteurs des revêtements avec ramification sauvage, I

Résumé

1. Introduction

Let $k$ be an algebraically closed field of characteristic $p$. Abhyankar’s Conjecture (Raynaud [5]) states that there exists a $G$-Galois cover $\phi: Y \to \mathbb{P}^1_k$ branched at only one point if and only if $G$ is a quasi-$p$ group which means that $G$ is generated by $p$-groups. An open problem is to determine which filtrations of higher ramification groups can be realized for the inertia groups of such a cover $\phi$.

Let $S$ be a chosen Sylow $p$-subgroup of $G$. In this note, I restrict to the case that $S$ has order $p$. Under this assumption, any inertia group of $\phi$ is of the form $I \cong \mathbb{Z}/p \rtimes \mu_m$ with $\gcd(p, m) = 1$. Furthermore, the filtration of higher ramification groups at a ramification point $\eta$ is determined by one integer $j$, namely by the lower jump or conductor; note that $j = \val(g(\pi_\eta) - \pi_\eta) - 1$ where $id \neq g \in S$ and $\pi_\eta$ is a uniformizer at $\eta$. Note that $\gcd(p, j) = 1$ and the order $n'$ of the prime-to-$p$ part of the center of $I$ equals $\gcd(j, m)$. When $G \neq \mathbb{Z}/p$, there is a nontrivial lower bound for $j$. In this case, under an additional hypothesis on $G$, I show the existence of such a cover $\phi$ with small conductor, Theorem 3.5.

The main idea of the proof is that it is possible to decrease the ramification data of a given $G$-Galois cover $\phi: Y \to \mathbb{P}^1_k$. The method is to use [4] to deform the original cover $\phi$ to a family of covers having a fibre $\phi_K$ with bad reduction. I analyze the special fibre of the semi-stable model of $\phi_K$ to find new covers...
of \( \mathbb{P}^1_k \) each branched at only one point. Under a condition on \( G \), one of these covers will be connected.

Theorem 2.8 compares the ramification information of these covers and of \( \phi_K \). This is motivated by [5,6].

Suppose \( f : Y \to X \) is a morphism of schemes, \( \xi \) is a point of \( X \), and \( \eta \in f^{-1}(\xi) \). The germ \( \hat{X}_\xi \) of \( X \) at \( \xi \) is the spectrum of the complete local ring of functions of \( X \) at \( \xi \) and \( \hat{f}_\eta : \hat{Y}_\eta \to \hat{X}_\xi \).

### 2. Degeneration of covers

Let \( R \simeq k[[t]] \) where \( k = \overline{k} \) has characteristic \( p > 2 \) and let \( K = \text{Frac}(R) \). In this section, all \( R \)-curves are proper, normal, reduced and flat over \( R \) with smooth and geometrically connected generic fibres. All covers of \( R \)-curves are flat and generically separable. We analyze the semi-stable model of the special fibre of a cover \( \phi \) of \( R \)-curves with bad reduction. The results follow those of Raynaud [5,6] where \( R \) has unequal characteristic. See also [7].

**Lemma 2.1.** Suppose that \( f : Y \to X \) is a cover of normal curves over \( R \) with \( X \) and \( Y \) reduced. Let \( x_R \) be an \( R \)-point of \( X \) which specializes to a smooth point \( x \) of \( X \). Let \( y \in f^{-1}(x) \) and suppose \( \hat{f}_y \) is étale outside \( x_R \). Let \( e \) be the ramification index of \( \hat{f}_{y,K} \) over the point \( x_K = x_R \times_R K \). If \( \gcd(e, p) = 1 \), then \( y \) is smooth and \( \hat{f}_{y,K} \) is tamely ramified at \( x \) with ramification index \( e \).

**Proof.** – The proof is the same as in unequal characteristic, which was proved in [5, 6.3.2] using Abhyankar’s Lemma. See also [7, 1.7] for a proof using Kato’s formula [2].

**Lemma 2.2.** Let \( f : Y \to X \) be a Galois cover of integral semi-stable \( R \)-curves. Let \( y_K \) be a rational point of \( Y_K \) specializing to a point \( y \) of \( Y_k \). Assume \( f : Y_K \to X_K \) is étale outside \( \hat{f}(y_K) \). Let \( \eta \) be the generic point of an irreducible component of \( Y_k \) which contains \( y \). Then \( I(y_K) \subset I(y) \) and \( I(\eta) \) is a \( p \)-group normal in the inertia group \( I(y) \) at \( y \) and in the stabilizer \( D(\eta) \) of this component.

**Proof.** – The proof is the same as the unequal characteristic case in [5, 6.3.3, 6.3.6].

**Lemma 2.3.** Let \( f : Y \to X \) be as in Lemma 2.2 with \( x \in X_k \) and \( y \in f^{-1}(x) \).

(i) Assume \( p \neq 2 \). Suppose \( x \) is a smooth point of \( X_k \). Suppose that \( f \) has at most one branch point \( x_R \) specializing to \( x \). Then \( y \) is a smooth point of \( Y_k \).

(ii) Suppose \( \hat{f}_{x,K} \) is étale. If \( x \) is a node of \( X_k \) then \( y \) is a node. If \( I(\eta_1) \) and \( I(\eta_2) \) are the inertia groups of the generic points of the components of \( \hat{Y} \) containing \( y \) then \( I(\eta_1), I(\eta_2) \) is normal in \( I(y) \) and contains the Sylow \( p \)-subgroup of \( I(y) \).

**Proof.** – (i) (The proof is similar to [7, 1.11]). If \( y \) is a node, let \( I' \) be the subgroup of \( I(y) \) which stabilizes each of the two components passing through \( y \). Since \( \hat{f}_y \) is Galois, \( I' \) is of index 2 and normal in \( I(y) \). Consider the Galois quotient \( \hat{f}_y : \hat{Y} \to \hat{X} \) of \( \hat{f}_y \) by \( I' \). Thus \( \hat{f}_y \) is a Galois cover of degree two from a singular to a smooth germ of a curve. It is generically étale over \( \hat{X}_{x,k} \) and the ramification index \( e \) of \( \hat{f}_y \) over \( x_k \) divides 2. Since \( p \neq 2 \), this contradicts Lemma 2.1.

(ii) See [7, 1.4, 1.9]. Here is the outline: \( y \) is a node since \( Y \) is semi-stable and the singularity can only worsen. The subgroup \( I' = (I(\eta_1), I(\eta_2)) \) is normal in \( I(y) \). As in part (i), take the quotient of \( \hat{f}_y \) by \( I' \).

The resulting morphism \( \hat{f}_y \) is generically étale. Applying a formula of Kato [2] to \( \hat{f}_y \) implies that it is tame and thus prime-to-\( p \). Thus \( I' \) contains the Sylow \( p \)-subgroup of \( I(y) \).

Now let \( \phi_K : Y_K \to \mathbb{P}^1_k \) be a flat \( G \)-Galois cover of proper, smooth, reduced, geometrically connected curves over \( \text{Spec}(K) \) with genus(\( Y_K \)) \( \geq 2 \). Let \( Y_{0,R} \) be the normalization of \( \mathbb{P}^1_k \) in \( Y_K \) and let \( \phi_{0,R} : Y_{0,R} \to \mathbb{P}^1_R \). Note that \( \phi_{0,k} \) can be generically inseparable and \( Y_{0,k} \) can be singular.

Here we assume that \( \phi_K \) is étale away from one (necessarily wild) branch point \( \infty_K \).

After a finite extension \( R' \) of \( R \), there exists a minimal semi-stable normal curve \( Y \) which is a blow-up of \( Y_{0,R} \) and has an action of \( G \) so that: the quotient map is a \( G \)-Galois cover \( \phi : Y \to X \); the irreducible components of \( Y_k \) are smooth; and the branch points of \( \phi \) specialize in distinct smooth points of \( X_k \).
curve $X$ is semi-stable and normal and $X_k$ is a tree of projective lines. We call $\phi : Y \to X$ the stable model of $\phi_K$, [5, 6.3]. Let $X_{\text{br}}$ be the component of $X_k$ into which $\infty_K$ specializes to a point $\infty_K$.

**Definition 2.4.** – If $Y_k$ is smooth and $\phi_k$ is generically étale then $\phi_K$ has good reduction.

**Lemma 2.5.** – The cover $\phi_K$ has good reduction if and only if $X_k$ is irreducible.

**Proof.** – If $\phi_K$ has good reduction, then $Y_k$ is connected by Zariski’s Theorem and smooth; thus $X_k$ is irreducible since $Y_k$ is. If $X_k$ is irreducible, then it is smooth. Since the branch points of $\phi_K$ specialize to distinct points of $X_k$ and since $p \neq 2$, Lemma 2.3(ii) indicates that every point $y$ of $Y_k$ is smooth. Since $Y_k$ is smooth and genus($Y_k$) $\geq 2$ the morphism $\phi_K : Y_k \to X_k$ is generically étale; see [6, 2.4.10].

**Definition 2.6.** – Suppose $\phi_K$ has bad reduction. An irreducible component $C$ of $X_k$ is terminal if $C \neq X_{\text{br}}$ and $C$ intersects the closure of $X_k - C$ in only one point.

**Proposition 2.7.** – Let $\phi : Y \to X$ be the stable model of $\phi_K$. If $\phi : Y \to X$ is generically étale over a component $C$ of $X_k$ then $C$ is terminal. Suppose that $\eta$ is the generic point of a terminal component $C$ of $X_k$. Then $|I(\eta)| = |S|$, so $\phi$ is generically étale over $C$.

**Proof.** – This proof is a modification of [5, 6.3.8], [6, 2.4.8], and [6, 3.1.2] to equal characteristic case. The crucial point is that (taking the initial component to be $X_{\text{br}}$) no wild branch point specializes to a component which needs to be contracted in the proof.

Suppose that $\phi_K$ does not have good reduction. By Lemma 2.5, $Y_k$ and $X_k$ are singular. Let $U \subset X_k$ be the union of the non-terminal components of the tree $X_k$. Choose a connected component $V$ of $\phi^{-1}(U)$.

With Proposition 2.7 and Lemmas 2.2, 2.3(ii), one can show that $I \subset D(V) \subset N_G(S)$. Let $B$ be the set of terminal components of $X_k$. For $b \in B$, let $P_b$ be the corresponding terminal component and let $\infty_b$ be the point of intersection of $P_b$ with $U$. For each $b \in B$, let $\sigma_b = j_b/m_b$ be the upper jump of the restriction of $\phi$ to $P_b$ over $\infty_b$. Let $\sigma = j/m$ be the upper jump of $\phi_K$ over $\infty_K$.

**Theorem 2.8** (Key formula). – $\sigma - 1 = \sum_{b \in B} (\sigma_b - 1)$.

**Proof.** – The proof parallels that of [6, (3.4.2)(5)] by constructing a $D(V)$-Galois auxiliary cover $\psi : Z \to X$ of semi-stable curves which has the same ramification as $\phi$ but is easier to analyze. The construction of $\psi$ parallels [6, 3.2], using [3] and [1, Theorem 4].

**3. Decreasing the conductor**

Let $\phi : Y \to \mathbb{P}^1_k$ be a $G$-Galois cover branched at only one point and having inertia $I \simeq \mathbb{Z}/p \rtimes \mu_m$ and conductor $j$. When $G \neq \mathbb{Z}/p$, there is a small set of values $j_{\text{min}}(I)$, depending only on $I$, consisting of the minimal possible conductors for $\phi$. Let $n$ be such that $m = n^r$ for $n^r$ as in Section 1.

**Definition 3.1.** – Define $j_{\text{min}}(I) = \{j_{\text{min}}(I, a) : 1 \leq a \leq n, \gcd(a, n) = 1\}$ where $j_{\text{min}}(I, a) = 2m + n'$ if $a = 1$ and $n = p - 1$ and $j_{\text{min}}(I, a) = m + an'$ otherwise.

The cover $\phi$ has a non-isotrivial deformation in equal characteristic $p$ if and only if $j \notin j_{\text{min}}(I)$, [4, Theorem 3.1.11]. If $j \notin j_{\text{min}}(I)$ then genus($Y_K$) $\geq 2$. Suppose $1 \leq a \leq n$ and $j \equiv an' \mod m$. If $G \neq \mathbb{Z}/p$ then $j \geq j_{\text{min}}(I, a)$, by [4, Lemma 1.4.3].

**Definition 3.2.** – Let $G(S) \subset G$ be the subgroup generated by all proper quasi-$p$ subgroups $G'$ such that $G' \cap S$ is a Sylow $p$-subgroup of $G'$. The group $G$ is $p$-pure if $G(S) \neq G$.

This condition was introduced in [5]. If $G$ is quasi-$p$ with $|S| = p$, then $G$ is $p$-pure if and only if $G$ is not generated by all proper quasi-$p$ subgroups $G' \subset G$ such that $S \subset G'$.

**Proposition 3.3.** – Let $\phi : Y \to X$ be the stable model of $\phi_K$. If $G$ is $p$-pure and has no (non-trivial) normal $p$-subgroups, then for some terminal component $P_b$ of $X_k$, the curve $Y_b = \phi^{-1}(P_b)$ is connected.

483
Theorem 3.4. — Let $G$ be a finite $p$-pure quasi-$p$ group whose Sylow $p$-subgroups have order $p 
eq 2$. Suppose there exists a $G$-Galois cover $\phi : Y \to P^1_k$ branched at only one point with inertia group $I \cong \mathbb{Z}/p \rtimes \mu_m$ and conductor $j \notin j_{\text{min}}(I)$. Then there exists a $G$-Galois cover $\phi_0 : Y_b \to P^1_k$ which is branched at only one point with inertia group $I_b \cong \mathbb{Z}/p \rtimes \mu_{m_b} \subset N_G(S)$ and conductor $j_b$ satisfying $j_b/m_b < j/m$.

Proof. — By [4, Theorem 3.3.7], for some proper connected variety $\mathcal{O}$, there exists a family of $G$-Galois covers $\phi_\mathcal{O} : Y_\mathcal{O} \to P^1_\mathcal{O}$ of flat, proper, semi-stable $\mathcal{O}$-curves branched at only one $\mathcal{O}$-point such that: for some $k$-point $\omega$, $\phi \cong \phi_\omega$; and for some $K$-point of $\mathcal{O}$ the pullback $\phi_K : Y_K \to P^1_K$ has bad reduction.

Consider the stable model $\phi : Y \to X$ for $\phi_K$. Since $\phi_K$ has bad reduction there are at least two terminal components of $X_k$. By Proposition 3.3, the cover is connected over one of the terminal components $P_b$. By Proposition 2.7, the restriction $\phi_b : Y_b \to P_b \cong P^1_k$ is separable. By Lemma 2.1, $\phi_b$ is branched only at $\infty_b$ since no ramification of $\phi_K$ specializes to $P_b$. Over $\infty_b$, the cover $\phi_b$ has some inertia group $I_b \cong \mathbb{Z}/p \rtimes \mu_{m_b} \subset N_G(S)$ and some conductor $j_b$. By Theorem 2.8, $\sigma_b = j_b/m_b < j/m = \sigma$. □

Theorem 3.5. — Let $G$ be a finite $p$-pure quasi-$p$ group whose Sylow $p$-subgroups have order $p \neq 2$. For some $I \cong \mathbb{Z}/p \rtimes \mu_m \subset G$ and some $j \in j_{\text{min}}(I)$, there exists a $G$-Galois cover $\phi : Y \to P^1_k$ of smooth connected curves branched at only one point over which it has inertia group $I$ and conductor $j$. In particular, $\text{genus}(Y) \leq 1 + \#G(p - 1)/2p$.

Proof. — By Abhyankar’s Conjecture [5, 6.5.3], for some $I$ of the form $\mathbb{Z}/p \rtimes \mu_{m'}$ and some $j'$, there exists a $G$-Galois cover $\phi : Y \to P^1_k$ with group $G$ which is branched at only one point with inertia group $I$ and conductor $j'$. If $j' \notin j_{\text{min}}(I)$, Theorem 3.4 implies there exists a $G$-Galois cover $\phi_b : Y_b \to P^1_k$ which is branched at only one point with inertia group $I_b \cong \mathbb{Z}/p \rtimes \mu_{m_b} \subset N_G(S)$ and conductor $j_b$ satisfying $j_b/m_b < j'/m'$. We reiterate this process until the inertia group $I_b = \mathbb{Z}/p \rtimes \mu_{m_b}$ and conductor $j_b$ satisfy $j_b/m_b \leq 2 + 1/(p - 1)$, which implies $j_b \in j_{\text{min}}(I)$. The condition on genus($Y$) follows directly from Definition 3.1 and the Riemann–Hurwitz formula. □

Example 1. — Let $p = 11$. The simple group $G = M_{11}$ is quasi-11. The only maximal subgroup containing $\mathbb{Z}/11$ is $\text{PSL}_2(11)$, so $G$ is 11-pure and $N_G(S) = \mathbb{Z}/11 \rtimes \mathbb{Z}/5$. By Theorem 3.5, there exists a $G$-Galois cover $\phi : Y \to P^1_k$ branched at only one point, either having inertia $\mathbb{Z}/11$ and conductor 2 or inertia $N_G(S)$ and conductor $6 \leq j \leq 9$.

References


484